Stability of operators and C_0-semigroups

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To my teachers

Rainer Nagel and Anna M. Vishnyakova

Zusammenfassung in deutscher Sprache

In dieser Arbeit betrachten wir die Potenzen T^n eines linearen, beschränkten Operators T und stark stetige Operatorhalbgruppen $(T(t))_{t\geq 0}$ auf einem Banachraum X. Dafür suchen wir nach Bedingungen, die "Stabilität" garantieren, d.h.

$$\lim_{n\to\infty} T^n = 0 \qquad \text{bzw.} \qquad \lim_{t\to\infty} T(t) = 0$$

bezüglich einer der natürlichen Topologie. Dazu gehen wir wie folgt vor.

In Kapitel 1 stellen wir die (nichttrivialen) funktionalanalytischen Methoden zusammen, wie z.B. das Jacobs–Glicksberg–de Leeuw Zerlegungstheorem, spektrale Abbildungssätze und eine inverse Laplacetransformation.

In Kapitel 2 diskutieren wir den "zeitdiskreten" Fall und beschreiben zuerst polynomiale Beschränktheit und Potenzbeschränkheit eines Operators T. In Abschnitt 2 wird die Stabilität bezüglich der starken Operatortopologie behandelt. Schwache und fast schwache Stabilität wird in den Abschnitten 3, 4 und 5 untersucht und durch abstrakte Charakterisierungen und konkrete Beispiele erläutet. Wir zeigen insbesondere, dass eine "typische" Kontraktion sowie ein "typischer" unitärer oder isometrischer Operator auf einem unendlich-dimensionalen separablen Hilbertraum fast schwach aber nicht schwach stabil ist.

Analog gehen wir in Kapitel 3 für eine C_0-Halbgruppe $(T(t))_{t\geq 0}$ vor. Zunächst wird Beschränktheit bzw. polynomiale Beschränktheit über die Resolvente des Generators oder den Kogenerator charakterisiert. Ein kurzes Resumé über gleichmäßige Stabilität folgt in Abschnitt 2. Für stark stabile Halbgruppen werden die klassischen Sätze von Foiaş–Sz.-Nagy and Arendt–Batty–Lyubich–Vũ zitiert und ergänzt. In den Abschnitten 4 bis 6 behandeln wir schwach stabile und fast schwach stabile Halbgruppen. Neben unterschiedlichen Charakterisierungen geben wir neue konkrete und abstrakte Beispiele (in Form von Kategoriensätzen) an.

Contents

Introduction

The real understanding involves, I believe, a synthesis of the discrete and continuous ...
L. Lovász, Discrete and Continuous: two sides of the same?

Systems evolving in time ("dynamical systems" for short) can be modeled using a discrete or a continuous time scale. The discrete model leads to a map φ and its powers φ^n on the state space Ω, while the continuous model is given by a (semi)flow $(\varphi_t)_{t\geq 0}$ on Ω. In the first situation, at least in finite dimensions, methods from discrete mathematics are used, while the second essentially needs analytic tools, e.g., from differential equations. This has the effect that frequently the common structure of the results gets out of sight. In this respect we want to quote L. Lovasz:

"There is a deep division (or at least so it appears) between the Continuous and Discrete Mathematics. ... How much we could lose if we let this chasm grow wider, and how much we can gain by building bridges over it." László Lovász, One Mathematics (1998)

In this thesis we study both discrete and continuous linear dynamical systems in Banach spaces and concentrate on "stability" of these systems. More presicely, we call a bounded linear operator or a C_0-semigroup $(T(t))_{t\geq 0}$ "stable", if

$$\lim_{n\to\infty} T^n = 0 \qquad \text{or} \qquad \lim_{t\to\infty} T(t) = 0, \qquad \text{respectively}$$

in some appropriate sense. This property is fundamental for most qualitative theories of linear and nonlinear dynamical systems. Although we treat the discrete (Chapter 2) and the continuous case (Chapter 3) separately, we emphasise the common structure of the results and ideas dispite the often different methods needed for their proofs. We try to give a reasonably complete picture of the situation by mentioning (most of) the relevant results. This helped, by the way, to identify a number of natural open problems. This thesis and future work on these problems can hopefully help to bridge the gap between the discrete and continuous situation, thus presenting two sides of the same reality.

In the following we summarise the content of this thesis.

In Chapter 1 we give an overview on some functional analytic tools needed later. Besides the classical decomposition theorems of Jacobs–Glicksberg–de Leeuw for compact semigroups we discuss several variants of the spectral mapping theorem. We then recall the powerful concept of the cogenerator of a C_0-semigroup. We give an elementary proof of the famous characterisation of Sz.-Nagy and Foiaş (see Theorem 0.39) of operators being cogenerators (see also Katz [**70**]). Finally, we present one of our main tools for the investigation of stability of C_0-semigroups. This is the Laplace inversion formula which can also be considered as an extension of the Dunford functional calculus for the exponential function. The version given in Theorem 0.42 appeared first in Eisner [**24**].

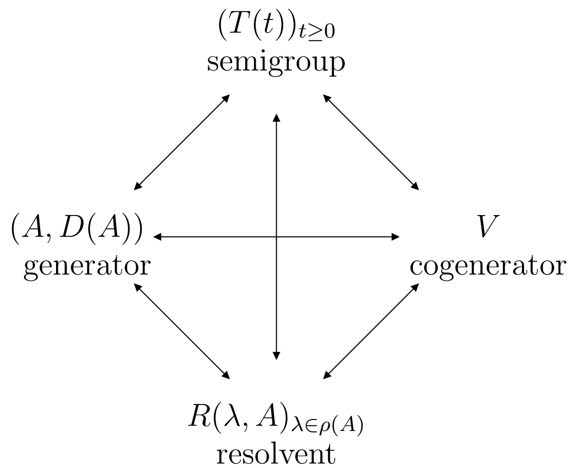
In Chapter 2 we investigate the powers of a bounded linear operator on a Banach space. As a first step, we characterise power boundedness on Hilbert spaces. Theorem 1.9 is a discrete version of the corresponding characterisation of bounded C_0-semigroups due to Gomilko, Shi and Feng (see Theorem 1.11 in Chapter 3). While the proof is more direct than its countinuous counterpart, the result seems to be new. Furthermore, we describe the possible growth of the powers of an operator T with spectral radius 1 (see Example 1.12). Surprisingly, polynomially bounded operators, i.e., operators whose powers T^n grow not faster than some polynomial in n, admit a simple characterisation in terms of the resolvent behaviour near the unit circle (Theorem 1.13).

Stability, i.e., $\lim T^n = 0$ in some sense, is the theme in the rest of this chapter. After some classical results, we discuss strong stability in terms of the resolvent (Theorem 2.13). The condition uses the L^2-norm of the resolvent on circles with radius greater than 1 and its growth if these circles converge to the unit circle. On Hilbert spaces this condition is necessary and sufficient for strong stability. We then give an analogous, but only sufficient condition for weak stability of operators (Theorem 3.12). Again this is a discrete version of the corresponding sufficient condition found by Chill and Tomilov [**15**] for C_0-semigroups. It is an open question whether the condition is also sufficient at least for Hilbert spaces.

We then introduce the concept of almost weak stability (see Definition 4.3) and give various equivalent conditions being partially classical and partially new, see Theorem 4.1. By this theorem we see that almost weak stability of operators is much easier to characterise than weak stability. In particular, if the operator has relatively compact orbits, almost weak stability is equivalent to "no point spectrum on the unit circle". We also present a concrete example of an almost weakly but not weakly stable operator (see Example 4.8).

Although the notion of almost weak stability seems to be very close to weak stability, it turns out to be very different as we see in Section 5. We relate our stability concepts to weakly and strongly mixing flows in ergodic theory and (via the spectral theorem) to Rajchman and non Rajchman measures in harmonic analysis. This leads to classes of almost weakly but not weakly stable operators. We then prove category theorems for almost weakly and weakly stable operators stating that a "typical" (in the sense of Baire) unitary operator, a "typical" isometry, and a "typical" contraction on a separable Hilbert space is almost weakly but not weakly stable (see Theorems 5.7, 5.12 and 5.15, respectively). These results give an operator theoretic counterpart to the classical theorems of Halmos [**51**] and Rohlin [**109**] for weakly and strongly mixing flows on a measure space.

In Chapter 3 we turn our attention to the time continuous case and consider C_0-semigroups $(T(t))_{t\geq 0}$ on Banach spaces. As in the previous chapter we discuss boundedness and stability of a semigroup and try to characterise these properties by the generator, its resolvent, and/or the cogenerator, respectively. We visualise the situation by the following diagram. We start with boundedness of C_0-semigroups. In particular, we discuss



the quite recent characterisation of generators of a bounded C_0-semigroup on a Hilbert space by Gomilko [**45**] and Shi, Feng [**115**]. Their integrability condition on the resolvent on vertical lines (Theorem 1.11) is the key to the resovent type approach for boundedness and stability of discrete and continuous semigroups. We also discuss the connection between power boundedness of the cogenerator and boundedness of the semigroup. Note that it is still open whether these two concepts are equivalent for C_0-semigroups on Banach spaces. We then characterise generators of polynomially bounded semigroups on Hilbert spaces (see Theorem 1.16) in terms of an integral resolvent condition. This is analogous to the result Gomilko, Shi and Feng and generalises their theorem as well as a result of Malejki [**87**] on polynomially bounded C_0-groups. We further give a characterisation of polynomial bounded semigroups in terms of a simple resolvent condition similar to the famous Kreiss condition (Theorem 1.18). This result is analogous to Theorem 1.13 in Chapter 2, however the proof needs more technics. Moreover, we again describe the possible growth of a C_0-semigroup with $\omega_0 = 0$ (see Example 1.15).

We then discuss uniform exponential stability of C_0-semigroups, a notion which is more difficult to characterise than its discrete analogue. Besides the classical results such as Gearhart's generalisation to Hilbert spaces of Liapunov's stability theorem and the Datko–Pazy theorem, we generalise Gearhart's theorem to Banach spaces (see Theorem 2.8) using the Laplace inversion formula. This result seems to be new, but we mention that there are other variants of this theorem, e.g., the one of Kaashoek and Verduyn Lunel [**67**] using a resolvent condition similar to the one appearing in Theorem 2.8.

Strong stability is the subject of the next section. Besides some classical results due to Sz.-Nagy and Foiaş, Lax and Phillips and the more recent theorem of Arendt, Batty [**3**], Lyubich and Vũ [**85**] we discuss the resolvent approach to strong stability developed by Tomilov [**124**]. This leads to a characterisation of strong stability on Hilbert spaces. However, it is not clear whether this characterisaion holds for semigroups on Banach spaces.

We further consider weak stability and its characterisations, partly classical by Foguel [**34**] and Sz.-Nagy, Foiaş [**119, 120**] and partly quite recent (see Theorem 4.14 due to Chill Tomilov [**15**] and Eisner, Farkas, Nagel, Sereny [**25**]). We emphasise that weak stability is much less understood than the strong and uniform analogue with still many open questions. For example, it is unknown for which classes of C_0-semigroups the resolvent condition in Theorem 4.14 is necessary and for which it is not. More generally, it is not clear how to characterise weak stability in terms of the resolvent of the generator. Another open question is to find a connection between weak stability of a C_0-semigroup and weak stability of its cogenerator.

We finally introduce the concept of almost weak stability (see Definition 5.3) closely related to weak stability, but occuring in many more situations. We give various equivalent conditions, being partly classical and partly new (see Theorem 5.1). This theorem is analogous to Theorem 4.1 in Chapter 2. We then give a concrete example of a C_0-semigroup coming from an ordinary differential equation which is almost weakly but not weakly stable.

In the last section we first discuss abstract examples of C_0-semigroups arising naturally in ergodic and measure theory. We finally present category theorems analogous to the discrete ones stating that a "typical" (in the sense of Baire) unitary C_0-group as well as a "typical" isometric C_0-semigroup on a separable infinite-dimensional Hilbert space is almost weakly but not weakly stable (see Theorems 6.7 and 6.11, respectively). The question whether such a category result also holds for contraction semigroups is still open.

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CHAPTER 1

Functional analytic tools

It this chapter we give an overview of some functional analytic tools needed later.

0.1. Preliminaries. We first recall some facts from functional analysis concerning the weak topology and weak compactness in Banach spaces. We refer to Dunford, Schwartz [**23**, Sections V.4-6] and Schaefer [**113**, Section IV.11].

We begin with the Eberlein–Šmulian theorem characterising weak compactness in Banach spaces.

THEOREM 0.1. *(Eberlein–Šmulian) For subsets of a Banach space weak compactness and weak sequential compactness coincide.*

Next, we recall the following version of the classical Banach–Alaoglu theorem.

THEOREM 0.2. *A Banach space is reflexive if and only if its closed unit ball is weakly compact.*

In particular, every bounded set of a Banach space is relatively weakly compact if and only if the Banach space is reflexive.

Another important property of weakly compact sets is expressed in the Kreĭn–Šmulian theorem.

THEOREM 0.3. *(Kreĭn–Šmulian) The closed convex hull of a weakly compact subset of a Banach space is weakly compact.*

The following theorem characterises metrisability of the weak topology.

THEOREM 0.4. *The weak topology on the closed unit ball of a Banach space X is metrisable if and only if the dual space X' is separable.*

We recall that the separability of X' implies the separability of X, while the converse does not always hold. However, for separable Banach spaces relatively weakly compact sets are metrisable by the following theorem.

THEOREM 0.5. *The weak topology on a weakly compact subset of a separable Banach space is metrisable.*

We now introduce basic notations which we will use in the following.

For a bounded linear operator T we denote by $\sigma(T)$, $P_\sigma(T)$, $R_\sigma(T)$, $r(T)$, $\rho(T)$ and $R(\lambda, T)$ its spectrum, point and residual spectrum, spectral radius, resolvent set and resolvent operator at the point $\lambda \in \rho(T)$, respectively.

For a C_0-semigroup $(T(t))_{t\geq 0}$ we will often use the simplified notation $T(\cdot)$. The growth bound of $T(\cdot)$ is denoted by $\omega_0(T)$.

For an (in general unbounded) operator A we denote by $s(A)$ its spectral bound and by

$$s_0(A) := \inf\{a \in \mathbb{R} : \ R(\lambda, A) \text{ is bounded on } \{\lambda : \ \operatorname{Re}\lambda > a\}\}$$

its *pseudo-spectral bound* (also called *abscissa of the uniform boundedness of the resolvent*). Recall that for a C_0-semigroup $T(\cdot)$ with the generator A the relation

$$s(A) \leq s_0(A) \leq \omega_0(T)$$

holds, but both inequalities can be strict, see Engel, Nagel [**31**, Example IV.3.4] and van Neerven [**99**, Example 4.2.9], respectively.

0.2. Relatively compact sets in $\mathcal{L}_\sigma(X)$. In this subsection we characterise relatively weakly compact sets of operators and give some important examples. In the following we denote the space of all bounded linear operators on a Banach space X endowed with the weak operator topology by $\mathcal{L}_\sigma(X)$.

LEMMA 0.6. *[see Engel, Nagel [**31**], Lemma V.2.7] For a set of operators $\mathcal{T} \subset \mathcal{L}(X)$, X a Banach space, the following assertions are equivalent.*

(a) $\mathcal{T}$ is relatively compact in $\mathcal{L}_\sigma(X)$.
(b) $\{Tx : \ T \in \mathcal{T}\}$ is relatively weakly compact in X for all $x \in X$.
(c) $\mathcal{T}$ is bounded, and $\{Tx : \ T \in \mathcal{T}\}$ is relatively weakly compact in X for all x in some dense subset of X.

PROOF. We follow Derndinger, Nagel, Palm [**21**, p. 191] and Engel, Nagel [**31**, pp. 512–514].

The implication $(a) \Rightarrow (b)$ follows directly from the continuity of the mapping $T \mapsto Tx$ for every $x \in X$, while for the proof of the converse implication $(b) \Rightarrow (a)$ based on Tychonoff's theorem we refer to Dugundji [**22**].

The implication $(b) \Rightarrow (c)$ follows immediately from the uniform boundedness principle.

$(c) \Rightarrow (b)$: Take $x \in X$ and $\{x_n\}_{n=1}^\infty \subset D$ converging to x, where D denotes the dense subset of X from (c). By the Eberlein–Šmulian theorem (Theorem 0.1), it is enough to show that every sequence $\{T_n x\}_{n=1}^\infty \subset \mathcal{T}x$ has a weakly convergent subsequence.

Take a sequence $\{T_n x\}_{n=1}^\infty \subset \mathcal{T}x$. For $x_1 \in D$, there exists a subsequence $\{n_{k,1}\}_{k=1}^\infty$ and a vector $z_1 \in X$ such that $T_{n_{k,1}}x \to z_1$ weakly as $k \to \infty$. Analogously, for $x_2 \in D$ there exists a subsequence $\{n_{k,2}\}_{k=1}^\infty$ and a vector $z_2 \in X$ such that $T_{n_{k,2}}x \to z_2$ weakly as $k \to \infty$, and so on.

By the standard diagonal procedure there is a subsequence $\{n_k\}_{k=1}^\infty$ such that $T_{n_k}x_m \to z_m$ weakly as $k \to \infty$ for every $m \in \mathbb{N}$.

We have

$$\begin{aligned}\|z_n - z_m\| &= \sup\{\langle z_n - z_m, y\rangle : \ y \in X',\ \|y\| = 1\} \\ &= \sup\{\lim_{k\to\infty}\langle T_{n_k}x_n - T_{n_k}x_m, y\rangle : \ y \in X',\ \|y\| = 1\} \le C\|x_n - x_m\|,\end{aligned}$$

where $C = \sup\{\|T\| : T \in \mathcal{T}\}$ for every $n, m \in \mathbb{N}$. So $\{z_n\}_{n=1}^{\infty}$ is a Cauchy sequence and therefore converges to some $z \in X$. By the standard 3ε–argument $T_{n_k}x \to z$ weakly as $k \to \infty$. □

We now give some examples of relatively weakly compact subsets of operators.

EXAMPLE 0.7.

(a) On a reflexive Banach space X any norm bounded family $\mathcal{T} \subseteq \mathcal{L}(X)$ is relatively weakly compact by the Banach-Alaoglu theorem.
(b) Let $\mathcal{T} \subseteq \mathcal{L}(L^1(\mu))$ be a norm bounded subset of positive operators on the Banach lattice $L^1(\mu)$, and suppose that $Tu \le u$ for some μ-almost everywhere positive $u \in L^1(\mu)$ and every $T \in \mathcal{T}$. Then $\mathcal{T}$ is relatively weakly compact since the order interval $[-u, u]$ is weakly compact, $\mathcal{T}$-invariant and generates a dense subset (see Schaefer [**114**, Thm. II.5.10 (f) and Prop. II.8.3]).
(c) Let S be a semitopological semigroup, i.e., a (multiplicative) semigroup S which is a topological space such that the multiplication is separately continuous (see Engel, Nagel [**31**, Section V.2]). Consider the space $C(S)$ of bounded, continuous (real- or complex-valued) functions over S. For $s \in S$ define the corresponding rotation operator $(L_s f)(t) := f(s \cdot t)$. A function $f \in C(S)$ is said to be *weakly almost periodic* if the set $\{L_s f : s \in S\}$ is relatively weakly compact in $C(S)$, see Berglund, Junghenn, Milnes [**9**, Def. 4.2.1]. The set of weakly almost periodic functions is denoted by $WAP(S)$. If S is a *compact* semitopological semigroup, then $C(S) = WAP(S)$ holds, see [**9**, Cor. 4.2.9]. This means that for a compact semitopological semigroup S the set $\{L_s : s \in S\}$ is always relatively weakly compact in $\mathcal{L}(C(S))$. We come back to this example later in the proof of Theorem 4.1 and Example 4.8 (Chapter 2) as well as in the proof of Theorem 5.1 and Example 5.8 (Chapter 3).

0.3. Semitopological semigroups and the abstract Jacobs–Glicksberg–de Leeuw decomposition. In this subsection we present a general approach being fundamental in our study of asymptotics using the theory of compact semigroups. We follow the abstract setting given in Engel, Nagel [**31**, Section V.2].

We will call a pair $(\mathcal{S}, \cdot)$ for a set $\mathcal{S}$ with an associative multiplication $\cdot$ an *abstract semigroup*. In the following we will often write only $\mathcal{S}$ and st instead of $s \cdot t$.

DEFINITION 0.8. An abstract semigroup $\mathcal{S}$ is called a *semitopological semigroup* if $\mathcal{S}$ is a topological space such that the multiplication is separately continuous, i.e., such that the maps $s \mapsto st$ and $s \mapsto ts$ are continuous for every $t \in \mathcal{S}$. If $\mathcal{S}$ is in addition compact, then we call S a *compact (semitopological) semigroup*.

The following classical theorem describes the structure of compact commutative semigroups.

THEOREM 0.9. *(Abstract Jacobs–Glicksberg–de Leeuw decomposition) Let $\mathcal{S}$ be a compact commutative semitopological semigroup. Then the following assertions hold.*

(a) $\mathcal{S}$ contains a unique minimal ideal $\mathcal{K}$ with idempotent q and $\mathcal{K} = q\mathcal{S}$.
(b) The minimal ideal $\mathcal{K}$ is a compact topological group. In particular, the multiplication and the inverse map are continuous.

We now consider the case where $\mathcal{S}$ is a subsemigroup of $\mathcal{L}(X)$ (X a Banach space) considered with the usual multiplication. The typical examples for topologies which are considered on $\mathcal{S}$ which make it a semitopological semigroup are the weak, strong and norm operator topologies.

If $\mathcal{S}$ is compact with respect to one of these topologies, the idempotent element q from Theorem 0.9 is a projection, $X = \ker(q) \oplus \operatorname{rg}(q)$ and both subspaces are $\mathcal{S}$-invariant. If we now restrict $\mathcal{S}$ to $\ker(q)$, then the minimal ideal of the restricted semigroup consists of the zero operator and hence

$$0 \in \mathcal{S}|_{\ker(q)}.$$

This will be the basic property for the study of stability of operators and C_0-semigroups.

0.4. Operators having relatively weakly compact orbits. We now present the mean ergodic theorem and apply the abstract setting above to discrete semigroups.

DEFINITION 0.10. An operator T on a Banach space X *has relatively weakly compact orbits* if the set $\mathcal{T} := \{T^n : n \in \mathbb{N}\}$ satisfies one of the equivalent conditions in Lemma 0.6.

A first asymptotic property of operators having relatively weakly compact orbits is the classical mean ergodic theorem.

THEOREM 0.11. *(Mean ergodic theorem, see, e.g., Yosida* [**137**, Theorem VIII.3.2], *Nagel* [**94**]*)*

Let T be a bounded operator on a Banach space X with relatively weakly compact orbits. Then T is mean ergodic, *i.e., we have strong convergence of the Cesàro means*

$$\lim_{n\to\infty} \frac{1}{n+1} \sum_{k=0}^{n} T^k x = Px \quad \textit{for all } x \in X, \tag{1}$$

where $P \in \mathcal{L}(X)$ is a projection onto $\operatorname{Fix}(T) := \ker(I - T)$.

The projection P is called the associated *mean ergodic projection*.

For mean ergodic operators (in particular, for operators with relatively weakly compact orbits) one has the following decomposition of the space X, see Yosida [**137**, Section VIII.3].

PROPOSITION 0.12. *Let T be a mean ergodic operator on a Banach space X with mean ergodic projection P satisfying* $\sup_{n\in\mathbb{N}} \|T\| < \infty$. *Then* $\ker P = \overline{\mathrm{rg}(I-T)}$ *and therefore*

$$X = \mathrm{Fix}(T) \oplus \overline{\mathrm{rg}(I-T)}$$

holds. Moreover, the projection P can be also described as

$$Px = \lim_{r\to 1+} (r-1)R(r,T)x \quad \text{for all } x \in X.$$

Note that the last formula follows from the Neumann series representation for the resolvent, formula (1) and the equivalence of the Abel and Cesàro means (see, e.g., Hardy [**57**], p. 136).

REMARK 0.13. There are many criteria to check mean ergodicity. For example, by Nagel [**93**] a power bounded operator is mean ergodic if and only if

$$\mathrm{Fix}(T) \text{ separates } \mathrm{Fix}(T').$$

In particular, a power bounded operator T is mean ergodic with the mean ergodic projection zero if and only if $\mathrm{Fix}(T') = \{0\}$.

We now present a consequence of the abstract decomposition presented in Subsection 0.3 being one of the basic results in the theory of operators having relatively weakly compact orbits.

THEOREM 0.14. *(Jacobs–Glicksberg–de Leeuw decomposition) Let X be a Banach space and let $T \in \mathcal{L}(X)$ have relatively weakly compact orbits. Then $X = X_r \oplus X_s$, where*

$$X_r := \overline{\mathrm{lin}}\big\{x \in X : \ Tx = \gamma x \text{ for some } \gamma \in \Gamma\big\}$$
$$X_s := \big\{x \in X : 0 \text{ is a weak accumulation point of } \{T^n x : \ n = 0,1,2,\ldots\}\big\}.$$

REMARK 0.15. : If $P_\sigma(T) \cap \Gamma \subset \{1\}$, where Γ denotes the unit circle, then the mean ergodic projection coincides with the projection from the Jacobs–Glicksberg–de Leeuw decomposition.

Finally, we state the decomposition theorem of Jacobs–Glicksberg–de Leeuw for the strong operator topology.

THEOREM 0.16. *Let X be a Banach space and $T \in \mathcal{L}(X)$ such that for every $x \in X$ the orbit $\{T^n x : n = 0,1,2\ldots\}$ is relatively compact in X. Then $X = X_r \oplus X_s$ for*

$$X_r := \ \overline{\mathrm{lin}}\{x \in X : \ Tx = \gamma x \text{ for some } \gamma \in \Gamma\},$$
$$X_s := \ \{x \in X : \ \|T^n x\| \to 0 \text{ as } n \to \infty\}.$$

The proof follows directly from Theorem 0.14, the fact that for compact sets weak convergence implies convergence and Lemma 2.4 (Chapter 2).

REMARK 0.17. Note that the relative compactness of every orbit is also a necessary condition for the decomposition given in Theorem 0.16 for power bounded operators by Lemma 0.6.

0.5. Jacobs–Glicksberg–de Leeuw decomposition for C_0-semigroups. Analogously to the previous subsection we apply the abstract setting of Section 0.3 to C_0-semigroups and present the continuous version of the mean ergodic theorem. Note that the results of this section are completely analogous to the discrete case considered in Section 0.4.

Definition 0.18. A C_0-semigroup $T(\cdot)$ on a Banach space X is called *relatively weakly compact* if the set $\mathcal{T} := \{T(t) : t \geq 0\} \subset \mathcal{L}(X)$ satisfies one of the equivalent conditions in Lemma 0.6.

The classical mean ergodic theorem gives information about the asymptotic behaviour of relatively weakly compact C_0-semigroups. We refer to Engel, Nagel [**31**, Section V.4] or Arendt, Batty, Hieber, Neubrander [**4**] for the proofs and a more detailed study of mean ergodic C_0-semigroup.

Theorem 0.19. *(Mean ergodic theorem for C_0-semigroups)*
Let $T(\cdot)$ be a relatively weakly compact semigroup a Banach space X. Then $T(\cdot)$ is mean ergodic*, i.e., we have strong convergence of the Cesàro means*

$$\lim_{t\to\infty} \frac{1}{t}\int_0^t T(s)x\, ds = Px \quad \text{for all } x \in X, \tag{2}$$

where $P \in \mathcal{L}(X)$ is a projection onto $\mathrm{Fix}(T(\cdot)) := \cap_{t\geq 0} \mathrm{Fix}(T(t))$.

Remark 0.20. Relatively weakly compact C_0-semigroups are even *totally ergodic*, i.e., one has strong convergence of the Cesàro means

$$\lim_{t\to\infty} \frac{1}{t}\int_0^t e^{is\tau}T(\tau)x\, d\tau = P_s x \quad \text{for all } x \in X,$$

for every $s \in \mathbb{R}$. This follows from the fact that the semigroup $(e^{ist}T(t))_{t\geq 0}$ is relatively weakly compact as well for every $s \in \mathbb{R}$.

The projection P is called the *mean ergodic projection* associated with $T(\cdot)$. As a consequence of mean ergodicity, one can prove the following decomposition of X.

Proposition 0.21. *Let $T(\cdot)$ be a mean ergodic C_0-semigroup on a Banach space X with generator A satisfying* $\lim_{t\to\infty} \frac{\|T(t)\|}{t} = 0$. *Then* $\ker P = \overline{\mathrm{rg}(A)}$ *and therefore the decomposition*

$$X = \ker(A) \oplus \overline{\mathrm{rg}(A)}$$

holds. Moreover, the projection P can also be obtained as

$$Px = \lim_{a\to 0+} aR(a, A)x \quad \text{for all } x \in X.$$

Remark 0.22. Note that one can verify the mean ergodicity of a bounded semigroup quite easily. Indeed, a bounded C_0-semigroup on a Banach space with generator A is mean ergodic if and only if

$$\ker(A) = \mathrm{Fix}(T(\cdot)) \quad \text{separates} \quad \ker(A') = \mathrm{Fix}(T'(\cdot)).$$

For the proof see Engel, Nagel [**31**, Thm. V.4.5]. Note further that for the generator A of a bounded C_0-semigroup

$$P_\sigma(A) \cap i\mathbb{R} \subset P_\sigma(A') \cap i\mathbb{R}$$

holds (see Engel, Nagel [**31**, Lemma V.2.20]). In particular, a bounded C_0-semigroup is mean ergodic with the mean ergodic projection zero if and only if its generator A satisfies the spectral condition $0 \notin P_\sigma(A')$.

The following theorem being a special case of the abstract Jacobs–Glicksberg–de Leeuw decomposition is one of the basic results in the stability theory of (relatively weakly compact) C_0-semigroups.

THEOREM 0.23. *(Jacobs–Glicksberg–de Leeuw decomposition for C_0-semigroups, see Engel, Nagel* [**31**]*, Thm. V.2.8) Let X be a Banach space and $T(\cdot)$ be a relatively weakly compact C_0-semigroup on X. Then $X = X_r \oplus X_s$, where*

$$\begin{aligned} X_r &:= \overline{\text{lin}}\big\{x \in X : \ T(t)x = e^{i\alpha t}x \text{ for some } \alpha \in \mathbb{R} \text{ and all } t \geq 0\big\} \\ X_s &:= \big\{x \in X : \ 0 \text{ is a weak accumulation point of } \{T(t)x : \ t \geq 0\}\big\}. \end{aligned}$$

We refer to Arendt, Batty, Hieber, Neubrander [**4**, Theorem 5.4.11] for an individual version of the above theorem.

REMARK 0.24. If $P_\sigma(A) \cap i\mathbb{R} = \{0\}$, then the mean ergodic projection coincides with the projection from the Glicksberg–Jacobs–de Leeuw decomposition.

We now state the decomposition theorem of Jacobs–Glicksberg–de Leeuw for C_0-semigroups with respect to the strong operator topology.

THEOREM 0.25. *Let X be a Banach space and $T(\cdot)$ be relatively compact in the strong operator topology, i.e., for every $x \in X$ the orbit $\{T(t)x, t \geq 0\}$ is relatively compact in X. Then $X = X_r \oplus X_s$ for*

$$\begin{aligned} X_r &:= \ \overline{\text{lin}}\{x \in X : \ T(t)x = e^{i\alpha t}x \text{ for some } \alpha \in \mathbb{R} \text{ and all } t \geq 0\}, \\ X_s &:= \ \{x \in X : \ \|T(t)x\| \to 0 \text{ as } t \to \infty\}. \end{aligned}$$

The proof follows as in the discrete case directly from Theorem 0.23, the fact that for compact sets weak convergence implies convergence, and Lemma 2.4 (Chapter 2).

For an individual version of Theorem 0.25 see Arendt, Batty, Hieber, Neubrander [**4**, Theorem 5.4.6].

REMARK 0.26. Note that relative compactness of every orbit is also a necessary condition for the decomposition given in Theorem 0.25 by Lemma 0.6 assuming boundedness of the semigroup.

The above decomposition holds, for instance, for bounded C_0-semigroups with generator having compact resolvent or for eventually compact C_0-semigroups, see Engel, Nagel [**31**, Corollary V.2.15].

0.6. Spectral mapping theorems. For the study of stability of C_0-semigroups it is very important to find a useful characterisation of the growth bound of a C_0-semigroup in terms of the generator. In particular, the equality

$$s(A) = \omega_0(T)$$

is of special interest. Since it does not hold in general, one looks for sufficient conditions implying the equality. A class of such conditions is given by so called spectral mapping theorems which are the subject of this subsection.

We begin with the classical spectral mapping theorem, the most natural one coming from the philosophy that a C_0-semigroup is an exponential function of its generator.

DEFINITION 0.27. Let $T(\cdot)$ be a C_0-semigroup on a Banach space X with generator A. We will say that the semigroups satisfies *the spectral mapping theorem* (or *the spectral mapping property*) if

$$\sigma(T(t)) \setminus \{0\} = e^{t\sigma(A)} \quad \text{for every } t > 0.$$

The spectral mapping theorem always holds for the point and residual spectrum, see Engel, Nagel [**31**, Theorem IV.3.7]:

PROPOSITION 0.28. *For a C_0-semigroup $T(\cdot)$ on a Banach space X and its generator A the identities*

$$P_\sigma(T(t)) \setminus \{0\} = e^{tP_\sigma(A)},$$
$$R_\sigma(T(t)) \setminus \{0\} = e^{tR_\sigma(A)}$$

hold for every $t \geq 0$.

The next theorem shows that the spectral mapping theorem holds for several important classes of C_0-semigroups.

THEOREM 0.29. *(see Engel, Nagel* [**31**, Cor. IV.3.12]*) The spectral mapping theorem holds for all eventually norm continuous C_0-semigroups, in particular for the following classes of semigroups:*

(i) uniformly continuous semigroups (or, equivalently, for semigroups with bounded generators),
(ii) eventually compact semigroups,
(iii) analytic semigroups,
(iv) eventually differentiable semigroups.

However, already for multiplication semigroups the spectral mapping theorem can fail.

EXAMPLE 0.30. Let $T(\cdot)$ be the multiplication semigroup on l^2, i.e.,

$$T(x_1, x_2, \ldots) = (e^{tq_1}x_1, e^{tq_2}x_2, \ldots)$$

for the sequence $(q_n)_{n=1}^\infty$ given by $q_n := \frac{1}{n} + in$. Then

$$1 \in \sigma(T(2\pi)) = \overline{\{e^{2\pi/n} : n \in \mathbb{N}\}},$$

while

$$1 \notin \{e^{2\pi/n} : n \in \mathbb{N}\} = e^{t\sigma(A)}.$$

So the spectral mapping theorem does not hold.

In the following, we mention some weaker forms of a spectral mapping theorem.

DEFINITION 0.31. Let $T(\cdot)$ be a C_0-semigroup on a Banach space X with generator A. We say that the semigroups satisfies *the weak spectral mapping theorem* (or *the weak spectral mapping property*) if

$$\sigma(T(t)) \setminus \{0\} = \overline{e^{t\sigma(A)}} \setminus \{0\} \quad \text{for every } t > 0.$$

Every multiplication semigroup on the spaces $C_0(\Omega)$ and $L^p(\Omega, \mu)$ satisfies the weak spectral mapping theorem, see Engel, Nagel [**31**, Prop. IV.3.13]. Consequently, we obtain (by the spectral theorem) that every semigroup of normal operators on a Hilbert space has the weak spectral mapping property.

Moreover, the weak spectral mapping theorem holds for every bounded C_0-group on a Banach space X, see Engel, Nagel [**31**, Prop. IV.3.13], and, more generally, for every C_0-group with so-called non-quasianalytic growth, see Huang [**61**] and Huang, Nagel [**96**] for details.

DEFINITION 0.32. Let $T(\cdot)$ be a C_0-semigroup on a Banach space X with generator A. We say that the semigroup satisfies *the weak circular spectral mapping theorem* (or *the weak circular spectral mapping property*) if

$$\Gamma \cdot \sigma(T(t)) \setminus \{0\} = \Gamma \cdot \overline{e^{t\sigma(A)}} \setminus \{0\} = \overline{e^{t\sigma(A)+i\mathbb{R}}} \setminus \{0\} \quad \text{for every } t > 0.$$

Concrete examples of C_0-semigroups satisfying the weak circular spectral mapping theorem arise, e.g., from neutral differential equations, flows on networks and delay equations. We refer to Greiner, Schwarz [**48**], Kramar, Sikolya [**74**] and Bátkai, Eisner, Latushkin [**6**], respectively.

Even the weak circular spectral mapping theorem allows characterisations of stability properties of C_0-semigroups in terms of the spectrum of its generator.

THEOREM 0.33. *For a C_0-semigroup $T(\cdot)$ on a Banach space X with generator A satisfying the weak circular spectral mapping theorem the equality*

$$\omega_0(T) = s(A)$$

holds. Moreover, if $(a + i\mathbb{R}) \cap \sigma(A) = \emptyset$ for some $a \in \mathbb{R}$, then there exist $M, \varepsilon > 0$ such that $X = X_1 \oplus X_2$ and

(1) $\|T(t)x\| \le Me^{(a-\varepsilon)t}$ for every $x \in X_1$ and $t \ge 0$,
(2) $\|T(t)x\| \ge \frac{1}{M}e^{(a+\varepsilon)t}$ for every $x \in X_2$ and $t \ge 0$.

The first assertion follows directly from the weak circular spectral mapping theorem and the formula $r(T(t)) = e^{t\omega_0(T)}$. For an elegant proof of the second part using spectral projection technique see Engel, Nagel [**31**, Prop. V.1.15].

The following theorem gives an exact characterisation of the spectrum of a C_0-semigroup in terms of the generator, see Nagel (ed.) [**95**, Theorems A-III.7.8 and 7.10].

THEOREM 0.34. *(Greiner's Spectral Mapping Theorem) Let $T(\cdot)$ be a C_0-semigroup on a Banach space X with generator A, $\lambda \in \mathbb{C}$ and $t > 0$. Then the following assertions are equivalent.*

(i) $e^{t\lambda} \in \rho(T(t))$.

(ii) $\Lambda := \{\lambda + \frac{2\pi k}{t} : k \in \mathbb{Z}\} \subset \rho(A)$ *and the resolvent of A is Cesáro bounded on Λ, i.e.,*

$$\sup_{N\in\mathbb{N}} \frac{1}{2N+1} \left\| \sum_{k=-N}^{N} R\left(\lambda + \frac{2\pi k}{t}, A\right) \right\| < \infty.$$

In addition, if X is a Hilbert space, then the conditions above are equivalent to

(ii)* $\Lambda := \{\lambda + \frac{2\pi k}{t} : k \in \mathbb{Z}\} \subset \rho(A)$ *and the resolvent of A is bounded on Λ, i.e.,*

$$\sup_{k\in\mathbb{N}} \left\| R\left(\lambda + \frac{2\pi k}{t}, A\right) \right\| < \infty.$$

0.7. Cogenerator. A powerful tool for the investigation of a C_0-semigroup is its cogenerator. The cogenerator can be obtained easily from the generator (see formula (3) below), it is a bounded operator, and, as we will see later, it reflects many properties of the semigroup.

It is defined as follows.

DEFINITION 0.35. Let A generate a C_0-semigroup $T(\cdot)$ on a Banach space X satisfying $1 \in \rho(A)$. The operator V defined by

$$V := (A+I)(A-I)^{-1} \in \mathcal{L}(X)$$

is called the *cogenerator* of $T(\cdot)$.

REMARK 0.36. The identity

$$V = 1 - 2R(1, A) \tag{3}$$

implies that $V - I$ has a densely defined inverse $(V - I)^{-1} = \frac{1}{2}(A - I)$. In particular,

$$A = (V+I)(V-I)^{-1} = 1 + 2(V-I)^{-1}$$

holds, i.e., the generator is also the (negative) Cayley transform of the cogenerator.

Note that the cogenerator determines the generator, and therefore the semigroup, uniquely. As a further consequence of (3) one has the following characterisation of the spectrum of V using the spectral mapping theorem for the resolvent, see Engel, Nagel [**31**, Theorem IV.1.13].

PROPOSITION 0.37. *The spectrum of the cogenerator V is*

$$\sigma(V) \setminus \{1\} = \left\{ \frac{\lambda+1}{\lambda-1} : \lambda \in \sigma(A) \right\}.$$

The same relation holds for the point spectrum, residual point spectrum and approximative point spectrum, respectively.

On Hilbert spaces there are many parallels between the properties of a C_0-semigroup and of its cogenerator. We begin with an easy proposition, see Foiaş, Sz.-Nagy [**120**, p. 141]. The complete equivalence is in Theorem 0.39 below.

PROPOSITION 0.38. *Let $T(\cdot)$ be a contractive C_0-semigroup on a Hilbert space H. Then its cogenerator is a contraction.*

PROOF. Denote by A the generator of $T(\cdot)$ and by V its cogenerator. By the Lumer–Phillips Theorem (see, e.g., Engel, Nagel [**31**, Theorem II.3.15]) A is dissipative and hence

$$\|(A+I)x\|^2 - \|(A-I)x\|^2 = 4\mathrm{Re}\,\langle Ax, x\rangle \le 0 \qquad \forall x \in D(A).$$

Therefore $\|Vx\| = \|(A+I)(A-I)^{-1}x\| \le \|(A-I)(A-I)^{-1}x\| = \|x\|$ for every $x \in H$. □

It is remarkable that for operators on Hilbert spaces there is a very simple characterisation of operators being the cogenerator of a contraction semigroup. This is the following theorem due to Foiaş, Sz.-Nagy [**120**, Theorem III.8.1]. Our proof (see also Katz [**70**]) is much simpler than the original one which uses a special functional calculus to construct the semigroup.

THEOREM 0.39. *Let V be a contraction on a Hilbert space H. Then V is a cogenerator of a contractive C_0-semigroup if and only if $1 \notin P_\sigma(V)$.*

PROOF. Since, by assumption, the operator $I - V$ is injective, we can define

$$A := -(I+V)(I-V)^{-1} \qquad \text{with} \quad D(A) := \mathrm{rg}(I-V).$$

Note that $A = I - 2(I-V)^{-1}$ holds.

We show first that $\mathrm{Re}\,\langle Ax, x\rangle \le 0$ for every $x \in D(A)$. Indeed, for $x \in D(A)$ and $y := (V-I)^{-1}x$ we have

$$\begin{aligned} \langle Ax, x\rangle &= \langle (I+V)(V-I)^{-1}x, x\rangle = \langle (I+V)y, (V-I)y\rangle \\ &= \|Vy\|^2 - \|y\|^2 + 2i \cdot \mathrm{Im}\,\langle y, Vy\rangle \end{aligned}$$

and therefore $\mathrm{Re}\,\langle Ax, x\rangle \le 0$.

We observe further that $(I-A)^{-1} = \frac{1}{2}(I-V)$ and therefore $1 \in \rho(A)$. Moreover, since V is mean ergodic, we have by Proposition 0.12 $\overline{\mathrm{rg}(I-V)} = H$, so the operator A is densely defined.

The assertion follows now directly from the Lumer–Phillips Theorem (see, e.g., Engel, Nagel [**31**, Theorem II.3.15]). □

Further, many properties of a contractive semigroup on a Hilbert space can be seen from its cogenerator. The following is again due to Foiaş, Sz.-Nagy, see [**120**, Sections III.8–9].

THEOREM 0.40. *Let $T(\cdot)$ be a contractive C_0-semigroup on a Hilbert space with cogenerator V. Then $T(\cdot)$ is normal, self-adjoint, isometric or unitary if and only if V is normal, self-adjoint, isometric or unitary, respectively.*

Note that the equivalence of all except the isometry property follows from the spectral theorem in its multiplicator form.

The cogenerator approach allows to transfer many properties of single operators to C_0-semigroups in a short and elegant way. However, it is not yet clear how this approach extends to C_0-semigroups on general Banach spaces.

0.8. Inverse Laplace transform formula for C_0-semigroups. Our main tool for the resolvent approach to stability of C_0-semigroups is an inverse Laplace transform formula for the semigroup.

THEOREM 0.41. *(see Kaashoek, Verduyn Lunel* [**67**]*, van Neerven* [**99**, Thm.1.3.3]*) Let $T(\cdot)$ be a C_0-semigroup on a Banach space X with generator A. Then*

$$\begin{aligned} T(t)x &= \lim_{N\to\infty} \frac{1}{2\pi N} \int_{-N}^{N} e^{(a+is)t} R(a+is,A)x ds \\ &= \lim_{N\to\infty} \frac{1}{2\pi N t} \int_{-N}^{N} e^{(a+is)t} R^2(a+is,A)x ds \end{aligned}$$

holds for all $a > s_0(A)$, $t > 0$ and $x \in X$.

We now consider a linear densely defined operator A satisfying $s_0(A) < \infty$ for which it is not known whether it is a generator or not.

The basis of our approach is the following condition

$$\langle R(a+i\cdot,A)^2 x, y\rangle \in L^1(\mathbb{R}) \quad \text{for all } x \in X,\ y \in X', \tag{4}$$

where $a > s_0(A)$. Indeed, this property allows us to construct the inverse Laplace transform of the resolvent of the operator A which actually yields a semigroup. Note that this semigroup need not be strongly continuous. (See also Kaiser, Weis [**68**] for a related result.)

THEOREM 0.42. *(Laplace inversion formula, see Eisner* [**24**]*) Let A be a densely defined linear operator on a Banach space X satisfying $s_0(A) < \infty$ and assume that the condition (4) holds for all $a > s_0(A)$. Then the bounded linear operators defined by $T(0) = I$ and*

$$T(t)x := \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{(a+is)t} R(a+is,A)x ds \tag{5}$$

$$= \frac{1}{2\pi t} \int_{-\infty}^{\infty} e^{(a+is)t} R(a+is,A)^2 x ds \qquad \text{for } t > 0, \tag{6}$$

where the improper integrals converge in norm, are independent of $a > s_0(A)$. In addition, the family $(T(t))_{t\geq 0}$ is a semigroup which is strongly continuous on $(0,\infty)$ and satisfies

$$\lim_{t\to 0+} T(t)x = x \quad \text{for all } x \in D(A^2). \tag{7}$$

Finally, we have

$$R(z,A)x = \int_0^\infty e^{-zt} T(t)x\, ds \qquad \text{for all } x \in D(A),\quad \operatorname{Re} z > s_0(A). \tag{8}$$

PROOF. We first prove that for a densely defined operator A with $s_0(A) < \infty$

$$\|R(z,A)x\| \to 0 \quad \text{as } |z| \to \infty,\ \operatorname{Re} z \geq a, \tag{9}$$

holds for every $a > s_0(A)$ and $x \in X$. Indeed, take $a > s_0(A)$. Then there exists a constant $M > 0$ such that $\|R(z,A)\| \leq M$ for all $z \in \mathbb{C}$ with $\operatorname{Re} z \geq a$. Take now $x \in D(A)$ and z with $\operatorname{Re} z \geq a$. Then

$$\|R(z,A)x\| = \frac{1}{|z|}\|x + R(z,A)Ax\| \leq \frac{1}{|z|}(\|x\| + M\|Ax\|),$$

and therefore we have

$$\|R(z,A)x\| \to 0, \quad |z| \to \infty,\ \operatorname{Re} z \geq a$$

for all $x \in D(A)$. Since $D(A)$ is dense in X and the resolvent of A is uniformly bounded on $\{z :\ \operatorname{Re} z \geq a\}$, this is true for all $x \in X$ and property (9) is proved.

Let us define now $T(0) := I$ and

$$T(t)x := \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{(a+is)t} R(a+is,A)x\,ds \tag{10}$$

(the inverse Laplace transform of the resolvent) for all $x \in X$, $t > 0$ and some $a > 0$. We prove that the integral on the right hand side of (10) converges for all $a > 0$ and all $x \in X$ and does not depend on $a > 0$. For a fixed $t > 0$ using $\frac{d}{dz}R(z,A) = -R(z,A)^2$, we obtain for any $r > 0$

$$\begin{aligned} it\int_{-r}^{r} e^{(a+is)t}R(a+is,A)x\,ds &= e^{(a+ir)t}R(a+ir,A)x - e^{(a-ir)t}R(a-ir,A)x \\ &\quad + i\int_{-r}^{r} e^{(a+is)t}R(a+is,A)^2x\,ds. \end{aligned}$$

By (9), the first two summands converge to zero if $r \to +\infty$. Therefore

$$t\int_{-\infty}^{\infty} e^{(a+is)t}R(a+is,A)x\,ds = \int_{-\infty}^{\infty} e^{(a+is)t}R(a+is,A)^2x\,ds, \tag{11}$$

and by condition (4) the integral on the right hand side converges. Indeed, for all $r, R \in \mathbb{R}$, all $x \in X$ and for $B^* = \{y \in X^* : \|y\| = 1\}$ we have, by the uniform boundedness principle, that

$$\begin{aligned} \left\|\int_{r}^{R} e^{ist}R(a+is,A)^2x\,ds\right\| &= \sup_{y\in B^*}\int_{r}^{R}\langle e^{ist}R(a+is,A)^2x, y\rangle ds \\ &\leq \sup_{y\in B^*}\|\langle R(a+i\cdot,A)^2x, y\rangle\|_1 \leq L_1(a)\|x\| \end{aligned}$$

holds for some constant $L_1(a)$ independend on x. This implies the convergence of the integral on the right hand side of (11).

Therefore the integral on the right hand side of (10) converges and

$$T(t)x = \frac{1}{2\pi t}\int_{-\infty}^{\infty} e^{(a+is)t}R(a+is,A)^2x\,ds \tag{12}$$

for every $x \in X$ and $t > 0$. We show next that $T(t)$ does not depend on $a > 0$. Indeed, by Cauchy's theorem we obtain

$$\int_{-r}^{r} e^{(a+is)t} R(a+is, A)^2 x ds - \int_{-r}^{r} e^{(b+is)t} R(b+is, A)^2 x ds$$
$$= - \int_{a}^{b} e^{\tau+ir} R(\tau+ir, A)^2 x d\tau + \int_{a}^{b} e^{\tau-ir} R(\tau-ir, A)^2 x d\tau$$

for all $a, b > s_0(A)$. By (9) and by $a, b > s_0(A)$ the right hand side converges to zero if $r \to +\infty$. So we have proved that $T(t)$ does not depend on $a > 0$ and formula (12) holds.

From (12) we obtain

$$|\langle T(t)x, y\rangle| \leq \frac{e^{at}}{2\pi t} \|\langle R(a+i\cdot, A)^2 x, y\rangle\|_1 \tag{13}$$

and, by the uniform boundedness principle, each $T(t)$ is a bounded linear operator satisfying

$$\|T(t)\| \leq \frac{Ce^{at}}{t}, \quad t > 0, \tag{14}$$

for some constant C depending on $a > s_0(A)$.

As in Kaiser, Weis [**68**, Lemma 4.2] we obtain that $T(t+s)x = T(t)T(s)x$ for all $x \in D(A^4)$. Since $D(A^4)$ is dense (see, e.g., Engel, Nagel [**31**, pp. 53–54]), the semigroup law holds for all $x \in X$. Let us prove that (8) holds for all $x \in D(A)$. Take $x \in D(A)$, $\operatorname{Re} z > s_0(A)$ and $a \in (s_0(A), \operatorname{Re} z)$. Then, by Fubini's Theorem and Cauchy's Integral Theorem for bounded analytic functions on a right half-plane, we have

$$\int_{0}^{\infty} e^{-zt} T(t) x dt$$
$$= \frac{1}{2\pi} \int_{0}^{\infty} e^{-zt} \int_{-\infty}^{\infty} e^{(a+is)t} R(a+is, A) x ds dt$$
$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} \left\{ \int_{0}^{\infty} e^{(a+is-z)t} dt \right\} \frac{R(a+is, A)Ax + x}{a+is} ds$$
$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{R(a+is, A)Ax + x}{(a+is)(z-a-is)} ds = \frac{R(z, A)Ax + x}{z} = R(z, A)x,$$

and equality (8) is proved.

Finally, we show strong continuity of our semigroup on $(0, \infty)$. Since by (14) the semigroup is uniformly bounded on all compact intervals in $(0, \infty)$, it is enough to show that (7) holds for all $x \in D(A^2)$. So take $x \in D(A^2)$ and $a > 0$. By Kaiser, Weis [**68**, Lemma 4.1 and 4.2] we have

$$T(t)x - x = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{(a+is)t} \frac{R(a+is, A)Ax}{a+is} ds$$

and $\|R(a+is, A)Ax\| \leq \frac{c\|A^2 x\|}{1+|a+is|}$ for some constant c. Therefore, by Lebesgue's theorem,

$$\lim_{t \to 0+} (T(t)x - x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{R(a+is, A)Ax}{a+is} ds \tag{15}$$

and the integral on the right hand side converges absolutely.

We now show that

$$\int_{-\infty}^{\infty} \frac{R(a+is,A)Ax}{a+is}ds = 0. \tag{16}$$

Again by Cauchy's Theorem and (9) we have

$$\left\| \int_{-r}^{r} \frac{R(a+is,A)Ax}{a+is}ds \right\| = \left\| \int_{-\pi/2}^{\pi/2} \frac{ire^{i\phi}}{a+re^{i\phi}} R(a+re^{i\phi},A)Ax \; d\phi \right\|$$
$$\leq \int_{-\pi/2}^{\pi/2} \|R(a+re^{i\phi},A)Ax\| d\phi \to 0, \quad r \to \infty.$$

So equality (16) is proved, and (15) implies (7) and the strong continuity of our semigroup on $(0,\infty)$. □

CHAPTER 2

Stability of linear operators

1. Power boundedness

In this section we investigate operators on power boundedness and on the related property of polynomial boundedness.

1.1. Preliminaries. We first introduce power bounded operators and show some easy properties.

DEFINITION 1.1. A linear operator T on a Banach space X is called *power bounded* if $\sup_{n\in\mathbb{N}} \|T^n\| < \infty$.

An immediate necessary spectral condition for power bounded operators is the following.

REMARK 1.2. The spectral radius $r(T)$ of a power bounded operator T on a Banach space X satisfies

$$r(T) = \inf_{n\in\mathbb{N}} (\|T^n\|)^{\frac{1}{n}} \leq 1,$$

and hence $\sigma(T) \subset \{z : \ |z| \leq 1\}$.

Note that $r(T) \leq 1$ does not imply power boundedness as can be seen from $T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ on $\mathbb{C}^2$. Moreover, we refer to Subsection 1.3 for more sophisticated examples and a quite complete description of the possible growth of the powers of an operator satisfying $r(T) \leq 1$, see Example 1.12. However, the case $r(T) < 1$ automatically implies a much stronger assertion.

PROPOSITION 1.3. *Let X be a Banach space and $T \in \mathcal{L}(X)$. The following assertions are equivalent.*

(a) $r(T) < 1$.

(b) $\|T^n\| \underset{n\to\infty}{\longrightarrow} 0$.

(c) T *is* uniformly exponentially stable, *i.e., there exist constants $M \geq 0$, $\varepsilon > 0$ such that $\|T^n\| \leq Me^{-\varepsilon n}$ for all $n \in \mathbb{N}$.*

The proof follows from the formula $r(T) = \lim_{n\in\mathbb{N}} (\|T^n\|)^{\frac{1}{n}}$.

REMARK 1.4. It is interesting that for power bounded operators on separable Banach spaces some more information on the spectrum is known. For example, Jamison proved that for every power bounded operator T on a separable Banach space X the boundary point spectrum $\sigma_P(T) \cap \Gamma$ is countable, see Jamison [**64**]. For more information on this phenomena, see e.g. Ransford [**107**].

We will see later that countability of the entire spectrum on the unit circle plays an important role for strong stability of the operator (see Subsection 2.3).

The following easy lemma is useful to understand power boundedness.

LEMMA 1.5. *Let T be power bounded on a Banach space X. Then there exists an equivalent norm on X such that T becomes a contraction.*

PROOF. Take $\|x\|_1 := \sup_{n\in\mathbb{N}\cup\{0\}} \|T^n x\|$ for every $x \in X$. □

REMARK 1.6. It is a difficult problem in operator theory to characterise those power bounded operators on Hilbert spaces which are similar to a contraction for a Hilbert space norm. Foguel [**35**] showed that this is not always true (see also Halmos [**55**]).

1.2. Characterisation via resolvent. We discuss in this subsection the connection between power boundedness and behaviour of the resolvent near the unit circle.

We begin with the following propostion which is analogous to one implication in the famous Hille–Yosida theorem for C_0-semigroups.

PROPOSITION 1.7. *Let T be a power bounded operator on a Banach space X. Then T satisfies the* strong Kreiss resolvent condition *(also called iterated resolvent condition), i.e., there exists a constant M such that*

$$\|R^n(\lambda, T)\| \leq \frac{M}{(|\lambda|-1)^n} \qquad \text{for all } n \in \mathbb{N} \text{ and } \lambda \text{ with } |\lambda| > 1. \tag{17}$$

It is surprising that the converse implication in Proposition 1.7 is not true, i.e., the discrete analogue of the Hille-Yosida theorem does not hold. For a counterexample with the maximal possible growth of $\|T^n\|$ being equal to $\sqrt{n}$, see Lubich, Nevanlinna [**81**]. For a systematic discussion of the strong Kreiss condition and the related uniform Kreiss condition we refer the reader to Gomilko, Zemánek [**46**], Montes-Rodríguez, Sánchez-Álvarez and Zemánek [**88**], see also Nagy, Zemánek [**97**] and Nevanlinna [**101**].

Note that the question whether the strong Kreiss resolvent condition implies power boundedness for operators on Hilbert spaces is still open.

Recall that condition (17) for $n = 1$ is called the *Kreiss resolvent condition* and plays an important role in numerical analysis. By the celebrated Kreiss matrix theorem, it is equivalent to power boundedness for operators acting on finite-dimensional spaces. On infinite-dimensional spaces this is no longer true, see Subsection 1.3 for details. We also mention here the *Ritt resolvent condition* (also called *Tadmor–Ritt resolvent condition*)

$$\|R(\lambda, T)\| \leq \frac{\text{const}}{|\lambda - 1|} \qquad \text{for all } \lambda \text{ with } |\lambda| > 1$$

which implies power boundedness, see Nagy, Zemánek [**97**]. However, this condition also implies $\sigma(T) \cap \Gamma \subset \{1\}$ and hence is far from being necessary for power boundedness of general operators. We refer to Shields [**116**], Lubich, Nevanlinna [**81**], Nagy, Zemánek

[**97**], Borovykh, Drissi, Spijker [**10**], Nevanlinna [**101**], Spijker, Tracogna, Welfert [**117**], Tsedenbayar, Zemánek [**122**] and Vitse [**125**]–[**127**] for systematic studies of operators satisfying Kreiss and Ritt resolvent conditions.

In the following we consider conditions not involving all powers of the resolvent and characterising power boundedness at least on Hilbert spaces.

We first state an easy but very useful lemma.

LEMMA 1.8. *Let X be a Banach space and $T \in \mathcal{L}(X)$. Then*

$$T^n = \frac{r^{n+1}}{2\pi}\int_0^{2\pi} e^{i\varphi(n+1)}R(re^{i\varphi},T)d\varphi = \frac{r^{n+2}}{2\pi(n+1)}\int_0^{2\pi} e^{i\varphi(n+1)}R^2(re^{i\varphi},T)d\varphi \tag{18}$$

for every $n \in \mathbb{N}$ and $r > r(T)$.

PROOF. The first part of (18) follows directly from the Dunford functional calculus and the second by integration by parts. □

The main result of this subsection is the following theorem which is a discrete analogue of a characterisation of bounded C_0-semigroups due to Gomilko [**45**] and Shi, Feng [**115**].

THEOREM 1.9. *Let X be Banach space and $T \in \mathcal{L}(X)$ with $r(T) \le 1$. Consider the following assertions.*

(a) $\limsup_{r\to 1+}(r-1)\int_0^{2\pi}\|R(re^{i\varphi},T)x\|^2 d\varphi < \infty$ *for all $x \in X$,*
$\limsup_{r\to 1+}(r-1)\int_0^{2\pi}\|R(re^{i\varphi},T')y\|^2 d\varphi < \infty$ *for all $y \in X'$;*

(b) $\limsup_{r\to 1+}(r-1)\int_0^{2\pi}|\langle R^2(re^{i\varphi},T)x,y\rangle| d\varphi < \infty$ *for all $x \in X$, $y \in X'$;*

(c) T is power bounded.

Then (a)⇒(b)⇒(c). Moreover, if X is a Hilbert space, then (a)⇔(b)⇔(c).

PROOF. By the Cauchy-Schwarz inequality we have

$$\begin{aligned}\int_0^{2\pi}|\langle R^2(re^{i\varphi},T)x,y\rangle|d\varphi &= \int_0^{2\pi}|\langle R(re^{i\varphi},T)x, R(re^{i\varphi},T')y\rangle|d\varphi \\ &\le \left(\int_0^{2\pi}\|R(re^{i\varphi},T)x\|^2 d\varphi\right)^{\frac{1}{2}}\left(\int_0^{2\pi}\|R(re^{i\varphi},T')y\|^2 d\varphi\right)^{\frac{1}{2}}\end{aligned}$$

for all $x \in X$, $y \in X'$ and $r > r(T)$. This proves the implication (a)⇒(b).

For the implication (b)⇒(c) take $n \in \mathbb{N}$ and $r > 1$. By Lemma 1.8 we have

$$|\langle T^n x, y\rangle| \le \frac{r^{n+2}}{2\pi(n+1)}\int_0^{2\pi}|\langle R^2(re^{i\varphi})x,y\rangle|d\varphi$$

for every $x \in X$ and $y \in X'$. By (b) and the uniform boundedness principle there exists a constant $M > 0$ such that

$$(r-1)\int_0^{2\pi}|\langle R^2(re^{i\varphi},T)x,y\rangle|d\varphi \le M\|x\|\|y\| \quad \text{for every } x \in X,\ y \in X' \text{ and } r > 1.$$

Therefore we obtain

$$|\langle T^n x, y\rangle| \le \frac{Mr^{n+2}}{2\pi(n+1)(r-1)}\|x\|\|y\|. \tag{19}$$

Take now $r := 1 + \frac{1}{n+1}$. Then $\frac{r^{n+2}}{(n+1)(r-1)} = \left(1 + \frac{1}{n+1}\right)^{n+2} \to e$ as $n \to \infty$ and we obtain by (19) that $\sup_{n\in\mathbb{N}} \|T^n\| < \infty$.

For the second part of the theorem assume that X is a Hilbert space and T is power bounded. Then, by Lemma 1.8 and Parseval's equality,

$$(r-1)\int_0^{2\pi} \|R(re^{i\varphi},T)x\|^2 d\varphi = \frac{r-1}{r}\sum_{n=0}^{\infty}\frac{\|T^n x\|}{r^n} = (1-s)\sum_{n=0}^{\infty} s^n\|T^n x\| \tag{20}$$

for $s := \frac{1}{r} < 1$. Note that the left hand side of (20) is the Abel mean of the sequence $\{\|T^n x\|\}_{n=0}^{\infty}$, so it is bounded by power boundedness of T. This proves the first part of (a).

Analogously we obtain the second part of (a) using the power boundedness of T'. □

REMARKS 1.10. 1.) As can be seen from the proof, Theorem 1.9 can be also formulated for a single weak orbit $\{\langle T^n x, y\rangle : n \in \mathbb{N}\}$. More precisely, for a fixed pair $x \in X$ and $y \in X'$ condition (a) implies (b) and (b) implies boundedness of the corresponding weak orbit and the converse implications hold for Hilbert spaces.

2.) Moreover, one can replace condition (a) by

$$\limsup_{r\to 1+}(r-1)\int_0^{2\pi} \|R(re^{i\varphi},T)x\|^p d\varphi < \infty,$$
$$\limsup_{r\to 1+}(r-1)\int_0^{2\pi} \|R(re^{i\varphi},T')y\|^q d\varphi < \infty$$

for some $p,\ q > 1$ (depending on x and y) with $\frac{1}{p} + \frac{1}{q} = 1$.

The important question to find a useful characterisation of power boundedness on Banach spaces remains open.

1.3. Polynomial boundedness. In this subsection we discuss the related notion of polynomial boundedness which surprisingly is much easier to characterise.

DEFINITION 1.11. A bounded linear operator T on a Banach space X is called *polynomially bounded* if $\|T^n\| \le p(n)$ for some polynomial p and all $n \in \mathbb{N}$.

Without loss of generality we will assume the polynomial to be of the form $p(t) = Ct^d$.

Note that a polynomially bounded operator T again satisfies $r(T) \le 1$. The following example shows that the converse implication is far from being true.

EXAMPLE 1.12. (Operators satisfying $r(T) \le 1$ with non-polynomial growth.) Consider the Hilbert space

$$H := l_a^2 = \left\{ \{x_n\}_{n=1}^{\infty} \subset \mathbb{C} : \sum_{n=1}^{\infty} |x_n|^2 a_n^2 < \infty \right\}$$

for a positive sequence $\{a_n\}_{n=1}^{\infty}$ satisfying

$$a_{n+m} \le a_n a_m \qquad \text{for all } n, m \in \mathbb{N} \tag{21}$$

with the natural scalar product. On H take the right shift operator $T(x_1, x_2, x_3, \ldots) := (0, x_1, x_2, \ldots)$. Then for $x = (x_1, x_2, \ldots) \in X$ we have by (21)

$$\|T^k x\|^2 = \|(0, \ldots, 0, x_1, x_2, \ldots)\|^2 = \sum_{n=1}^{\infty} a_{n+k}^2 |x_n|^2 \leq a_k^2 \sum_{n=1}^{\infty} a_n^2 |x_n|^2 = a_k^2 \|x\|^2$$

for every $k \in \mathbb{N}$. Moreover, $\|T^k e_1\| = \|e_k\| = a_{k+1} = \frac{a_{k+1}}{a_1}\|e_1\|$, where e_k denotes a sequence having the k-th component equal to 1 and others equal to zero. Therefore we have the norm estimate

$$\frac{a_{k+1}}{a_1} \leq \|T^k\| \leq a_k,$$

which implies that the powers of T have the same growth as the sequence $\{a_n\}_{n=1}^{\infty}$. Now every sequence having the natural property (21) and growing faster than every polynomial but slower than any exponential function with positive exponent yields an operator growing non-polynomially and satisfying $r(T) \leq 1$.

As a concrete example of such sequence consider

$$a_n := (n+5)^{\ln(n+5)} = e^{\ln^2(n+5)}.$$

The assertion about the growth is clear and we only need to check condition (21). For $n, m \geq 6$ we have to prove that $\ln^2(n+m) \leq \ln^2 n + \ln^2 m$. This follows from the following two properties of the function $x \mapsto \ln^2 x$:

$$f(2x) \leq 2f(x), \tag{22}$$

$$f''(x) < 0, \tag{23}$$

satisfied for $x \geq 6$. Indeed, from the conditions above we see that the inequality

$$f(x+y) - f(x) \leq f(y)$$

holds for $x = y$ and for a fixed y the derivative of the left hand side is negative, so the inequality holds for all $x, y \geq 6$. To finish this example we mention that for the function $f : x \mapsto \ln^2 x$ condition (23) is immediate and condition (22) follows from the fact that the inequality $\ln^2(2x) \leq 2\ln^2(x)$ is equivalent to $2x \leq x^{\sqrt{2}}$ which is satisfied for $x \geq 6$. So we constructed an operator whose powers grow as $n^{\ln n}$.

Analogously one can construct an operator with powers growing as $n^{\ln^\alpha n}$ for any $\alpha \geq 1$.

Note further that one can also use the idea of Zabczyk (in the continuous case) using matrices with increasing dimensions (see, e.g., Engel, Nagel [**31**, Counterexample IV.3.4] or the original paper of Zabczyk [**138**]) to construct an operator with spectral radius 1 having non-polynomially growing powers. Note that with this construction one does not have information about the actual growth of the powers. We do not go into the details.

The following characterisation of polynomial boundedness uses the resolvent of T in a neighbourhood of the unit circle, see Eisner, Zwart [**29**]. See also Lubich, Nevanlinna [**81**] for a related result concerning all powers of the resolvent.

THEOREM 1.13. *Let T be a bounded operator on a Banach space X with $r(T) \leq 1$. If*

$$\limsup_{|z|\to 1+} (|z|-1)^d \|R(z,T)\| < \infty \quad \text{for some } d \geq 0, \tag{24}$$

then

$$\|T^n\| \leq Cn^d \quad \text{for some } C > 0 \text{ and all } n \in \mathbb{N}. \tag{25}$$

Moreover, if (25) holds for $d = k$, then (24) holds with $d = k+1$.

PROOF. Assume that condition (24) holds and take $n \in \mathbb{N}$ and $r > 1$. By Lemma 1.8 and (24) we have

$$\|T^n\| \leq \frac{r^{n+1}}{2\pi} \int_0^{2\pi} \|R(re^{i\varphi})\| d\varphi \leq \frac{Mr^{n+1}}{(r-1)^k}$$

for $M := \limsup_{|z|\to 1+} (|z|-1)^d \|R(z,T)\|$. Taking $r := 1 + \frac{1}{n}$ we obtain $\|T^n\| \leq 2Men^k$ and the first part of the theorem is proved.

For the second part we assume that condition (25) holds for $d = k$. Take $n \in \mathbb{N}$, $r > 1$, $\varphi \in [0, 2\pi)$, and $q := \frac{1}{r} < 1$. Then

$$\begin{aligned} \|R(re^{i\varphi}, T)\| &\leq \sum_{n=0}^{\infty} \frac{\|T^n\|}{r^{n+1}} \leq Cq \sum_{n=0}^{\infty} n^k q^n \leq C \sum_{n=0}^{k-1} n^k + C \sum_{n=k}^{\infty} n^k q^n \\ &\leq C \sum_{n=0}^{k-1} n^k + C\tilde{C} q^k \frac{d^k}{dq^k} \sum_{n=0}^{\infty} q^n \leq C \sum_{n=0}^{k-1} n^k + \frac{C\tilde{C} k!}{(1-q)^{k+1}}, \end{aligned}$$

where $\tilde{C}$ is such that $n^k \leq \tilde{C} \cdot n(n-1)\dots(n-k+1)$ for all $n \geq k$. For $k = 0$ we suppose the first sum on the right hand side to be equal to zero. Substituting q by $\frac{1}{r}$ we obtain condition (24) for $d = k+1$. □

REMARK 1.14. Notice that condition (24) for $0 \leq d < 1$ already implies $r(T) < 1$ and hence uniform exponential stability by the inequality $\|R(\lambda, T)\| \geq \frac{1}{\operatorname{dist}(\lambda, \sigma(T))}$. So for $0 \leq d < 1$ Theorem 1.13 does not give the best information about the growth of the semigroup. Nevertheless, for $d = 1$, i.e., for the above mentioned Kreiss resolvent condition, the growth stated in Theorem 1.13 is the best possible and the exponent d in (25) cannot be decreased in general, see Shields [**116**]. For $d > 1$ it is not clear whether Theorem 1.13 gives the best possible growth exponent.

2. Strong stability

In this section we consider a weaker stability concept than the norm stability discussed in Proposition 1.3 and replace uniform by pointwise convergence.

2.1. Preliminaries. We now introduce strongly stable operators and present some fundamental properties of them.

DEFINITION 2.1. An operator T on a Banach space X is called *strongly stable* if $\|T^n x\| \underset{n\to\infty}{\longrightarrow} 0$ for every $x \in X$.

Our aim is to investigate and characterise this property. In particular, one looks for characterisations not involving all powers of the operator.

The following example is, in a certain sense, typical for Hilbert spaces as we will see in Theorem 2.7.

EXAMPLE 2.2. Consider $H := l^2(\mathbb{N}, H_0)$ for a Hilbert space H_0 and $T \in \mathcal{L}(H)$ defined by

$$T(x_1, x_2, x_3, \ldots) := (x_2, x_3, \ldots). \tag{26}$$

The operator T is called the *left shift* on H. It is evident that T is strongly stable.

Analogously, the operator defined by formula (26) is also strongly stable on the spaces $c_0(\mathbb{N}, X)$, $l^p(\mathbb{N}, X)$ for a Banach space X and $1 \le p < \infty$, but not on $l^\infty(\mathbb{N}, X)$.

The following property of strongly stable operators is an easy consequence of the uniform boundedness principle.

REMARK 2.3. Every strongly stable operator T on a Banach space X is power bounded which in particular implies $\sigma(T) \subset \{z : |z| \le 1\}$. Moreover, the property $P_\sigma(T) \cap \Gamma = \emptyset$ is necessary for strong stability.

We now present an elementary property which is very helpful to show strong stability.

LEMMA 2.4. *Let X be a Banach space, $T \in \mathcal{L}(X)$ be power bounded and $x \in X$.*

(a) If there exists a subsequence $\{n_k\}_{k=1}^\infty \subset \mathbb{N}$ such that $\|T^{n_k}x\| \to 0$, then $\|T^n x\| \to 0$.
(b) If T is a contraction, then there exists $\lim_{n\to\infty} \|T^n x\|$.

PROOF. The second assertion follows from the fact that for a contraction the sequence $\{\|T^n x\|\}_{n=1}^\infty$ is non-increasing. For the first one take $\varepsilon > 0$, $M := \sup_{n\in\mathbb{N}} \|T^n\|$ and $k \in \mathbb{N}$ such that $\|T^{n_k}x\| \le \varepsilon$. Then we have

$$\|T^n x\| \le \|T^{n-n_k}\| \|T^{n_k}x\| \le M\varepsilon$$

for every $n \ge n_k$, and (a) proved. □

REMARK 2.5. Assertion (b) in the above lemma is no longer true for power bounded operators. This can be easily seen from the following example. Consider the Hilbert space of all l^2-sequences endowed with the norm

$$\|x\| := \left(\sum_{n=1}^{\infty}\left[|x_{2n-1}|^2 + \frac{1}{4}|x_{2n}|^2\right]\right)^{\frac{1}{2}}.$$

On this space consider the right shift operator which is obviously power bounded. We see that for the vector $e_1 = (1, 0, 0, \ldots)$ we have $\|T^{2n-1}e_1\| = 1$ and $\|T^{2n}e_1\| = \frac{1}{2}$ for every $n \in \mathbb{N}$ which implies

$$\frac{1}{2} = \liminf_{n\to\infty} \|T^n e_1\| \neq \limsup_{n\to\infty} \|T^n e_1\| = 1.$$

Finally, we state a surprising and nontrivial result of Müller [**89**] on the asymptotic behaviour of operators which are not uniformly exponentially stable.

THEOREM 2.6. *[V. Müller] Let T be a bounded linear operator on a Banach space X with $r(T) \geq 1$. For every $\varepsilon \in (0, 1)$ and $\{\alpha_n\}_{n=1}^{\infty} \subset [0, 1]$ converging monotonically to 0 there exists $x \in X$ with $\|x\| = 1$ such that*

$$\|T^n x\| \geq (1-\varepsilon)\alpha_n \quad \text{for every } n \in \mathbb{N}.$$

In other words, the orbits of a strongly but not uniformly exponentially stable operator decrease arbitrary slowly. We refer to Müller [**89**] and [**90**] for more phenomena.

2.2. Hilbert spaces. The following classical result of Foiaş [**33**] and de Branges–Rovnyak [**11**] (see also Sz.-Nagy, Foiaş [**120**, p. 95]) shows that for contractions on Hilbert spaces Example 2.2 represents the general situation

THEOREM 2.7. *Let T be a strongly stable contraction on a Hilbert space H with $r(T) = 1$. Then T is unitarily isomorphic to a left shift, i.e., there is a Hilbert space H_0 and a unitary operator $U : H \to H_1$ for some closed subspace $H_1 \subset l^2(\mathbb{N}, H_0)$ such that UTU^{-1} is the left shift on $l^2(\mathbb{N}, H_0)$.*

We note that for contractions on Hilbert spaces Theorem 2.6 follows from the above theorem since the assertion of Theorem 2.6 obviously holds for a left shift.

An important question is to find necessary and sufficient conditions for strong stability of C_0-semigroups on Banach spaces. Up to now, there is no complete answer to this question. Only on Hilbert spaces there is a resolvent condition for strong stability which we will discuss in Subsection 2.4.

2.3. Sufficient spectral conditions. In this subsection we present sufficient conditions for strong stability in terms of the spectrum of the operator on the unit circle.

The following theorem is the basis for a spectral characterisation of strong stability, see Katznelson, Tzafriri [**71**].

THEOREM 2.8. *(Katznelson-Tzafriri, 1986) Let T be power bounded operator on a Banach space X. Then $\|T^{n+1} - T^n\| \to 0$ as $n \to \infty$ if and only if $\sigma(T) \cap \Gamma \subset \{1\}$.*

For a short and elegant proof of this theorem see Vũ [**128**].

An immediate corollary concerning strong stability is the following.

COROLLARY 2.9. *Let T be power bounded operator on a Banach space X with $\sigma(T) \cap \Gamma \subset \{1\}$. Then $\|T^n x\| \to 0$ as $n \to \infty$ for every $x \in \overline{\mathrm{rg}(I-T)}$.*

On the basis of the Katznelson–Tzafriri theorem, Arendt, Batty [**3**] (in the continuous case) and independently Lyubich, Vũ [**85**] proved the following beautiful result. We remark that the authors used completely different methods.

THEOREM 2.10. *(Arendt–Batty–Lyubich–Vũ, 1988) Let T be power bounded on a Banach space X. Assume that*

(i) $P_\sigma(T') \cap \Gamma = \emptyset$;
(ii) $\sigma(T) \cap \Gamma$ is countable.

Then T is strongly stable.

Note that for operators with relatively weakly compact orbits (in particular, for power bounded operators on reflexive Banach spaces) condition (i) is equivalent to $P_\sigma(T) \cap \Gamma = \emptyset$.

REMARK 2.11. As one of many possible applications of the above theorem we present the following stability result for positive operators: Let T be a positive power bounded operator on a Banach lattice. Then T is strongly stable if $P_\sigma(T') \cap \Gamma = \emptyset$ and $\sigma(T) \cap \Gamma \neq \Gamma$. Note that in this case the boundary spectrum $\sigma(T) \cap \Gamma$ is necessarily a finite union of roots of unity. The proof follows from the Perron–Frobenius theory stating that the boundary spectrum of a positive operator is multiplicatively cyclic (see Schaefer [**114**, Section V.4]) and Theorem 2.10.

The following result is a generalisation of the Arendt–Batty–Lyubich–Vũ theorem for completely non-unitary contractions on Hilbert spaces (for the definition of completely non-unitary operators see Remark 3.9).

THEOREM 2.12. *(Foiaş and Sz.-Nagy, see* [**120**, Prop. II.6.7]*) Let T be a completely non-unitary contraction on a Hilbert space H. If*

$$\sigma(T) \cap \Gamma \text{ has Lebesgue measure } 0,$$

then T and T^* are both strongly stable.

See also Kérchy, van Neerven [**72**] for related results.

2.4. Characterisation via resolvent. In this subsection we pursue a resolvent approach to stability of operators introduced by Tomilov [**124**].

Our main result is the following being a discrete analogue to the spectral characterisation given by Tomilov. For related results and discussion we refer to Tomilov [**124**].

THEOREM 2.13. *Let X be Banach space and $T \in \mathcal{L}(X)$ with $r(T) \leq 1$ and $x \in X$. Consider the following assertions.*

(a) $\lim_{r\to 1+}(r-1)\int_0^{2\pi}\|R(re^{i\varphi},T)x\|^2 d\varphi = 0$ *and*
$\limsup_{r\to 1+}(r-1)\int_0^{2\pi}\|R(re^{i\varphi},T')y\|^2 d\varphi < \infty$ *for all* $y\in X'$*;*
(b) $\|T^n x\| \underset{n\to\infty}{\longrightarrow} 0$.
Then (a) implies (b). Moreover, if X is a Hilbert space, then (a)⇔(b).

In particular, condition (a) for all $x \in X$ implies strong stability of T and, in the case of Hilbert space, is equivalent to it.

Proof. To prove the first part of the theorem we take $x\in X$, $n\in\mathbb{N}$ and $r>1$. By Lemma 1.8 and the Cauchy-Schwarz inequality we have

$$\begin{aligned}|\langle T^n x, y\rangle| &\le \frac{r^{n+2}}{2\pi(n+1)}\int_0^{2\pi}|\langle R^2(re^{i\varphi})x,y\rangle|d\varphi\\ &\le \frac{r^{n+2}}{2\pi(n+1)}\left(\int_0^{2\pi}\|R(re^{i\varphi},T)x\|^2 d\varphi\right)^{\frac{1}{2}}\left(\int_0^{2\pi}\|R(re^{i\varphi},T')y\|^2 d\varphi\right)^{\frac{1}{2}}\end{aligned}$$

for every $y\in X'$. By (a) and the uniform boundedness principle there exists a constant $M>0$ such that

$$(r-1)\int_0^{2\pi}\|R(re^{i\varphi},T')y\|^2 d\varphi \le M\|y\|^2 \quad \text{for every } y\in X' \text{ and } r>0.$$

Therefore, we obtain

$$\|T^n x\| \le \frac{Mr^{n+2}}{2\pi(n+1)(r-1)}\left((r-1)\int_0^{2\pi}\|R(re^{i\varphi},T)x\|^2 d\varphi\right)^{\frac{1}{2}}. \tag{27}$$

For $r := 1+\frac{1}{n+1}$, we obtain $\frac{r^{n+2}}{(n+1)(r-1)} = \left(1+\frac{1}{n+1}\right)^{n+2} \to e$ as $n\to\infty$, hence $\lim_{n\to\infty}\|T^n x\| = 0$ by (27).

Assume now that X is a Hilbert space and T is strongly stable. Then by Lemma 1.8 and Parseval's equality

$$(r-1)\int_0^{2\pi}\|R(re^{i\varphi},T)x\|^2 d\varphi = (r-1)\sum_{n=0}^{\infty}\frac{\|T^n x\|}{r^{n+1}} = (1-s)\sum_{n=0}^{\infty}s^n\|T^n x\|$$

for $s := \frac{1}{r} < 1$. The left hand side is the Abel mean of $\{\|T^n x\|\}$. Therefore it converges to zero as $s\to 1$ by the strong stability of T. This proves the first part of (a).

The second part of (a) follows from Theorem 1.9. □

By Theorem 1.9 we obtain immediately the following characterisation of strongly stable operators on Hilbert spaces.

Corollary 2.14. *Let T be a power bounded operator on a Hilbert space H and $x\in X$. Then $\|T^n x\|\to 0$ if and only if*

$$\lim_{r\to 1+}(r-1)\int_0^{2\pi}\|R(re^{i\varphi},T)x\|^2 d\varphi = 0. \tag{28}$$

In particular, T is strongly stable if and only if (28) holds for every $x\in H$ or for every x in a dense set of H.

It is an open question whether the converse direction in Theorem 2.13 and the assertion of Corollary 2.14 hold for arbitrary Banach spaces. More generally, it is not clear what kind of resolvent conditions characterise strong stability of operators on Banach spaces.

3. Weak stability

We now consider stability of operators in the weak operator topology. Surprisingly, this property is much more difficult to characterise than strong and uniform stability.

3.1. Preliminaries. We begin with the definition and some properties of weakly stable operators.

DEFINITION 3.1. Let X be a Banach space. An operator $T \in \mathcal{L}(X)$ is called *weakly stable* if $\langle T^n x, y\rangle \underset{n\to\infty}{\longrightarrow} 0$ for every $x \in X$ and $y \in X'$.

Note that by the uniform boundedness principle every weakly stable operator T on a Banach space is power bounded and hence $\sigma(T) \subset \{z : |z| \le 1\}$. Moreover, the spectral conditions $P_\sigma(T)\cap\Gamma = \emptyset$ and $R_\sigma(T)\cap\Gamma = P_\sigma(T')\cap\Gamma = \emptyset$ are necessary for weak stability.

EXAMPLE 3.2. (a) The left and right shifts are weakly stable on the spaces $c_0(\mathbb{Z}, X)$ and $l^p(\mathbb{Z}, X)$ for any Banach space X and $1 \le p < \infty$. Note that these operators are isometries and therefore not strongly stable.

(b) Consider $H := l^2$ and the multiplication operator T given by

$$T(x_n)_{n=1}^\infty := (a_n x_n)_{n=1}^\infty$$

for some bounded sequence $(a_n)_{n=1}^\infty$. Then T is weakly stable if and only if $|a_n| < 1$ for every $n \in \mathbb{N}$. However, in this case T is automatically strongly stable.

(c) The situation becomes different for a multiplication operator on $L^2(\mathbb{R})$ with respect to the Lebesgue measure μ. Let T be defined as $(Tf)(s) := a(s)f(s)$ for some bounded measurable function $a : \mathbb{R} \to X$. Then T is strongly stable if and only if $|a(s)| < 1$ for almost all s, while weak stability is more involved. Indeed, the operator T is weakly stable if and only if $|a(s)| \le 1$ for almost all s and $\int_c^d a^n(s)ds \to 0$ as $n \to \infty$ for every interval $[a, b] \subset \mathbb{R}$ (consider the dense set of linear combinations of characteristic functions). This is, e.g., the case for $a(s) = e^{i\alpha s^\gamma}$, $0 \ne \alpha, \gamma \in \mathbb{R}$.

For more examples see Section 5.

We now present a simple condition implying weak stability and begin with the following definition.

DEFINITION 3.3. A subsequence $\{n_j\}_{j=1}^\infty$ of $\mathbb{N}$ is called *relatively dense* if there exists a number $\ell > 0$ such that for every $n \in \mathbb{N}$ the set $\{n, n+1, \ldots, n+\ell\}$ intersects $\{n_j\}_{j=1}^\infty$ (see Bart, Goldberg [**5**] for the terminology).

For example, $k\mathbb{N} + m$ for natural numbers k and m is a relatively dense subsequence of $\mathbb{N}$.

THEOREM 3.4. *Let X be a Banach space and $T \in \mathcal{L}(X)$. Suppose that $T^{n_j} \to 0$ weakly as $j \to \infty$ for some relatively dense subsequence $\{n_j\}_{j=1}^\infty$. Then T is weakly stable.*

PROOF. Define $\ell := \sup_{n\in\mathbb{N}}(n_{j+1} - n_j)$ which is finite by assumption and fix $x \in X$ and $y \in X'$. For $n \in \{n_j, \dots, n_{j+\ell}\}$ we have

$$\langle T^n x, y\rangle = \langle T^{n-n_j} x, T'^{n_j} y\rangle. \tag{29}$$

Note that $T^{n-n_j}x$ belongs to the finite set $\{x, Tx, \dots, T^\ell x\}$. By assumption $(T')^{n_j} y \to 0$ as $j \to \infty$ in the weak*-topology and (29) implies $\langle T^n x, y\rangle \to 0$. □

REMARK 3.5. We will see later that one cannot drop the relative density in Theorem 3.4 or even replace it by the assumption of density 1, see Section 4.

We now present the following characterisation of weak convergence in terms of (strong) convergence of the Cesàro means of subsequences.

THEOREM 3.6. *Let X be a Banach space and $T \in \mathcal{L}(X)$. Consider the following assertions.*

(i) $T^n x$ converges weakly as $n \to \infty$ for every $x \in X$;
(ii) For every $x \in X$ and every increasing sequence $\{n_k\}_{k=1}^\infty \subset \mathbb{N}$ with positive lower density, the limit

$$\lim_{N\to\infty} \frac{1}{N} \sum_{k=1}^{N} T^{n_k} x \text{ exists in norm for every } x \in X; \tag{30}$$

(iii) Property (30) holds for every $x \in X$ and every increasing sequence $\{n_k\}_{k=1}^\infty \subset \mathbb{N}$.

Then (i) $\Leftrightarrow$ (ii) $\Leftarrow$ (iii), and they all are equivalent provided X is a Hilbert space and T is a contraction.

Assertion (*iii*) in Theorem 3.6 is called the *Blum-Hanson property* and arises in ergodic theory. For an overview on this property and some recent results and applications to some classical problems in operator theory (e.g., to the quasisimilarity problem) see Müller, Tomilov [**92**].

We now present a result of Müller [**90**] on possible decay of weak orbits.

THEOREM 3.7. *(Müller) Let T be an operator on a Banach space X with $r(T) \geq 1$ and $\{a_n\}_{n=1}^\infty$ be a positive sequence satisfying $a_n \to 0$. Then there exist $x \in X$, $y \in X'$ such that*

$$\operatorname{Re}\langle T^{n_j} x, y\rangle \geq a_j \qquad \forall j \in \mathbb{N} \tag{31}$$

holds for some increasing sequence $\{n_j\}_{j=1}^\infty \subset \mathbb{N}$.

Surprisingly, the inequality (31) does not hold for all $n \in \mathbb{N}$ in general, i.e., the weak version of Theorem 2.6 is not true. For an example and other phenomena see Müller [**90**].

3.2. Contractions on Hilbert spaces. In this subsection we present classical decomposition theorems for contractions on Hilbert spaces having direct connection to weak stability.

The first theorem is due to Sz.-Nagy, Foiaş [**119**], see also [**120**].

THEOREM 3.8. *(Sz.-Nagy, Foiaş, 1960) Let T be a contraction on a Hilbert space H. Then H is the orthogonal sum of two T- and T^*-invariant subspaces H_1 and H_2 such that*

(a) H_1 is the maximal subspace on which the restriction T_1 of T is unitary;
(b) the restrictions of T and T^* to H_2 are weakly stable.

We present the proof given by Foguel, see [**34**].

PROOF. Define

$$H_1 := \{x \in H : \ \|T^n x\| = \|T^{*n} x\| = \|x\| \quad \text{for all } n \in \mathbb{N}\}.$$

We first prove that for every $x \in H_1$ and $n \in \mathbb{N}$ one has $T^{*n}T^n x = T^n T^{*n} x = x$. If $x \in H_1$, then $\|x\|^2 = \langle T^n x, T^n x\rangle = \langle T^{*n}T^n x, x\rangle \leq \|T^{*n}T^n x\|\|x\| \leq \|x\|^2$ holds. Therefore, by the equality in the Cauchy-Schwarz inequality and positivity of $\|x\|^2$ we have $T^{*n}T^n x = x$. Analogously one shows $T^n T^{*n} x = x$. On the other hand, every x with these two properties belongs to H_1. So we proved that

$$H_1 = \{x \in H : \ T^{*n}T^n x = T^n T^{*n} x = x \quad \text{for all } n \in \mathbb{N}\} \tag{32}$$

is the maximal (closed) subspace on which T is unitary. The T- and T^*-invariance of H_1 follows from the definition of H_1 and the fact that $T^*T = TT^*$ on H_1.

For (b) take $x \in H_2 := H_1^\perp$. Note that H_2 is T- and T^*-invariant since H_1 is. Suppose that $T^n x$ does not converge to zero as $n \to \infty$. This means that there exists $\varepsilon > 0$ and a subsequence $\{n_j\}_{j=1}^\infty$ such that $|\langle T^{n_j} x, y\rangle| \geq \varepsilon$ for every $j \in \mathbb{N}$.

Since every bounded set in a reflexive Banach space is relatively weakly compact by the Banach-Alaoglu theorem, and since weak compactness on Banach spaces coincides with weak sequential compactness by the Eberlein-Šmulian theorem (Theorem 0.1 in Chapter 1) there exists a weakly converging subsequence of $\{T^{n_j}x\}_{j=1}^\infty$. We will denote this subsequence again by $\{n_j\}_{j=1}^\infty$ and its limit by x_0. To achieve a contradiction we will show below that actually $x_0 = 0$. Since H_2 is T-invariant and closed, x_0 belongs to H_2.

For a fixed $k \in \mathbb{N}$ we have

$$\begin{aligned}\|T^{*k}T^kT^n x - T^n x\|^2 &= \|T^{*k}T^{k+n}x\|^2 - 2\langle T^{*k}T^{k+n}x, T^n x\rangle + \|T^n x\|^2 \\ &\leq \|T^{k+n}x\|^2 - 2\|T^{k+n}x\|^2 + \|T^n x\|^2 = \|T^n x\|^2 - \|T^{k+n}x\|^2.\end{aligned}$$

The right hand side converges to zero as $n \to \infty$ since the sequence $\{\|T^n x\|\}_{n=1}^\infty$ is monotone decreasing and therefore convergent. So $\|T^{*k}T^kT^n x - T^n x\| \to 0$ as $n \to \infty$.

We return now to the above subsequence $\{T^{n_j}x\}_{j=1}^\infty$ converging weakly to x_0. Then $T^{*k}T^kT^{n_j}x \to T^{*k}T^k x_0$ weakly. On the other hand, by the considerations above $T^{*k}T^kT^{n_j}x \to x_0$ weakly and therefore $T^{*k}T^k x_0 = x_0$. Analogously one shows that

$T^kT^{*k}x_0 = x_0$ and hence $x_0 \in H_1$. Since $H_1 \cap H_2 = \{0\}$, we obtain $x_0 = 0$, the desired contradiction.

Analogously, the powers of the restriction of T^* on H_2 converge weakly to zero. □

REMARK 3.9. The restriction of T to the subspace H_2 in Theorem 3.8 is *completely non-unitary* (c.n.u. for short), i.e., there is no nontrivial subspace of H_2 on which the restriction of T becomes unitary. In other words, Theorem 3.8 states that every Hilbert space contraction can be decomposed into unitary and c.n.u. part and the c.n.u. part is weakly stable.

For a systematic study of completely non-unitary operators as well as an alternative proof of Theorem 3.8 using unitary dilation theory see the monograph of Sz.-Nagy and Foiaş [**120**].

On the other hand, we have the following decomposition into the weakly stable and the weakly unstable part due to Foguel [**34**]. We present a simplified proof of it.

THEOREM 3.10. *(Foguel, 1963) Let T be a contraction on a Hilbert space H. Define*

$$W := \{x \in H : \lim_{n\to 0} \langle T^n x, x\rangle = 0\}.$$

Then

$$W = \{x \in H : \lim_{n\to 0} T^n x = 0 \text{ weakly } \} = \{x \in H : \lim_{n\to 0} T^{*n} x = 0 \text{ weakly } \}$$

is a closed T- and T^*-invariant subspace of H and the restriction of T to $W^\perp$ is unitary.

PROOF. We first take $x \in W$ and show that $T^n x \to 0$ weakly. By Theorem 3.8 we may assume that $x \in H_1$. If we take $S := \overline{\text{lin}}\{T^n x : n = 0, 1, 2 \ldots\}$, then it is enough to show that $\langle T^n x, y\rangle \to 0$ as $n \to \infty$ for all $y \in S$, since for all $y \in S^\perp$ we automatically have $\langle T^n x, y\rangle = 0$. For $y := T^k x$ we obtain

$$\langle T^n x, y\rangle = \langle T^{*k}T^n x, x\rangle = \langle T^{n-k}x, y\rangle \to 0 \quad \text{for } k \le n \to \infty,$$

where we used that the restriction of T to H_1 is unitary. From the density of $\{T^n x : n = 0, 1, 2, \ldots\}$ in S, it follows that $\langle T^n x, y\rangle \to 0$ for every $y \in S$ and therefore $T^n x \to 0$ weakly. Analogously, one shows that $T^{*n}x \to 0$ weakly. The converse implication, the closedness and the invariance of W are evident.

The last assertion of the theorem follows directly from Theorem 3.8. □

Combining Theorem 3.8 and Theorem 3.10 we obtain the following decomposition.

THEOREM 3.11. *Let T be a contraction on a Hilbert space H. Then H is the orthogonal sum of three closed T- and T^*-invariant subspaces H_1, H_2 and H_3 such that the restrictions T_1, T_2 and T_3 satisfy*

(1) T_1 is unitary and has no weakly stable orbit;
(2) T_2 is unitary and weakly stable;
(3) T_3 is weakly stable and completely non-unitary.

We see from the above theorem that a characterisation of weak stability for unitary operators is of special importance. For an abstract approach using the spectral theorem see Subsection 5.2. A direct characterisation, not involving the spectral theorem, is still unknown.

3.3. Characterisation via resolvent. In this subsection we present a resolvent approach to weak stability being a discrete analogue of the resolvent approach for C_0-semigroups due to Chill, Tomilov [**15**].

THEOREM 3.12. *Let T be a bounded operator on a Banach space X with $r(T) \leq 1$. Assume that*

$$(r-1)\int_0^{2\pi} |\langle R^2(re^{i\varphi},T)x,y\rangle|\,d\varphi \underset{r\to 1+}{\longrightarrow} 0 \tag{33}$$

holds for some $x \in X$ and $y \in X'$. Then $\langle T^n x, y\rangle \to 0$.

In particular, if condition (33) holds for all $x \in X$ and $y \in X'$, then T is weakly stable.

PROOF. Let $x \in X$ and $y \in X'$ satisfy (33). By formula (18) we have

$$\langle T^n x, y\rangle \leq \frac{r^{n+2}}{2\pi(r-1)(n+1)}(r-1)\int_0^{2\pi} |\langle R^2(re^{i\varphi},T)x,y\rangle|\,d\varphi$$

for all $n \in \mathbb{N}$ and $r > 1$. Taking $r := 1 + \frac{1}{n+1}$ we obtain by (33) that $\langle T^n x, y\rangle \to 0$ as $n \to \infty$. □

REMARK 3.13. A (simple) necessary and sufficient resolvent condition for weak stability is still unknown. In particular, even for unitary operators on Hilbert spaces it is not clear whether condition (33) is necessary.

4. Almost weak stability

In this section we consider a stability concept which is analogous to weak mixing in ergodic theory (see Halmos [**53**]). This notion of stability is weaker and is much easier to investigate than the weak stability. In fact, a full characterisation is available as we will see below. In large parts we modify and extend the treatment in Eisner, Farkas, Nagel, Sereny [**25**].

4.1. Characterisation.

The main result of this section is the following.

THEOREM 4.1. *Let T be an operator on a Banach space X such that $\{T^n : n \in \mathbb{N}\}$ is relatively weakly compact in $\mathcal{L}(X)$. Then the following assertions are equivalent.*

(i) $0 \in \overline{\{T^n x : n \in \mathbb{N}\}}^{\sigma}$ *for every $x \in X$;*
(i') $0 \in \overline{\{T^n : n \in \mathbb{N}\}}^{\mathcal{L}_\sigma}$*;*
(ii) For every $x \in X$ there exists a subsequence $\{n_j\}_{j=1}^{\infty} \subset \mathbb{N}$ such that $T^{n_j}x \to 0$ weakly;
(iii) For every $x \in X$ there exists a subsequence $\{n_j\}_{j=1}^{\infty} \subset \mathbb{N}$ with density 1 such that $T^{n_j}x \to 0$ weakly;
(iv) $\frac{1}{n+1}\sum_{k=0}^{n} |\langle T^k x, y\rangle| \underset{n\to\infty}{\longrightarrow} 0$ *for all $x \in X$ and $y \in X'$;*
(v) $\lim_{r\to 1+}(r-1)\int_0^{2\pi} |\langle R(re^{i\varphi}, T)x, y\rangle|^2 d\varphi = 0$ *for all $x \in X$ and $y \in X'$;*
(vi) $\lim_{r\to 1+}(r-1)R(re^{i\varphi}, T)x = 0$ *for all $x \in X$ and $0 \le \varphi < 2\pi$;*
(vii) $P_\sigma(T) \cap \Gamma = \emptyset$*, i.e., T has no eigenvalues on the unit circle.*

In addition, if X' is separable, then the following conditions are equivalent to the conditions above.

(ii) There exists a subsequence $\{n_j\}_{j=1}^{\infty} \subset \mathbb{N}$ such that $T^{n_j} \to 0$ weakly;*
(iii) There exists a subsequence $\{n_j\}_{j=1}^{\infty} \subset \mathbb{N}$ with density 1 such that $T^{n_j} \to 0$ weakly.*

We recall that the density of a sequence $\{n_j\}_{j=1}^{\infty} \subset \mathbb{N}$ is

$$d := \lim_{n\to\infty} \frac{\#\{j : n_j \le n\}}{n}$$

if the limit exists. (Note that 1 is the greatest possible number here.)

The following elementary lemma (see Petersen [**105**, p. 65]) will be needed in the proof of Theorem 4.1.

LEMMA 4.2. *(Koopman–von Neumann, 1932) For a bounded sequence $\{a_n\}_{n=1}^{\infty} \subset [0,\infty)$ the following assertions are equivalent.*

(a) $\frac{1}{n}\sum_{k=1}^{n} a_k \to 0$ *as $n \to \infty$;*
(b) There exists a subsequence $\{n_j\}_{j=1}^{\infty}$ of $\mathbb{N}$ with density 1 such that $a_{n_k} \to 0$ as $k \to \infty$.

PROOF. (Theorem 4.1).

The implications (i')⇒(i) and (ii)⇒(vii) are trivial.

(i)⇒(ii) follows from the equivalence of weak compactness and weak sequential compactness in Banach spaces (see Theorem 0.1 in Chapter 1).

The implication (vii)⇒(i') is a consequence of Theorem 0.14 in Chapter 1 and the construction in its proof.

Therefore, the equivalences (i)⇔(i')⇔(ii)⇔(vii) are proved.

(vi)⇒(vii): Assume that there exists $0 \neq x \in X$ and $\varphi \in [0, 2\pi)$ such that $Tx = e^{i\varphi}x$. Then we have $\|R(re^{i\varphi})x\| = (r-1)^{-1}\|x\|$ and (vi) does not hold.

The converse implication follows from Proposition 0.12 and the fact that for every $0 \le \varphi < 2\pi$ the operator $e^{i\varphi}T$ has weakly relatively compact orbits as well. So (vi)⇔(vii).

(i')⇒(iii): Take $S := \overline{\{T^n x : n \geq 0\}}^{\mathcal{L}_\sigma(X)} \subset \mathcal{L}(X)$ with the usual multiplication and the weak operator topology. It becomes a compact semi-topological semigroup (for the definition and basic properties of compact semi-topological semigroups see Subsection 0.3 in Chapter 1). By (i') we have $0 \in S$. Define the operator $\tilde{T} : C(S) \to C(S)$ by

$$(\tilde{T}f)(R) := f(TR), \quad f \in C(S),\ R \in S.$$

Note that $\tilde{T}$ is a contraction on $C(S)$.

By Example 0.7 (c) the set $\{f(T(t)\,\cdot) : t \geq 0\}$ is relatively weakly compact in $C(S)$ for every $f \in C(S)$. It means that every set $\{\tilde{T}^n f : n \geq 0\}$ is relatively weakly compact, i.e., $\tilde{T}$ has relatively weakly compact orbits.

Denote by $\tilde{P}$ the mean ergodic projection of $\tilde{T}$. We have $\operatorname{Fix}(\tilde{T}) = \langle \mathbf{1} \rangle$. Indeed, for $f \in \operatorname{Fix}(\tilde{T})$ one has $f(T^n I) = f(I)$ for all $n \geq 0$ and therefore f must be constant. Hence $\tilde{P}f$ is constant for every $f \in C(S)$. By definition of the ergodic projection

$$(\tilde{P}f)(0) = \lim_{n\to\infty} \frac{1}{n+1} \sum_{k=0}^{n} (\tilde{T}^k f)(0) = f(0). \tag{34}$$

Thus we have

$$(\tilde{P}f)(R) = f(0) \cdot \mathbf{1}, \qquad f \in C(S),\ R \in S. \tag{35}$$

Take now $x \in X$. By Theorem 0.5 in Chapter 1 and its proof (see Dunford, Schwartz [**23**], p. 434), the weak topology on the orbit $\{T^n x : n \geq 0\}$ is metrisable and coincides with the topology induced by some sequence $\{y_n\}_{n=1}^\infty \subset X' \setminus \{0\}$. Consider $f_x \in C(S)$ defined by

$$f_x(R) := \sum_{n\in\mathbb{N}} \frac{1}{2^n} \left| \left\langle Rx, \tfrac{y_n}{\|y_n\|} \right\rangle \right|, \qquad R \in S.$$

By (35) we obtain

$$0 = \lim_{n\to\infty} \frac{1}{n+1} \sum_{k=0}^{n} (\tilde{T}^k f_x)(I) = \lim_{n\to\infty} \frac{1}{n+1} \sum_{k=0}^{n} f_x(T^k).$$

Lemma 4.2 applied to the sequence $\{f_x(T^n I)\}_{n=0}^\infty \subset \mathbb{R}_+$ yields a subsequence $(n_j)_{j=1}^\infty$ of $\mathbb{N}$ with density 1 such that

$$f_x(T^{n_j}) \to 0 \quad \text{as } j \to \infty.$$

By definition of f_x and by the fact that the weak topology on the orbit is induced by $\{y_n\}_{n=1}^\infty$ we have that

$$T^{n_j}x \xrightarrow{\sigma} 0 \quad \text{as } j \to \infty,$$

and (iii) is proved.

(iii)⇒(iv) follows directly from Lemma 4.2.

(iv)⇒(vii) is clear.

(iv)⇔(v): We note first that the set $\{T^n : n \geq 0\}$ is bounded in $\mathcal{L}(X)$. Take $x \in X$, $y \in X'$ and let $r > 1$. By (18) and the Parseval's equality we have

$$\int_0^{2\pi} |\langle R(re^{i\varphi}, T)x, y\rangle|^2 \, d\varphi = 2\pi \sum_{n=0}^{\infty} \frac{|\langle T^n x, y\rangle|^2}{r^{n+1}}.$$

We obtain, by the equivalence of Abel and Cesàro limits (see, e.g., Hardy [**57**, p. 136] or van Casteren [**12**]), that

$$\begin{aligned} \lim_{r\to 1+} (r-1) \int_0^{2\pi} |\langle R(re^{i\varphi}, T)x, y\rangle|^2 \, d\varphi &= 2\pi \lim_{s\to 1-} (1-s) \sum_{n=0}^{\infty} s^n |\langle T^n x, y\rangle|^2 \\ &= \pi \lim_{n\to\infty} \frac{1}{n+1} \sum_{k=0}^{n} |\langle T^k x, y\rangle|^2. \end{aligned} \tag{36}$$

Note that for a bounded sequence $\{a_n\}_{n=1}^{\infty} \subset \mathbb{R}_+$ with $C := \sup_{n\in\mathbb{N}} a_n$ we have

$$\left(\frac{1}{C(n+1)} \sum_{k=0}^{n} a_k^2\right)^2 \leq \left(\frac{1}{n+1} \sum_{k=0}^{n} a_k\right)^2 \leq \frac{1}{n+1} \sum_{k=0}^{n} a_k^2$$

for every $n \in \mathbb{N}$, where for the second part we used the Cauchy-Schwarz inequality. This together with (36) gives the equivalence of (iv) and (v).

For the additional part of the theorem suppose X' to be separable. Then so is X, and we can take dense subsets $\{x_n \neq 0 : \; n \in \mathbb{N}\} \subseteq X$ and $\{y_m \neq 0 : \; m \in \mathbb{N}\} \subseteq X'$. Consider the functions

$$f_{n,m} : S \to \mathbb{R}, \qquad f_{n,m}(R) := \left|\left\langle R\tfrac{x_n}{\|x_n\|}, \tfrac{y_m}{\|y_m\|}\right\rangle\right|, \quad n, m \in \mathbb{N},$$

which are continuous and uniformly bounded in $n, m \in \mathbb{N}$. Define the function

$$f : S \to \mathbb{R}, \qquad f(R) := \sum_{n,m\in\mathbb{N}} \frac{1}{2^{n+m}} f_{n,m}(R).$$

Then clearly $f \in C(S)$. Thus, as in the proof of the implication (i')⇒(iii), i.e., using (34) we obtain

$$\frac{1}{n+1} \sum_{k=0}^{n} f(T^n I) \underset{n\to\infty}{\longrightarrow} 0.$$

Hence, applying Lemma 4.2 to the bounded sequence $\{f(T^n)\}_{n=0}^{\infty} \subset \mathbb{R}_+$ gives the existence of a subsequence $\{n_j\}_{j=1}^{\infty}$ of $\mathbb{N}$ with density 1 such that $f(T^{n_j}) \to 0$ as $j \to \infty$. In particular, $|\langle T^{n_j} x_n, y_m\rangle| \to 0$ for all $n, m \in \mathbb{N}$ as $j \to \infty$, which, together with the boundedness of $\{T^n\}_{n=1}^{\infty}$, proves the implication (i')⇒(iii*). The implications (iii*)⇒(ii*)⇒(ii') are straightforward, hence the proof is complete. □

The above theorem shows that the property

"no eigenvalues of T on the unit circle" implies properties like (iii) on the asymptotic behaviour of the orbits of T. Motivated by this we introduce the following terminology.

DEFINITION 4.3. We will call an operator on a Banach space with relatively weakly compact orbits *almost weakly stable* if it satisfies condition (iii) in Theorem 4.1.

HISTORICAL REMARK 4.4. Theorem 4.1 and especially the implication (vii)$\Rightarrow$(iii) has a long history. It goes back to ergodic theory and von Neumann's spectral mixing theorem for flows, see Halmos [**53**], Mixing Theorem, p. 39. This has been generalised to operators on Banach spaces by many authors, see, e.g., Nagel [**94**], Jones, Lin [**65, 66**] and Krengel [**75**], pp. 108–110. Note that the equivalence (vii)$\Leftrightarrow$(iv) for contractions on Hilbert spaces follows from the so-called generalised Wiener theorem, see Goldstein [**43**].

For a continuous analogue of the above characterisation see Theorem 5.1 in Chapter 3.

REMARK 4.5. We emphasise that the conditions appearing in Theorem 4.1 are of quite different nature. Conditions (i)–(iv), (ii*) and (iii*) describe the behaviour of the powers of T, while conditions (v)–(vii) consider the resolvent of T in a neigbourhood of the unit circle. Among them condition (vii) apparently is the simplest to verify.

REMARK 4.6. Surprisingly, the equivalence (i')$\Leftrightarrow$(v) in Theorem 4.1 is a weak analogue of the characterisation of strong stability given in Corollary 2.14.

One also can formulate Theorem 4.1 for single orbits. This is the following result partially due to Jan van Neerven (oral communication).

COROLLARY 4.7. *Let T be an operator on a Banach space X and $x \in X$. Assume that the orbit $\{T^n x : n = 0, 1, 2, \dots\}$ is relatively weakly compact in X and the restriction of T to $\overline{\text{lin}}\{T^n x : n = 0, 1, 2, \dots\}$is power bounded. Then there is a holomorphic continuation of the function $R(\cdot, T)x$ to $\{\lambda : |\lambda| > 1\}$ denoted by $R_x(\cdot)$ and the following assertions are equivalent.*

(i) $0 \in \overline{\{T^n x : n \in \mathbb{N}\}}^{\sigma}$;

(ii) There exists subsequence $\{n_j\}_{j=1}^{\infty} \subset \mathbb{N}$ such that $T^{n_j} x \to 0$ weakly;

(iii) There exists a subsequence $\{n_j\}_{j=1}^{\infty} \subset \mathbb{N}$ with density 1 such that $T^{n_j} x \to 0$ weakly;

(iv) $\frac{1}{n+1} \sum_{k=0}^{n} |\langle T^k x, y\rangle| \underset{n\to\infty}{\longrightarrow} 0$ *for all* $y \in X'$;

(v) $\lim_{r\to 1+}(r-1) \int_0^{2\pi} |\langle R_x(re^{i\varphi}), y\rangle|^2 d\varphi = 0$ *for all* $y \in X'$;

(vi) $\lim_{r\to 1+}(r-1) R_x(re^{i\varphi}) = 0$ *for all* $0 \le \varphi < 2\pi$;

(vii) The restriction of T on $\overline{\text{lin}}\{T^n x : n \in \mathbb{N} \cup \{0\}\}$ has no unimodular eigenvalue.

PROOF. For the first part of the theorem we just define

$$R_x(\lambda) := \sum_{n=0}^{\infty} \frac{T^n x}{\lambda^{n+1}} \quad \text{whenever } |\lambda| > 1.$$

This implies the representation

$$T^n x = \frac{r^{n+1}}{2\pi} \int_0^{2\pi} e^{(n+1)i\varphi} R_x(re^{i\varphi}) d\varphi \quad \text{for all } n \in \mathbb{N}.$$

Denote now by Z the closed linear span of the orbit $\{T^n x : n = 0, 1, 2, \dots\}$. Then Z is a T-invariant closed subspace of X and we can restrict T to it. The restriction, which we will denote by T_Z, has relatively weakly compact orbits by Lemma 0.6 in Chapter 1. The equivalence of the assertions follows from the canonical decomposition $X' = Z' \oplus Z^0$ with $Z^0 := \{y \in X' : \langle z, y\rangle = 0 \text{ for all } z \in Z\}$ and Theorem 4.1 applied to the restricted operator. □

4.2. A concrete example. As we will see from the abstract examples in the next section, almost weak stability does not imply weak stability. In this subsection we present a concrete example of a (positive) operator being almost weakly but not weakly stable. (For definitions and basic theory of positive operators we refer to the monograph of Schaefer [**114**].)

Example 4.8. Consider the operator $T_0(1)$ from Example 5.8 in Chapter 3 being a positive operator on the Banach lattice $C(\Omega)$ for some $\Omega \subset \mathbb{C}$. The relative weak compactness of the orbits follows from the relative weak compactness of the semigroup $T_0(\cdot)$. The almost weak stability of $T_0(1)$ is a consequence of $P_\sigma(T_0(1)) \cap \Gamma = \emptyset$ and Theorem 4.1. Further, the facts that the semigroup $T_0(\cdot)$ is not weakly stable and that $\mathbb{N}$ is relatively dense set in $\mathbb{R}_+$ together with Theorem 4.4 in Chapter 3 imply that the operator $T_0(1)$ is not weakly stable.

So we proved the following result answering in negative a question of Emelyanov [**30**].

Theorem 4.9. *There is a locally compact space Ω and a positive contraction T on $C_0(\Omega)$ which is almost weakly but not weakly stable.*

5. Abstract examples

In this section we present abstract examples arising in ergodic and measure theory which show that weak stability is not equivalent to almost weak stability. Moreover, we will show that a "typical" (in the sense of Baire) contraction as well as a "typical" isometric or unitary operator on a separable Hilbert space is almost weakly but not weakly stable.

5.1. Ergodic theory. We discuss the analogues between weak and almost weak stability and the concepts of strongly and weakly mixing in ergodic theory. We begin with some definitions.

A measurable measure-preserving transformation φ on a probability space $(\Omega, \mathcal{M}, \mu)$ is called *strongly mixing* if

$$\lim_{n\to\infty} \mu(\varphi^{-n}(A) \cap B) = \mu(A)\mu(B)$$

for any two measurable sets $A, B \in \mathcal{M}$. The transformation φ is called *weakly mixing* if for all $A, B \in \mathcal{M}$ we have

$$\lim_{N\to\infty} \frac{1}{N+1} \sum_{n=0}^{N} |\mu(\varphi^{-n}(A) \cap B) - \mu(A)\mu(B)| = 0.$$

These concepts play an essential role in ergodic theory, and we refer to the monographs Cornfeld, Fomin, Sinai [**18**], Krengel [**75**], Petersen [**105**], or Halmos [**53**] for further information. Strong mixing implies weak mixing, but the converse implication does not hold in general. We note that examples of weakly but not strongly mixing transformations are not easy to construct; for some years an available construction was even an open question. See Lind [**80**] for an example and Petersen [**105**, p. 209] for a method of constructing such transformations.

It is a classical procedure to define, for the transformation φ on $(\Omega, \mathcal{M}, \mu)$, an isometry T on each of the Banach spaces $X = L^p(\Omega, \mu)$ $(1 \le p < \infty)$ by the formula

$$(Tf)(\omega) := f(\varphi(\omega)), \quad \omega \in \Omega, f \in L^p(\Omega, \mu).$$

The operator T has relatively weakly compact orbits by the theorem of Banach-Alaoglu for $p > 1$ and by Example 0.7 (b) with $u = \mathbf{1}$ for $p = 1$. Furthermore, it is well-known (see, e.g., Halmos [**53**], pp. 37–38) that

$$\varphi \text{ is strongly mixing } \iff \lim_{n\to\infty} \langle T^n f, g\rangle = \langle Pf, g\rangle \text{ for all } f \in X,\ g \in X',$$

and

$$\varphi \text{ is weakly mixing} \iff \lim_{N\to\infty} \frac{1}{N+1} \sum_{n=0}^{N} |\langle T^n f, g\rangle - \langle Pf, g\rangle| = 0 \text{ for all } f \in X,\ g \in X',$$

where P is the projection onto $\operatorname{Fix} T$ given by $Pf := \int_\Omega f\, d\mu \cdot \mathbf{1}$ for all $f \in X$. Note that in both cases $\operatorname{Fix} T = \langle \mathbf{1} \rangle$ holds.

We now consider any transformation φ being weakly but not strongly mixing and the corresponding operator T. Then $X = X_0 \oplus \langle \mathbf{1} \rangle$ with

$$X_0 := \left\{ f \in X \ : \ \int_\Omega f \, d\mu = 0 \right\}$$

a closed and T-invariant subspace. The restriction T_0 of T to X_0 also has relatively weakly compact orbits. Hence, by $P\sigma(T_0) \cap \Gamma = \emptyset$, the operator T_0 is almost weakly stable. On the other hand, T_0 is not weakly stable since φ is not strongly mixing.

So we saw that every weakly but not strongly mixing flow induces an almost weakly but not weakly stable operator.

5.2. Rajchman measures. In this subsection we discuss parallels between operator theory and measure theory based on the spectral theorem (see, e.g., Halmos [**54**]). We first consider an example which will help us to understand the general situation.

EXAMPLE 5.1. Let μ be some finite measure on the unit circle Γ and define the multiplication operator T on the space $H := L^2(\Gamma, \mu)$ by

$$(Tf)(z) := zf(z), \quad z \in \Gamma.$$

Then T is unitary on X and hence not strongly stable. We are interested in weak stability of T.

We see that $z \in P_\sigma(T)$ if and only if $\mu(\{z\}) > 0$, and hence, by the Jacobs–Glicksberg–de Leeuw decomposition (Theorem 0.14 in Chapter 1),

$$T \text{ is almost weakly stable} \iff \mu \text{ is continuous.}$$

To characterise weak stability of T, we first observe that

$$\langle T^n f, f \rangle = \int_0^{2\pi} e^{in\varphi} |f(e^{i\varphi})|^2 d\mu(\varphi)$$

holds for every $f \in H$. In particular, if T is weakly stable, then

$$a_n(\mu) := \int_0^{2\pi} e^{in\varphi} d\mu(\varphi) \to 0 \quad \text{as } n \to \infty, \tag{37}$$

where a_n are the Fourier coefficients of μ.

Conversely, if condition (37) holds, then $\langle T^n f, f \rangle \to 0$ as $n \to \infty$ for every function f having constant absolute value. Since the linear span of $\{z \mapsto z^n, n \in \mathbb{Z}\}$ is dense in H and T is contractive, we obtain that $\langle T^n f, f \rangle \to 0$ as $n \to \infty$ for every $f \in H$. So T is weakly stable by Theorem 3.10.

Note that a unitary operator is weakly stable if and only if its inverse is, hence in condition (37) $n \to \infty$ can be replaced by $n \to -\infty$.

This proves the following proposition (see Lyons [**83**] for the first equivalence).

PROPOSITION 5.2. *T is weakly stable* $\iff a_n(\mu) \underset{n\to\infty}{\longrightarrow} 0 \iff a_n(\mu) \underset{|n|\to\infty}{\longrightarrow} 0$.

Measures with this property have their own name.

DEFINITION 5.3. A measure on Γ is called *Rajchman* if its Fourier coefficients converge to zero.

Proposition 5.2 states that the operator T is weakly stable if and only if the measure μ is Rajchman.

For an overview on Rajchman measures we refer to Lyons [**83, 84**]. We only remark that every absolutely continuous measure is Rajchman by the Riemann-Lebesgue lemma and every Rajchman measure is continuous by Wiener's theorem. Furthermore, there are continuous measures which are not Rajchman and Rajchman measures which are not absolutely continuous, see Lyons [**83**]. By our considerations above, each continuous non-Rajchman measure induces an almost weakly but not weakly stable unitary operator. For a specific example of a unitary group with non-Rajchman spectral measures we refer to Engel, Nagel [**31**, p. 316]. Taking $T := T(1)$ in this example we obtain an example of an almost weakly but not weakly stable unitary operator.

By the spectral theorem, one can reduce the general situation to the previous example.

Indeed, consider a Hilbert space H and a contraction T on it. By Theorem 3.8 the restriction T_1 of T to the subspace $H_1 := \{x : \|T^n x\| = \|T^{*n}x\| = \|x\| \; \forall n \in \mathbb{N}\}$ is unitary and the restriction to $H_1^\perp$ is weakly stable. So T is weakly stable if and only if T_1 is weakly stable.

We now apply the spectral theorem to T_1. This gives for each $x \in H_1$ a measure μ_x on Γ such that the restriction of T to $\overline{\text{lin}}\{T^n x : n = 0, 1, 2, \dots\}$ acts as the multiplication operator $M_z f(z) := zf(z)$ on $L^2(\Gamma, \mu_x)$ and we are in the context of Example 5.1. So we see that the orbit $\{T^n x : n \in \mathbb{N}\}$ converges to zero if and only if the measure μ_x is Rajchman.

This gives a measure theoretic approach to weak stability. However, since the result is based on the spectral theorem, it is very difficult to apply in concrete situations.

5.3. Category theorems. In this subsection we will show that for a separable infinite-dimensional Hilbert space H a "typical" (in the Baire category sense) unitary operator, a "typical" isometry and a "typical" contraction on H is almost weakly but not weakly stable. This gives an operator–theoretic analogue to the classical theorems of Halmos and Rohlin from ergodic theory stating that a "typical" flow is weakly but not strongly mixing, see Halmos [**53**, pp. 77–80] or the original papers by Halmos [**51**] and Rohlin [**109**].

We follow in this section Eisner, Sereny [**26**] and assume the underlying Hilbert space H to be separable and infinite-dimensional.

5.3.1. ***Unitary operators***. Denote the set of all unitary operators on H by $\mathcal{U}$. The following density result for periodic operators is a first step in our construction.

PROPOSITION 5.4. *For every $n \in \mathbb{N}$ the set of all periodic unitary operators with period greater than n is dense in $\mathcal{U}$ endowed with the norm topology.*

Proof. Take $U \in \mathcal{U}$, $N \in \mathbb{N}$ and $\varepsilon > 0$. By the spectral theorem H is isomorphic to $L^2(\Omega, \mu)$ for some locally compact space Ω and finite measure μ and U is unitarily equivalent to a multiplication operator $\tilde{U}$ with

$$(\tilde{U}f)(\omega) = \varphi(\omega)f(\omega) \quad \text{for almost all } \omega \in \Omega,$$

for some measurable $\varphi : \Omega \to \Gamma := \{z \in \mathbb{C} : \ |z| = 1\}$.

We approximate the operator $\tilde{U}$ as follows. Consider the set

$$\Gamma_N := \{e^{2\pi i \cdot \frac{p}{q}} : \ p, q \in \mathbb{N} \text{ relatively prime }, \ q > N\}$$

which is dense in Γ. Take a finite set $\{\alpha_j\}_{j=1}^n \subset \Gamma_N$ such that $\arg(\alpha_{j-1}) < \arg(\alpha_j)$ and $|\alpha_j - \alpha_{j-1}| < \varepsilon$ hold for all $2 \leq j \leq n$. Define

$$\psi(\omega) := \alpha_{j-1}, \quad \forall \omega \in \varphi^{-1}(\{z \in \Gamma : \ \arg(\alpha_{j-1}) \leq \arg(z) \leq \arg(\alpha_j)\}),$$

and denote by $\tilde{P}$ the multiplication operator with ψ. The operator $\tilde{P}$ is periodic with period greater than N and satisfies

$$\|\tilde{U} - \tilde{P}\| = \sup_{\omega \in \Omega} |\varphi(\omega) - \psi(\omega)| \leq \varepsilon,$$

hence the proposition is proved. □

For the second step we need the following lemma.

Lemma 5.5. *Let H be a separable infinite-dimensional Hilbert space. Then there exists a sequence $\{T_n\}_{n=1}^\infty$ of almost weakly stable unitary operators satisfying $\|T_n - I\| \to 0$ as $n \to \infty$.*

Proof. By the isomorphy of all separable infinite-dimensional Hilbert spaces there exists a unitary operator $U : H \to L^2(\mathbb{R})$, where $L^2(\mathbb{R})$ is taken with the Lebesgue measure.

Take $n \in \mathbb{N}$ and define $\tilde{T}_n$ on $L^2(\mathbb{R})$ by

$$(\tilde{T}_n f)(s) := e^{\frac{iq(s)}{n}} f(s), \quad s \in \mathbb{R}, \quad f \in L^2(\mathbb{R}),$$

where $q : \mathbb{R} \to [0,1]$ is strictly monotone. Then all $\tilde{T}_n$ are almost weakly stable by the theorem of Jacobs–Glicksberg–de Leeuw and we have

$$\|\tilde{T}_n - I\| = \sup_{s \in \mathbb{R}} |e^{\frac{iq(s)}{n}} - 1| \leq |e^{\frac{i}{n}} - 1| \to 0, \quad n \to \infty.$$

To finish the proof we only need to define $T_n := U^* \tilde{T}_n U$ on H. □

We now introduce the appropriate topology. We say that a sequence $\{T_n\} \subset \mathcal{L}(H)$ converges to $T \in \mathcal{L}(H)$ in the strong*-topology if $T_n \to T$ and $T_n^* \to T^*$ strongly (for details see, e.g., Takesaki [**121**, p. 68]). Further we consider the space $\mathcal{U}$ of all unitary operators on H endowed with this topology. Note that $\mathcal{U}$ is a complete metric space with respect to the metric given by

$$d(U,V) := \sum_{j=1}^{\infty} \frac{\|Ux_j - Vx_j\| + \|U^*x_j - V^*x_j\|}{2^j \|x_j\|} \quad \text{for } U, V \in \mathcal{U},$$

and $\{x_j\}_{j=1}^{\infty}$ some dense subset of H. Further we denote by $\mathcal{S}_{\mathcal{U}}$ the set of all weakly stable unitary operators on H and by $\mathcal{W}_{\mathcal{U}}$ the set all almost weakly stable unitary operators on H.

We now show the following density property for $\mathcal{W}_{\mathcal{U}}$.

PROPOSITION 5.6. *The set $\mathcal{W}_{\mathcal{U}}$ of all almost weakly stable unitary operators is dense in $\mathcal{U}$.*

PROOF. By Proposition 5.4 it is enough to approximate periodic unitary operators by almost weakly stable unitary operators. Let U be a periodic unitary operator and let N be its period. Take $\varepsilon > 0$, $n \in \mathbb{N}$ and $x_1, \dots, x_n \in H \setminus \{0\}$. We have to find an almost weakly stable unitary operator T with $\|Ux_j - Tx_j\| \le \varepsilon$ and $\|U^*x_j - T^*x_j\| \le \varepsilon$ for all $j = 1, \dots, n$.

By $U^N = I$ and the spectral theorem, $\sigma(U) \subset \left\{1, e^{\frac{2\pi i}{N}}, \dots, e^{\frac{2\pi(N-1)i}{N}}\right\}$ and

$$H = \mathrm{Fix}(U) \oplus \mathrm{Fix}(e^{\frac{-2\pi i}{N}}U) \oplus \dots \oplus \mathrm{Fix}(e^{\frac{-2\pi(N-1)i}{N}}U) \tag{38}$$

holds. So we can assume without loss of generality that x_j are orthogonal eigenvectors of U.

We first construct a periodic unitary operator S satisfying $Ux_j = Sx_j$ for all $j = 1, \dots, n$ and having infinite-dimensional eigenspaces only. For this purpose define the n-dimensional U- and U^*-invariant subspace $H_0 := \mathrm{lin}\{x_j\}_{j=1}^{n}$ and the operator S on H_0 as the restriction of U on H_0. Further we consider an orthogonal decomposition

$$H = \bigoplus_{k=0}^{\infty} H_k, \quad \dim(H_k) = n \ \text{ for all } k \in \mathbb{N}.$$

Fix k and denote by $\{y_j\}_{j=1}^{n}$ an orthonormal basis of H_k. Define $Sy_j := \frac{Sx_j}{\|x_j\|}$ and extend S linearly to H_k.

This operator S is unitary and periodic with period being a divisor of N. So a decomposition analogous to (38) is valid for S. Moreover, $Ux_j = Sx_j$ and $U^*x_j = S^*x_j$ hold for all $j = 1, \dots, n$ and all eigenspaces of S are infinite dimensional. Therefore by Lemma 5.5 for every $j = 1, \dots, N$ there exists a weakly stable unitary operator T_j on $F_j := \mathrm{Fix}(e^{\frac{-2\pi ji}{N}}S)$ satisfying $\|T_j - S_{|F_j}\| = \|T_j - e^{\frac{2\pi ij}{N}}I\| < \varepsilon$. Finally we define the desired operator $T := T_j$ on F_j for every $j = 1 \dots, n$ and the proposition is proved. □

We can now prove the following category theorem for weakly and almost weakly stable unitary operators. To do so we extend the argument used in the proof of the corresponding category theorems for flows in ergodic theory (see Halmos [**53**, pp. 77–80]).

THEOREM 5.7. *The set $\mathcal{S}_{\mathcal{U}}$ of weakly stable operators is of first category and the set $\mathcal{W}_{\mathcal{U}}$ of almost weakly stable operators is residual in $\mathcal{U}$ with the *-strong topology.*

PROOF. We first prove that $\mathcal{S}$ is of first category in $\mathcal{U}$. Fix $x \in H$ with $\|x\| = 1$ and consider the closed sets

$$M_k := \left\{ U \in \mathcal{U} : \ |\langle U^k x, x\rangle| \le \frac{1}{2} \right\}.$$

Let $U \in \mathcal{U}$ be weakly stable. Then there exists $n \in \mathbb{N}$ such that $U \in M_k$ for all $k \ge n$, i.e., $U \in \cap_{k\ge n} M_k$. So we obtain

$$\mathcal{S}_{\mathcal{U}} \subset \bigcup_{n=1}^{\infty} N_n, \tag{39}$$

where $N_n := \cap_{k\ge n} M_k$. Since the sets N_n are closed, it remains to show that $\mathcal{U} \setminus N_n$ is dense for every n.

Fix $n \in \mathbb{N}$ and let U be a periodic unitary operator. Then $U \notin M_k$ for some $k \ge n$ and therefore $U \notin N_n$. Since by Proposition 5.4 the periodic unitary operators are dense in $\mathcal{U}$, $\mathcal{S}$ is of first category.

To show that $\mathcal{W}_{\mathcal{U}}$ is residual we take a dense subspace $D = \{x_j\}_{j=1}^{\infty}$ of H and define the open sets

$$W_{jkn} := \left\{ U \in \mathcal{U} : \ |\langle U^n x_j, x_j\rangle| < \frac{1}{k} \right\}.$$

Then the sets $W_{jk} := \cup_{n=1}^{\infty} W_{jkn}$ are also open.

We show that

$$\mathcal{W}_{\mathcal{U}} = \bigcap_{j,k=1}^{\infty} W_{jk} \tag{40}$$

holds.

The inclusion "$\subset$" follows from the definition of almost weak stability. To prove the converse inclusion we take $U \notin \mathcal{W}_{\mathcal{U}}$ and $n \in \mathbb{N}$. Then there exists $x \in H$ with $\|x\| = 1$ and $\varphi \in \mathbb{R}$ such that $Ux = e^{i\varphi}x$ and therefore $|\langle U^n x, x\rangle| = 1$. Take $x_j \in D$ with $\|x_j - x\| \le \frac{1}{4}$. Then

$$\begin{aligned} |\langle U^n x_j, x_j\rangle| &= |\langle U^n(x-x_j), x-x_j\rangle + \langle U^n x, x\rangle - \langle U^n x, x-x_j\rangle - \langle U^n(x-x_j), x\rangle| \\ &\ge 1 - \|x-x_j\|^2 - 2\|x-x_j\| > \frac{1}{3}. \end{aligned}$$

So $U \notin W_{j3}$ which implies $U \notin \cap_{j,k=1}^{\infty} W_{jk}$. Therefore (40) holds and $\mathcal{W}_{\mathcal{U}}$ is residual as a countable intersection of open dense sets. □

5.3.2. ***Isometries***. We now consider the space $\mathcal{I}$ of all isometries on H endowed with the strong topology and prove analogous category results as above. We again assume H to be separable and infinite-dimensional. Note that $\mathcal{I}$ is a complete metric space with respect to the metric given by the formula

$$d(T,S) := \sum_{j=1}^{\infty} \frac{\|Tx_j - Sx_j\|}{2^j \|x_j\|} \quad \text{for } T, S \in \mathcal{I},$$

where $\{x_j\}_{j=1}^{\infty}$ is a fixed dense subset of H.

Further we denote by $\mathcal{S}_{\mathcal{I}}$ the set of all weakly stable isometries on H and by $\mathcal{W}_{\mathcal{I}}$ the set all almost weakly stable isometries on H.

The basis of our results in this section is the following classical theorem on isometries on Hilbert spaces.

THEOREM 5.8. *(Wold decomposition, see* [**120**, Theorem 1.1]*.) Let V be an isometry on a Hilbert space H. Then H can be decomposed into an orthogonal sum $H = H_0 \oplus H_1$ of V-invariant subspaces such that the restriction of V on H_0 is unitary and the restriction of V on H_1 is a unilateral shift, i.e., there exists a subspace $Y \subset H_1$ with $V^n Y \perp V^m Y$ for all $n \neq m$, $n, m \in \mathbb{N}$, such that $H_1 = \oplus_{n=1}^{\infty} V^n Y$ holds.*

As a first application of this decomposition and the following easy lemma, see also Peller [**104**], we obtain the density of periodic operators in $\mathcal{I}$. (Note that periodic isometries are unitary.)

LEMMA 5.9. *Let Y be a Hilbert space and let R be the right shift on $H := l^2(\mathbb{N}, Y)$. Then there exists a sequence $\{T_n\}_{n=1}^{\infty}$ of periodic unitary operators on H converging strongly to R.*

PROOF. We define the operators T_n by

$$T_n(x_1, x_2, \ldots, x_n, \ldots) := (x_n, x_1, x_2, \ldots, x_{n-1}, x_{n+1}, \ldots).$$

Every T_n is unitary and has period n. Moreover, for an arbitrary $x = (x_1, x_2, \ldots) \in H$ we have

$$\|T_n x - Rx\|^2 = \|x_n\|^2 + \sum_{k=n}^{\infty} \|x_{k+1} - x_k\|^2 \underset{n\to\infty}{\longrightarrow} 0,$$

and the lemma is proved. □

PROPOSITION 5.10. *The set of all periodic operators is dense in $\mathcal{I}$.*

PROOF. Let V be an isometry on H. Then by Theorem 5.8 the orthogonal decomposition $H = H_0 \oplus H_1$ holds, where the restriction V_0 on H_0 is unitary and the space H_1 is unitarily equivalent to $l^2(\mathbb{N}, Y)$. The restriction V_1 of V on H_1 corresponds (by this equivalence) to the right shift operator on $l^2(\mathbb{N}, Y)$. By Proposition 5.4 and Lemma 5.9 we can approximate both operators V_0 and V_1 by unitary periodic ones and the assertion follows. □

As a further consequence of the Wold decomposition we obtain the density of almost weakly stable operators in $\mathcal{I}$.

PROPOSITION 5.11. *The set $\mathcal{W}_{\mathcal{I}}$ of almost weakly stable isometries is dense in $\mathcal{I}$.*

PROOF. Let V be an isometry on H and let V_0 and V_1 be the corresponding restrictions of V to the orthogonal subspaces H_0 and H_1 from Theorem 5.8. By Lemma 5.9 the operator V_1 can be approximated by unitary operators on H_1. The assertion now follows from Proposition 5.6. □

Using the same idea as in the proof of Theorem 5.7 one obtains with the help of Propositions 5.10 and 5.11 the following category result for weakly and almost weakly stable isometries.

THEOREM 5.12. *The set $\mathcal{S}_\mathcal{I}$ of all weakly stable isometries is of first category and the set $\mathcal{W}_\mathcal{I}$ of all almost weakly stable isometries is residual in $\mathcal{I}$.*

5.3.3. ***Contractions***. We now extend the above category results to the case of contractive operators.

Let the Hilbert space H be as before and denote by $\mathcal{C}$ the set of all contractions on H endowed with the weak operator topology. Note that $\mathcal{C}$ is a complete metric space with respect to the metric given by the formula

$$d(T,S) := \sum_{i,j=1}^{\infty} \frac{|\langle Tx_i, x_j\rangle - \langle Sx_i, x_j\rangle|}{2^{i+j}\|x_i\|\|x_j\|} \quad \text{for} \quad T,S \in \mathcal{C},$$

where $\{x_j\}_{j=1}^{\infty}$ is a fixed dense subset of H with each $x_j \neq 0$.

By Takesaki [**121**, p. 99], the set of all unitary operators is dense in $\mathcal{C}$ (see also Peller [**104**] for a much stronger assertion). Combining this with Propositions 5.4 and 5.6 we have the following fact.

PROPOSITION 5.13. *The set of all periodic unitary operators and the set of all almost weakly stable unitary operators are both dense in $\mathcal{C}$.*

The following well-known property is a key for the further results (cf. Halmos [**56**, p. 14]).

LEMMA 5.14. *Let $\{T_n\}_{n=1}^{\infty}$ be a sequence of contractions on a Hilbert space H converging weakly to an isometry S. Then $T_n \to S$ strongly.*

PROOF. For each $x \in H$ we have

$$\begin{aligned} \|T_n x - Sx\|^2 &= \langle T_n x - Sx, T_n x - Sx\rangle = \|Sx\|^2 + \|T_n x\|^2 - 2Re\,\langle T_n x, Sx\rangle \\ &\leq 2\,\langle Sx, Sx\rangle - 2Re\,\langle T_n x, Sx\rangle = 2\mathrm{Re}\,\langle (S-T_n)x, Sx\rangle \underset{n\to\infty}{\longrightarrow} 0, \end{aligned}$$

and the lemma is proved. □

We now state the category result for contractions, but note that its proof differs from the corresponding proofs in the previous sections.

THEOREM 5.15. *The set $\mathcal{S}_\mathcal{C}$ of all weakly stable contractions is of first category and the set $\mathcal{W}_\mathcal{C}$ of all almost weakly stable contractions is residual in $\mathcal{C}$.*

PROOF. To prove the first statement we fix $x \in X$, $\|x\| = 1$, and define as before the sets

$$N_n := \left\{ T \in \mathcal{C} : \ |\langle T^k x, x\rangle| \leq \frac{1}{2} \text{ for all } k \geq n \right\}.$$

Let $T \in \mathcal{C}$ be weakly stable. Then there exists $n \in \mathbb{N}$ such that $T \in N_n$, and we obtain

$$\mathcal{S}_\mathcal{C} \subset \bigcup_{n=1}^{\infty} N_n. \tag{41}$$

It remains to show that the sets N_n are nowhere dense. Fix $n \in \mathbb{N}$ and let U be a periodic unitary operator. We show that U does not belong to the closure of N_n. Assume the opposite, i.e., that there exists a sequence $\{T_k\}_{k\in\mathbb{N}} \subset N_n$ satisfying $T_k \to U$ weakly. Then, by Lemma 5.14, $T_k \to U$ strongly and therefore $U \in N_n$ by the definition of N_n. This contradicts the periodicity of U. By the density of the set of unitary periodic operators in $\mathcal{C}$ we obtain that N_n is nowhere dense and therefore $\mathcal{S}_\mathcal{C}$ is of first category.

To show the residuality of $\mathcal{W}$ we again take a dense subset $D = \{x_j\}_{j=1}^\infty$ of H and define

$$W_{jk} := \left\{ T \in \mathcal{C} : \ |\langle T^n x_j, x_j \rangle| < \frac{1}{k} \text{ for some } n \in \mathbb{N} \right\}.$$

As in the proof of Theorem 5.7 the equality

$$\mathcal{W}_\mathcal{C} = \bigcap_{j,k=1}^{\infty} W_{jk} \tag{42}$$

holds.

Fix $j, k \in \mathbb{N}$. We have to show that the complement W_{jk}^c of W_{jk} is nowhere dense. We note that

$$W_{jk}^c = \left\{ T \in \mathcal{C} : \ |\langle T^n x_j, x_j \rangle| \geq \frac{1}{k} \text{ for all } n \in \mathbb{N} \right\}.$$

Let U be a unitary almost weakly stable operator. Assume that there exists a sequence $\{T_m\}_{m=1}^\infty \subset W_{ijk}^c$ satisfying $T_m \to U$ weakly. Then, by Lemma 5.14, $T_m \to U$ strongly and therefore $U \in W_{jk}^c$. This contradicts the almost weak stability of U. Therefore the set of all unitary almost weakly stable operators does not intersect the closure of W_{jk}^c. By Proposition 5.13 all sets W_{jk}^c are nowhere dense and therefore $\mathcal{W}_\mathcal{C}$ is residual. □

REMARK 5.16. As a consequence of Theorem 5.15 we see that the set of all strongly stable operators as well as the set of all operators T satisfying $r(T) < 1$ are also of first category in $\mathcal{C}$.

OPEN QUESTION 5.17. Do the above category theorems for weakly and almost weakly stable isometries and contractions hold on reflexive Banach spaces?

We note that on non-reflexive Banach spaces these results do not need to be true. Indeed, every almost weakly stable contraction on the space l^1 is authomatically weakly and even strongly stable by Schur's lemma, see, e.g., Conway [**17**, Prop. V.5.2], and Lemma 2.4.

CHAPTER 3

Stability of C_0-semigroups

1. Boundedness

In this section we consider boundedness and the related notion of polynomial boundedness for C_0-semigroups.

1.1. Preliminaries. We start with the definition of bounded C_0-semigroups and their elementary properties.

DEFINITION 1.1. A C_0-semigroup $T(\cdot)$ on a Banach space X is called *bounded* if $\sup_{t\geq 0}\|T(t)\| < \infty$.

REMARK 1.2. Every bounded semigroup $T(\cdot)$ satisfies $\omega_0(T) \leq 0$ and hence $\sigma(A) \subset \{z: \mathrm{Re}\,(z) \leq 0\}$. However, the spectral condition $\sigma(A) \subset \{z: \mathrm{Re}\,(z) \leq 0\}$ does not imply boundedness of the semigroup, which can be seen from the matrix semigroup $T(\cdot)$ given by $T(t) = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$ on $\mathbb{C}^2$. Moreover, we refer to Subsection 1.3 for more sophisticated examples and a general description of possible growth.

REMARK 1.3. It is interesting that, analogous to power bounded operators (see Subsection 1.1), some more information on the spectrum of the generator of a bounded C_0-semigroup is known if X is separable. For example, Jamison [**64**] proved that if a C_0-semigroup on a separable Banach space is bounded, then the point spectrum of its generator on the imaginary axis has to be countable. For more results in this direction, see e.g. Ransford [**107**].

We will see later that countability of the whole spectrum of the generator on the imaginary axis plays an important role for strong stability of the semigroup, see Subsection 3.3.

The following simple lemma is useful to understand boundedness.

LEMMA 1.4. *Let $T(\cdot)$ be a bounded semigroup on a Banach space X. Then there exists an equivalent norm on X such that $T(\cdot)$ becomes a contraction semigroup.*

PROOF. Take $\|x\|_1 := \sup_{t\geq 0}\|T(t)x\|$ for every $x \in X$. □

REMARK 1.5. Packel [**102**] showed that not all bounded C_0-semigroups on Hilbert spaces are similar to a contraction semigroup for a Hilbert space norm. His example was a modification of the corresponding examples of Foguel [**35**] and Halmos [**55**] for the discrete case.

However, Sz.-Nagy [**118**] proved that every bounded C_0-group is similar to a unitary one, see also van Casteren [**12**]. We also mention here Vũ and Yao [**131**] who proved that every bounded uniformly continuous quasi-compact C_0-semigroup on a Hilbert space is similar to a contraction semigroup.

We now present an interesting result due to Guo and Zwart showing that boundedness of a C_0-semigroup is equivalent to absolute Cesàro-boundedness of the semigroup and its adjoint. This is the following theorem, see Zwart [**139**] and Guo, Zwart [**50**, Thm. 8.2] for the case $p = q = 2$. See also van Casteren [**12**] for the case of bounded C_0-groups on Hilbert spaces.

THEOREM 1.6. *For a C_0-semigroup $T(\cdot)$ on a Banach space X the following assertions are equivalent.*
(a) $T(\cdot)$ is bounded;
(b) For all $x \in X$ and $y \in X'$

$$\sup_{t\geq 0} \frac{1}{t}\int_0^t \|T(s)x\|^2 ds < \infty,$$
$$\sup_{t\geq 0} \frac{1}{t}\int_0^t \|T'(s)y\|^2 ds < \infty$$

hold.

REMARK 1.7. The second part of condition (b) in the above theorem cannot be omitted, i.e., absolute Cesàro-boundedness of is not equivalent to boundedness. Van Casteren [**12**] gave an examlpe of an unbounded C_0-group on a Hilbert space satisfying the first part of condition (b).

We note that Theorem 1.6 can be also formulated for single orbits as follows.

PROPOSITION 1.8. *For a C_0-semigroup $T(\cdot)$ on a Banach space X, $x \in X$ and $y \in X'$ the weak orbit $\{\langle T(t)x, y\rangle : t \geq 0\}$ is bounded if and only if there exist $p, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$ such that*

$$\sup_{t\geq 0} \frac{1}{t}\int_0^t \|T(s)x\|^p ds < \infty,$$
$$\sup_{t\geq 0} \frac{1}{t}\int_0^t \|T'(s)y\|^q ds < \infty.$$

Since the semigroup is, in most cases, not known explicitly, one looks for other characterisations. This is the aim of the rest of the section.

1.2. Characterisation via cogenerator in Hilbert spaces. In this subsection we discuss the connection between boundedness of a C_0-semigroup and power boundedness of its cogenerator. This yields a bridge between continuous and discrete semigroups. We recall that the cogenerator can be easily obtained from the resolvent of the generator and hence does not involve any explicit knowledge of the semigroup.

As we saw in Subsection 0.7, a C_0-semigroup on a Hilbert space is contractive if and only if its cogenerator is contractive. Guo, Zwart gave an analogous characterisation for boundedness which however needs more assumptions.

THEOREM 1.9. *Let A generate a bounded C_0-semigroup $T(\cdot)$ on a Hilbert space H. Assume further that A^{-1} generates a bounded C_0-semigroup as well. Then the cogenerator of $T(\cdot)$ is power bounded.*

Another result of Guo, Zwart shows the equivalence of boundedness of an analytic C_0-semigroup to power boundedness of its generator.

THEOREM 1.10. *Let $T(\cdot)$ be an analytic C_0-semigroup on a Hilbert space H with cogenerator V. Then $T(\cdot)$ is bounded on $\mathbb{R}_+$ if and only if V is power bounded.*

We refer to Guo, Zwart [**50**] for the proofs and further results.

It is naturally to ask whether there is an analogue of the above theorems for semigroups on Banach spaces. As far as we know, there is still no satisfactory answer to this question.

1.3. Characterisation via resolvent. In this subsection we discuss a resolvent type characterisation of boundedness for C_0-semigroups involving only the first and the second power of its generator.

The main result of this subsection is the following generation theorem.

THEOREM 1.11. *(Gomilko [**45**], Shi and Feng [**115**]) Let A be a densely defined operator on a Banach space X satisfying $s_0(A) \leq 0$. Consider the following assertions.*

(a) For every $x \in X$ and $y \in X'$

$$\sup_{a>0} a \int_{-\infty}^{\infty} \|R(a+is, A)x\|^2 ds < \infty,$$

$$\sup_{a>0} a \int_{-\infty}^{\infty} \|R(a+is, A')y\|^2 ds < \infty;$$

(b) $\sup_{a>0} a \int_{-\infty}^{\infty} |\langle R^2(a+is, A)x, y\rangle| ds < \infty$ for all $x \in X$, $y \in X'$;

(c) A generates a bounded C_0-semigroup on X.

Then (a)⇒(b)⇒(c). Moreover, if X is a Hilbert space, then (a)⇔(b)⇔(c).

Gomilko first proved the non-trivial direction (b)⇒(c) using the Hille–Yosida theorem. Then Shi and Feng presented an alternative proof using an explicit construction of the semigroup by formula (6) in Subsection 0.8. The direction (c)⇒(a) for Hilbert spaces follows easily from the Plancherel Theorem.

Note that van Casteren [**12**] presented an analogous characterisation for bounded C_0-groups on Hilbert spaces much earlier. He also showed that the second part of condition (a) cannot be omitted.

REMARK 1.12. One can replace the L^2-norms in condition (a) by the L^p-norm in the first inequality and L^q-norm in the second one for some $p, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$, possibly depending on x and y. However, for the converse implication, if X is a Hilbert space, one needs $p=q=2$.

A direct consequence of the construction given in the proof of Theorem 1.11 by Shi and Feng is the following result for single orbits.

PROPOSITION 1.13. *Let X be Banach space and A generate a C_0-semigroup $T(\cdot)$ with $s_0(A)\le 0$, $x\in X$ and $y\in X'$. Consider the following assertions.*
(a) For some $p,q>1$ with $\frac{1}{p}+\frac{1}{q}=1$

$$\limsup_{a\to 0+}\ a\int_{-\infty}^{\infty}\|R(a+is,A)x\|^p ds<\infty,$$
$$\limsup_{a\to 0+}\ a\int_{-\infty}^{\infty}\|R(a+is,A')y\|^q ds<\infty;$$

(b) $\limsup_{a\to 0+}\ a\int_{-\infty}^{\infty}|\langle R^2(a+is,A)x,y\rangle|ds<\infty$ *for all $x\in X$, $y\in X'$;*
(c) $\{\langle T(t)x,y\rangle : t\ge 0\}$ *is bounded.*
Then (a)⇒(b)⇒(c). Moreover, if X is a Hilbert space, then (a)⇔(b)⇔(c) for $p=q=2$.

In particular, conditions (a) and (b) holding for all $x\in X$ and $y\in X'$ both imply boundedness of $T(\cdot)$ and are equivalent to it in the case when X is a Hilbert space for $p=q=2$.

Note that conditions (a) and (b) in Theorem 1.11 are both not necessary for boundedness by Corollary 2.11.

The question to find analogous necessary and sufficient resolvent conditions for C_0-semigroups on Banach spaces is still open.

1.4. Polynomial boundedness. In this subsection we introduce and discuss polynomial boundedness of C_0-semigroups being related to boundedness. Surprisingly, this notion can be characterised much easier.

DEFINITION 1.14. A semigroup $T(\cdot)$ on a Banach space X is called *polynomially bounded* if $\|T(t)\|\le p(t)$ for some polynomial p and all $t\ge 0$.

In the following we will assume $\|T(t)\|\le K(1+t^d)$ for some constants $d\ge 0$ (being not necessary integer), $K\ge 1$ and all $t\ge 0$.

Note that every polynomially bounded semigroup $T(\cdot)$ again satisfies $\omega_0(T)\le 0$. The following example shows that the converse implication does not hold.

EXAMPLE 1.15. (C_0-semigroups satisfying $\omega_0(T)\le 0$ with non-polynomial growth) We present a construction of a C_0-semigroup with a given growth analogous to Example 1.12 in the discrete case.

Consider the Hilbert space

$$H:=L^2_{a^2}=\{f:\mathbb{R}_+\to\mathbb{C}\text{ measurable}: \int_0^{\infty}|f(s)|^2a^2(s)ds<\infty\}$$

for some positive continuous function a satisfying $a(0) \geq 1$ and

$$a(t+s) \leq a(t)a(s) \qquad \text{for all } t, s \in \mathbb{R}_+ \tag{43}$$

with the natural scalar product. On H take the right shift semigroup $T(\cdot)$.

We first check strong continuity of $T(\cdot)$. For the characteristic function f on an interval $[a, b]$ and $t < b - a$ one has

$$\|T(t)f - f\| = \int_a^{a+t} a^2(s)ds + \int_b^{b+t} a^2(s)ds \underset{t \to 0+}{\longrightarrow} 0.$$

Further, for $f \in X$ we have by (43)

$$\begin{aligned} \|T(t)f\|^2 &= \int_t^\infty |f(s-t)|^2 a^2(s)ds = \int_0^\infty |f(s)|^2 a^2(s+t)ds \\ &\leq a^2(t) \int_0^\infty |f(s)|^2 a^2(s)ds = a^2(t)\|f\|^2 \end{aligned}$$

for every $t \geq 0$ and therefore $\|T(t)\| \leq a(t)$. By a density argument the semigroup $T(\cdot)$ is strongly continuous.

Moreover, for the characteristic functions f_n of the intervals $[0, 1/n]$ we have

$$\|T(t)f_n\|^2 = \int_t^{t+\frac{1}{n}} a^2(s)ds = \frac{\int_t^{t+\frac{1}{n}} a^2(s)ds}{\int_0^{\frac{1}{n}} a^2(s)ds} \|f_n\|^2$$

and hence

$$\|T(t)\|^2 \geq \frac{\frac{1}{n}\int_t^{t+\frac{1}{n}} a^2(s)ds}{\frac{1}{n}\int_0^{\frac{1}{n}} a^2(s)ds} \underset{n \to \infty}{\longrightarrow} \frac{a^2(t)}{a^2(0)}$$

So we obtain the following norm estimate

$$\frac{a(t)}{a(0)} \leq \|T(t)\| \leq a(t)$$

and hence the semigroup $T(\cdot)$ has the same growth as the function a. Note that if $a(0) = 1$, then $\|T(t)\| = a(t)$ for all $t \geq 0$. Now every function a satisfying (43) which grows faster than every polynomial but slower than any exponential function with positive exponent gives an example of a non-polynomially growing C_0-semigroup $T(\cdot)$ satisfying $\omega_0(T) \leq 0$.

As a concrete example of such a function consider again

$$a(t) := (t+6)^{\ln(t+6)} = e^{\ln^2(t+6)}.$$

We saw in Example 1.12 that this function satisfies condition (43). Hence we constructed a C_0-semigroup on a Hilbert space which grows like $t^{\ln t}$.

Analogously, using this method one can construct a C_0-semigroup growing as $t^{\ln^\alpha t}$ for any $\alpha \geq 1$.

The idea to use the natural condition (43) is due to Sen-Zhong Huang.

Finally, we mention that one can also use the idea of Zabczyk taking matrices with increasing dimensions (see, e.g., Engel, Nagel [**31**, Counterexample IV.3.4] or the original paper of Zabczyk [**138**]) to construct a C_0-semigroup with growth bound zero having non-polynomial growth. However, one does not have information about the actual growth with this construction. We do not go into the details.

The following theorem gives a characterisation of operators which generate a polynomially bounded C_0-semigroup, see Eisner [**24**]. This generalises Theorem 1.11 for bounded C_0-semigroups and theorems of Malejki [**87**] (see also Kiselev [**73**]) for the case of C_0-groups. The proof is based on an explicit construction of the semigroup given by Shi, Feng [**115**] for the bounded case, see Theorem 1.11.

THEOREM 1.16. *(Eisner [**24**]) Let X be a Banach space and A be a densely defined operator on X with $s(A) \le 0$ and $d \ge 0$. If the condition*

$$\int_{-\infty}^{\infty} |\langle R(a+is,A)^2 x, y\rangle| ds \le \frac{M}{a}(1+a^{-d})\|x\|\|y\| \qquad \forall x \in X,\ \forall y \in X' \tag{44}$$

holds for all $a > 0$, then A is the generator of a C_0-semigroup $(T(t))_{t\ge 0}$ which does not grow faster than t^d, i.e.,

$$\|T(t)\| \le K(1+t^d) \tag{45}$$

for some constant K and all $t > 0$. Conversely, if X is a Hilbert space, then growth condition (45) implies (44) for the parameter $d_1 := 2d$.

PROOF. We will contruct the semigroup explicitly.

Let us first prove that by condition (44) we have $s_0(A) \le 0$. Since $\frac{d}{dz}R(z,A) = -R^2(z,A)$ we have for all $a > 0$, $x \in X$ and $y \in X^*$,

$$\langle R(a+is,A)x, y\rangle = \langle R(a,A)x, y\rangle - \ i\int_0^s \langle R(a+i\tau,A)^2 x, y\rangle d\tau. \tag{46}$$

By the absolute convergence of the integral on the right hand side we obtain that $\langle R(a+is,A)x,y\rangle \to 0$ if $s \to \infty$. From (46) and condition (44) it follows that

$$\|R(a+is,A)\| \le \frac{M}{a}(1+a^{-d}),$$

hence $s_0(A) \le 0$ holds.

Define now

$$T(t)x = \frac{1}{2\pi t}\int_{-\infty}^{\infty} e^{(a+is)t} R(a+is,A)^2 x ds$$

for all $x \in X$. By the assumption the operators $T(\cdot)$ are well-defined and form a semigroup by by Lemma 0.42. Let us estimate the norm of $T(t)$. From representation (6) and condition (44) we have

$$\begin{aligned} |\langle T(t)x, y\rangle| &\le \frac{e^{at}}{2\pi t}\int_{-\infty}^{\infty} |\langle R(a+is,A)^2 x, y\rangle| ds \\ &\le \frac{Me^{at}}{2\pi t a}(1+a^{-d})\|x\|\|y\|. \end{aligned}$$

Taking $a := t^{-1}$ we obtain for $K := \frac{Me}{2\pi}$ the desired estimate

$$\|T(t)\| \leq K(1 + t^d). \tag{47}$$

This and Lemma 0.42 imply the strong continuity of $T(\cdot)$.

For the converse implication, let $T(\cdot)$ be a C_0-semigroup on a Hilbert space H with generator A satisfying $\|T(t)\| \leq K(1 + t^d)$. By Parseval's inequality and integration by parts we have

$$\begin{aligned}\int_{-\infty}^{\infty} \|R(a+is, A)x\|^2 ds &= \int_0^{\infty} e^{-2at}\|T(t)x\|^2 dx \\ &\leq K^2 \int_0^{\infty} e^{-2at}(1+t^d)^2\|x\|^2 dt \leq \frac{M}{a}(1+a^{-d})\|x\|^2\end{aligned}$$

and analogously

$$\int_{-\infty}^{\infty} \|R(a+is, A')y\|^2 ds \leq \frac{M}{a}(1+a^{-d})\|y\|^2$$

for some constant $M > 0$ and every $x, y \in H$. Now, by the Cauchy–Schwarz inequality, we obtain

$$\begin{aligned}\int_{-\infty}^{\infty} |\langle R(a+is, A)x, y\rangle| ds &\leq \left(\int_{-\infty}^{\infty} \|R(a+is, A)x\|^2 ds\right)^{\frac{1}{2}} \left(\int_{-\infty}^{\infty} \|R(a+is, A')y\|^2 ds\right)^{\frac{1}{2}} \\ &\leq \frac{M}{a}(1+a^{-d})\|x\|\|y\|, \qquad \forall x, y \in H,\end{aligned}$$

and the theorem is proved. □

Remark 1.17. Kiselev [**73**] showed that the exponent $2d$ in the implication (45)⇒ (44) is sharp for the case of C_0-groups.

If it is already known that A generates a C_0-semigroup, then it becomes much easier to check whether the semigroup is polynomially bounded at least for a large set of semigroups.

Following Eisner, Zwart [**29**], we define that an operator A has a *p-integrable resolvent* if for some/all $a, b > s_0(A)$ the following conditions hold

$$\int_{-\infty}^{\infty} \|R(a+is, A)x\|^p ds < \infty \qquad \forall x \in X, \tag{48}$$

$$\int_{-\infty}^{\infty} \|R(b+is, A')y\|^q ds < \infty \qquad \forall x \in X', \tag{49}$$

where $1 < p, q < \infty$ with $\frac{1}{p} + \frac{1}{q} = 1$. Note that in particular condition (4) from Subsection 0.8 is satisfied for such semigroups by the Cauchy–Schwarz inequality.

Plancherel's theorem applied to the functions $t \mapsto e^{-at}T(t)x$ and $t \mapsto e^{-at}T^*(t)y$ for sufficiently large $a > 0$ implies that every generator of a C_0-semigroup on a Hilbert space has 2-integrable resolvent. Moreover, for generators on a Banach space with Fourier type $p > 1$ condition (48) is satisfied automatically. Finally, every generator of an analytic semigroup (in particular, every bounded operator) on an arbitrary Banach space has p-integrable resolvent for every $p > 1$. Intuitively, the property of having p-integrable resolvent for some p means good properties of the generator A or/and good properties of the space X.

THEOREM 1.18. *(Eisner, Zwart [**29**]) Let A be the generator of a C_0-semigroup $(T(t))_{t\geq 0}$ having p-integrable resolvent for some $p>1$. Assume that $\mathbb{C}_0^+ = \{\lambda: \ Re\lambda > 0\}$ is contained in the resolvent set of A and there exist $a_0>0$ and $M>0$ such that the following conditions hold.*

(a) $\|R(\lambda, A)\| \leq \frac{M}{(Re\lambda)^d}$ for all λ with $0 < Re\lambda < a_0$ and for some $d \geq 0$;

(b) $\|R(\lambda, A)\| \leq M$ for all λ with $Re\lambda \geq a_0$.

Then $\|T(t)\| \leq K(1+t^{2d-1})$ holds for some constant $K>0$ and all $t\geq 0$.

Conversely, if $(T(t))_{t\geq 0}$ is a C_0-semigroup on a Banach space with

$$\|T(t)\| \leq K(1+t^{\gamma})$$

for some constants $\gamma \geq 0$, $K>0$ and all $t\geq 0$, then for every $a_0>0$ there exists a constant $M>0$ such that the resolvent of the generator satisfies conditions (a) and (b) above for $d=\gamma+1$.

PROOF. The second part of the theorem follows easily from the representation

$$R(\lambda, A)x = \int_0^{\infty} e^{-\lambda t} T(t)x dt.$$

The idea of the proof of the first part is based on the inverse Laplace transform representation of the semigroup presented in Subsection 0.8 and the technique from Zwart [**140**], and Eisner, Zwart [**28**].

We first note that by conditions (a) and (b) we obtain $s_0(A) \leq 0$.

Next, since the function $\omega \mapsto R(a+i\omega, A)x$ is an element of $L^p(\mathbb{R}, X)$ for all $x\in X$, we conclude by the uniform boundedness theorem that there exists a constant $M_0>0$ such that

$$\|R(a+i\cdot, A))x\|_{L^p(\mathbb{R},X)} \leq M_0\|x\| \tag{50}$$

for all $x\in X$. Similarly, one obtains the dual result, i.e.,

$$\|R(b+i\cdot, A')y\|_{L^q(\mathbb{R},X')} \leq \tilde{M}_0\|y\| \tag{51}$$

for all $y\in X'$.

Take now $0<r<a_0$. By the resolvent equality we have

$$R(r+i\omega), A)x = [I + (a-r)R(r+i\omega, A)]\, R(a+i\omega, A)x.$$

Hence

$$\begin{aligned} \|R(r+i\omega, A)x\| &\leq [1+|a-r|\|(R(r+i\omega, A)\|]\, \|R(a+i\omega, A)x\| \\ &\leq \left[1+|a-r|\frac{M}{r^d}\right] \|R(a+i\omega, A)x\|, \end{aligned}$$

where we have used (a). Combining this with the estimate (50), we find that

$$\begin{aligned} \|R(r+i\cdot, A)x\|_{L^p(\mathbb{R},X)} &\leq \left[1+|a-r|\frac{M}{r^d}\right] M_0\|x\| \\ &\leq M_1\left[1+\frac{1}{r^d}\right] \|x\|. \end{aligned} \tag{52}$$

Similarly, we find that

$$\|R(r+i\cdot,A')y\|_{L^q(\mathbb{R},X')} \le \tilde{M}_1\left[1+\frac{1}{r^d}\right]\|y\|. \tag{53}$$

From estimates (52) and (53) we obtain

$$\begin{aligned}
&\int_{-\infty}^{\infty} |\langle R(r+i\omega,A)^2x,y\rangle| d\omega \\
&\qquad = \int_{-\infty}^{\infty} |\langle R(r+i\omega,A)x, R(r+i\omega,A)'y\rangle| d\omega \\
&\qquad \le \|R(r+i\cdot,A)x\|_{L^p(\mathbb{R},X)}\|R(r+i\cdot,A')y\|_{L^q(\mathbb{R},X')} \\
&\qquad \le M_1\tilde{M}_1\|x\|\|y\|\left[1+\frac{1}{r^d}\right]^2. \qquad (54)
\end{aligned}$$

Convergence of the integral on the right hand side of (54) implies that the inverse formula for the semigroup

$$T(t)x = \frac{1}{2\pi t}\int_{-\infty}^{\infty} e^{(r+is)t}R(r+is,A)^2x ds$$

holds for all $x \in X$ by Theorem 0.42. Notice that the condition $r > s_0(A)$ is essential. Combining this formula with (54) we obtain

$$\begin{aligned}
|\langle T(t)x,y\rangle| &\le \frac{1}{2\pi t}\int_{-\infty}^{\infty} e^{rt}|\langle R(r+i\omega,A)^2x,y\rangle| d\omega \\
&\le \frac{1}{2\pi t}e^{rt}M_1\tilde{M}_1\|x\|\|y\|\left[1+\frac{1}{r^d}\right]^2. \qquad (55)
\end{aligned}$$

Since this holds for all $0 < r < a_0$, we may choose $r := \frac{1}{t}$ for t large enough, which gives

$$|\langle T(t)x,y\rangle| \le \frac{1}{2\pi t}eM_1\tilde{M}_1\|x\|\|y\|\left[1+t^d\right]^2. \tag{56}$$

So for large t the norm of the semigroup is bounded by Ct^{2d-1} for some constant C. Since any C_0-semigroup is uniformly bounded on compact time intervals, the result follows. □

As mentioned above, every generator on a Hilbert space has 2-integrable resolvent, hence we have the following immediate corollary.

COROLLARY 1.19. *Let A generate a C_0-semigroup $(T(t))_{t\ge 0}$ on the Hilbert space H. If A satisfies conditions (a) and (b) of Theorem 1.18 for some $d \ge 0$ and $a_0 > 0$, then there exists $K > 0$ such that $\|T(t)\| \le K[1+t^{2d-1}]$ for all $t \ge 0$.*

REMARK 1.20. Notice that conditions (a) and (b) for $0 \le d < 1$ already imply $s_0(A) < 0$ (use the power series expansion for the resolvent). On the other hand, for generators with p-integrable resolvent the equality $\omega_0(T) = s_0(A)$ holds by Corollary 2.11. Combining these facts we obtain that in this case the semigroup is even uniformly exponentially stable. On the other hand, the exponential stability follows from the Theorem 1.18 only for $d < \frac{1}{2}$. So for $\frac{1}{2} \le d < 1$ Theorem 1.18 does not give the best information about the growth of the semigroup. Nevertheless, for $d = 1$ the growth stated in Theorem 1.18 is best possible, i.e., the exponent $2d-1$ cannot be decreased in general (see Eisner, Zwart [**28**]). For $d > 1$ it is not clear whether Theorem 1.18 gives the best possible constant γ.

Note that the parameter $d = \gamma + 1$ in the converse implication of Theorem 1.18 is optimal for $\gamma \in \mathbb{N}$. Indeed, for $X := \mathbb{C}^n$ and

$$A := \begin{pmatrix} 0\,1\,0\,...\,0 \\ 0\,0\,1\,...\,0 \\ ... \\ 0\,0\,0\,...\,0 \end{pmatrix}$$

conditions (a) and (b) in Theorem 1.18 are fulfilled for $d = n$ and the semigroup generated by A grows exactly as t^{n-1}.

By Corollary 1.19 we see that the class of generators of polynomially bounded semigroups on a Hilbert space coincides with the class of generators of C_0-semigroups satisfying the resolvent conditions (a) and (b). For semigroups on Banach spaces this is not true since there exist C_0-semigroups with $w_0(T) > s_0(A)$ (see Engel, Nagel [**31**, Examples IV.3.2 and IV.3.3]).

As a corollary of Theorem 1.18 we have the following characterisation of polynomially bounded C_0-groups in terms of the resolvent of the generator.

Theorem 1.21. *Let A be the generator of a C_0-group $(T(t))_{t\in\mathbb{R}}$. Assume that A has p-integrable resolvent for some $p > 1$. Then the group $(T(t))_{t\in\mathbb{R}}$ is polynomially bounded if and only if the following conditions on the operator A are satisfied.*

(a) $\sigma(A) \subset i\mathbb{R}$;
(b) *There exist $a_0 > 0$ and $d \geq 0$ such that $\|R(\lambda, A)\| \leq \frac{M}{|Re\lambda|^d}$ for some constant M and all λ with $0 < |Re\lambda| < a_0$;*
(c) *$R(\lambda, A)$ is uniformly bounded on $\{\lambda : \ |Re\lambda| \geq a_0\}$.*

Proof. It is enough to show that the operator $-A$ also has p-integrable resolvent whenever A satisfies (a)–(c). Take any $a > 0$. Then by (*b*) or (*c*), respectively, $R(\lambda, A)$ is bounded on the vertical line $-a + i\mathbb{R}$. By the resolvent equation we obtain

$$\|R(-a+is, A)x\| \leq [1 + 2a\|R(-a+is, A)\|]\|R(a+is, A)x\|,$$

and therefore the function $s \mapsto \|R(-a+is, A)x\|$ also belongs to $L^p(\mathbb{R})$. The rest follows immediately from Theorem 1.18. □

Again, this yields a characterisation of polynomially bounded C_0-groups on Hilbert spaces. Note that the relation between the growth of the group and the growth of the resolvent appearing in (b) of Theorem 1.21 is the same as in Theorem 1.18.

2. Uniform exponential stability

In this section we consider the concept of uniform exponential stability of C_0-semigroups which turns out to be more complex and much more difficult to characterise in terms of the generator than the corresponding notion for operators (compare Proposition 1.3).

Uniform exponential stability is defined as follows.

DEFINITION 2.1. A C_0-semigroup $T(\cdot)$ is called *uniformly exponentially stable* if there exist $M \geq 1$ and $\epsilon > 0$

$$\|T(t)\| \leq Me^{-\varepsilon t} \quad \text{for all } t \geq 0$$

or, equivalently, $\omega_0(T) < 0$.

For C_0-semigroups on finite-dimensional Banach spaces there is a simple characterisation in terms of the spectrum of the generator given by the classical Lyapunov theorem.

The aim of this section is to characterise uniformly exponentially stable C_0-semigroups on infinite-dimensional Banach and Hilbert spaces. We will see that many different results are known, but only in Hilbert spaces a simple spectral theoretic characterisation holds.

2.1. Spectral characterisation. An elementary description of uniformly exponentially stable C_0-semigroups is the following theorem which is the basis for many further results in this section.

THEOREM 2.2. *(see Engel, Nagel* [**31**, Prop. V.1.7]*) For a C_0-semigroup $T(\cdot)$ on a Banach space X the following assertions are equivalent.*

(i) $r(T(t_0)) < 1$ *for some* $t_0 > 0$*;*
(ii) $\|T(t_0)\| < 1$ *for some* $t_0 > 0$*;*
(iii) $\|T(t)\| \underset{t\to\infty}{\longrightarrow} 0$*;*
(iv) $T(\cdot)$ *is uniformly exponentially stable.*

The proof of the nontrivial implication $(i) \Rightarrow (iv)$ is based on the formula $r(T(t)) = e^{t\omega_0(T)}$.

The theorem above shows that, in particular, stability in the norm operator topology already implies uniform exponential stability.

Moreover, as a consequence of (i), we obtain the following characterisation of uniform exponential stability in terms of the generator, provided some spectral mapping theorem holds. We recall that the weak circular spectral mapping theorem is the condition

$$\Gamma\sigma(T(t)) = \Gamma\overline{e^{t\sigma(A)}} = \overline{e^{t(\sigma(A)+i\mathbb{R})}} \quad \text{for some/all } t > 0.$$

For details on spectral mapping theorems see Subsection 0.6.

COROLLARY 2.3. *Let A generate a C_0-semigroup $T(\cdot)$ on a Banach space X satisfying the weak circular spectral mapping theorem. Then $s(A) = \omega_0(T)$. In particular, $T(\cdot)$ is uniform exponential stable if and only if $s(A) < 0$, i.e., the spectrum of the generator is contained in a halfplane $\{z : Re(z) \leq -a\}$ for some $a > 0$.*

2.2. Theorem of Datko–Pazy. The main result of this subsection is the classical theorem characterising uniform exponential stability of a C_0-semigroup in terms of the integrability of its orbits.

THEOREM 2.4. *(Datko–Pazy) Let $T(\cdot)$ be a C_0-semigroup on a Banach space X. Then $T(\cdot)$ is uniformly exponentially stable if and only if for some $p \geq 1$*

$$\int_0^\infty \|T(t)x\|^p dt < \infty \quad \text{for all } x \in X.$$

Datko [**19**] proved this theorem for $p = 2$ and Pazy [**103**, Theorem 4.4.1] extended it to the case $p \geq 1$.

There are various generalisations of the Datko–Pazy theorem. As an example we present two theorems due to van Neerven and Rolewicz.

THEOREM 2.5. *(van Neerven [**99**, Cor. 3.1.6]) Let $T(\cdot)$ be a C_0-semigroup on a Banach space X, $p \geq 1$ and $\beta \in L^1_{loc}(\mathbb{R}_+)$ a positive function satisfying*

$$\int_0^\infty \beta(t)dt = \infty.$$

If

$$\int_0^\infty \beta(t)\|T(t)x\|^p dt < \infty \quad \text{for all } x \in X,$$

then $T(\cdot)$ is uniformly exponentially stable.

THEOREM 2.6. *(Rolewicz, see van Neerven [**99**, Theorem 3.2.2]) Let $T(\cdot)$ be a C_0-semigroup on a Banach space X. If there exists a strictly positive increasing function ϕ on $\mathbb{R}_+$ such that*

$$\int_0^\infty \phi(\|T(t)x\|)dt < \infty \quad \text{for all } x \in X,\ \|x\| \leq 1,$$

then $T(\cdot)$ is uniformly exponentially stable.

For further discussion of the above result we refer to van Neerven [**99**, pp. 110–111].

The following theorem is a weak version of the Datko–Pazy theorem due to Weiss [**132**].

THEOREM 2.7. *(Weiss) Let $T(\cdot)$ be a C_0-semigroup on a Banach space X. If for some $p \geq 1$*

$$\int_0^\infty |\langle T(t)x, y\rangle|^p dt < \infty \quad \text{for all } x \in X \text{ and } \ y \in X',$$

then $T(\cdot)$ is uniformly exponentially stable.

For a generalisation of this result see Tomilov [**123**]. Moreover, we refer to Weiss [**133**] for a discrete version.

For further generalisations of the Datko–Pazy theorem see e.g. Vũ [**130**], van Neerven [**99**, Sections 3.3-4].

2.3. Theorem of Gearhart. For C_0-semigroups on Hilbert spaces there is a simple characterisation of uniform exponential stability in terms of the generator's resolvent on the right half plane. This is the classical theorem of Gearhart [**37**] (see also Prüss [**106**], Huang [**60**] and Greiner's theorems in Nagel (ed.) [**95**, A-III.7.8 and 7.10]) which can be considered as a generalisation of the Lyapunov theorem to the infinite dimensional case. To this theorem and its generalisations we dedicate this subsection.

We begin with the following result on uniform exponential stability using the inverse Laplace transform method presented in Subsection 0.8.

THEOREM 2.8. *Let A generate a C_0-semigroup $T(\cdot)$ on a Banach space X such that the resolvent of A is bounded on the right half plane $\{z : \operatorname{Re} z > 0\}$. Assume further that*

$$\int_{-\infty}^{\infty} |\langle R^2(is,A)x,y\rangle| ds < \infty \qquad \text{for all } x \in X \text{ and } y \in X'. \tag{57}$$

Then $T(\cdot)$ is uniformly exponentially stable.

PROOF. We first observe that, since the resolvent is bounded on the right half plane, then it is also bounded on a half plane $\{z : \operatorname{Re} z > -\delta\}$ for some $\delta > 0$ by the power series expansion of the resolvent and hence $s_0(A) < 0$. Therefore, by condition (57) and Theorem 0.41 we have

$$\langle T(t)x,y\rangle = \frac{2\pi}{t}\int_{-\infty}^{\infty} e^{ist}\langle R^2(is,A)x,y\rangle ds \qquad \text{for all } x \in X,\ y \in X'.$$

Therefore, by the uniform boundedness principle

$$|\langle T(t)x,y\rangle| \leq \frac{2\pi}{t}\int_{-\infty}^{\infty} |\langle R^2(is,A)x,y\rangle| ds \leq \frac{2\pi}{t} M\|x\|\|y\|$$

for some constant M and all $x \in X$, $y \in X'$. This implies $\|T(t)\| \leq \frac{2\pi M}{t} \longrightarrow 0$ as $t \to \infty$. □

Note that this result generalises a result of Xu and Feng [**136**].

We now present the Gearhart theorem. The proof we give here is based on the theorem above, i.e., on the properties of the inverse Laplace transform. Note that the idea of this proof seems to be due to W. Arendt, see also Engel, Nagel [**31**, Theorem V.1.11].

THEOREM 2.9. *(Gearhart) Let A generate a C_0-semigroup $T(\cdot)$ on a Hilbert space H. Then $T(\cdot)$ is uniformly exponentially stable if and only if there exists a constant $M > 0$ such that*

$$\|R(\lambda,A)\| < M \quad \text{for all } \lambda \text{ with } Re(\lambda) > 0.$$

PROOF. As in the proof of Theorem 2.8 we see that by the power series expansion for the resolvent $s_0(A) < 0$ holds. By Theorem 2.8 it suffices to check condition (57).

Take $a > \max \omega_0(T)$ and $x, y \in H$. By the Plancherel theorem applied to the function $t \mapsto e^{-at}T(t)x$ we obtain

$$\int_{-\infty}^{\infty} \|R(a+is,A)x\|^2 ds = \int_0^{\infty} e^{-2at}\|T(t)x\|^2 dt < \infty.$$

Further, by the inequality $s_0(A) < 0$ and the resolvent identity

$$\|R(is,A)x\| = \|[I + aR(is,A)]R(a+is,A)x\| \leq [1 + aM]\|R(a+is,A)x\|$$

holds for every $s \in \mathbb{R}$ and hence

$$\int_{-\infty}^{\infty} \|R(is,A)x\|^2 ds < \infty.$$

Applying the same arguments to the operator A and the function $t \mapsto T^*(t)y$ as well as the Cauchy–Schwarz inequality we finally obtain

$$\begin{aligned}\int_{-\infty}^{\infty} |\langle R^2(is,A)x, y\rangle| ds &= \int_{-\infty}^{\infty} |\langle R(a+is,A)x, R(a-is,A^*)y\rangle| ds \\ &\leq \left(\int_{-\infty}^{\infty} \|R(is,A)x\|^2 ds\right)^{\frac{1}{2}} \left(\int_{-\infty}^{\infty} \|R(is,A^*)y\|^2 ds\right)^{\frac{1}{2}} < \infty,\end{aligned}$$

and the theorem is proved. □

Remark 2.10. The assumption that the resolvent is bounded on the left half plane cannot be replaced by the existence of the resolvent only on the left half plane. For an example of a semigroup on a Hilbert space satisfying $s(A) < s_0(A)$ see e.g. Engel, Nagel [**31**, Counterexample IV.3.4].

By the rescaling procedure one obtains the following corollary, see also Kaashoek and Verduyn Lunel [**67**] for a similar assertion (but a different method of proof).

Corollary 2.11. *Let A generate a C_0-semigroup $(T(t))_{t\geq 0}$ on a Banach space X. If for some $\delta > 0$ the integrability condition*

$$\int_{-\infty}^{\infty} |\langle R^2(a+is,A)x, y\rangle| ds < \infty \qquad \text{for all } x, y \in H$$

holds for all $s_0(A) < a < s_0(A) + \delta$, then $s_0(A) = \omega_0(T)$.

In particular, we see that for C_0-semigroups on Hilbert spaces the equality

$$s_0(A) = \omega_0(T)$$

holds. This is not true for C_0-semigroups on Banach spaces. For an example of a C_0-semigroup on a Banach space satisfying $s_0(A) < \omega_0(T)$ see van Neerven [**99**, Ex. 4.2.9]. So the integrability assumption on the resolvent on vertical lines cannot be omitted. It would be interesting to know whether this condition can be weakened or what kind of other additional assumptions on the resolvent imply the property $s_0(A) = \omega_0(T)$.

Remark 2.12. As an application of Gearhart's theorem we present the following property of positive C_0-semigroups on Hilbert lattices: A positive C_0-semigroup with generator A on a Hilbert lattice is uniformly exponentially stable if and only if $[0,\infty) \subset \rho(A)$. This follows immediately from Gearhart's theorem and the fact that

$$\|R(a+is,A)\| \leq \|R(a,A)\| \quad \text{for all } s \in \mathbb{R}$$

holds for every $a > s(A)$, see e.g. Engel, Nagel [**31**, p.355]. For basic definitions and properties of positive semigroups we refer to Nagel (ed.) [**95**].

For further generalisations of Gearhart's theorem see e.g. Herbst [**58**], Huang [**62**], Weis, Wrobel [**135**].

At the end of this subsection we present an easy characterisation of a hyperbolic type decomposition for semigroups on Hilbert spaces based on Gearhart's theorem.

COROLLARY 2.13. *(Kaashoek, Verduyn Lunel [**67**]) Let $T(\cdot)$ be a C_0-semigroup on a Hilbert space H and $a \in \mathbb{R}$. Then there exists a decomposition $X = X_0 \oplus X_1$ such that*

(1) $\|T(t)x\| \leq Me^{(a-\varepsilon)t}$ for every $x \in X_1$,
(2) $\|T(t)x\| \geq \frac{1}{M}e^{(a+\varepsilon)t}$ for every $x \in X_2$

if and only if the vertical line $a + i\mathbb{R} \subset \rho(A)$ and the resolvent of A is bounded on $a + i\mathbb{R}$.

Note that one of the subspaces X_1 and X_2 may be zero. Note further that one can obtain an analogous characterisation for semigroups on Banach spaces replacing boundedness by Cesàro boundedness of the resolvent.

3. Strong stability

We now consider a weaker concept than the uniform exponential stability which is strong stability.

3.1. Preliminaries. We start by introducing strongly stable C_0-semigroups and then present some fundamental properties.

DEFINITION 3.1. A C_0-semigroup $T(\cdot)$ on a Banach space X is called *strongly stable* if $\|T(t)x\| \underset{t\to\infty}{\longrightarrow} 0$ for every $x \in X$.

The following example (*a*) is, in a certain sense, typical on Hilbert spaces, see Theorem 3.8 below.

EXAMPLE 3.2. (a) (Shift semigroup) Consider $H := L^2(\mathbb{R}_+, H_0)$ for a Hilbert space H_0 and $T(\cdot)$ defined by

$$T(t)f(s) := f(s+t), \quad f \in H,\ t, s \geq 0. \tag{58}$$

The semigroup $T(\cdot)$ is called the *left shift semigroup* on H and is strongly stable. Note that the spectrum of its generator is the whole left halfplane.

The same semigroup on the spaces $C_0(\mathbb{R}_+, X)$ and $L^p(\mathbb{R}_+, X)$, X a Banach space, is also strongly stable for $1 \leq p < \infty$, but not for $p = \infty$.

(b) (Multiplication semigroup, see Engel, Nagel [**31**, p. 323]) Consider $X := C_0(\Omega)$ for a locally compact space Ω and the operator A given by

$$Af(s) := q(s)f(s), \quad f \in X,\ s \in \Omega$$

with the maximal domain $D(A) = \{f \in X : qf \in X\}$, where q is a continuous function on Ω. The operator A generates the C_0-semigroup given by

$$T(t)f(s) = e^{tq(s)}f(s), \quad f \in X,\ s \in \Omega,$$

if and only if $\mathrm{Re}\,(q)$ is bounded from above. The semigroup is bounded if $\mathrm{Re}\,(q(s)) \leq 0$ for every $s \geq 0$. Moreover, the semigroup is strongly stable if and only if $\mathrm{Re}\,(q(s)) < 0$ for every $s \in \Omega$. Indeed, if $\mathrm{Re}\,(q(s)) < 0$, then

$$\|T(t)f\| \leq \sup_{s\in K} e^{t\mathrm{Re}\,(q(s))}\|f\| \underset{t\to\infty}{\longrightarrow} 0$$

for every function f with compact support K. By the density of these functions $T(\cdot)$ is strongly stable. Conversely, if $q(s_0) \in i\mathbb{R}$, then $\|T(t)f\| \geq |f(s_0)|$ for every $f \in X$ and hence $T(\cdot)$ is not strongly stable.

Note that $\sigma(A) = \overline{q(\Omega)}$ and therefore every closed set contained in the closed left halfplane is possible for the spectrum of a strongly stable C_0-semigroup.

The following property of strongly stable semigroups follows directly from the uniform boundedness principle.

REMARK 3.3. Every strongly stable C_0-semigroup $T(\cdot)$ is bounded, hence $\sigma(A) \subset \{z : \operatorname{Re} z \leq 0\}$ holds for the generator A. Note that condition $P_\sigma(A) \cap i\mathbb{R} = \emptyset$ is also necessary for strong stability by the spectral mapping theorem for the point spectrum, see Proposition 0.28.

We now state an elementary but useful property to show strong stability.

LEMMA 3.4. *Let $T(\cdot)$ be a bounded C_0-semigroup on a Banach space X and let $x \in X$.*

(a) If there exists an unbounded sequence $\{t_n\}_{n=1}^\infty \subset \mathbb{R}_+$ such that $\|T(t_n)x\| \to 0$, then $\|T(t)x\| \to 0$.

(b) If $T(\cdot)$ is a contraction semigroup, then $\lim_{t\to\infty} \|T(t)x\|$ exists.

PROOF. The second part follows from the fact that for contraction semigroups the function $t \mapsto \|T(t)x\|$ is non-increasing. To verify the first one assume that $\|T(t_n)x\| \to 0$. Take $\varepsilon > 0$, $n \in \mathbb{N}$ such that $\|T(t_n)x\| < \varepsilon$ and $M := \sup_{t\geq 0} \|T(t)\|$. We obtain

$$\|T(t)x\| \leq \|T(t-t_n)\| \|T(t_n)x\| < M\varepsilon$$

for every $t \geq t_n$, and (a) is proved. □

An immediate corollary is the the following.

COROLLARY 3.5. *Let $T(\cdot)$ be a bounded C_0-semigroup on a Banach space X and $x \in X$. Then the following assertions are equivalent.*

(a) $\|T(t)x\| \underset{t\to\infty}{\longrightarrow} 0$;

(b) $\displaystyle\frac{1}{t}\int_0^t \|T(s)x\|^p ds \underset{t\to\infty}{\longrightarrow} 0$ for some/all $p \geq 1$.

In particular, $T(\cdot)$ is strongly stable if and only if (b) holds for every $x \in X$.

The following theorem, similar to Corollary 3.5, gives an equivalent description of strong stability without assuming boundedness, see Zwart [**139**] and Guo, Zwart [**50**].

THEOREM 3.6. *For a C_0-semigroup $T(\cdot)$ on a Banach space X and $x \in X$ the following assertions are equivalent.*

(a) $T(t)x \underset{t\to\infty}{\longrightarrow} 0$;

(b) For some/all $p, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$

$$\frac{1}{t}\int_0^t \|T(s)x\|^p ds \underset{t\to\infty}{\longrightarrow} 0,$$

$$\sup_{t\geq 0} \frac{1}{t}\int_0^t \|T'(s)y\|^q ds < \infty \quad \text{for all } y \in X'.$$

In particular, $T(\cdot)$ is strongly stable if and only if (b) holds for all $x \in X$.

Note that for bounded C_0-semigroups the second part of condition (b) holds automatically.

We now state a continuous analogue of the result of Müller (Theorem 2.6 in Chapter 2) on the asymptotic behaviour of semigroups which are not uniformly exponentially stable, see van Neerven [**99**, Lemma 3.1.7].

THEOREM 3.7. *[van Neerven] Let $T(\cdot)$ be a C_0-semigroup on a Banach space X with $\omega_0(T) \geq 0$. Then for every $\varepsilon \in (0,1)$ and every function $\alpha : \mathbb{R}_+ \to [0,1]$ converging monotonically to 0 there exists $x \in X$ with $\|x\| = 1$ such that*

$$\|T(t)x\| \geq (1-\varepsilon)\alpha(t) \quad \text{for every } t \geq 0.$$

Theorem 3.7 means that strongly stable semigroups which are not exponentially stable possess arbitrary slowly decreasing orbits.

3.2. Strong stability on Hilbert spaces. By the following classical result of Lax and Phillips being analogous to Theorem 2.7 (Chapter 2), Example 3.2 (a) represents the general situation for contractive strongly stable C_0-semigroups on Hilbert spaces.

THEOREM 3.8. *(Lax, Phillips* [**79**, p. 67]*, see also Lax* [**78**, pp. 450–451]*) Let $T(\cdot)$ be a strongly stable contraction semigroup on a Hilbert space H with $\omega_0(T) = 0$. Then T is unitarily isomorphic to a left shift, i.e., there is a Hilbert space H_0 and a unitary operator $U : H \to H_1$ for some closed subspace $H_1 \subset L^2(\mathbb{R}_+, H_0)$ such that $UT(\cdot)U^{-1}$ is the left shift on $L^2(\mathbb{R}_+, H_0)$.*

Notice that for contraction semigroups on Hilbert spaces Theorem 3.7 follows from the above theorem since the assertion of Theorem 3.7 obviously holds for a left shift semigroup.

By the following cogenerator approach we build a bridge between strong stability of C_0-semigroups and operators. We begin with the classical theorem on contraction semigroups on Hilbert spaces based on the dilation theory developed by Foiaş and Sz.-Nagy, see their monograph [**120**].

THEOREM 3.9. *(Foiaş, Sz.-Nagy* [**120**, Prop. III.9.1]*) Let $T(\cdot)$ be a contraction semigroup on a Hilbert space H with cogenerator V. Then*

$$\lim_{t\to\infty} \|T(t)x\| = \lim_{n\to\infty} \|V^n x\|$$

holds for every $x \in H$. In particular, $T(\cdot)$ is strongly stable if and only if its cogenerator V is strongly stable.

For further parallels between a contraction C_0-semigroup on a Hilbert space and its cogenerator see Foiaş, Sz.-Nagy [**120**, Sections III.8-9].

Theorem 3.9 was partially generalised by Guo and Zwart to bounded C_0-semigroups.

THEOREM 3.10. *(Guo, Zwart,* [**50**]*) Let $T(\cdot)$ be a bounded semigroup on a Hilbert space H with power bounded cogenerator V. If $T(\cdot)$ is strongly stable, then so is V.*

Note that power boundedness of V in the above theorem is satisfied if A^{-1} exists and generates a bounded C_0-semigroup as well, see Guo, Zwart [**50**].

For a necessary and sufficient condition for strong stability on Hilbert spaces using the resolvent of the generator we refer to Subsection 3.4.

3.3. Spectral conditions. In this subsection we present spectral conditions on the generator implying strong stability of the semigroup.

We begin with the following theorem based on a theorem of Ingham for the Laplace transform, for a short and elegant proof see Chill, Tomilov [**16**].

THEOREM 3.11. *Let $T(\cdot)$ be a bounded C_0-semigroup with generator A. If $i\mathbb{R} \subset \rho(A)$, then $T(\cdot)$ is strongly stable.*

This result is surprising in view of the fact that the equality $s(A) = \omega_0(T)$ fails in general, see, e.g., Engel, Nagel [**31**, Counterexample IV.2.7].

The converse implication in the above theorem is not true. Moreover, as we saw in Example 3.2 (b), the boundary spectrum of the generator of a strongly stable C_0-semigroup can be an arbitrary closed subset of $i\mathbb{R}$.

We now present the famous stability theorem for C_0-semigroups proved by Arendt, Batty [**3**] and Lyubich, Vũ [**85**] independently. The proof of Arendt and Batty is based on the Laplace transform method while the proof on Lyubich and Vũ uses so-called isometric limit semigroups (see also Engel, Nagel [**31**, Theorem V.2.21]). We saw the discrete version of this result in Subsection 2.3 (Chapter 2).

THEOREM 3.12. *(Arendt–Batty–Lyubich–Vũ) Let $T(\cdot)$ be a bounded semigroup on a Banach space X with generator A. Assume that*

(i) $P_\sigma(A') \cap i\mathbb{R} = \emptyset$;
(ii) $\sigma(A) \cap i\mathbb{R}$ is countable.

Then $T(\cdot)$ is strongly stable.

The following example shows the power of the above theorem.

EXAMPLE 3.13. Let A generate a bounded eventually norm continuous positive semigroup $T(\cdot)$ on a Banach lattice X. Then $T(\cdot)$ is strongly stable if and only if $0 \notin P_\sigma(A')$.

The proof follows from several facts. First, the Perron–Frobenius theory implies that the spectrum of the generator A of a positive bounded semigroup on the imaginary axis is additively cyclic, i.e., $i\alpha \in \sigma(A)$ implies $i\alpha\mathbb{Z} \in \sigma(A)$ for real α (see Nagel (ed.) [**95**, Theorem C-III.2.10 and Proposition C-III.2.9]). Moreover, the spectrum of the generator of an eventually norm continuous semigroup is bounded on every vertical line (see Engel, Nagel [**31**, Theorem II.4.18]). The combination of this two facts leads $\sigma(A) \cap i\mathbb{R} \subset \{0\}$. The theorem of Arendt–Batty–Lyubich–Vũ finishes the argument.

We note that in the above result the condition $0 \notin P_\sigma(A')$ cannot be replaced by $0 \notin P_\sigma(A)$. This can be easily seen for $A := T_l - I$ on l^1, where T_l is the left shift operator on l^1. However, for relatively weakly compact C_0-semigroups these conditions are indeed equivalent by the mean ergodic theorem, see Subsection 0.5 (Chapter 1).

As in the discrete case, the theorem of Arendt–Batty–Lyubich–Vu can be generalised for completely non-unitary contraction semigroups on Hilbert spaces.

THEOREM 3.14. *(see Foiaş and Sz.-Nagy* [**120**, II.6.7] *and Kérchy, van Neerven, see* [**72**]*) Let $T(\cdot)$ be a completely non-unitary contractive C_0-semigroup on a Hilbert space with generator A. If*

$$\sigma(A) \cap i\mathbb{R} \quad \textit{has Lebesgue measure } 0,$$

then $T(\cdot)$ and $T^*(\cdot)$ are both strongly stable.

See also Kérchy, van Neerven, see [**72**] for related results.

3.4. Characterisation via resolvent. We now present a powerful resolvent approach for strong stability introduced by Tomilov [**124**]. Opposite to the spectral approach discussed in the previous subsection, resolvent conditions are necessary and sufficient at least for C_0-semigroups on Hilbert spaces.

THEOREM 3.15. *(Tomilov* [**124**]*) Let A generate a C_0-semigroup $T(\cdot)$ on a Banach space X satisfying $s_0(A) \leq 0$ and $x \in X$. Consider the following assertions:*
(a) $\lim_{a\to 0+} a\int_{-\infty}^{\infty} \|R(a+is,A)x\|^2 ds = 0$,
$\limsup_{a\to 0+} a\int_{-\infty}^{\infty} \|R(a+is,A')y\|^2 ds < \infty$ *for all* $y \in X'$*;*
(b) $\|T(t)x\| \underset{t\to\infty}{\longrightarrow} 0$.
Then (a) implies (b). Moreover, if X is a Hilbert space, then (a)⇔(b).

In particular, condition (a) for all $x \in X$ implies strong stability of T and, in the case of Hilbert space, is equivalent to it.

PROOF. We first prove the first part of the theorem. By Theorem 0.42 and the Cauchy-Schwarz inequality we have

$$\begin{aligned} |\langle T(t)x,y\rangle| &\leq \frac{e^{at}}{2\pi t}\int_{-\infty}^{\infty} |\langle R^2(a+is,A)x,y\rangle| ds \\ &\leq \frac{e^{at}}{2\pi t}\left(\int_{-\infty}^{\infty}\|R(a+is,A)x\|^2 ds\right)^{\frac{1}{2}}\left(\int_{-\infty}^{\infty}\|R(a+is,A')y\|^2 ds\right)^{\frac{1}{2}} \end{aligned}$$

for every $t > 0$, $a > 0$ and $y \in X'$. By (a) and the uniform boundedness principle there exists a constant $M > 0$ such that

$$a\int_{-\infty}^{\infty}\|R(a+is,A')y\|^2 ds \leq M\|y\|^2 \quad \text{for every } y \in X' \text{ and } a > 0.$$

Therefore, we have

$$\|T(t)x\| \leq \frac{Me^{at}}{2\pi ta}\left(a\int_{-\infty}^{\infty}\|R(a+is,A)x\|^2 ds\right)^{\frac{1}{2}}. \tag{59}$$

By choosing $a := \frac{1}{t}$, we obtain by (59) $\lim_{t\to\infty}\|T(t)x\| = 0$.

Assume now that $T(\cdot)$ is strongly stable on a Hilbert space X. By Parseval's equality

$$a\int_{-\infty}^{\infty}\|R(a+is,A)x\|^2 ds = a\int_0^{\infty} e^{-2at}\|T(t)x\|^2 dt$$

and left hand side is the Abel mean of the function $t \mapsto \|T(t)x\|$. Therefore it converges to zero as $a \to 0+$ by the strong stability of T. This proves the first part of (a).

The second part of (a) follows from Theorem 1.11. □

By Theorem 1.11 and the above result one immediately obtains the following characterisation of bounded C_0-semigroups on Hilbert spaces.

Corollary 3.16. *Let A generate a bounded semigroup $T(\cdot)$ on a Hilbert space H and $x \in X$. Then $\|T(t)x\| \to 0$ if and only if*

$$\lim_{a \to 0+} a \int_{-\infty}^{\infty} \|R(a+is, T)x\|^2 d\varphi = 0. \tag{60}$$

In particular, $T(\cdot)$ is strongly stable if and only if (60) holds for every x in a dense set of H.

It is still an open question whether the above characterisation holds for C_0-semigroups on Banach spaces.

4. Weak stability

In this section we consider stability of C_0-semigroups in the weak operator topology. Surprisingly, this turns out to be much more difficult than the strong and uniform analogues.

4.1. Preliminaries. We begin with the definition and some examples of weakly stable semigroups.

DEFINITION 4.1. A C_0-semigroup $T(\cdot)$ on a Banach space X is called *weakly stable* if $\langle T(t)x, y\rangle \underset{t\to\infty}{\longrightarrow} 0$ for every $x \in X$ and $y \in X'$.

Note that by the uniform boundedness principle every weakly stable semigroup $T(\cdot)$ on a Banach space is bounded, hence $\omega_0(T) \leq 0$ by Remark 1.2. In particular, the spectrum of the generator A belongs to the closed left half plane. Moreover, the spectral conditions $P_\sigma(A) \cap i\mathbb{R} = \emptyset$ and $R_\sigma(A) \cap i\mathbb{R} = P_\sigma(A') \cap i\mathbb{R} = \emptyset$ are necessary for weak stability.

EXAMPLE 4.2. (a) The left and right shift semigroups are weakly stable isometries (and hence not strongly stable) on the spaces $C_0(\mathbb{R}, X)$ and $L^p(\mathbb{R}, X)$ for a Banach space X and $1 < p < \infty$.

(b) The right shift semigroup on $L^p(\mathbb{R}_+, X)$ for a Banach space X defined by

$$(T(t)f)(s) = \begin{cases} f(s-t), & s \geq t, \\ 0, & s < t \end{cases}$$

is also an isometric semigroup (hence not strongly stable) which is weakly stable for $1 < p < \infty$. When $p = 2$ and X is a Hilbert space, this semigroup is called the *(continuous) unilateral shift*, see, e.g., Sz.-Nagy and Foiaş [**120**, p. 150]. Note that the adjoint semigroup of an unilateral shift is the left shift on the same space and hence is strongly stable. Therefore, there is no subspace on which the restriction of an unilateral shift becomes unitary. We will see in Subsection 4.2 that unilateral shifts represent the general situation of isometric completely non-unitary weakly stable semigroups on Hilbert spaces.

(c) Consider $H := L^2(\mathbb{R})$ and the multiplication semigroup $T(\cdot)$ given by

$$T(t)f(s) := e^{tq(s)}f(s)$$

for some bounded measurable function q. Then $T(\cdot)$ is strongly stable if and only if $\mathrm{Re}\,(q(s)) < 0$ a.e. The characterisation of weak stability is as in the discrete case more technical. Indeed, a bounded semigroup $T(\cdot)$ (i.e., $\mathrm{Re}\,(q(s)) \leq 0$ a.e.) is weakly stable if and only if $\int_a^b e^{tq(s)}ds \to 0$ as $t \to \infty$ for every $[a, b] \subset \mathbb{R}$. This is the case for e.g. $q(s) = i\alpha s^\beta$ for any $\alpha, \beta \in \mathbb{R} \setminus \{0\}$.

For more examples see Section 6.

We now present a simple condition implying weak stability of C_0-semigroups using the following concept.

Definition 4.3. A sequence $\{t_n\}_{n=1}^\infty$ is called *relatively dense* in $\mathbb{R}_+$ if there exists a number $\ell > 0$ such that every sub-interval of $\mathbb{R}_+$ of length ℓ intersects $\{t_n : n \in \mathbb{N}\}$ (see Bart, Goldberg [**5**] for the terminology).

It turns out that weak convergence to zero for such a sequence already implies weak stability.

Theorem 4.4. *Let $T(\cdot)$ be a C_0-semigroup and suppose that $T(t_n) \to 0$ weakly as $n \to \infty$ for some relatively dense sequence $\{t_n\}_{j=1}^\infty$. Then $T(\cdot)$ is weakly stable.*

Proof. Without loss of generality, by passing to a subsequence if necessary, we assume that $\{t_n\}_{n=1}^\infty$ is monotone increasing and set $\ell := \sup_{n\in\mathbb{N}}(t_{n+1} - t_n)$, which is finite by assumption. Since every C_0-semigroup is bounded on compact time intervals and $(T(t_n))_{n\in\mathbb{N}}$ is weakly converging, hence bounded, we obtain that the semigroup $(T(t))_{t\geq 0}$ is bounded.

Fix $x \in X$, $y \in X'$. For $t \in [t_n, t_{n+1}]$ we have

$$\langle T(t)x, y\rangle = \langle T(t - t_n)x, T'(t_n)y\rangle,$$

where $(T'(t))_{t\geq 0}$ is the adjoint semigroup. We note that by assumption $T'(t_n)y \to 0$ in the weak*-topology.

Further, the set $K_x := \{T(s)x : 0 \leq s \leq \ell\}$ is compact in X and $T(t - t_n)x \in K_x$ for every $n \in \mathbb{N}$. Since pointwise convergence is equivalent to the uniform convergence on compact sets (see, e.g., Engel, Nagel [**31**, Prop. A.3]), we see that $\langle T(t)x, y\rangle \to 0$. □

Remark 4.5. Taking $t_n = n$ in the above theorem, we see that a C_0-semigroup $T(\cdot)$ is weakly stable if and only if the operator $T(1)$ is weakly stable. This builds a bridge between weak stability of discrete and continuous semigroups.

Remark 4.6. We will see later that one cannot drop the relative density assumption in Theorem 4.4 or even replace it by the assumption of density 1, see Section 5.

We finally present very recent results of Vladimír Müller (oral communication) concerning possible decay of weak orbits.

Theorem 4.7. *(Müller) Let $T(\cdot)$ be a weakly stable C_0-semigroup on a Banach space X with $\omega_0(T) \geq 0$ and $a : \mathbb{R}_+ \to \mathbb{R}_+$ decreasing to zero. Then there exist $x \in X$, $y \in X'$ such that*

$$|\langle T(t)x, y\rangle| \geq a(t) \qquad \text{for all } t \geq 0.$$

In other words, the orbits of a weakly stable C_0-semigroup decrease arbitrary slowly.

V. Müller also proved that the weak stability assumption cannot be replaced by the almost weak stability in the above theorem. This is very surprising in view of Theorem 3.7. However, for every C_0-semigroup $T(\cdot)$ with $\omega_0(T) \geq 0$ there exist $x \in X$, $y \in X'$ and an increasing sequence $\{t_j\}_{j=1}^\infty$ such that $|\langle T(t_j)x, y\rangle| \geq a(t_j)$ for all $j \in \mathbb{N}$ by Theorem 3.7 in Chapter 2.

4.2. Contraction semigroups on Hilbert spaces. In this subsection we present some classical theorems on the decomposition of contractive C_0-semigroups on Hilbert spaces with respect to different qualitative behaviour.

We begin with a decomposition into unitary and completely non-unitary parts due to Foiaş and Sz.-Nagy [**119**].

THEOREM 4.8. *(Foiaş, Sz.-Nagy) Let $T(\cdot)$ be a contraction semigroup on a Hilbert space H. Then H is the orthogonal sum of two $T(\cdot)$- and $T^*(\cdot)$-invariant subspaces H_1 and H_2 such that*
(a) H_1 is the maximal subspace on which the restriction $T_1(\cdot)$ of $T(\cdot)$ is unitary;
(b) the restrictions of $T(\cdot)$ and $T^*(\cdot)$ to H_2 are weakly stable.

We present the proof of Foguel given in [**34**] being analogous to the proof of the discrete version of this result (Theorem 3.8, Chapter 2).

PROOF. Define

$$H_1 := \{x \in H : \ \|T(t)x\| = \|T^*(t)x\| = \|x\| \quad \text{for all } t \geq 0\}.$$

Observe that for every $0 \neq x \in H_1$ and $t \geq 0$

$$\|x\|^2 = \langle T(t)x, T(t)x\rangle = \langle T^*(t)T(t)x, x\rangle \leq \|T^*(t)T(t)x\|\|x\| \leq \|x\|^2.$$

Therefore, by the equality in the Cauchy-Schwarz inequality and the positivity of $\|x\|^2$, we obtain $T^*(t)T(t)x = x$. Analogously, $T(t)T^*(t)x = x$. On the other hand, every x with these two properties belongs to H_1. So we proved the equality

$$H_1 = \{x \in H : \ T^*(t)T(t)x = T(t)T^*(t)x = x \quad \text{for all } t \geq 0\} \tag{61}$$

which shows, in particular, that H_1 is the maximal (closed) subspace on which $T(\cdot)$ is unitary. The $T(t)$- and $T^*(t)$-invariance of H_1 follows from the definition of H_1 and the equality $T^*(t)T(t) = T(t)T^*(t)$ on H_1.

To show (b) take $x \in H_2 := H_1^\perp$. We first note that H_2 is $T(t)$- and $T^*(t)$-invariant since H_1 is so. Suppose now that $T(t)x$ does not converge weakly to zero as $t \to \infty$, or, equivalently, that there exists $y \in H$, $\varepsilon > 0$ and a sequence $\{t_n\}_{n=1}^\infty$ such that $|\langle T(t_n)x, y\rangle| \geq \varepsilon$ for every $n \in \mathbb{N}$.

Observe that, since every bounded set in a reflexive Banach space is relatively weakly compact by the Banach-Alaoglu theorem and since weak compactness on Banach spaces coincides with weak sequential compactness by the Eberlein-Šmulian theorem (see Theorem 0.1, Chapter 1), there exists a weakly converging subsequence of $\{T(t_n)x\}_{n=1}^\infty$. For convenience we denote the subsequence again by $\{t_n\}_{n=1}^\infty$ and its limit by x_0. The closedness and $T(t)$-invariance of H_2 imply that $x_0 \in H_2$.

For a fixed $t_0 \geq 0$ we obtain

$$\begin{aligned} \|T^*(t_0)T(t_0)T(t)x - T(t)x\|^2 &= \|T^*(t_0)T(t+t_0)x\|^2 - 2\langle T^*(t_0)T(t+t_0)x, T(t)x\rangle \\ +\|T(t)x\|^2 &\leq \|T(t+t_0)x\|^2 - 2\|T(t+t_0)x\|^2 + \|T(t)x\|^2 \\ &= \|T(t)x\|^2 - \|T(t+t_0)x\|^2. \end{aligned}$$

The right hand side converges to zero as $t \to \infty$ since the function $t \mapsto \|T(t)x\|$ is monotone decreasing on $\mathbb{R}_+$. Therefore we obtain $\|T^*(t_0)T(t_0)T(t)x - T(t)x\| \to 0$ as $t \to \infty$.

We now remember that $T(t_n)x \to x_0$ weakly as $n \to \infty$. This implies immediately that $T^*(t_0)T(t_0)T(t_n)x \to T^*(t_0)T(t_0)x_0$ weakly. By the considerations above, we have on the other hand that $T^*(t_0)T(t_0)T(t_n)x \to x_0$ weakly and therefore $T^*(t_0)T(t_0)x_0 = x_0$. One shows analogously that $T(t_0)T^*(t_0)x_0 = x_0$ and $x_0 \in H_1$. By $H_1 \cup H_2 = \{0\}$ this implies $x_0 = 0$, which is a contradiction.

Analogously one shows that the restriction of $T^*(\cdot)$ to H_2 converge weakly to zero as well. □

REMARK 4.9. The restriction of $T(\cdot)$ to the subspace H_2 in Theorem 4.8 is *completely non-unitary* (c.n.u. for short), i.e., there is no subspace of H_2 on which the restriction of $T(\cdot)$ becomes unitary. In other words, Theorem 4.8 states that every Hilbert space contraction can be decomposed into unitary and c.n.u. part and the c.n.u. part is weakly stable.

For a systematic study of completely non-unitary semigroups as well as an alternative proof of Theorem 4.8 using unitary dilation theory see the monograph of Sz.-Nagy and Foiaş [**120**].

On the other hand, the following theorem gives a decomposition into weakly stable and weakly unstable part due to Foguel [**34**]. We now present a simplified proof of it.

THEOREM 4.10. *(Foguel) Let $T(\cdot)$ be a contraction semigroup on a Hilbert space H. Define*

$$W := \{x \in H : \lim_{t\to 0}\langle T(t)x, x\rangle = 0\}.$$

Then

$$W = \{x \in H : \lim_{t\to 0} T(t)x = 0 \text{ weakly}\} = \{x \in H : \lim_{t\to 0} T^*(t)x = 0 \text{ weakly}\},$$

W is a closed $T(\cdot)$- and $T^*(\cdot)$-invariant subspace of H and the restriction of $T(\cdot)$ to $W^\perp$ is unitary.

PROOF. We first show that $T(t)x \to 0$ weakly for a fixed $x \in W$. By Theorem 4.8 we may assume that $x \in H_1$. If we take $S := \overline{\text{lin}}\{T(t)x : t \geq 0\}$, then by the decomposition $H = S \oplus S^\perp$ it is enough to show that $\langle T(t)x, y\rangle \to 0$ for all $y \in S$. For $y := T(t_0)x$ we obtain

$$\langle T(t)x, y\rangle = \langle T^*(t_0)T(t)x, x\rangle = \langle T(t-t_0)x, y\rangle \to 0 \quad \text{for } t_0 \leq t \to \infty,$$

since the restriction of $T(\cdot)$ to H_1 is unitary. By the density of $\text{lin}\{T(t)x : t \geq 0\}$ in S we obtain that $\langle T(t)x, y\rangle \to 0$ for every $y \in S$ and therefore $T(t)x \to 0$ weakly. Analogously, $T^*(t)x \to 0$ weakly. The converse implication, the closedness and the invariance of W are evident.

The last assertsion of the theorem follows directly from Theorem 4.8. □

Combining Theorem 4.8 and Theorem 4.10 we obtain the following decomposition into three orthogonal subspaces.

THEOREM 4.11. *Let $T(\cdot)$ be a contraction semigroup on a Hilbert space H. Then H is the orthogonal sum of three closed $T(\cdot)$- and $T^*(\cdot)$-invariant subspaces H_1, H_2 and H_3 such that the restrictions $T_1(\cdot)$, $T_2(\cdot)$ and $T_3(\cdot)$ satisfy the following.*

(1) $T_1(\cdot)$ is unitary and has no weakly stable orbit;
(2) $T_2(\cdot)$ is unitary and weakly stable;
(3) $T_3(\cdot)$ is weakly stable and completely non-unitary.

As in the discrete case (see Subsection 3.2 in Chapter 2), we see from the above theorem that a characterisation of weak stability for contraction semigroups on Hilbert spaces is of special importance. In Subsection 6.2 we will obtain an abstract answer using the spectral theorem. However, it is an open question to find a direct characterisation not involving the spectral theorem.

At the end of this subsection we present the following classical result describing description of the part $T_3(\cdot)$ in the above theorem if the semigroup consists of isometries.

THEOREM 4.12. *(Wold decomposition, see Foiaş, Sz.-Nagy* [**120**, Theorem III.9.3]*) Let $T(\cdot)$ be an isometric C_0-semigroup on a Hilbert space H. Then H can be decomposed into an orthogonal sum $H = H_0 \oplus H_1$ of $T(\cdot)$-invariant subspaces such that the restriction of $T(\cdot)$ on H_0 is a unitary semigroup and the restriction of $T(\cdot)$ on H_1 is unitarily equivalent to the unilateral shift on $L^2(\mathbb{R}_+, H_0)$ for a Hilbert space H_0. In addition,* $\dim H_0 = \dim(\operatorname{rg} V)^\perp$ *where V is the cogenerator of $T(\cdot)$.*

OPEN QUESTIONS 4.13. 1) It would be interesting to know whether there is a "general" form for weakly stable contraction semigroups, i.e., a weak analogue of Theorem 3.8.

2) No relation between weak stability of a contractive C_0-semigroup and weak stability of its cogenerator seems to be known.

4.3. Characterisation via resolvent. In this subsection we present a resolvent approach originally presented by Chill, Tomilov [**15**], see also Eisner, Farkas, Nagel, Sereny [**25**].

The main result gives some sufficient conditions for weak stability.

THEOREM 4.14. *(Chill, Tomilov* [**15**] *and Eisner, Farkas, Nagel, Sereny* [**25**]*) Let $T(\cdot)$ be a C_0-semigroup on a Banach space X with generator A satisfying $s_0(A) \le 0$. For $x \in X$ and $y \in X'$ fixed, consider the following assertions.*

(a) $$\int_0^1 \int_{-\infty}^{\infty} |\langle R^2(a+is,A)x,y\rangle|\, ds\, da < \infty.$$

(b) $$\lim_{a\to 0+} a \int_{-\infty}^{\infty} |\langle R^2(a+is,A)x,y\rangle|\, ds = 0.$$

(c) $$\lim_{t\to\infty} \langle T(t)x,y\rangle = 0$$

Then (a) $\Rightarrow$ (b) $\Rightarrow$ (c). In particular, if $T(\cdot)$ is bounded and (a) or (b) holds for all x from a dense subset of X and all y from a dense subset of X', then $(T(t))_{t\ge 0}$ is weakly stable.

Proof. First we show that (a) implies (b).

From the theory of Hardy spaces we know that the function $f : (0,1) \mapsto \mathbb{R}_+$ defined by

$$f(a) := \int_{-\infty}^{\infty} |\langle R^2(a+is,A)x,y\rangle|\, ds$$

is monotone decreasing for $a > 0$ (see Rosenblum, Rovnyak [**111**] for the theory of Hardy spaces). Assume now that (b) is not true. Then there exists a monotone decreasing null sequence $\{a_n\}_{n=1}^{\infty}$ such that

$$a_n f(a_n) \geq c \tag{62}$$

holds for some $c > 0$ and all $n \in N$.

Take now any $n, m \in \mathbb{N}$ such that $a_n \leq \frac{a_m}{2}$. By (62) and monotonicity of f we have

$$\int_{a_n}^{a_m} f(a)da \geq \sum_{k=n}^{m-1} (a_k - a_{k+1}) f(a_k) \geq \frac{c}{a_n}(a_m - a_n) = c\left(\frac{a_m}{a_n} - 1\right) \geq c$$

holds. This contradicts (a) and the implication $(a) \Rightarrow (b)$ is proved.

It remains to show that (b) implies (c).

By (b) we have for every $a > 0$

$$\int_{-\infty}^{\infty} |\langle R^2(a+is,A)x,y\rangle|\, ds < \infty.$$

Moreover, condition $s_0(A) \leq 0$ implies that the function $\lambda \mapsto \langle R^2(\lambda,A)x,y\rangle$ is bounded on every half-plane $\{\lambda : \operatorname{Re}\lambda \geq a\}$. Therefore, it belongs to the Hardy space $H^1(\{\lambda : \operatorname{Re}\lambda > a\})$ and

$$\int_{-\infty}^{\infty} |\langle R^2(a+is,A)x,y\rangle|\, ds < \infty$$

holds for all $a > 0$. This allows us to represent the semigroup as the inverse Laplace transform for all $a > \max\{0, \omega_0(T)\}$, where $\omega_0(T)$ is the growth bound of $(T(t))_{t\geq 0}$. Indeed, from e.g. Kaashoek, Verduyn Lunel [**67**] or Kaiser, Weis [**68**] it follows that

$$\langle T(t)x,y\rangle = \frac{1}{2\pi t}\int_{-\infty}^{\infty} \mathrm{e}^{(a+is)t}\langle R^2(a+is,A)x,y\rangle\, ds. \tag{63}$$

A standard application of Cauchy's theorem extends the validity of (63) to all $a > 0$. We now take $t = \frac{1}{a}$ to obtain

$$|\langle T(t)x,y\rangle| \leq a\int_{-\infty}^{\infty} |\langle R^2(a+is,A)x,y\rangle|\, ds \to 0$$

as $a \to 0+$, so $t = \frac{1}{a} \to \infty$.

The last part of the theorem follows from the fact that pointwise convergence of a an operator sequence is equivalent to the boundedness of the sequence and convergence on a dense subset of the Banach space. □

Remark 4.15. Convergence of the integrals in (b) and hence in (a) in Theorem 4.14 for all $x \in X$ and $y \in X'$ would imply $s_0(A) = \omega_0(T)$ by Corollary 2.11, and therefore is not necessary for weak stability of C_0-semigroups on Banach spaces.

A useful necessary and sufficient resolvent condition for weak stability is still unknown. In particular, it is not clear whether condition (b) in Theorem 4.14 holds for all x and y from dense subsets for weakly stable C_0-semigroups on Banach spaces (and even for unitary groups on Hilbert spaces).

5. Almost weak stability

In this section we consider a weaker concept than weak stability which is much easier to characterise and occurs under very general conditions. We will follow Eisner, Farkas, Nagel, Sereny [**25**].

5.1. Characterisation. In the following we will concentrate only on relatively weakly compact semigroups on Banach spaces. Since every weakly stable semigroup has this property, this is not a too strong restriction with respect to our aims.

We begin with a characterisation which motivates the definition of almost weakly stable C_0-semigroups.

THEOREM 5.1. *Let $(T(t))_{t\geq 0}$ be a relatively weakly compact C_0-semigroup on a Banach space X with generator A. The following assertions are equivalent.*

(i) $0 \in \overline{\{T(t)x : t \geq 0\}}^{\sigma}$ *for every $x \in X$;*
(i') $0 \in \overline{\{T(t) : t \geq 0\}}^{\mathcal{L}_\sigma}$*;*
(ii) For every $x \in X$ there exists a sequence $\{t_n\}_{n=1}^\infty$ with $t_n \to \infty$ such that $T(t_n)x \xrightarrow{\sigma} 0$;
(iii) For every $x \in X$ there exists a set $M \subset \mathbb{R}_+$ with density 1 such that $T(t)x \xrightarrow{\sigma} 0$, as $t \in M$, $t \to \infty$;
(iv) $\frac{1}{t}\int_0^t |\langle T(s)x, y\rangle|\, ds \underset{t\to\infty}{\longrightarrow} 0$ *for all $x \in X$, $y \in X'$;*
(v) $\lim_{a\to 0+} a \int_{-\infty}^{\infty} |\langle R(a+is, A)x, y\rangle|^2\, ds = 0$ *for all $x \in X$, $y \in X'$;*
(vi) $\lim_{a\to 0+} aR(a+is, A)x = 0$ *for all $x \in X$ and $s \in \mathbb{R}$;*
(vii) $P\sigma(A) \cap i\mathbb{R} = \emptyset$*, i.e., A has no purely imaginary eigenvalues.*

If, in addition, X' is separable, then the conditions above are also equivalent to

(ii) There exists a sequence $\{t_n\}_{n=1}^\infty$ with $t_n \to \infty$ such that $T(t_n) \xrightarrow{\sigma} 0$;*
(iii) There exists a set $M \subset \mathbb{R}_+$ with density 1 such that $T(t) \xrightarrow{\sigma} 0$, $t \in M$ and $t \to \infty$.*

Recall that the (asymptotic) density of a measurable set $M \subset \mathbb{R}_+$ is

$$d(M) := \lim_{t\to\infty} \frac{1}{t}\lambda([0,t] \cap M),$$

whenever the limit exists (here λ is the Lebesgue measure on $\mathbb{R}$). Note that 1 is the greatest possible density.

We will use the following elementary lemma (the proof is analogous to the discrete case, see Petersen [**105**, p. 65]).

LEMMA 5.2. *(Koopman–von Neumann, 1932) Let $f : \mathbb{R}_+ \to \mathbb{R}_+$ be continuous and bounded. The following assertions are equivalent.*

(a) $\frac{1}{t}\int_0^t f(s)\, ds \to 0$ *as $t \to \infty$;*
(b) There exists a set $M \subset \mathbb{R}_+$ with density 1 such that $f(t) \to 0$, $t \to \infty$ and $t \in M$.

PROOF. (Theorem 5.1). The proof of the implication (i') $\Rightarrow$ (i) is trivial. The implication (i) $\Rightarrow$ (ii) holds since in Banach spaces weak compactness and weak sequential compactness coincide by the Eberlein–Šmulian theorem (Theorem 0.1 in Chapter 1).

If (vii) does not hold, then (ii) cannot be true by the spectral mapping theorem for the point spectrum (see Proposition 0.28 in Chapter 1), hence (ii) $\Rightarrow$ (vii).

The implication (vii) $\Rightarrow$ (i') is the main consequence of the Jacobs–Glicksberg–de Leeuw decomposition (Theorem 0.23 in Chapter 1) and follows from the construction in its proof, see Engel, Nagel [**31**], p. 313.

This proves the equivalences (i) $\Leftrightarrow$ (i') $\Leftrightarrow$ (ii) $\Leftrightarrow$ (vii).

(vi) $\Leftrightarrow$ (vii): Since the semigroup $(T(t))_{t\geq 0}$ is bounded and mean ergodic by the mean ergodic theorem (Theorem 0.19 in Chapter 1), we have by Proposition 0.21 in Chapter 1 that the decomposition $X = \ker A \oplus \operatorname{rg} A$ holds and the limit

$$Px := \lim_{a\to 0+} aR(a, A)x$$

exists for all $x \in X$ with a projection P onto $\ker A$. Therefore, $0 \notin P\sigma(A)$ if and only if $P = 0$. Take now $s \in \mathbb{R}$. The semigroup $(\mathrm{e}^{ist}T(t))_{t\geq 0}$ is also relatively weakly compact and hence mean ergodic. Repeating the argument for this semigroup we obtain (vi) $\Leftrightarrow$ (vii).

(i') $\Rightarrow$ (iii): Let $S := \overline{\{T(t) : t \geq 0\}}^{\mathcal{L}_\sigma} \subseteq \mathcal{L}(X)$ which is a compact semi-topological semigroup if considered with the usual multiplication and the weak operator topology. By (i) we have $0 \in S$. Define the operators $\tilde{T}(t) : C(S) \to C(S)$ by

$$(\tilde{T}(t)f)(R) := f(T(t)R), \quad f \in C(S),\ R \in S.$$

By Nagel (ed.) [**95**], Lemma B-II.3.2, $(\tilde{T}(t))_{t\geq 0}$ is a C_0-semigroup on $C(S)$.

By Example 0.7 (c) in Chapter 1 the set $\{f(T(t)\,\cdot) : t \geq 0\}$ is relatively weakly compact in $C(S)$ for every $f \in C(S)$. It means that every orbit $\{\tilde{T}(t)f : t \geq 0\}$ is relatively weakly compact, and, by Lemma 0.6 in Chapter 1, $(\tilde{T}(t))_{t\geq 0}$ is a relatively weakly compact semigroup.

Denote by $\tilde{P}$ the mean ergodic projection of $(\tilde{T}(t))_{t\geq 0}$. We have $\operatorname{Fix}(\tilde{T}) = \bigcap_{t\geq 0} \operatorname{Fix}(\tilde{T}(t)) = \langle \mathbf{1} \rangle$. Indeed, for $f \in \operatorname{Fix}(\tilde{T})$ one has $f(T(t)I) = f(I)$ for all $t \geq 0$ and therefore f must be constant. Hence $\tilde{P}f$ is constant for every $f \in C(S)$. By definition of the ergodic projection

$$(\tilde{P}f)(0) = \lim_{t\to\infty} \frac{1}{t}\int_0^t \tilde{T}(s)f(0)\, ds = f(0). \tag{64}$$

Thus we have

$$(\tilde{P}f)(R) = f(0)\cdot \mathbf{1}, \qquad f \in C(S),\ R \in S. \tag{65}$$

Take now $x \in X$. By Theorem 0.5 in Chapter 1 and its proof (see Dunford, Schwartz [**23**, p. 434]), the weak topology on the orbit $\{T(t)x : t \geq 0\}$ is metrisable and coincides with the topology induced by some sequence $\{y_n\}_{n=1}^\infty \subset X' \setminus \{0\}$. Consider $f_{x,n} \in C(S)$ defined by

$$f_{x,n}(R) := |\langle Rx, \tfrac{y_n}{\|y_n\|}\rangle|, \qquad R \in S,$$

and $f_x \in C(S)$ defined by

$$f_x(R) := \sum_{n\in\mathbb{N}} \frac{1}{2^n} f_{x,n}(R), \qquad R \in S.$$

By (65) we obtain

$$0 = \lim_{t\to\infty} \frac{1}{t}\int_0^t \tilde{T}(s) f_{x,y}(I)\, ds = \lim_{t\to\infty} \frac{1}{t}\int_0^t f_x(T(s))\, ds.$$

Lemma 5.2 applied to the continuous and bounded function $\mathbb{R}_+ \ni t \mapsto f(T(t)I)$ yields a set $M \subset \mathbb{R}$ with density 1 such that

$$f_x(T(t)) \to 0 \quad \text{as } t \to \infty, \ \ t \in M.$$

By definition of f_x and by the fact that the weak topology on the orbit is induced by $\{y_n\}_{n=1}^\infty$ we have in particular that

$$T(t)x \overset{\sigma}{\to} 0 \quad \text{as } t \to \infty, \ \ t \in M.$$

This proves (iii).

(iii) $\Rightarrow$ (iv) follows directly from Lemma 5.2.

(iv) $\Rightarrow$ (vii) holds by the spectral mapping theorem for the point spectrum, see Proposition 0.28 in Chapter 1.

(iv) $\Leftrightarrow$ (v): Clearly, the semigroup $(T(t))_{t\geq 0}$ is bounded. Take $x \in X$, $y \in X'$ and let $a > 0$. By the Plancherel theorem applied to the function $t \mapsto \mathrm{e}^{-at}\langle T(t)x, y\rangle$ we have

$$\int_{-\infty}^{\infty} |\langle R(a+is, A)x, y\rangle|^2\, ds = 2\pi \int_0^\infty \mathrm{e}^{-2at} |\langle T(t)x, y\rangle|^2\, dt.$$

We obtain by the equivalence of Abel and Cesàro limits (see, e.g., Hardy [**57**], p. 136)

$$\begin{aligned}\lim_{a\to 0+} a \int_{-\infty}^{\infty} |\langle R(a+is, A)x, y\rangle|^2\, ds &= 2\pi \lim_{a\to 0+} a \int_0^\infty \mathrm{e}^{-2at} |\langle T(s)x, y\rangle|^2\, ds \\ &= \pi \lim_{t\to\infty} \frac{1}{t}\int_0^t |\langle T(s)x, y\rangle|^2\, ds. \end{aligned} \tag{66}$$

Note that for a bounded continuous function $f : \mathbb{R}_+ \to \mathbb{R}_+$ with $C := \sup f(\mathbb{R}_+)$ we have

$$\left(\frac{1}{Ct}\int_0^t f^2(s)\, ds\right)^2 \leq \left(\frac{1}{t}\int_0^t f(s)\, ds\right)^2 \leq \frac{1}{t}\int_0^t f^2(s)\, ds,$$

which together with (66) gives the equivalence of (iv) and (v).

For the additional part of the theorem suppose X' to be separable. Then so is X, and we can take dense subsets $\{x_n \neq 0 : \ n \in \mathbb{N}\} \subseteq X$ and $\{y_m \neq 0 : \ m \in \mathbb{N}\} \subseteq X'$. Consider the functions

$$f_{n,m} : S \to \mathbb{R}, \qquad f_{n,m}(R) := \left|\left\langle R\tfrac{x_n}{\|x_n\|}, \tfrac{y_m}{\|y_m\|}\right\rangle\right|, \quad n, m \in \mathbb{N},$$

which are continuous and uniformly bounded in $n, m \in \mathbb{N}$. Define the function

$$f : S \to \mathbb{R}, \qquad f(R) := \sum_{n,m\in\mathbb{N}} \frac{1}{2^{n+m}} f_{n,m}(R).$$

Then clearly $f \in C(S)$. Thus, as in the proof of the implication (i') $\Rightarrow$ (iii), i.e., using (64) we obtain

$$\frac{1}{t}\int_0^t f(T(s)I)\,ds \underset{t\to\infty}{\longrightarrow} 0.$$

Hence, applying Lemma 5.2 to the continuous and bounded function $\mathbb{R}_+ \ni t \mapsto f(T(t)I)$, we obtain the existence of a set M with density 1 such that $f(T(t)) \to 0$ as $t \to \infty$, $t \in M$. In particular, $|\langle T(t)x_n, y_m\rangle| \to 0$ for all $n, m \in \mathbb{N}$ as $t \to \infty$, $t \in M$, which, together with the boundedness of $(T(t))_{t\geq 0}$, proves the implication (i') $\Rightarrow$ (iii*). The implications (iii*) $\Rightarrow$ (ii*) $\Rightarrow$ (ii') are straightforward, hence the proof is complete. □

The above theorem shows that starting from

"no purely imaginary eigenvalues of the generator",

one arrives at properties like (iii) on the asymptotic behaviour of the orbits of the semigroup. This justifies the following name for this property.

Definition 5.3. We call a relatively weakly compact C_0-semigroup *almost weakly stable* if it satisfies condition (iii) in Theorem 5.1.

Historical remark 5.4. Theorem 5.1 and especially the implication (vii) $\Rightarrow$ (iii) were first proved for discrete semigroups and has a long history, see Remark 4.4 in Chapter 2. The conditions (i), (iii) and (iv) were studied by Hiai [**59**] also for strongly measurable semigroups. He related it to the discrete case as well. See also Kühne [**76, 77**]. The implication (vii)$\Rightarrow$(i) appears also in Ruess, Summers [**112**] in a more abstract context. Note that the equivalence (vii)$\Leftrightarrow$(iv) for contraction semigroups on Hilbert spaces is a consequence of the Wiener theorem, see Goldstein [**42**].

Remark 5.5. The conditions in Theorem 5.1 are of quite different natures. Conditions (i)–(iv) as well as (ii*) and (iii*) give information on the behaviour of the semigroup, while conditions (v)–(vii) deal with the resolvent of the generator near the imaginary axis. Among them condition (vii) apparently is the simplest to verify.

Remark 5.6. It is surprising that the equivalence (i') $\Leftrightarrow$ (v) in Theorem 5.1 is a weak analogue to the Tomilov's characterisation of strong stability given in Corollary 3.16.

One can also formulate Theorem 5.1 for single orbits. This is the following result partially due to Jan van Neerven (private communication).

Corollary 5.7. *Let A generate a C_0-semigroup $T(\cdot)$ on a Banach space X and $x \in X$. Assume that the orbit $\{T(t)x : t \geq 0\}$ is relatively weakly compact in X and that the restriction of $T(\cdot)$ to $\overline{\text{lin}}\{T(t)x : t \geq 0\}$ is bounded. Then there is a holomorphic continuation of the resolvent function $R(\cdot, T)x$ to $\{\lambda : Re(\lambda) > 0\}$ denoted by $R_x(\cdot)$ and the following assertions are equivalent.*

(i) $0 \in \overline{\{T(t)x : t \geq 0\}}^{\sigma}$;

(ii) There exists a sequence $\{t_n\}_{n=1}^{\infty}$ converging to ∞ such that $T(t_n)x \to 0$ weakly;

(iii) *There exists a set* $M \subset \mathbb{R}_+$ *with density* 1 *such that* $T(t)x \to 0$ *weakly as* $t \to \infty$, $t \in M$;

(iv) $\frac{1}{t}\int_0^t |\langle T(s)x, y\rangle| \underset{t\to\infty}{\longrightarrow} 0$ *for all* $y \in X'$;

(v) $\lim_{a\to 0+} a\int_{-\infty}^{\infty} |\langle R_x(a+is), y\rangle|^2 ds = 0$ *for all* $y \in X'$;

(vi) $\lim_{a\to 0+} aR_x(a+is) = 0$ *for all* $s \in \mathbb{R}$;

(vii) *The restriction of* A *to* $\overline{\text{lin}}\{T(t)x : t \geq 0\}$ *has no purely imaginary eigenvalue.*

Proof. For the first part of the theorem we just define

$$R_x(\lambda) := \int_0^\infty e^{-\lambda t} T(t)x dt \quad \text{whenever } Re(\lambda) > 1.$$

Denote now by Z the closed linear span of the orbit $\{T(t)x : t \geq 0\}$. Then Z is a $T(\cdot)$-invariant closed subspace of X and we can restrict $T(\cdot)$ to it. The restriction, which we will denote by $T_Z(\cdot)$, is by Lemma 0.6 relatively weakly compact as well. By the uniqueness of the Laplace transform we obtain that $R(A_Z, \lambda)x = R_x(\lambda)$ for every λ with $Re(\lambda) > 0$, where A_Z denotes the generator of $T_Z(\cdot)$.

The rest follows from the canonical decomposition $X' = Z' \oplus Z^0$ with $Z^0 := \{y \in X' : \langle z, y\rangle = 0 \text{ for all } z \in Z\}$ and Theorem 5.1. □

5.2. Example. In the next section we will see different classes of examples showing that almost weak stability does not imply weak stability. First we present a concrete example of a (positive) C_0-semigroup which is almost weakly but not weakly stable. We again follow Eisner, Farkas, Nagel, Sereny [**25**].

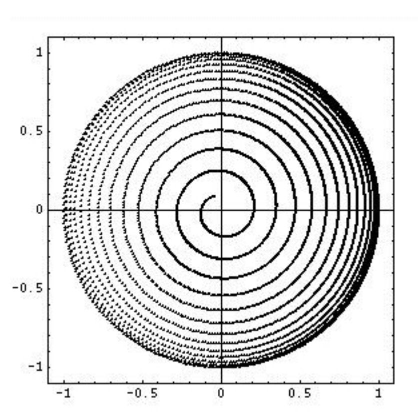
Example 5.8. As in Nagel (ed.) [**95**], p. 206, we start from a flow on $\mathbb{C}\backslash\{0\}$ with the following properties:

1) The orbits starting in z with $|z| \neq 1$ spiral towards the unit circle Γ;
2) 1 is the fixed point of φ and $\Gamma\backslash\{1\}$ is a homoclinic orbit, i.e., $\lim_{t\to-\infty} \varphi_t(z) = \lim_{t\to\infty} \varphi_t(z) = 1$ for every $z \in \Gamma$.

A concrete example comes from the differential equation in polar coordinates $(r, \omega) = (r(t), \omega(t))$:

$$\begin{cases} \dot{r} &= 1 - r, \\ \dot{\omega} &= 1 + (r^2 - 2r\cos\omega), \end{cases}$$

see the following picture.



Take $x_0 \in \mathbb{C}$ with $0 < |x_0| < 1$ and denote by $S_{x_0} := \{\varphi_t(x_0) : t \geq 0\}$ the orbit starting from x_0. Then $S := S_{x_0} \cup \Gamma$ is compact for the usual topology of $\mathbb{C}$.

We define a multiplication on S as follows. For $x = \varphi_t(x_0)$ and $y = \varphi_s(x_0)$ we put

$$xy := \varphi_{t+s}(x_0).$$

For $x \in \Gamma$, $x = \lim_{n\to\infty} x_n$, $x_n = \varphi_{t_n}(x_0) \in S_{x_0}$ and $y = \varphi_s(x_0) \in S_{x_0}$, we define $xy = yx := \lim_{n\to\infty} x_n y$. Note that by $|x_n y - \varphi_s(x)| = |\varphi_s(x_n) - \varphi_s(x)| \leq C|x_n - x| \underset{n\to\infty}{\longrightarrow} 0$ the definition is correct and satisfies

$$xy = \varphi_s(x).$$

For $x, y \in \Gamma$ we define $xy := 1$. This multiplication on S is separately continuous and makes S a semi-topological semigroup (see Engel, Nagel [**31**], Sec. V.2).

Consider now the Banach space $X := C(S)$. By Example 0.7 (c) the set

$$\{f(s\,\cdot) : s \in S\} \subset C(S)$$

is relatively weakly compact for every $f \in C(S)$. By definition of the multiplication on S this implies that

$$\{f(\varphi_t(\cdot)) : t \geq 0\}$$

is relatively weakly compact in $C(S)$. Consider the semigroup induced by the flow, i.e.,

$$(T(t)f)(x) := f(\varphi_t(x)), \quad f \in C(S),\ x \in S.$$

By the above, each orbit $\{T(t)f : t \geq 0\}$ is relatively weakly compact in $C(S)$ and hence, by Lemma 0.6, $(T(t))_{t\geq 0}$ is weakly compact. Note that the strong continuity of $(T(t))_{t\geq 0}$ follows, as shown in Nagel (ed.) [**95**], Lemma B-II.3.2, from the separate continuity of the flow.

Next, we take $X_0 := \{f \in C(S) : f(1) = 0\}$ and identify it with the Banach lattice $C_0(S \setminus \{1\})$. Then both subspaces in the decomposition $C(S) = X_0 \oplus \langle \mathbf{1} \rangle$ are invariant under $(T(t))_{t\geq 0}$. Denote by $(T_0(t))_{t\geq 0}$ the semigroup restricted to X_0 and by A_0 its generator. The semigroup $(T_0(t))_{t\geq 0}$ is still relatively weakly compact.

Since $\mathrm{Fix}(T_0) := \bigcap_{t\geq 0} \mathrm{Fix}(T_0(t)) = \{0\}$, we have that $0 \notin P\sigma(A_0)$. Moreover, $P\sigma(A_0) \cap i\mathbb{R} = \emptyset$ holds, which implies by the Jacobs-Glicksberg-de Leeuw theorem that $(T_0(t))_{t\geq 0}$ is almost weakly stable.

To see that $(T_0(t))_{t\geq 0}$ is not weakly stable it is enough to consider $\delta_{x_0} \in X_0'$. Since

$$\langle T_0(t)f, \delta_{x_0} \rangle = f(\varphi(t, x_0)),\ f \in X_0,$$

$f(\Gamma)$ always belongs to the closure of $\{\langle T_0(t)f, \delta_{x_0}\rangle : t \geq 0\}$ and hence the semigroup $(T_0(t))_{t\geq 0}$ can not be weakly stable.

We summarise the above as follows.

Theorem 5.9. *There exist a locally compact space Ω and a positive, relatively weakly compact C_0-semigroup on $C_0(\Omega)$ which is almost weakly but not weakly stable.*

The above theorem becomes particularly interesting in view of the following results; for details and discussion see Chill, Tomilov [**16**].

Theorem 5.10 (Groh, Neubrander [**49**, Theorem. 3.2]; Chill, Tomilov [**16**, Theorem. 7.7]). *For a bounded, positive, mean ergodic C_0-semigroup $(T(t))_{t\geq 0}$ on a Banach lattice X with generator $(A, D(A))$, the following assertions hold.*

(i) If $X \cong L^1(\Omega, \mu)$, then $P\sigma(A') \cap i\mathbb{R} = \emptyset$ is equivalent to the strong stability *of $(T(t))_{t\geq 0}$.*

(ii) If $X \cong C(K)$, K compact, then $P\sigma(A') \cap i\mathbb{R} = \emptyset$ is equivalent to the uniform exponential stability *of $(T(t))_{t\geq 0}$.*

Example 5.8 above shows that (ii) does not hold in spaces $C_0(\Omega)$, Ω locally compact.

6. Abstract examples

In this section we still compare weak and almost weak stability. We first discuss abstract examples arising from ergodic and measure theory showing that almost weak stability does not imply weak stability. Finally, we present category theorems analogous to the classical discrete results of Halmos and Rohlin in the ergodic theory and show that a "typical" unitary C_0-group as well as a "typical" isometric C_0-semigroup on a separable Hilbert space is almost weakly but not weakly stable.

6.1. Ergodic theory. There is a close connection between the notions of weak and strong mixing for flows in the ergodic theory and almost weak and weak stability for C_0-semigroups. In order to explain it, we follow Eisner, Farkas, Nagel, Sereny [**25**] and begin with some definitions.

A measurable measure-preserving semiflow $(\varphi_t)_{t\geq 0}$ on a probability space $(\Omega, \mathcal{M}, \mu)$ is called *strongly mixing* if $\lim_{t\to\infty} \mu(\varphi_t^{-1}(A) \cap B) = \mu(A)\mu(B)$ for any two measurable sets $A, B \in \mathcal{M}$. The semiflow $(\varphi_t)_{t\geq 0}$ is called *weakly mixing* if for all $A, B \in \mathcal{M}$ we have

$$\lim_{t\to\infty} \frac{1}{t} \int_0^t |\mu(\varphi_s^{-1}(A) \cap B) - \mu(A)\mu(B)|\, ds = 0.$$

These concepts play an essential role in ergodic theory, and we refer to the monographs Cornfeld, Fomin, Sinai [**18**], Krengel [**75**], Petersen [**105**], or Halmos [**53**] for further information. Clearly, strong mixing implies weak mixing, but the converse implication does not hold in general. However, examples of weakly but not strongly mixing semiflows are not easy to construct; see Lind [**80**] for an example and Petersen [**105**], p. 209 for a method of constructing such semiflows.

The semiflow $(\varphi_t)_{t\geq 0}$ on $(\Omega, \mathcal{M}, \mu)$ induces a semigroup of isometries $(T(t))_{t\geq 0}$ on each of the Banach spaces $X = L^p(\Omega, \mu)$ $(1 \leq p < \infty)$ by defining

$$(T(t)f)(\omega) := f(\varphi_t(\omega)), \quad \omega \in \Omega, f \in L^p(\Omega, \mu).$$

This semigroup is strongly continuous (see Krengel [**75**], §1.6, Thm. 6.13) and relatively weakly compact (use Example 0.7 (b) with $u = \mathbf{1}$ for $p = 1$). It is well-known (see, e.g., Halmos [**53**], pp. 37–38) that

$$(\varphi_t)_{t\geq 0} \text{ is strongly mixing } \iff \lim_{t\to\infty} \langle T(t)f, g\rangle = \langle Pf, g\rangle \text{ for all } f \in X,\ g \in X',$$

and

$$(\varphi_t)_{t\geq 0} \text{ is weakly mixing} \iff \lim_{t\to\infty} \frac{1}{t} \int_0^t |\langle T(s)f, g\rangle - \langle Pf, g\rangle|\, ds = 0 \text{ for all } f \in X,\ g \in X',$$

where P is the projection onto $\mathrm{Fix}(T)$ given by $Pf := \int_\Omega f\, d\mu \cdot \mathbf{1}$ for all $f \in X$. Note that in both cases $\mathrm{Fix}(T) = \langle \mathbf{1}\rangle$ holds.

Take now any semiflow $(\varphi_t)_{t\geq 0}$ which is weakly but not strongly mixing. Observe that $X = X_0 \oplus \langle \mathbf{1}\rangle$, where

$$X_0 := \left\{ f \in X \,:\, \int_\Omega f\, d\mu = 0 \right\}$$

is closed and $(T(t))_{t\geq 0}$-invariant. We denote the restriction of $(T(t))_{t\geq 0}$ to X_0 by $(T_0(t))_{t\geq 0}$ and its generator by A_0. The semigroup $(T_0(t))_{t\geq 0}$ is still relatively weakly compact and, since $P\sigma(A)\cap i\mathbb{R}=\emptyset$, it is almost weakly stable. On the other hand, $(T_0(t))_{t\geq 0}$ is not weakly stable since $(\varphi_t)_{t\geq 0}$ is not strongly mixing.

Analogously, each strongly mixing flow induces a weakly stable C_0-semigroup on X_0.

6.2. Rajchman measures. In this subsection we investigate weak stability for contraction C_0-semigroups on Hilbert spaces with help of the spectral theorem and spectral measures.

We consider first an example which, as we will see later, is important for understanding the general situation.

Example 6.1. Let μ be some finite measure on $\mathbb{R}$ and define on the space $H := L^2(\mathbb{R},\mu)$ the multiplication operator A by

$$(Af)(s) := isf(s) \quad \text{on } \mathbb{R}$$

with the maximal domain $D(A) := \{f\in H : g\in H \text{ for } g(s) := isf(s)\}$. The C_0-group $T(\cdot)$ generated by A is given by

$$(T(t)f)(s) := e^{ist}f(s) \quad s,t\in\mathbb{R},\ f\in H.$$

It is unitary and hence not strongly stable. We are interested in weak stability of $T(\cdot)$.

Note that $is\in P_\sigma(A)$ for a real s if and only if $\mu\{s\}>0$. Therefore we obtain by the Jacobs–Glicksberg–de Leeuw decomposition (Theorem 0.23 in Chapter 1) that

$$T(\cdot) \text{ is almost weakly stable} \iff \mu \text{ is continuous.}$$

On the other hand, we see that

$$\langle T(t)f,f\rangle = \int_{-\infty}^{\infty} e^{ist}|f(s)|^2 d\mu(s)$$

holds for every $f\in H$. In particular, if $T(\cdot)$ is weakly stable, then

$$\mathcal{F}\mu(t) := \int_{-\infty}^{\infty} e^{ist} d\mu(s) \to 0 \quad \text{as } t\to\infty, \tag{67}$$

where $\mathcal{F}\mu$ denotes the Fourier transform of μ.

Conversely, if (67) holds, then $\langle T(t)f,f\rangle\to 0$ as $t\to\infty$ for every function f having constant absolute value. Since the linear span of $\{e^{it\cdot}\}_{n=-\infty}^{\infty}$ is dense in H and $T(\cdot)$ is contractive, $\langle T(t)f,f\rangle\to 0$ as $t\to\infty$ for every $f\in H$, so by Theorem 4.10 $T(\cdot)$ is weakly stable. Note further that a unitary group is weakly stable for $t\to+\infty$ if and only if it is weakly stable for $t\to-\infty$.

This proves the following proposition (see Lyons [**83**]).

Proposition 6.2. *$T(\cdot)$ is weakly stable* $\iff \mathcal{F}\mu(t) \underset{t\to\infty}{\longrightarrow} 0 \iff \mathcal{F}\mu(t) \underset{|t|\to\infty}{\longrightarrow} 0.$

In harmonic analysis, this property of the measure μ got its own name. Indeed, a measure on $\mathbb{R}$ is called *Rajchman* if its Fourier transform converges to zero at infinity.

For a brief historical overview on Rajchman measures and their properties we again refer to Lyons [**83, 84**] and Goldstein [**39, 40**] (the second author uses the term "Riemann-Lebesgue measures"), but mention the following. Absolutely continuous measures are always Rajchman by the Riemann-Lebesgue lemma and all Rajchman measures are continuous. However, there are continuous measures which are not Rajchman and Rajchman measures which are not absolutely continuous (see Lyons [**84**] and Goldstein [**40**]).

It is now a consequence of the considerations above that each continuous non-Rajchman measure gives rise to an almost weakly but not weakly stable unitary C_0-group. In Engel, Nagel [**31**, p. 316] a concrete example is given of a unitary group, even with bounded generator, for which the corresponding spectral measures are not Rajchman.

To connect Rajchman measures on $\mathbb{R}$ to Rajchman measures on Γ considered in Example 5.1 in Chapter 2 we note that by the considerations above for every Rajchman measure μ on $\mathbb{R}$ the image of μ under the map $s \mapsto e^{is}$ is a Rajchman measure on Γ.

Using the spectral theorem for unitary operators on Hilbert spaces, the general question concerning weak (and almost weak) stability can be reduced to the previous example.

Indeed, consider an arbitrary contraction semigroup $T(\cdot)$ on a Hilbert space H. By Theorem 4.8 the restriction $T_1(\cdot)$ of $T(\cdot)$ to the subspace $W := \{x : \|T(t)x\| = \|T^*(t)x\| = \|x\| \ \forall t \geq 0\}$ is unitary and the restriction to $W^\perp$ is weakly stable. Therefore it remains to investigate the unitary (semi)group $T_1(\cdot)$ on weak stability.

Applying the spectral theorem (see, e.g., Halmos [**54**]) to A_1 we obtain for each $x \in H_1$ a measure μ_x on $\mathbb{R}$ such that the restriction of A_1 to $\overline{\text{lin}}\{T(t)x : n = 0, 1, 2, \dots\}$ acts as the multiplication operator $M_{is}f(s) := isf(s)$ on $L^2(\mathbb{R}, \mu_x)$. Now, applying Example 6.1, we see that

$$T(t)x \underset{t\to\infty}{\longrightarrow} 0 \ \text{ weakly } \iff \mu_x \text{ is Rajchman.}$$

Note further that by Theorem 5.1 $T(\cdot)$ is almost weakly stable if and only if μ_x is continuous for every x.

This gives a measure theoretic approach to weak stability. However, the construction uses the spectral theorem and therefore is very difficult to apply in concrete situations.

6.3. Category theorems. In this section we show that a "typical" (in the Baire category sense) unitary group as well as a "typical" isometric C_0-semigroup on a Hilbert space is almost weakly but not weakly stable. This gives an analogue to the famous category theorems of Rohlin and Halmos for "typical" discrete flows in ergodic theory, see Halmos [**53**, pp. 77–80] or the original papers by Halmos [**51**] and Rohlin [**109**].

We follow here Eisner, Sereny [**27**]. We also remark that the results and most proofs are analogous to the discrete ones presented in Subsection 5.3.1 in Chapter 2.

Throughout this section we assume the underlying Hilbert space to be separable and infinite-dimensional.

6.3.1. ***Unitary case.*** We begin with unitary groups on a (separable infinite-dimensional) Hilbert space H. The set of all unitary C_0-groups on H will be denoted by $\mathcal{U}$.

A first step in our construction is the following density result.

PROPOSITION 6.3. *For every $n \in \mathbb{N}$ the set of all periodic unitary C_0-groups with period greater than n is dense in $\mathcal{U}$ endowed with the norm topology uniform on compact time intervals.*

PROOF. Take $U(\cdot) \in \mathcal{U}$ and $n \in \mathbb{N}$. By the spectral theorem H is isomorphic to $L^2(\Omega, \mu)$ for some locally compact space Ω and finite measure μ and $U(\cdot)$ is unitary equivalent to a multiplication semigroup $\tilde{U}(\cdot)$ with

$$(\tilde{U}(t)f)(\omega) = e^{itq(\omega)}f(\omega), \quad \forall \omega \in \Omega,\ t \geq 0,\ f \in L^2(\Omega, \mu)$$

for some measurable $q : \Omega \to \mathbb{R}$.

We approximate the semigroup $\tilde{U}(\cdot)$ as follows. For $k > n$ define

$$q_k(\omega) := \frac{2\pi j}{k}, \quad \forall \omega \in q^{-1}\left(\left[\frac{2\pi j}{k}, \frac{2\pi(j+1)}{k}\right]\right),\ j \in \mathbb{Z}.$$

The multiplication operator with $e^{itq_k(\cdot)}$, $t \in \mathbb{R}$, is denoted by $\tilde{V}_k(t)$. The unitary group $\tilde{V}_k(\cdot)$ is periodic with period greater than or equal to k and therefore n. Moreover,

$$\begin{aligned} \|\tilde{U}(t)f - \tilde{V}_k(t)f\| &= \int_\Omega |e^{itq(\omega)} - e^{itq_k(\omega)}|^2 \|f(\omega)\|^2 d\omega \\ &\leq 2|t| \sup_\omega |q(\omega) - q_k(\omega)| \cdot \|f\|^2 = \frac{4\pi|t|}{k}\|f\|^2 \end{aligned}$$

holds. So $\|\tilde{U}(t) - \tilde{V}_k(t)\| \to 0$ as $k \to \infty$ uniformly in t on compact intervals and the proposition is proved. □

REMARK 6.4. By a modification of the proof of Proposition 6.3 one can show that for every $n \in \mathbb{N}$ the set of all periodic unitary groups with period greater than n with bounded generators is dense in $\mathcal{U}$ endowed with the strong topology uniform on compact time intervals.

For the second step we need the following lemma.

LEMMA 6.5. *Let H be a separable infinite-dimensional Hilbert space. Then there exists a sequence $\{(U_n(t))_{t\in\mathbb{R}}\}_{n=1}^\infty$ of almost weakly stable unitary groups with bounded generator satisfying $\|U_n(t) - I\| \to 0$ as $n \to \infty$ uniformly in t in compact intervals.*

PROOF. By isomorphy of all separable infinite-dimensional Hilbert spaces we can assume without loss of generality that $H = L^2(\mathbb{R})$ with respect to the Lebesgue measure.

Take $n \in \mathbb{N}$ and define $U_n(\cdot)$ on $L^2(\mathbb{R})$ by

$$(U_n(t)f)(s) := e^{\frac{itq(s)}{n}} f(s), \quad s \in \mathbb{R}, \quad f \in L^2(\mathbb{R}),$$

where $q : \mathbb{R} \to [0, 1]$ is a strictly monotone increasing function.

Then all $U_n(\cdot)$ are almost weakly stable by Theorem 5.1 and we have

$$\|U_n(t)-I\| = \sup_{s\in\mathbb{R}} |e^{\frac{itq(s)}{n}}-1| \le [\text{for } t\le \pi n] \le |e^{\frac{it}{n}}-1| \le \frac{2t}{n} \to 0, \quad n\to\infty,$$

uniformly in t in compact intervals. □

The topology we use on the space $\mathcal{U}$ is the topology coming from the metric

$$d(U(\cdot),V(\cdot)) := \sum_{n,j=1}^{\infty} \frac{\sup_{t\in[-n,n]} \|U(t)x_j - V(t)x_j\|}{2^j\|x_j\|} \quad \text{for } U,V\in\mathcal{U},$$

where $\{x_j\}_{j=1}^{\infty}$ is some fixed dense subset of H. Note that this topology corresponds to the strong convergence uniform on compact time intervals in $\mathbb{R}$ and is a continuous analogue of the so-called strong*-topology for operators, see, e.g., Takesaki [**121**, p. 68]).

We further denote by $\mathcal{S}_{\mathcal{U}}$ the set of all weakly stable unitary groups on H and by $\mathcal{W}_{\mathcal{U}}$ the set all almost weakly stable unitary groups on H.

The following result shows density of $\mathcal{W}_{\mathcal{U}}$ in $\mathcal{U}$.

Proposition 6.6. *The set $\mathcal{W}_{\mathcal{U}}$ of all almost weakly stable unitary groups with bounded generator is dense in $\mathcal{U}$.*

Proof. By Proposition 6.3 it is enough to approximate periodic unitary groups by almost weakly unitary groups. Let $U(\cdot)$ be a periodic unitary group with period τ. Take $\varepsilon>0$, $n\in\mathbb{N}$, $x_1,\dots,x_n\in H\setminus\{0\}$ and $t_0>0$. We have to find an almost weakly stable unitary group $V(\cdot)$ with $\|U(t)x_j-V(t)x_j\|\le\varepsilon$ for all $j=1,\dots,n$ and all t with $|t|\le t_0$.

By Engel, Nagel [**31**, Theorem IV.2.26] we have

$$H=\overline{\oplus^{\perp}_{k\in\mathbb{Z}} \ker\left(A-\frac{2\pi i k}{\tau}\right)},$$

where A denotes the generator of $U(\cdot)$. So we can assume without loss of generality that $\{x_j\}_{j=1}^{n}$ is an orthonormal system of eigenvalues of A.

Define now the $T(\cdot)$-invariant subspace $H_0:=\text{lin}\{x_1,\dots,x_n\}$ and $B:=A$ on H_0. Further, since H is separable, the decomposition

$$H=\oplus^{\perp}_{k\in\mathbb{N}} H_k$$

holds, where $\dim H_k=\dim H_0$ for every $k\in\mathbb{N}$. For a fixed orthonormal basis $\{e_j^k\}_{j=1}^n$ of each H_k we define $Be_j^k:=Bx_j$ and extend B to a bounded linear operator on H.

From the construction follows that

$$H=\ker\left(B-\frac{2\pi i\lambda_1}{\tau}\right)\oplus^{\perp}\dots\oplus^{\perp}\ker\left(B-\frac{2\pi i\lambda_n}{\tau}\right),$$

where $\frac{2\pi i\lambda_j}{\tau}$ is the eigenvalue of A (and therefore of B) corresponding to the eigenvector x_j.

Denote $X_j:=\ker\left(B-\frac{2\pi i\lambda_1}{\tau}\right)$ for every $j=1,\dots,n$. On every X_j the operator B is equal to $\frac{2\pi i\lambda_j}{\tau}I$. Note further that all X_j are infinite-dimensional. By Lemma 6.5 for every j there exists an almost weakly stable unitary group $T_j(\cdot)$ on X_j such that $\|T_j(t)-e^{\frac{2\pi t i\lambda_j}{\tau}}I\|<\varepsilon$ for every t with $|t|\le t_0$. Denote now by $T(\cdot)$ the orthogonal sum of $T_j(\cdot)$ which is a weakly stable unitary group with bounded generator. Moreover, we obtain that

$$\|T(t)x_j - U(t)x_j\| = \|T(t)x_j - e^{\frac{2\pi i t\lambda_j}{\tau}} x_j\| \le \varepsilon$$

for every t with $|t| \le t_0$ and the proposition is proved. □

We now prove a category theorem for weakly and almost weakly stable unitary groups being analogous to its discrete version given in Subsection 5.3.1 in Chapter 2.

THEOREM 6.7. *The set $\mathcal{S}_\mathcal{U}$ of weakly stable unitary groups is of first category and the set $\mathcal{W}_\mathcal{U}$ of almost weakly stable unitary groups is residual in $\mathcal{U}$.*

PROOF. We first prove that $\mathcal{S}$ is of first category in $\mathcal{U}$. Fix $x \in H$ with $\|x\| = 1$ and consider

$$M_t := \left\{ U(\cdot) \in \mathcal{U} : \ |\langle U(t)x, x\rangle| \le \frac{1}{2} \right\}.$$

Note that all sets M_t are closed.

For every weakly stable $U(\cdot) \in \mathcal{U}$ there exists $t > 0$ such that $U \in M_s$ for all $s \ge t$, i.e., $U(\cdot) \in N_t := \cap_{s\ge t} M_t$. So we obtain

$$\mathcal{S}_\mathcal{U} \subset \bigcup_{t>0} N_t. \tag{68}$$

Since all N_t are closed, it remains to show that $\mathcal{U} \setminus N_t$ is dense for every t.

Fix $t > 0$ and let $U(\cdot)$ be a periodic unitary group. Then $U(\cdot) \notin M_s$ for some $s \ge t$ and therefore $U(\cdot) \notin N_t$. Since by Proposition 6.3 periodic unitary groups are dense in $\mathcal{U}$, the set $\mathcal{S}$ is of first category.

To show that $\mathcal{W}_\mathcal{U}$ is residual we take a dense subspace $D = \{x_j\}_{j=1}^\infty$ of H and define

$$W_{jkt} := \left\{ U(\cdot) \in \mathcal{U} : \ |\langle U(t)x_j, x_j\rangle| < \frac{1}{k} \right\}.$$

All these sets are open, and therefore the sets $W_{jk} := \cup_{t>0} W_{jkt}$ are also open.

We now show the equality

$$\mathcal{W}_\mathcal{U} = \bigcap_{j,k=1}^{\infty} W_{jk}. \tag{69}$$

The inclusion "⊂" follows from the definition of almost weak stability. To prove the converse inclusion we take $U(\cdot) \notin \mathcal{W}_\mathcal{U}$ and $t > 0$. Then there exists $x \in H$ with $\|x\| = 1$ and $\varphi \in \mathbb{R}$ such that $U(t)x = e^{it\varphi}x$ for all $t > 0$, what implies $|\langle U(t)x, x\rangle| = 1$. Take $x_j \in D$ with $\|x_j - x\| \le \frac{1}{4}$. Then we have

$$\begin{aligned} |\langle U(t)x_j, x_j\rangle| &= |\langle U(t)(x - x_j), x - x_j\rangle + \langle U(t)x, x\rangle - \langle U(t)x, x - x_j\rangle - \langle U(t)(x - x_j), x\rangle| \\ &\ge 1 - \|x - x_j\|^2 - 2\|x - x_j\| > \frac{1}{3}. \end{aligned}$$

So $U(\cdot) \notin W_{j3}$ which implies $U(\cdot) \notin \cap_{j,k=1}^\infty W_{jk}$, and equality (69) holds. Therefore $\mathcal{W}_\mathcal{U}$ is residual as a dense countable intersection of open sets. □

6.3.2. ***Isometric case***. In this subsection we consider the space $\mathcal{I}$ of all isometric C_0-semigroups on H endowed with the strong topology uniform on compact time intervals and prove analogous category results as in the previous subsection. We again assume H to be separable and infinite-dimensional. Note that $\mathcal{I}$ is a complete metric space with respect to the metric given by the formula

$$d(T(\cdot), S(\cdot)) := \sum_{n,j=1}^{\infty} \frac{\sup_{t\in[0,n]} \|T(t)x_j - S(t)x_j\|}{2^j \|x_j\|} \quad \text{for } T(\cdot), S(\cdot) \in \mathcal{I},$$

where $\{x_j\}_{j=1}^{\infty}$ is a fixed dense subset of H.

We further denote by $\mathcal{S}_{\mathcal{I}}$ the set of all weakly stable and by $\mathcal{W}_{\mathcal{I}}$ the set all almost weakly stable isometric C_0-semigroups on H.

The main tool for our results in this subsection is the classical Wold decomposition, compare Theorem 4.12. As a first application of Wold's decomposition and the following easy lemma, see also Peller [**104**], we obtain a density result for periodic C_0-semigroups in $\mathcal{I}$. (Note that every isometric C_0-semigroup being periodic is authomatically unitary.)

LEMMA 6.8. *Let Y be a Hilbert space and let $R(\cdot)$ be the right shift semigroup on $H := l^2(\mathbb{N}, Y)$. Then there exists a sequence $\{U_n(\cdot)\}_{n=1}^{\infty}$ of periodic unitary operators on H converging strongly to $R(\cdot)$ uniformly on compact time intervals.*

PROOF. For every $n \in \mathbb{N}$ we define $U_n(\cdot)$ by

$$(U_n(t)f)(s) := \begin{cases} f(s), & s \geq n; \\ R_n(t)f(s), & s \in [0, n], \end{cases}$$

where $R_n(\cdot)$ denotes the n-periodic right shift on the space $L^2([0, n], Y)$. Then every $U_n(\cdot)$ is a C_0-semigroup on $L^2(\mathbb{R}_+, Y)$ which is isometric and n-periodic, and therefore unitary.

Fix $f \in L^2(\mathbb{R}_+, Y)$ and $T > 0$. Then for $t \leq T$ and $n > T$ we have

$$\begin{aligned} \|U_n(t)f - R(t)f\|^2 &= \int_n^\infty \|f(s) - f(s+n-t)\|^2 ds + \int_0^t \|f(s+n-t)\|^2 ds \\ &\leq \int_n^\infty \|f(s)\|^2 ds + \int_{n-t}^\infty \|f(s)\|^2 ds + \int_{n-t}^n \|f(s)\|^2 ds \\ &= 2\int_{n-t}^\infty \|f(s)\|^2 ds \leq 2\int_{n-T}^\infty \|f(s)\|^2 ds \underset{n\to\infty}{\longrightarrow} 0 \end{aligned}$$

uniformly on $t \in [0, T]$ and the lemma is proved. □

As a further consequence of the Wold decomposition and results in the previous subsection we obtain the following density results for periodic C_0-semigroups in $\mathcal{I}$.

PROPOSITION 6.9. *The set of all periodic unitary C_0-groups is dense in $\mathcal{I}$.*

PROOF. Let $V(\cdot)$ be an isometric semigroup on H. Then by Theorem 4.12 the orthogonal decomposition $H = H_0 \oplus H_1$ holds, where the restriction $V_0(\cdot)$ of $V(\cdot)$ to H_0 is unitary, H_1 is unitarily equivalent to $L^2(\mathbb{R}_+, Y)$ for some Y and the restriction $V_1(\cdot)$ of

$V(\cdot)$ on H_1 corresponds by this equivalence to the right shift semigroup on $L^2(\mathbb{R}_+, Y)$. By Proposition 6.3 and Lemma 6.8 we can approximate both semigroups $V_0(\cdot)$ and $V_1(\cdot)$ by unitary periodic ones and the assertion follows. □

Also the following density result for almost weakly stable semigroups is a consequence of Wold's decomposition and the results of the previous subsection.

PROPOSITION 6.10. *The set $\mathcal{W}_\mathcal{I}$ of almost weakly stable isometries is dense in $\mathcal{I}$.*

PROOF. Let V be an isometry on H, H_0, H_1 the orthogonal subspaces from Theorem 4.12 and V_0 and V_1 the corresponding restrictions of V. By Lemma 6.8 the operator V_1 can be approximated by unitary operators on H_1. The assertion now follows from Proposition 6.6. □

We now obtain, using the same idea as in the proof of Theorem 6.7 as well as Propositions 6.9 and 6.10, the following category theorem for weakly and almost weakly stable isometric C_0-semigroups.

THEOREM 6.11. *The set $\mathcal{S}_\mathcal{I}$ of all weakly stable isometric C_0-semigroups is of first category and the set $\mathcal{W}_\mathcal{I}$ of all almost weakly stable isometric C_0-semigroups is residual in $\mathcal{I}$.*

REMARKS 6.12. 1) It is not clear whether the same category phenomenon also holds for weakly and almost weakly stable *contractive* C_0-semigroups. However, we strongly conjecture that it is true, i.e., that the continuous analogue of Theorem 5.7 in Chapter 2 holds.

2) It is also very interesting to know whether there are some category theorems for Rajchman and non-Rajchman measures.

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