On Fano varieties of low Picard number with torus action

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INTRODUCTION

This thesis contributes to the explicit classification of Fano varieties of Picard number one and two.

By a Fano variety we mean a normal projective variety X admitting an ample anticanonical divisor $-\mathcal{K}_X$. Smooth Fano varieties are well understood up to dimension three. There is a single one-dimensional Fano variety, the projective line. In dimension two the smooth Fano varieties are the classically known del Pezzo surfaces: The product of two projective lines and the blow-ups of the projective plane in at most eight points in general position. In dimension three we have the classifications by Iskovskikh [52, 53] and Mori–Mukai [63]. For higher dimensions there are partial results. For instance the smooth toric Fano varieties are classified up to dimension nine [11, 59, 66, 69]. In the singular case, the situation is less explored. As a landmark, we have in dimension two the classifications by Alexeev/Nikulin [1] and Nakayama [64] of the log terminal del Pezzo surfaces X of Gorenstein index $g \leq 2$. Here, log terminal means discrepancies greater than -1 and g is the smallest positive integer with $g\mathcal{K}_X$ Cartier; so, $g = 1$ merely means that X is Gorenstein. In dimension three, the classification problem for singular Fano varieties is still widely open. An intensely studied class are the Mori–Fano threefolds, that means terminal $\mathbb{Q}$-factorial Fano threefolds of Picard number one. See in particular Prokhorov's classifications for higher index and degree cases [71–74].

Once we restrict to Fano varieties with many symmetries, the singular case is more accessible. An example class are toric Fano varieties. Here we mention Kasprzyk's classification of the canonical toric Fano threefolds [56], comprising in particular the toric Mori–Fano threefolds. In Chapter 1 we consider *fake weighted projective spaces.* They are the $\mathbb{Q}$-factorial toric Fano varieties of Picard number one. Equivalently, a d-dimensional fake weighted projective space is a quotient of $\mathbb{C}^{d+1}\backslash\{0\}$ by a diagonal action of $\mathbb{C}^* \times \Gamma$, where Γ is a finite abelian group and the factor $\mathbb{C}^*$ acts with positive weights. Via this description, fake weighted projective spaces form a natural generalization of the well known class of weighted projective spaces. They appear in toric Mori theory as the fibers of elementary contractions; see [76], as well as [27, 36]. Fake weighted projective spaces form an interesting example class for the general question of effectively bounding geometric data of a Fano variety in terms of its singularities. For instance, in the case of Gorenstein index $g = 1$, Nill [65] provides a sharp bound for the *degree* of a d-dimensional fake weighted projective space, i.e. the self intersection number $(-\mathcal{K})^d$ of its anticanonical divisor.

Our first result extends Nill's bound to fake weighted projective spaces of any Gorenstein index. For any integer $g \geq 1$ we define the *g-Sylvester sequence* $s_{g,1}, s_{g,2}, \dots$ and the *truncated g-Sylvester sequence* $t_{g,1}, t_{g,2}, \dots$ as

$$s_{g,1} \ := \ g+1, \qquad s_{g,k+1} \ := \ s_{g,k}(s_{g,k}-1)+1, \qquad t_{g,k} \ := \ s_{g,k}-1.$$

Moreover, for any $g \geq 1$ and any $d \geq 2$ we define a $(d+1)$-tuple of positive integers by

$$Q_{d,g} := \left(\frac{2t_{g,d}}{s_{g,1}}, \dots, \frac{2t_{g,d}}{s_{g,d-1}}, 1, 1 \right).$$

Theorem 1. *See Theorem 1.1.1. There are sharp upper bounds on the anticanonical degree $(-\mathcal{K})^d$ of a fake weighted projective space Z, only depending on its dimension d and its Gorenstein index g:*

(i) *If $(d, g) = (2, 1)$, then we have $(-\mathcal{K})^2 \leq 9$. Equality holds if and only if Z is isomorphic to the projective plane $\mathbb{P}^2$.*

(ii) *In all other cases the anticanonical degree is bounded from above by*

$$(-\mathcal{K})^d \ \leq \ \frac{2\,t_{g,d}^2}{g^{d+1}}.$$

Equality holds if and only if $Z \cong \mathbb{P}(3, 1, 1, 1)$ or $Z \cong \mathbb{P}(Q_{d,g})$ holds.

The combinatorial counterpart to fake weighted projective spaces are the *Fano simplices*, i.e. lattice simplices with primitive vertices, containing the origin in their interior. The Goreinstein index of a Fano simplex Δ is the Gorenstein index of the corresponding fake weighted projective space. It can be expressed combinatorially as the smallest positive integer g such that the g-fold of the dual polytope Δ^* is a lattice simplex. The anticanonical degree of Z is precisely the *normalized volume* of Δ^*, i.e. the $d!$-fold of the euclidean volume of Δ^*. Our second result concerns the volume of Δ itself. For simplices of Gorenstein index one, i.e. *reflexive* simplices, Nill [65, Thm. A] provides sharp upper bounds on the normalized volume in terms of the dimension d of the simplex. We extend Nill's bound to Fano simplices of arbitrary Gorenstein index. We write $\Delta = \Delta(P)$ with the $d \times (d+1)$ matrix P having the vertices of Δ as its columns.

Theorem 2. *See Theorem 1.1.2. There are sharp upper bounds on the normalized volume of a Fano simplex Δ, only depending on its dimension d and its Gorenstein index g:*

(i) *Assume* $(d,g) = (2,1)$. *We have the following upper bound on the normalized volume of* Δ, *which is attained if and only if* $\Delta \cong \Delta(P)$ *holds:*

$$\mathrm{Vol}(\Delta) \ \leq \ 9, \qquad\qquad P \ = \ \begin{bmatrix} 1 & 1 & -2 \\ 0 & 3 & -3 \end{bmatrix}.$$

(ii) *In all other cases the normalized volume of* Δ *is bounded from above by*

$$\mathrm{Vol}(\Delta) \ \leq \ \frac{2t_{g,d}^2}{g^2}.$$

Equality holds if and only if we have $\Delta \cong \Delta(P)$, *where* P *is one of the following:*

$$P \ = \ \begin{bmatrix} 1 & 1 & 1 & -5 \\ 0 & 2 & 2 & -4 \\ 0 & 0 & 6 & -6 \end{bmatrix},$$

$$P \ = \ \begin{bmatrix} 1 & 0 & \dots & 0 & \frac{(s_{g,1}-g)}{s_{g,1}}\frac{t_{g,d}}{g} & -\frac{(s_{g,1}+g)}{s_{g,1}}\frac{t_{g,d}}{g} \\ 0 & 1 & \ddots & \vdots & \vdots & \vdots \\ \vdots & \ddots & \ddots & 0 & \vdots & \vdots \\ \vdots & & \ddots & 1 & \frac{(s_{g,d-1}-g)}{s_{g,d-1}}\frac{t_{g,d}}{g} & -\frac{(s_{g,1}+g)}{s_{g,1}}\frac{t_{g,d}}{g} \\ 0 & \dots & \dots & 0 & \frac{t_{g,d}}{g} & -\frac{t_{g,d}}{g} \end{bmatrix}.$$

Another invariant of a Fano simplex Δ is its *multiplicity* $\mathrm{mult}(\Delta)$, i.e. the index of the sublattice generated by the vertices of Δ. This is precisely the order of the torsion part of the divisor class group of the fake weighted projective space Z associated with Δ. For Fano simplices having only the origin as an interior lattice point Averkov, Kasprzyk, Lehmann and Nill [3, 55] provide sharp upper bounds on the multiplicity in terms of the dimension. These simplices correspond to fake weighted projective space with at most canonical singularities. Our third result provides multiplicity bounds for arbitrary Fano simplices in terms of the Gorenstein index and the dimension.

Theorem 3. *See Theorem 1.1.3. There are upper bounds on the multiplicity of any Fano simplex* Δ, *only depending on its dimension* d *and its Gorenstein index* g:

(i) *Assume* $d = 3$ *and* $g \in \{1,2\}$. *We have the following upper bound on the multiplicity of* Δ, *which is attained if and only if* $\Delta \cong \Delta(P)$ *holds:*

$$\mathrm{mult}(\Delta) \ \leq \ 16g^2, \qquad\qquad P \ = \ \begin{bmatrix} 1 & 4g-3 & 4g-3 & 5-8g \\ 0 & 4g & 0 & -4g \\ 0 & 0 & 4g & -4g \end{bmatrix}.$$

(ii) *Assume* $(d,g) = (4,1)$. *We have the following upper bound on the multiplicity of* Δ, *which is attained if and only if* $\Delta \cong \Delta(P)$ *holds:*

$$\mathrm{mult}(\Delta) \ \leq \ 128, \qquad\qquad P \ = \ \begin{bmatrix} 1 & 1 & 1 & 1 & -7 \\ 0 & 2 & 2 & 2 & -6 \\ 0 & 0 & 8 & 0 & -8 \\ 0 & 0 & 0 & 8 & -8 \end{bmatrix}.$$

(iii) *In all other cases the multiplicity of Δ is bounded from above by*

$$\operatorname{mult}(\Delta) \;\le\; \frac{3t_{g,d-1}^2}{g}.$$

If equality holds, then we either have $(d,g)=(3,3)$ and $\Delta \cong \Delta(P)$ holds, where

$$P \;=\; \begin{bmatrix} 1 & 1 & 5 & -7 \\ 0 & 12 & 0 & -12 \\ 0 & 0 & 12 & -12 \end{bmatrix},$$

or there are positive integers $a_1,\dots,a_{d-1} \in \mathbb{Z}_{\ge 1}$ such that $\Delta \cong \Delta(P)$ holds, where P is the matrix:

$$\begin{bmatrix} 1 & 0 & \dots & 0 & \frac{(s_{g,1}-g)}{s_{g,1}}\frac{t_{g,d-1}}{g} & a_1 & -\left(\frac{(s_{g,1}+2g)}{s_{g,1}}\frac{t_{g,d-1}}{g}+a_1\right) \\ 0 & 1 & \ddots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \ddots & \ddots & 0 & \vdots & \vdots & \vdots \\ \vdots & & \ddots & 1 & \frac{(s_{g,d-2}-g)}{s_{g,d-2}}\frac{t_{g,d-1}}{g} & a_{d-2} & -\left(\frac{(s_{g,d-2}+2g)}{s_{g,d-2}}\frac{t_{g,d-1}}{g}+a_{d-2}\right) \\ 0 & \dots & \dots & 0 & \frac{t_{g,d-1}}{g} & a_{d-1} & -\left(\frac{t_{g,d-1}}{g}+a_{d-1}\right) \\ 0 & \dots & \dots & 0 & 0 & 3t_{g,d-1} & -3t_{g,d-1} \end{bmatrix}$$

Moreover, if g is odd, then for $k=1,\dots,d-2$ we may choose

$$a_k \;=\; \frac{(s_{g,k}-g)}{s_{g,k}}\frac{t_{g,d-1}}{g}, \qquad\qquad a_{d-1} \;=\; \frac{t_{g,d-1}}{g}.$$

For our fourth result we consider the *Mahler volume* [62] of a (not necessarily Fano) rational *IP simplex*, i.e. a rational simplex Δ, that has the origin in its interior. The Mahler volume of Δ is the product $\operatorname{MV}(\Delta) := \operatorname{Vol}(\Delta)\operatorname{Vol}(\Delta^*)$. We obtain sharp upper bounds that only depend on the dimension and the Gorenstein index. For the Gorenstein index of a rational IP simplex see Definition 1.2.3.

Theorem 4. *See Theorem 1.1.4. Let Δ a d-dimensional IP simplex of Gorenstein index g. Then we have*

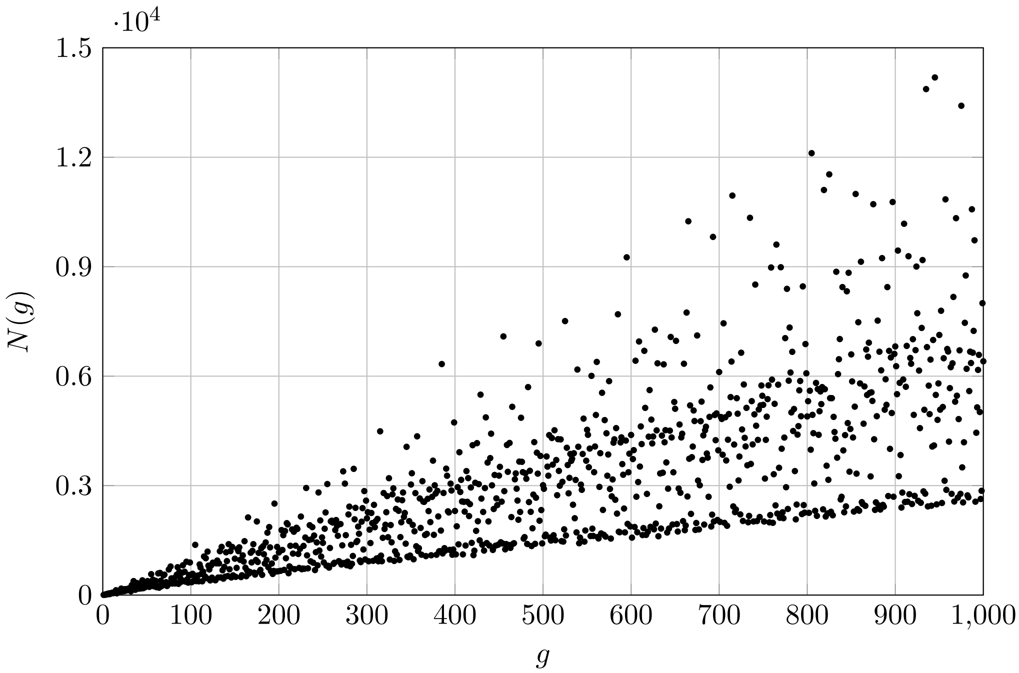
$$\operatorname{MV}(\Delta) \;\le\; \frac{t_{g,d+1}^2}{g^{d+2}}.$$

Equality holds if and only if there is $H \in \operatorname{GL}(d,\mathbb{Q})$ such that $\Delta \cong H\cdot\Delta(P)$ holds, where

$$P \;=\; \begin{bmatrix} 1 & 0 & \dots & 0 & -\frac{t_{g,d+1}}{s_{g,1}} \\ 0 & \ddots & \ddots & \vdots & \vdots \\ \vdots & \ddots & \ddots & 0 & \vdots \\ 0 & \dots & 0 & 1 & -\frac{t_{g,d+1}}{s_{g,d}} \end{bmatrix}.$$

We come to the explicit classification of Fano simplices. Conrads [28] provides an algorithm for the classification of reflexive simplices, i.e. Fano simplices of Gorenstein index one, and carries out the classification up to dimension four. Hättig, Hafner, Hausen and Springer [39] present an efficient classification procedure without the restriction on the Gorenstein index, but only for simplices of dimension two. This procedure is completely automated and the authors carry out the classification of Fano triangles up to Gorenstein index 200. We generalize and speed up the procedure from [39], which allows us to efficiently classify Fano simplices of any dimension and any Gorenstein index. This allows us to carry out the following classifications; the complete classification data, as well as the Julia code [22] to produce these results can be found at [13].

Theorem 5. *See Theorem 1.1.5. Up to isomorphy there are* 2,992,229 *Fano triangles of Gorenstein index* $g \leq 1000$*. The number of triangles* $N(g)$ *for given Gorenstein index* g *develops as follows:*



Theorem 6. *See Theorem 1.1.6. Up to isomorphy there are* 9,368,501 *Fano simplices of dimension three and Gorenstein index* $g \leq 30$*. The number of simplices* $N(g)$ *for given Gorenstein index* g *develops as follows:*

g	1	2	3	4	5	6	7	8
$N(g)$	48	435	1,703	3,042	7,506	14,527	16,627	21,789

g	9	10	11	12	13	14	15	16
$N(g)$	39,288	61,295	54,404	100,670	59,500	157,071	269,037	121,530
g	17	18	19	20	21	22	23	24
$N(g)$	133,559	319,176	173,707	473,732	523,939	401,328	332,612	695,989
g	25	26	27	28	29	30		
$N(g)$	515,042	565,225	824,950	1,007,089	513,356	1,960,325		

Theorem 7. *See Theorem 1.1.7. Up to isomorphy there are* 87,532 *Fano simplices of dimension four and Gorenstein index $g \leq 2$. Of those,* 1,561 *are of Gorenstein index $g = 1$. The remaining* 85,971 *simplices are of Gorenstein index $g = 2$.*

By the correspondence between Fano simplices and fake weighted projective spaces, Theorems 5 – 7 are also classifications of fake weighted projective spaces of respective dimension and Gorenstein index. Our classification procedure relies on the interplay between d-dimensional Fano simplices of Gorenstein index g and partitions of $1/g$ into $d+1$ unit fractions. These unit fractions impose strong divisibility properties on the data defining our simplices, making the procedure efficient. Theorems 1 – 4 are also essentially a result of the connection between (lattice) simplices and unit fraction partitions.

We turn to varieties with a torus action of complexity one. We first consider threefolds coming with an effective action of a two-dimensional torus. In this setting, the Mori–Fano threefolds have been classified by Bechtold, Hausen, Huggenberger and Nicolussi [21], using the so-called anticanonical complex: a generalization of the Fano polytope associated with a toric Fano variety. Hische and Wrobel [48, 49] successfully applied this approach to the case of higher complexity as well. A classification algorithm for Gorenstein canonical Fano varieties with a torus action of complexity one has been proposed by Ilten, Mishna and Trainor [51], using the approach via polyhedral divisors [2]. However, already in the three-dimensional case, feasability becomes a serious question. In Chapter 2 we classify the non-toric $\mathbb{Q}$-factorial log terminal Gorenstein Fano threefolds X of Picard number one that come with an effective action of a two-dimensional torus. We use the Cox ring based approach to rational varieties with a torus action of complexity one developed in [41, 46]. The Cox ring of a normal projective variety X with finitely generated divisor class group $\mathrm{Cl}(X)$ is defined as

$$\mathcal{R}(X) \;=\; \bigoplus_{\mathrm{Cl}(X)} \Gamma(X, \mathcal{O}_X(D)),$$

where we refer to [6] for the details. For our Fano threefolds X of Picard number one acted on by a two-dimensional torus, the divisor class group $\mathrm{Cl}(X)$ is of the form $\mathbb{Z} \oplus \Gamma$ with a finite abelian torsion part Γ and the Cox ring $\mathcal{R}(X)$ is a finitely generated complete intersection ring with a very specific system of trinomial relations. The variety X can be uniquely reconstructed from the list of generator degrees in $\mathrm{Cl}(X)$ and the defining relations of the Cox ring $\mathcal{R}(X)$ which allows us to encode X via these Cox ring data in a compact manner. Moreover, our variety X comes with an embedding into a fake weighted projective space, which dictates many geometrical properties of X. Similar to the toric case, the Gorenstein property of X leads to identities involving unit fractions, which eventually yield strong bounds on the Cox ring data, making a computational treatment viable.

Theorem 8. *See Classification 2.1.1. We obtain* 538 *families of non-toric, $\mathbb{Q}$-factorial, Gorenstein, log terminal Fano threefolds of Picard number one acted on effectively by a two-dimensional torus. Listed according to the possible divisor class groups, we have:*

Divisor class group	Sporadic varieties	True families
$\mathbb{Z}$	242	3 one-dimensional
$\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$	163	4 one-dimensional
$\mathbb{Z} \times (\mathbb{Z}/2\mathbb{Z})^2$	46	5 one-dimensional, 1 two-dimensional
$\mathbb{Z} \times (\mathbb{Z}/2\mathbb{Z})^3$	6	1 one-dimensional
$\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z}$	4	1 one-dimensional
$\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/6\mathbb{Z}$	1	0
$\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$	26	1 one-dimensional
$\mathbb{Z} \times (\mathbb{Z}/3\mathbb{Z})^2$	1	0
$\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z}$	18	1 one-dimensional
$\mathbb{Z} \times \mathbb{Z}/5\mathbb{Z}$	4	0
$\mathbb{Z} \times \mathbb{Z}/6\mathbb{Z}$	8	0
$\mathbb{Z} \times \mathbb{Z}/8\mathbb{Z}$	2	0

Moreover, every non-toric, $\mathbb{Q}$-factorial, Gorenstein, log terminal Fano threefold of Picard number one with an effective action of a two-dimensional torus is isomorphic to precisely one member of these 538 *families.*

Note that being Gorenstein and log terminal, all varieties from Theorem 8 are canonical. The overlap with the classification of non-toric Mori–Fano threefolds coming with an action of a two-dimensisonal torus given in [21] consists precisely of the smooth quadric in $\mathbb{P}_4$. The defining data of each of our 538 families are presented in the Classification lists 2.12.1 – 2.12.19 and can also be found in the file [15]. This file also contains geometric invariants such as genus, codimension, anticanonical self intersection and Hilbert series.

Finally we study Fano fourfolds with an effective action of a three-dimensional torus. We focus on the case of Picard number two. In this situation, Kleinschmidt [58] gave a complete description of all smooth toric varieties, which leads in particular to complete classifications of the Fano ones in any dimension. Via linear Gale duality, Kleinschmidt's approach can be turned into a study of two-dimensional combinatorial structures, see [19, Prop. 1.11]. The latter point of view applies as well to torus actions of higher complexity, i.e. higher maximal orbit codimension [41, 42, 46]. This allows for instance to extend Kleinschmidt's description to smooth varieties with a torus action of complexity one and gives complete classifications of smooth Fano varieties with torus action of complexity one in any dimension [35]. Further work in this spirit concerns smooth intrinsic quadrics, general arrangement varieties and intrinsic Grassmannians [34, 42, 75]. We leave the smooth setting and consider more generally locally factorial varieties, meaning that every Weil divisor is locally principal. Whereas in the toric case smoothness and local factoriality coincide, the latter setting turns out to be much more general for torus actions of complexity one; for instance, the varieties need not be log terminal any more and we find infinite series of non-isomorphic Fanos in fixed dimensions. We settle the case of dimension four, complexity one and a Cox ring defined by a single relation. Our main result considerably extends the corresponding one in the smooth case [35, Thm. 1.2].

Theorem 9. *See Theorem 3.1.1. There are* 447 *sporadic cases and* 106 *infinite series of locally factorial Fano fourfolds of Picard number two coming with an effective action of a three-dimensional torus and a Cox ring defined by a single relation.*

Our varieties in question are uniquely determined by the generator degrees and the relation in their Cox ring. Classification lists 3.10.1 – 3.10.11 provide the complete and redundancy free presentation of the specifying data for Theorem 3.1.1. A data file containing the complete classification data is also available at [18].

CHAPTER
ONE

LATTICE SIMPLICES AND FAKE WEIGHTED PROJECTIVE SPACES

We give sharp upper bounds on the anticanonical degree of fake weighted projective spaces, only depending on the dimension and the Gorenstein index. Furthermore, we present sharp upper bounds on the volume, Mahler volume and multiplicity for Fano simplices, also only depending on dimension and Gorenstein index. These bounds rely on the interplay between lattice simplices and unit fraction partitions. Moreover, we present an efficient procedure for explicitly classifying Fano simplicies of any dimension and Gorenstein index and we carry out the classification up to dimension four for various Gorenstein indices. This chapter is organized as follows. In Section 1.1 we present the main results of this chapter. Section 1.2 covers the basics on fake weighted projective spaces and IP simplices. In Section 1.3 we associate with every IP simplex a unit fraction partition of its Gorenstein index. The main result of this section is Proposition 1.3.3, which relates the volume and the multiplicity of a (Fano) simplex to its unit fraction partition. Section 1.4 is dedicated to providing sharp bounds on unit fraction partitions. The main result of that section is Theorem 1.4.2, which is the foundation for proving Theorems 1.1.1 – 1.1.4. Section 1.5 contains the proofs of Theorems 1.1.1 – 1.1.4. Section 1.6 contains our classification procedure for Fano simplices. In Section 1.7 we present and discuss our classification results. The results of this chapter are published in [12, 14].

1.1 Main results

A d-dimensional *fake weighted projective space* is a quotient $Z = (\mathbb{C}^{d+1}\setminus\{0\})/G$ by a diagonal action of $G := \mathbb{C}^* \times \Gamma$, where Γ is a finite abelian group and the factor $\mathbb{C}^*$ acts via positive weights. The case for trivial Γ delivers the weighted projective spaces. If moreover the weights are all equal to one, then Z is a classical projective space. Any fake weighted projective space Z is normal, $\mathbb{Q}$-factorial, of Picard number one and is a Fano variety, i.e. its anticanonical divisor $-\mathcal{K}$ is ample. Apart from the classical projective spaces, all fake weighted projective spaces are singular, but have at most abelian quotient singularities. In the case of Gorenstein index one, Nill [65] provides a bound for the *degree* of a d-dimensional fake weighted projective space Z, i.e. the self intersection number $(-\mathcal{K})^d$ of its anticanonical divisor. These degree bounds also hold more generally for any toric Fano variety with at most canonical singularities, see [9].

For our first result in this chapter we extend Nill's bound to fake weighted projective spaces of any Gorenstein index. For any integer $g \geq 1$ we define the *g-Sylvester sequence* $s_{g,1}, s_{g,2}, \ldots$ and the *truncated g-Sylvester sequence* $t_{g,1}, t_{g,2}, \ldots$ as

$$s_{g,1} \; := \; g+1, \qquad s_{g,k+1} \; := \; s_{g,k}(s_{g,k}-1)+1, \qquad t_{g,k} \; := \; s_{g,k}-1.$$

For $g = 1, 2, 3$ the beginning of the sequences $(s_{g,k})_k$ are the following

$$(s_{1,k})_k \; = \; 2, 3, 7, \ldots \qquad (s_{2,k})_k \; = \; 3, 7, 43, \ldots \qquad (s_{3,k})_k \; = \; 4, 13, 157, \ldots$$

Moreover, for any $g \geq 1$ and any $d \geq 2$ define a $(d+1)$-tuple of positive integers by

$$Q_{d,g} := \left(\frac{2t_{g,d}}{s_{g,1}}, \ldots, \frac{2t_{g,d}}{s_{g,d-1}}, 1, 1 \right).$$

For $g = 1, 2, 3$ and $d = 2, 3$ the corresponding $(d+1)$-tuples $Q_{g,d}$ are given by:

$$\begin{array}{llll} Q_{2,1} \; = \; (2,1,1), & Q_{2,2} \; = \; (4,1,1), & Q_{2,3} \; = \; (6,1,1), \\ Q_{3,1} \; = \; (6,4,1,1), & Q_{3,2} \; = \; (28,12,1,1), & Q_{3,3} \; = \; (78,24,1,1). \end{array}$$

Theorem 1.1.1. *There are sharp upper bounds on the anticanonical degree $(-\mathcal{K})^d$ of a fake weighted projective space Z, only depending on its dimension d and its Gorenstein index g.*

(i) *If $(d, g) = (2, 1)$, then we have $(-\mathcal{K})^2 \leq 9$. Equality holds if and only if Z is isomorphic to the projective plane $\mathbb{P}^2$.*

(ii) *In all other cases the anticanonical degree is bounded from above by*

$$(-\mathcal{K})^d \; \leq \; \frac{2\, t_{g,d}^2}{g^{d+1}}.$$

Equality holds if and only if $Z \cong \mathbb{P}(3, 1, 1, 1)$ or $Z \cong \mathbb{P}(Q_{d,g})$ holds.

We turn to *Fano simplices*, ie. lattice simplices Δ with primitive vertices, containing the origin in their interior. They form the combinatorial counterpart to fake weighted projective spaces, see Proposition 1.2.1. The Goreinstein index of a Fano simplex Δ is the Gorenstein index of the corresponding fake weighted projective space. It can be expressed combinatorially as the smallest positive integer g such that the g-fold of the dual polytope Δ^* is a lattice simplex. The anticanonical degree of Z is precisely the *normalized volume* of Δ^*, i.e. the $d!$-fold of the euclidean volume of Δ^*. Our second result concerns the volume of Δ itself. For simplices of Gorenstein index $g = 1$, i.e. *reflexive* simplices, Nill [65, Thm. A] provides sharp upper bounds on the normalized volume in terms of the dimension of the simplex. We extend these bounds to Fano simplices of arbitrary Gorenstein index. We write $\Delta = \Delta(P)$ with the $d \times (d+1)$ matrix P having the vertices of Δ as its columns.

Theorem 1.1.2. *There are sharp upper bounds on the normalized volume of a Fano simplex Δ, only depending on its dimension d and its Gorenstein index g:*

(i) *Assume $(d,g) = (2,1)$. We have the following upper bound on the normalized volume of Δ, which is attained if and only if $\Delta \cong \Delta(P)$ holds:*

$$\mathrm{Vol}(\Delta) \ \leq \ 9, \qquad\qquad P \ = \ \begin{bmatrix} 1 & 1 & -2 \\ 0 & 3 & -3 \end{bmatrix}.$$

(ii) *In all other cases the normalized volume of Δ is bounded from above by*

$$\mathrm{Vol}(\Delta) \ \leq \ \frac{2t_{g,d}^2}{g^2}.$$

Equality holds if and only if we have $\Delta \cong \Delta(P)$, where P is one of the following:

$$P \ = \ \begin{bmatrix} 1 & 1 & 1 & -5 \\ 0 & 2 & 2 & -4 \\ 0 & 0 & 6 & -6 \end{bmatrix},$$

$$P \ = \ \begin{bmatrix} 1 & 0 & \dots & 0 & \frac{(s_{g,1}-g)}{s_{g,1}}\frac{t_{g,d}}{g} & -\frac{(s_{g,1}+g)}{s_{g,1}}\frac{t_{g,d}}{g} \\ 0 & 1 & \ddots & \vdots & \vdots & \vdots \\ \vdots & \ddots & \ddots & 0 & \vdots & \vdots \\ \vdots & & \ddots & 1 & \frac{(s_{g,d-1}-g)}{s_{g,d-1}}\frac{t_{g,d}}{g} & -\frac{(s_{g,1}+g)}{s_{g,1}}\frac{t_{g,d}}{g} \\ 0 & \dots & \dots & 0 & \frac{t_{g,d}}{g} & -\frac{t_{g,d}}{g} \end{bmatrix}.$$

Another invariant of a Fano simplex Δ is its *multiplicity*, ie. the order of the sublattice generated by the vertices of Δ. This is also the order of the torsion part of the divisor class group of the fake weighted projective space Z corresponding to Δ. For Fano simplices having only the origin as an interior lattice point, for instance reflexive ones, [3, Thm. 1.1] provides sharp upper bounds on the multiplicity in terms of the dimension. In our third result we provide multiplicity bounds for arbitrary Fano simplices in terms of the Gorenstein index and the dimension.

Theorem 1.1.3. *There are upper bounds on the multiplicity of any Fano simplex Δ, only depending on its dimension d and its Gorenstein index g:*

(i) *Assume $d = 3$ and $g \in \{1,2\}$. We have the following upper bound on the multiplicity of Δ, which is attained if and only if $\Delta \cong \Delta(P)$ holds:*

$$\mathrm{mult}(\Delta) \ \leq \ 16g^2, \qquad\qquad P \ = \ \begin{bmatrix} 1 & 4g-3 & 4g-3 & 5-8g \\ 0 & 4g & 0 & -4g \\ 0 & 0 & 4g & -4g \end{bmatrix}.$$

(ii) *Assume $(d,g) = (4,1)$. We have the following upper bound on the multiplicity of Δ, which is attained if and only if $\Delta \cong \Delta(P)$ holds:*

$$\mathrm{mult}(\Delta) \;\le\; 128, \qquad\qquad P \;=\; \begin{bmatrix} 1 & 1 & 1 & 1 & -7 \\ 0 & 2 & 2 & 2 & -6 \\ 0 & 0 & 8 & 0 & -8 \\ 0 & 0 & 0 & 8 & -8 \end{bmatrix}.$$

(iii) *In all other cases the multiplicity of Δ is bounded from above by*

$$\mathrm{mult}(\Delta) \;\le\; \frac{3t_{g,d-1}^2}{g}.$$

If equality holds, then we either have $(d,g) = (3,3)$ and $\Delta \cong \Delta(P)$ holds, where

$$P \;=\; \begin{bmatrix} 1 & 1 & 5 & -7 \\ 0 & 12 & 0 & -12 \\ 0 & 0 & 12 & -12 \end{bmatrix},$$

or there are positive integers $a_1, \ldots, a_{d-1} \in \mathbb{Z}_{\ge 1}$ such that $\Delta \cong \Delta(P)$ holds, where P is the matrix:

$$\begin{bmatrix} 1 & 0 & \ldots & 0 & \frac{(s_{g,1}-g)}{s_{g,1}}\frac{t_{g,d-1}}{g} & a_1 & -\left(\frac{(s_{g,1}+2g)}{s_{g,1}}\frac{t_{g,d-1}}{g} + a_1\right) \\ 0 & 1 & \ddots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \ddots & \ddots & 0 & \vdots & \vdots & \vdots \\ \vdots & & \ddots & 1 & \frac{(s_{g,d-2}-g)}{s_{g,d-2}}\frac{t_{g,d-1}}{g} & a_{d-2} & -\left(\frac{(s_{g,d-2}+2g)}{s_{g,d-2}}\frac{t_{g,d-1}}{g} + a_{d-2}\right) \\ 0 & \ldots & \ldots & 0 & \frac{t_{g,d-1}}{g} & a_{d-1} & -\left(\frac{t_{g,d-1}}{g} + a_{d-1}\right) \\ 0 & \ldots & \ldots & 0 & 0 & 3t_{g,d-1} & -3t_{g,d-1} \end{bmatrix}$$

Moreover, if g is odd, then for $k = 1, \ldots, d-2$ we may choose

$$a_k \;=\; \frac{(s_{g,k}-g)}{s_{g,k}}\frac{t_{g,d-1}}{g}, \qquad\qquad a_{d-1} \;=\; \frac{t_{g,d-1}}{g}.$$

For our fourth result we consider the *Mahler volume* [62] of a (not necessarily Fano) rational *IP simplex*, ie. a rational simplex Δ, that has the origin in its interior. The Mahler volume of Δ is the product $\mathrm{MV}(\Delta) := \mathrm{Vol}(\Delta)\mathrm{Vol}(\Delta^*)$. We obtain sharp upper bounds that only depend on the dimension and the Gorenstein index. For the Gorenstein index of a rational IP simplex see Definition 1.2.3.

Theorem 1.1.4. *Let Δ a d-dimensional IP simplex of Gorenstein index g. Then we have*

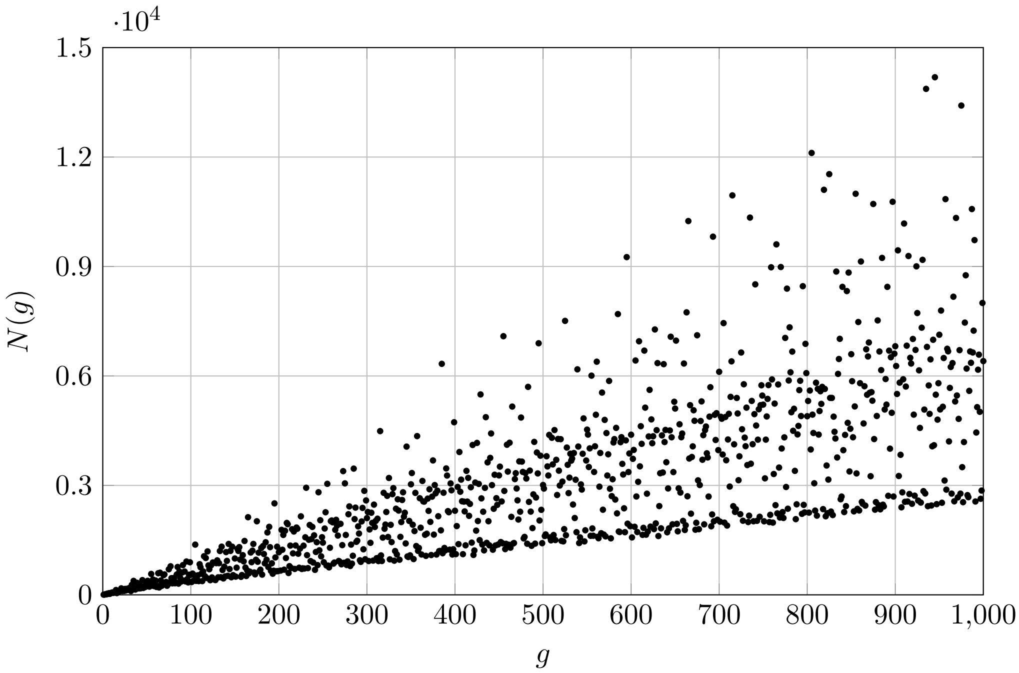
$$\mathrm{MV}(\Delta) \;\le\; \frac{t_{g,d+1}^2}{g^{d+2}}.$$

Equality holds if and only if there is $H \in \mathrm{GL}(d, \mathbb{Q})$ *such that* $\Delta \cong H \cdot \Delta(P)$ *holds, where*

$$P \;=\; \begin{bmatrix} 1 & 0 & \dots & 0 & -\frac{t_{g,d+1}}{s_{g,1}} \\ 0 & \ddots & \ddots & \vdots & \vdots \\ \vdots & \ddots & \ddots & 0 & \vdots \\ 0 & \dots & 0 & 1 & -\frac{t_{g,d+1}}{s_{g,d}} \end{bmatrix}.$$

We come to the explicit classification of Fano simplices. Hättig, Hafner, Hausen and Springer [39] present an efficient procedure for the classification of Fano triangles with fixed Gorenstein index based on unit fraction partitions. This procedure is completely automated and the authors carry out the classification of Fano triangles up to Gorenstein index 200. We generalize and speed up their procedure, which allows us to efficiently classify Fano simplices of any given dimension and Gorenstein index. This allows us to carry out the following classifications; the complete classification data, as well as the Julia code [22] to produce these results can be found at [13].

Theorem 1.1.5. *Up to isomorphy there are* 2,992,229 *Fano triangles of Gorenstein index* $g \leq 1000$. *The number of triangles* $N(g)$ *for given Gorenstein index* g *develops as follows:*



Theorem 1.1.6. *Up to isomorphy there are* 9,368,501 *Fano simplices of dimension three and Gorenstein index* $g \leq 30$. *The number of simplices* $N(g)$ *for given Gorenstein index* g *develops as follows:*

g	1	2	3	4	5	6	7	8
$N(g)$	48	435	1,703	3,042	7,506	14,527	16,627	21,789
g	9	10	11	12	13	14	15	16
$N(g)$	39,288	61,295	54,404	100,670	59,500	157,071	269,037	121,530
g	17	18	19	20	21	22	23	24
$N(g)$	133,559	319,176	173,707	473,732	523,939	401,328	332,612	695,989
g	25	26	27	28	29	30		
$N(g)$	515,042	565,225	824,950	1,007,089	513,356	1,960,325		

Theorem 1.1.7. *Up to isomorphy there are* 87,532 *Fano simplices of dimension four and Gorenstein index* $g \le 2$. *Of those,* 1,561 *are of Gorenstein index* $g = 1$. *The remaining* 85,971 *simplices are of Gorenstein index* $g = 2$.

1.2 Fake weighted projective spaces and simplices

We recall basic properties of fake weighted projective spaces and fix our notation, see also [65, Sec. 3]. The reader is assumed to be familiar with the very basics of toric geometry [30, 37]. Throughout, N is a rank d lattice for some $d \in \mathbb{Z}_{\ge 2}$. Its dual lattice is denoted by $M = \mathrm{Hom}(N, \mathbb{Z})$ with pairing $\langle \cdot\,, \cdot \rangle \colon M \times N \to \mathbb{Z}$. We write $N_\mathbb{Q} := N \otimes_\mathbb{Z} \mathbb{Q}$ and $M_\mathbb{Q} := M \otimes_\mathbb{Z} \mathbb{Q}$. Polytopes $\Delta \subseteq N_\mathbb{Q}$ are assumed to be full-dimensional. The *normalized volume* of a d-dimensional polytope Δ is $\mathrm{Vol}(\Delta) = d!\,\mathrm{vol}(\Delta)$, where $\mathrm{vol}(\Delta)$ denotes its euclidean volume. Suppose the origin $\mathbf{0} \in N_\mathbb{Q}$ is contained in the interior of Δ. Then the *dual* of Δ is the polytope

$$\Delta^* \ := \ \{u \in M_\mathbb{Q};\ \langle u, v\rangle \ge -1 \text{ for all } v \in \Delta\} \ \subseteq \ M_\mathbb{Q}.$$

For a facet F of Δ we denote by $u_F \in M_\mathbb{Q}$ the unique linear form with $\langle u_F, v \rangle = -1$ for all $v \in F$. We have

$$\Delta^* \ = \ \mathrm{conv}(\, u_F;\ F \text{ facet of } \Delta\,), \qquad \Delta \ = \ \{v \in N_\mathbb{Q};\ \langle u_F, v\rangle \ge -1,\ F \text{ facet of } \Delta\}.$$

A *lattice polytope* $\Delta \subseteq N_\mathbb{Q}$ is a polytope whose vertices are lattice points in N. An *IP polytope* is a lattice polytope that contains the origin $\mathbf{0} \in N_\mathbb{Q}$ in its interior. A *Fano* polytope is an IP polytope whose vertices are primitive lattice points. We regard two lattice polytopes $\Delta \subseteq N_\mathbb{Q}$ and $\Delta' \subseteq N'_\mathbb{Q}$ as isomorphic if there is a lattice isomorphism $\varphi \colon N \to N'$ mapping Δ bijectively to Δ'.

For an elementary proof of the following Proposition we refer to [39, Sec. 2].

Proposition 1.2.1. *The fake weighted projective spaces are precisely the toric varieties $Z = Z(\Delta)$ associated with the face fan of Fano simplices $\Delta \subseteq N_{\mathbb{Q}}$.*

Example 1.2.2. As a running example, we consider the two-dimensional Fano simplex Δ with the vertices

$$v_0 \ = \ (1,0), \qquad v_1 \ = \ (1,4), \qquad v_2 \ = \ (-7,-12).$$

The corresponding fake weighted projective plane $Z = Z(\Delta)$ has the divisor class group

$$\mathrm{Cl}(Z) \ \cong \ \mathbb{Z} \oplus \mathbb{Z}/4\mathbb{Z}.$$

Under this isomorphism the classes of the three torus-invariant divisors D_0, D_1, D_2 of Z are given by

$$[D_0] \ = \ (4,\bar{3}), \qquad [D_1] \ = \ (3,\bar{1}), \qquad [D_2] \ = \ (1,\bar{0}).$$

Denote by $C(4) \subseteq \mathbb{C}$ the group of 4-th roots of unity. The variety Z can be realized as the quotient of $\mathbb{C}^3 \backslash \{0\}$ by the action of $G = \mathbb{C}^* \times C(4)$ given by

$$(t,\eta) \cdot (z_0, z_1, z_2) \ = \ (t^4\eta^3 z_0, t^3\eta z_1, t z_2).$$

Two fake weighted projective spaces are isomorphic if and only if the corresponding Fano simplices are isomorphic. The weighted projective spaces among them correspond to Fano simplices whose vertices generate the lattice. Many geometric properties of a fake weighted projective space can be read off the corresponding simplex. Here we focus our attention on the Gorenstein index and the anticanonical degree.

Definition 1.2.3. Let $\Delta \subseteq N_{\mathbb{Q}}$ an IP polytope.

(i) The *index of rationality of* Δ is the positive integer

$$g_{\mathbb{Q}}(\Delta) \ := \ \min\{\, k \in \mathbb{Z}_{\geq 1};\ k\Delta \text{ is a lattice polytope}\,\}.$$

(ii) The *Gorenstein index of* Δ is the positive integer

$$g(\Delta) \ := \ g_{\mathbb{Q}}(\Delta) \cdot g_{\mathbb{Q}}(\Delta^*).$$

(iii) Assume Δ is a lattice simplex. Denote by $u_0, \ldots, u_d \in M_{\mathbb{Q}}$ the vertices of the dual $\Delta^* \subseteq M_{\mathbb{Q}}$. We call u_k the *k-th Gorenstein form* of Δ. We define the *k-th local Gorenstein index* g_k of Δ as the smallest positive integer such that $g_k u_k \in M$ holds.

Remark 1.2.4. If $\Delta \subseteq N_{\mathbb{Q}}$ is an IP lattice simplex with local Gorenstein indices $g_0, \ldots, g_d$, then we have $g(\Delta) = \mathrm{lcm}(g_0, \ldots, g_d)$.

Lemma 1.2.5. *The Gorenstein index of any fake weighted projective space $Z = Z(\Delta)$ coincides with the Gorenstein index $g(\Delta)$ of the corresponding Fano simplex $\Delta \subseteq N_{\mathbb{Q}}$.*

Proof. The dual polytope Δ^* coincides with the polytope $\Delta_{(-\mathcal{K})}$ associated with $-\mathcal{K}$:

$$\Delta_{(-\mathcal{K})} \ = \ \mathrm{conv}(\, m \in M_{\mathbb{Q}};\ \chi^m \in \Gamma(X, \mathcal{O}_X(-\mathcal{K}))\,).$$

The assertion thus follows from [30, Thm. 4.2.8]. □

Lemma 1.2.6. *See for instance* [37, p. 111]. *Let $Z = Z(\Delta)$ a d-dimensional fake weighted projective space. Then we have $(-\mathcal{K}_Z)^d \;=\; \mathrm{Vol}(\Delta^*)$.*

Example 1.2.7. We continue Example 1.2.2. The dual of Δ is the rational simplex Δ^* with the vertices

$$u_0 \;=\; \left(1,-\frac{1}{2}\right), \qquad u_1 \;=\; \left(-1,\frac{2}{3}\right), \qquad u_2 \;=\; (-1,0)\,.$$

Thus Δ has local Gorenstein indices $(g_0,g_1,g_2) = (2,3,1)$ and Gorenstein index $g(\Delta) = \mathrm{lcm}(g_0,g_1,g_2) = 6$. The group of Cartier divisor classes of Z is the intersection of the subgroups of $\mathrm{Cl}(Z)$ generated by the torus-invariant divisor classes:

$$\langle [D_0]\rangle \cap \langle [D_1]\rangle \cap \langle [D_2]\rangle \;=\; \langle (48,\bar{0})\rangle \;\subseteq\; \mathrm{Cl}(Z) \;=\; \mathbb{Z}\oplus\mathbb{Z}/4\mathbb{Z}.$$

An anticanonical divisor of Z is given by the sum of the torus-invariant divisors. In $\mathrm{Cl}(Z)$ we have

$$[-\mathcal{K}] \;=\; [D_0]+[D_1]+[D_2] \;=\; (8,\bar{0}).$$

The 6-fold of $\mathcal{K}$ is the smallest multiple that is Cartier. Thus Z has Gorenstein index $g = 6$.

Any weighted projective space $\mathbb{P}(q_0,\ldots,q_d)$ is up to an isomorphism uniquely determined by its weights $(q_0,\ldots,q_d)$. More generally we assign weights to any IP simplex $\Delta \subseteq N_{\mathbb{Q}}$.

Definition 1.2.8. See [28, 65]. A *weight system* Q of length d is a $(d+1)$-tuple of positive rational numbers $Q = (q_0,\ldots,q_d)$. The *total weight* of a weight system Q is the rational number $|Q| := q_0+\cdots+q_d$. A weight system Q is called *reduced* if it consists of integers and $\gcd(q_0,\ldots,q_d) = 1$ holds. A reduced weight system is called *well-formed* if $\gcd(q_j;\ j=0,\ldots,d,\ j\neq i) = 1$ holds for all $i = 0,\ldots,d$. Any weight system Q can be written as $\lambda(Q)\cdot Q^{\mathrm{red}}$ with a unique reduced weight system Q^{red} and a unique positive rational number $\lambda(Q)$. We call $\lambda(Q)$ the *factor* of Q and Q^{red} the *reduction* of Q.

Definition 1.2.9. See [28, 65]. To any IP simplex $\Delta \subseteq N_{\mathbb{Q}}$ with vertices $v_0,\ldots,v_d$ we associate a weight system by

$$Q_\Delta \;:=\; (q_0,\ldots,q_d), \qquad q_i \;:=\; |\det(\,v_j;\ j=0,\ldots,d,\ j\neq i\,)|.$$

Remark 1.2.10. Let $\Delta \subseteq N_{\mathbb{Q}}$ a d-dimensional IP simplex with vertices $v_0,\ldots,v_d$ and weight system $Q_\Delta = (q_0,\ldots,q_d)$.

(i) For the total weight we have $|Q_\Delta| = \mathrm{Vol}(\Delta)$.

(ii) If Δ is a Fano simplex, then Q_Δ^{red} is well-formed.

(iii) We have $\sum_{i=0}^d q_i v_i = 0$ and $Q_\Delta^{\mathrm{red}} = (q_0'\ldots,q_d')$ is the unique reduced weight system satisfying $\sum_{i=0}^d q_i' v_i = 0$.

(iv) For any $H \in \mathrm{GL}(d,N_{\mathbb{Q}})$ we have $Q_{H\Delta} = |\det(H)|\,Q_\Delta$. In particular, the weight systems of isomorphic IP simplices coincide up to order.

For an IP lattice simplex $\Delta \subseteq N_{\mathbb{Q}}$ we denote by $N(\Delta) \subseteq N$ the sublattice generated by the vertices of Δ. If $\Delta \subseteq N_{\mathbb{Q}}$ is any IP simplex and $\Delta' := g_{\mathbb{Q}}(\Delta)\,\Delta$, then we have

$$\lambda(\Delta) \;:=\; \lambda(Q_\Delta) \;=\; \frac{[N : N(\Delta')]}{g_{\mathbb{Q}}(\Delta)^d}.$$

In case Δ is a Fano simplex, we write $\mathrm{mult}(\Delta) := \lambda(\Delta)$ and call it the *multiplicity* of Δ. It coincides with the cardinality of the torsion part of the class group $\mathrm{Cl}(Z)$ of the associated fake weighted projective space $Z = Z(\Delta)$.

Example 1.2.11. For the two-dimensional Fano simplex Δ from Example 1.2.2 and Example 1.2.7 we have

$$Q_\Delta \;=\; (16, 12, 4), \qquad |Q_\Delta| \;=\; 32, \qquad \lambda(Q_\Delta) \;=\; 4, \qquad Q_\Delta^{\mathrm{red}} \;=\; (4, 3, 1).$$

For the sublattice $N(\Delta) \subseteq \mathbb{Z}^2$, generated by the vertices of Δ, and it's index we have

$$N(\Delta) \;=\; \langle\, (1, 0),\, (0, 4)\,\rangle, \qquad \lambda(\Delta) \;=\; [\mathbb{Z}^2 : N(\Delta)] \;=\; 4.$$

The following Proposition is a reformulation of [23, Prop. 2]. Compare also [28, 4.4–4.6].

Proposition 1.2.12. *To any reduced weight system Q of length d there exists a d-dimensional IP lattice simplex $\Delta(Q) \subseteq \mathbb{Q}^d$, unique up to an isomorphism, with $Q_{\Delta(Q)} = Q$. For any IP simplex $\Delta \in \mathbb{Q}^d$ with $(Q_\Delta)_{\mathrm{red}} = Q$ there is a linear map $H \in \mathrm{GL}(d, \mathbb{Q})$ whose determinant satisfies $|\det(H)| = \lambda(\Delta)$, such that $\Delta = H\,\Delta(Q)$ holds.*

Restricting to well-formed weight systems, we obtain the following Corollary to Proposition 1.2.12. Compare also [10, Thm. 5.4.5].

Corollary 1.2.13. *To any well-formed weight system Q of length d there exists a d-dimensional Fano simplex $\Delta(Q) \subseteq N_{\mathbb{Q}}$, unique up to an isomorphism, with $Q_{\Delta(Q)} = Q$. Any fake weighted projective space $Z = Z(\Delta)$ with $Q_\Delta^{\mathrm{red}} = Q$ is isomorphic to the quotient of the weighted projective space $\mathbb{P}(Q)$ by the action of the finite group $N/N(\Delta)$ corresponding to the inclusion $N(\Delta) \subseteq N$.*

As an immediate consequence, we can relate the Gorenstein index and the anticanonical degree of a fake weighted projective space $Z(\Delta)$ to those of the weighted projective space $\mathbb{P}(Q_\Delta^{\mathrm{red}})$.

Corollary 1.2.14. *Let $Z = Z(\Delta)$ a d-dimensional fake weighted projective space and let $Z' = \mathbb{P}(Q_\Delta^{\mathrm{red}})$ the corresponding weighted projective space. Then the Gorenstein index of Z is a multiple of the Gorenstein index of Z'. Moreover we have $\lambda(\Delta)(-\mathcal{K}_Z)^d = (-\mathcal{K}_{Z'})^d$. In particular, $(-\mathcal{K}_Z)^d = (-\mathcal{K}_{Z'})^d$ holds if and only if Z is isomorphic to Z'.*

Proof. By Proposition 1.2.12 there is a square matrix H in a lattice basis of N with determinant $\lambda(\Delta)$ such that $\Delta = H\Delta(Q)$ holds. Dualizing yields $\Delta(Q)^* = H^*\Delta^*$, where H^* denotes the transpose of H. Applying Lemma 1.2.5 and Lemma 1.2.6 yields the assertions. $\square$

Example 1.2.15. We continue Example 1.2.11. The vertices v_0', v_1', v_2' of the Fano simplex Δ' associated with the weighted projective plane $Z' = \mathbb{P}(4,3,1) = \mathbb{P}(Q_\Delta^{\mathrm{red}})$ and the vertices u_0', u_1', u_2' of it's dual simplex $(\Delta')^*$ are given by

$$v_0' \;=\; (1,0), \qquad v_1' \;=\; (0,1), \qquad v_2' \;=\; (-4,-3),$$

$$u_0' \;=\; (1,-1), \qquad u_1' \;=\; \left(-1, \frac{5}{3}\right), \qquad u_2' \;=\; (-1,-1).$$

Thus Δ' has Gorenstein index $g(\Delta') = 3$. The Gorenstein indices of Δ and Δ' satisfy $g(\Delta) = 6 = 2 \cdot 3 = 2g(\Delta')$. The simplex Δ is the image of Δ' under the linear map $\mathbb{Z}^2 \to \mathbb{Z}^2$ given by the matrix

$$H \;=\; \begin{bmatrix} 1 & 1 \\ 0 & 4 \end{bmatrix}.$$

We can recover $Z = Z(\Delta)$ as the quotient of $\mathbb{P}(4,3,1)$ by the action of the group $C(4)$ of 4-th roots of unity given in homogeneous coordinates by

$$\eta \cdot [z_0, z_1, z_2] \;=\; [\eta^3 z_0, \eta z_1, z_2].$$

Using Lemma 1.2.6, for the degrees of Z and Z' we obtain

$$(-\mathcal{K}_{Z'})^2 \;=\; \mathrm{Vol}((\Delta')^*) \;=\; \frac{16}{3} \;=\; 4 \cdot \frac{4}{3} \;=\; \lambda(\Delta)\mathrm{Vol}(\Delta^*) \;=\; \lambda(\Delta)(-\mathcal{K}_Z)^2.$$

1.3 Unit fraction partitions

We associate with every IP simplex a unit fraction partition of its Gorenstein index, see Proposition 1.3.2. The main result of this section is Proposition 1.3.3, which relates the volume and the factor of an IP simplex to its unit fraction partition.

Definition 1.3.1. Let $g \in \mathbb{Z}_{\geq 1}$. A tuple $A = (\alpha_1, \ldots, \alpha_n) \in \mathbb{Z}_{\geq 1}^n$ is called a *unit fraction partition* (ufp for short) *of* g of length n, if the following holds:

$$\frac{1}{g} \;=\; \sum_{k=1}^{n} \frac{1}{\alpha_k}.$$

A tuple $A = (\alpha_1, \ldots, \alpha_n) \in \mathbb{Z}_{\geq 1}^n$ is called a *unit fraction partition* if it is a ufp of g for some $g \in \mathbb{Z}_{\geq 1}$. For a unit fraction partition $A = (\alpha_1, \ldots, \alpha_n)$ of g we call

$$t_A \;:=\; \mathrm{lcm}(\alpha_1, \ldots, \alpha_n), \qquad \lambda(A) \;:=\; \gcd(g, \alpha_1, \ldots, \alpha_n), \qquad A^{\mathrm{red}} \;:=\; A/\lambda(A)$$

the *total weight*, the *factor* and the *reduction* of A, respectively. A unit fraction partition A is called *reduced* if it coincides with its reduction. It is called *well-formed* if $\alpha_i \mid \mathrm{lcm}(\alpha_j\,;\, j \neq i)$ holds for all $i = 1, \ldots, n$.

Proposition 1.3.2. *Let $\Delta \subseteq N_{\mathbb{Q}}$ a d-dimensional IP simplex of Gorenstein index g with weight system $Q_\Delta = (q_0, \dots, q_d)$. Then*

$$A(\Delta) \ := \ \left(\frac{g|Q_\Delta|}{q_0}, \dots, \frac{g|Q_\Delta|}{q_d}\right)$$

is a unit fraction partition of g of length $d+1$. We call it the unit fraction partition of g associated with Δ.

Proof. The entries of $A(\Delta)$ are positive. We show that they are integers. Denote by $v_0, \dots, v_d \in N_{\mathbb{Q}}$ the vertices of Δ. For $0 \leq i \leq d$ let $F_i = \operatorname{conv}(v_0, \dots, \hat{v}_i, \dots, v_d)$ the i-th facet of Δ, where $\hat{v}_i$ means that v_i is omitted. For all $i = 0, \dots, d$ we have

$$0 \ = \ \sum_{j=0}^{d} q_j g \langle u_{F_i}, v_j \rangle \ = \ g\langle u_{F_i}, v_i \rangle q_i - g \sum_{\substack{j=0,\\ j \neq i}}^{d} q_j \ = \ (g\langle u_{F_i}, v_i\rangle + 1) q_i - g|Q_\Delta|.$$

By definition of the Gorenstein index, $g\langle u_{F_i}, v_i \rangle = \langle g_{\mathbb{Q}}(\Delta^*) u_{F_i}, g_{\mathbb{Q}}(\Delta) v_i \rangle$ is an integer. Thus q_i divides $g|Q_\Delta|$, which means that $A(\Delta)$ consists of integers. Now summing over the reciprocals of $A(\Delta)$ we see that it is in fact a ufp of g. □

The following Proposition establishes a connection between geometric properties of an IP simplex Δ and its associated unit fraction partition. It can be seen as an extension of [65, Prop. 4.5] to the case of non-reflexive IP simplices.

Proposition 1.3.3. *Let $\Delta \subseteq N_{\mathbb{Q}}$ a d-dimensional IP simplex of Gorenstein index $g(\Delta) = g$ with associated unit fraction partition $A(\Delta) = (\alpha_0, \dots, \alpha_d)$ of g. Then $A(\Delta) = A(\Delta^*)$ holds and we have:*

(i)
$$\operatorname{Vol}(\Delta)\operatorname{Vol}(\Delta^*) \ = \ \frac{1}{g^{d+1}} \alpha_0 \cdots \alpha_d,$$

(ii)
$$\lambda(\Delta^*)\operatorname{Vol}(\Delta) \ = \ \lambda(\Delta)\operatorname{Vol}(\Delta^*) \ = \ \frac{1}{g^d} \frac{\alpha_0 \cdots \alpha_d}{\operatorname{lcm}(\alpha_0, \dots, \alpha_d)},$$

(iii)
$$\lambda(\Delta)\lambda(\Delta^*) \ = \ \frac{1}{g^{d-1}} \frac{\alpha_0 \cdots \alpha_d}{\operatorname{lcm}(\alpha_0, \dots, \alpha_d)^2}.$$

Note that the left hand side of equations (i)–(iii) in Proposition 1.3.3 only depends on the simplex Δ, while the right hand side only depends on the unit fraction partition $A(\Delta)$.

Example 1.3.4. We continue Example 1.2.15. The Fano simplex Δ has Gorenstein index $g = 6$ and weight system $Q_\Delta = (16, 12, 4)$. It's unit fraction partition is given by

$$A(\Delta) \ = \ (12, 16, 48).$$

This is a unit fraction partition of $g = 6$. Indeed, we have

$$\frac{1}{6} \;=\; \frac{1}{12} + \frac{1}{16} + \frac{1}{48}.$$

The IP simplices Δ and Δ have normalized volumes and factors

$$\mathrm{Vol}(\Delta) \;=\; 32, \qquad \lambda(\Delta) \;=\; 4, \qquad \mathrm{Vol}(\Delta^*) \;=\; \frac{4}{3}, \qquad \lambda(\Delta^*) \;=\; \frac{1}{6}.$$

Plugging these values into the formulas given in Proposition 1.3.3, we obtain

$$\begin{aligned}
\mathrm{Vol}(\Delta)\mathrm{Vol}(\Delta^*) \;&=\; 32 \cdot \frac{4}{3} \;=\; \frac{128}{3} \;=\; \frac{1}{6^3} \cdot 12 \cdot 16 \cdot 48 \;=\; \frac{1}{g^{d+1}}\, \alpha_0 \cdots \alpha_d,\\
\lambda(\Delta^*)\mathrm{Vol}(\Delta) \;=\; \lambda(\Delta)\mathrm{Vol}(\Delta^*) \;&=\; \frac{16}{3} \;=\; \frac{1}{6^2} \cdot \frac{12 \cdot 16 \cdot 48}{48} \;=\; \frac{1}{g^d} \frac{\alpha_0 \cdots \alpha_d}{\mathrm{lcm}(\alpha_0, \ldots, \alpha_d)},\\
\lambda(\Delta)\lambda(\Delta^*) \;&=\; 4 \cdot \frac{1}{6} \;=\; \frac{2}{3} \;=\; \frac{1}{6^3} \cdot \frac{12 \cdot 16 \cdot 48}{48^2} \;=\; \frac{1}{g^{d-1}} \frac{\alpha_0 \cdots \alpha_d}{\mathrm{lcm}(\alpha_0, \ldots, \alpha_d)^2}.
\end{aligned}$$

For the proof of Proposition 1.3.3, we need the following Lemma 1.3.6, which is originally [65, Prop. 3.6].

Definition 1.3.5. See [65, Def. 3.4]. For any weight system $Q = (q_0, \ldots, q_d)$ set

$$m_Q \;:=\; \frac{|Q|^{d-1}}{q_0 \cdots q_d}.$$

Lemma 1.3.6. *See* [65, Prop. 3.6]. *For any d-dimensional IP simplex Δ we have*

$$Q_{\Delta^*} \;=\; m_{Q_\Delta} Q_\Delta.$$

Proof of Proposition 1.3.3. By Lemma 1.3.6 the weight systems Q_Δ and Q_{Δ^*} differ only by a factor. Moreover, the simplices Δ and Δ^* have the same Gorenstein index. Thus the associated unit fraction partitions $A(\Delta)$ and $A(\Delta^*)$ coincide. Item (i) is an immediate consequence of (ii) and (iii). We prove (ii). Remark 1.2.10 (i) together with Lemma 1.3.6 yields

$$\mathrm{Vol}(\Delta^*) \;=\; |Q_{\Delta^*}| \;=\; \frac{|Q_\Delta|^d}{q_0 \cdots q_d}.$$

We multiply this by the multiplicity $\lambda(\Delta)$ and use the identity $\lambda(\Delta) = g|Q_\Delta|/t_{A(\Delta)}$ to obtain

$$\lambda(\Delta)\mathrm{Vol}(\Delta^*) \;=\; \frac{g|Q_\Delta|}{t_{A(\Delta)}} \frac{|Q_\Delta|^d}{q_0 \cdots q_d} \;=\; \frac{1}{g^d} \frac{\alpha_0 \cdots \alpha_d}{\mathrm{lcm}(\alpha_0, \ldots, \alpha_d)}.$$

Switching the roles of Δ and Δ^* and using the fact that they have the same unit fraction partition, we obtain $\lambda(\Delta)\mathrm{Vol}(\Delta^*) = \lambda(\Delta^*)\mathrm{Vol}(\Delta)$. We prove (iii). Let $Q_\Delta^{\mathrm{red}} = (q_0', \ldots, q_d')$. With Lemma 1.3.6 we obtain:

$$\lambda(\Delta^*) \;=\; \lambda(\Delta) m_{Q_\Delta} \;=\; \lambda(\Delta) \frac{|Q_\Delta|^{d-1}}{q_0 \cdots q_d} \;=\; \frac{1}{\lambda(\Delta)} \frac{|Q_\Delta^{\mathrm{red}}|^{d-1}}{q_0' \cdots q_d'} \;=\; \frac{1}{\lambda(\Delta)} m_{Q_\Delta^{\mathrm{red}}},$$

Multiplying both sides by $\lambda(\Delta)$ yields the identity $\lambda(\Delta)\lambda(\Delta^*) = m_{Q_\Delta^{\mathrm{red}}}$. We obtain:

$$m_{Q_\Delta^{\mathrm{red}}} \;=\; \frac{1}{|Q_\Delta^{\mathrm{red}}|^2}\frac{|Q_\Delta^{\mathrm{red}}|^{d+1}}{q_0'\cdots q_d'} \;=\; \frac{\lambda(\Delta)^2}{|Q_\Delta|^2}\frac{|Q_\Delta|^{d+1}}{q_0\cdots q_d} \;=\; \frac{1}{g^{d-1}}\frac{\alpha_0\cdots\alpha_d}{\mathrm{lcm}(\alpha_0,\ldots,\alpha_d)^2}.$$

□

It will be convenient to assign unit fraction partitions directly to weight systems and vice versa.

Definition 1.3.7. The *index of a weight system* $Q = (q_0, \ldots, q_d)$ is the positive integer

$$g(Q) \;:=\; \min\left(k \in \mathbb{Z}_{\geq 1};\; k|Q|/q_i \in \mathbb{Z} \text{ for all } i = 0, \ldots, d \right).$$

Remark 1.3.8. The index $g(Q_\Delta)$ of the weight system Q_Δ of an IP simplex Δ is always a divisor of it's Gorenstein index $g(\Delta)$. They might coincide, however frequently $g(Q_\Delta)$ is a true divisor of $g(\Delta)$.

Example 1.3.9. We continue Example 1.3.4. The weight system of Δ is $Q_\Delta = (16, 12, 4)$. It has index $g(Q_\Delta) = 3$, which is a proper divisor of the Gorenstein index of Δ.

Proposition 1.3.10. *Let $Q = (q_0, \ldots, q_d)$ a weight system of length d and let $A = (\alpha_0, \ldots, \alpha_d)$ a unit fraction partition of $g \in \mathbb{Z}_{\geq 1}$ of length $d+1$. Set*

$$A(Q) \;:=\; \left(\frac{g|Q|}{q_0}, \ldots, \frac{g|Q|}{q_d}\right), \qquad\qquad Q(A) \;:=\; \left(\frac{t_A}{\alpha_0}, \ldots, \frac{t_A}{\alpha_d}\right).$$

Then $A(Q)$ is a reduced unit fraction partition of $g(Q)$ and $Q(A)$ is a reduced weight system of length d and index $g(Q(A)) = g/\lambda(A)$. Moreover, we have

$$Q(A(Q)) \;=\; Q^{\mathrm{red}}, \qquad\qquad A(Q(A)) \;=\; A^{\mathrm{red}}$$

and this correspondence respects well-formedness.

Example 1.3.11. We continue Example 1.3.9. We have the weight system and the uf-partition of $g(\Delta) = 6$:

$$Q \;=\; Q_\Delta \;=\; (16, 12, 4), \qquad A \;=\; A(\Delta) \;=\; (12, 16, 48).$$

The weight system Q has index $g(Q) = 3$. Total weight, factor and reduction of A are given by

$$t_A \;=\; 48, \qquad \lambda_A \;=\; 2, \qquad A_{\mathrm{red}} \;=\; (6, 8, 24).$$

With respect to Proposition 1.3.10, we obtain the unit fraction partition and the weight system

$$A(Q) \;=\; (6, 8, 24) \;=\; A_{\mathrm{red}}, \qquad Q(A) \;=\; (4, 3, 1) \;=\; Q_{\mathrm{red}}.$$

For the proof of Proposition 1.3.10 we need the following Lemmas.

Lemma 1.3.12. *For $g, \alpha_1, \ldots, \alpha_n \in \mathbb{Z}$ set*

$$G(g; \alpha_1, \ldots, \alpha_n) \ := \ \begin{bmatrix} (\alpha_1 - g) & -g & \cdots & -g \\ -g & (\alpha_2 - g) & \ddots & \vdots \\ & & \ddots & \\ \vdots & \ddots & (\alpha_{n-1} - g) & -g \\ -g & \cdots & -g & (\alpha_n - g) \end{bmatrix}.$$

Then

$$\det(G(g; \alpha_1, \ldots, \alpha_n)) \ = \ \alpha_1 \cdots \alpha_n - g \sum_{i=1}^{n} \prod_{j \neq i} \alpha_j.$$

Proof. We prove the Lemma by induction on n. The cases $n = 1$ and $n = 2$ are verified by direct computation. Let $n \geq 3$. Subtracting the second to last row of $G := G(g; \alpha_1, \ldots, \alpha_n)$ from the last row, we obtain

$$\det(G) \ = \ \alpha_n \det(G') + \alpha_{n-1} \det(G''),$$

where $G' = G(g; \alpha_1, \ldots, \alpha_{n-1})$ and $G'' = G(g; \alpha_1, \ldots, \alpha_{n-2}, 0)$. By the induction hypothesis we have

$$\det(G') \ = \ \alpha_1 \cdots \alpha_{n-1} - g \sum_{i=1}^{n-1} \prod_{j \neq i} \alpha_j, \qquad \det(G'') \ = \ -g\alpha_1 \cdots \alpha_{n-2}.$$

Plugging these into the equation for $\det(G)$ yields the assertion. $\square$

Lemma 1.3.13. *For any unit fraction partition $(\alpha_1, \ldots, \alpha_n)$ of g and any $1 \leq k < n$ we have*

$$\det(G(g; \alpha_1, \ldots, \alpha)) \ \geq \ 1.$$

Proof. For any $1 \leq k < n$ we have $1/\alpha_1 + \cdots + 1/\alpha_k < 1/g$. Multiplying both sides by $g\, \alpha_1 \cdots \alpha_k$ and subtracting the left hand side we obtain

$$0 \ < \ \alpha_1 \cdots \alpha_k - g \sum_{i=1}^{k} \prod_{j \neq i} \alpha_j \ = \ \det(G(g; \alpha_1, \ldots, \alpha_k)).$$

Since the determinant of $G(g; \alpha_1, \ldots, \alpha_k)$ is an integer, it must be at least one. $\square$

Proof of Proposition 1.3.10. We show that $A(Q)$ is a reduced unit fraction partition of $g(Q)$. As q_i divides $g(Q)\,|Q|$, the tuple $A(Q)$ consists of positive integers. Summing over the reciprocals of $A(Q)$ shows that it is a ufp of $g(Q)$. Assume $A(Q)$ is not reduced and let A' its reduction. Then A' is a ufp of g' for some $g' < g(Q)$. This means that each q_i divides $g'|Q|$, which contradicts the minimality of the index $g(Q)$. Thus $A(Q)$

is reduced. The fact that $Q(A)$ is a reduced weight system of index $g/\lambda(A)$ follows directly from the definition of t_A. We prove the last assertion of Proposition 1.3.10. Let $Q = (q_0, \dots, q_d)$ a weight system of length d and index g and write $A(Q) = (\alpha_0, \dots, \alpha_d)$. To show that $Q(A(Q)) = Q^{\mathrm{red}}$ holds we consider the matrix $G = G(g; \alpha_0, \dots, \alpha_d)$ as defined in Lemma 1.3.12. Both Q and $Q(A(Q))$ are contained in its kernel and the latter weight system is reduced. So it suffices to show that G is of rank d. This follows from Lemma 1.3.13, as the minor of G, obtained by deleting the last row and column, equals $\det(G(g; \alpha_0, \dots, \alpha_{d-1}))$. Now let $A = (\alpha_0, \dots, \alpha_d)$ a ufp of g of length $d+1$. Write $Q(A) = (q_0, \dots, q_d)$ and let $A(Q) = (\alpha'_0, \dots, \alpha'_d)$. This is a ufp of $g(Q)$. Note that each q_i divides $g\,|Q|$ as well as $g(Q)|Q|$. The minimality of the index of Q implies that $g(Q)$ divides g. With $\lambda := g/g(Q)$ we obtain

$$\lambda \alpha'_i \;=\; \frac{g}{g(Q)} \frac{g(Q)|Q|}{q_i} \;=\; \frac{g\, t_{A(Q)} \alpha_i}{g\, t_{A(Q)}} \;=\; \alpha_i,$$

which yields $A = \lambda A(Q)$. As $A(Q)$ is reduced, we obtain $A(Q) = A^{\mathrm{red}}$. Now let $Q = (q_0, \dots, q_d)$ a reduced weight system of length d and write $A(Q) = (\alpha_0, \dots, \alpha_d)$. We have $q_i = t_{A(Q)}/\alpha_i$. The weight system Q is well-formed if and only if for all $i = 0, \dots, d$ we have

$$\prod_{j \neq i} \alpha_j \;=\; t_{A(Q)} \gcd\left(\prod_{k \neq i,j} \alpha_k;\ j \neq i \right).$$

This in turn is equivalent to the well-formedness of $A(Q)$. □

1.4 Sharp bounds on unit fraction partitions

For a unit fraction partition $A = (\alpha_1, \dots, \alpha_n)$ of length n we consider

$$F_k(A) \;:=\; \frac{\alpha_1 \cdots \alpha_n}{\operatorname{lcm}(\alpha_1, \dots, \alpha_n)^{n-k}}.$$

For $k = n-2, n-1, n$ these are the right hand side expressions in the identities from Proposition 1.3.3. We give sharp bounds on these expressions among all unit fraction partitions of g and completely describe the unit fraction partitions attaining those bounds.

Definition 1.4.1. The *g-Sylvester sequence* $S_g = (s_{g,1}, s_{g,2}, \dots)$ and the *truncated g-Sylvester sequence* $T_g = (t_{g,1}, t_{g,2}, \dots)$ for a positive integer g are given by

$$s_{g,1} \;:=\; g+1, \qquad s_{g,k+1} \;:=\; s_{g,k}(s_{g,k}-1)+1, \qquad t_{g,k} := s_{g,k} - 1.$$

Theorem 1.4.2. *Let $g \geq 1$ and let $n \geq 3$. Let $A = (\alpha_1, \dots, \alpha_n)$ a unit fraction partition of g with $\alpha_1 \leq \cdots \leq \alpha_n$. For the value of F_k on A the following hold:*

(i) *$F_n(A) \leq t_{g,n}^2/g$ and equality holds if and only if $A = (s_{g,1}, \dots, s_{g,n-1}, t_{g,n})$.*

(ii) *Assume* $k = n-1$. *If* $(n,g) = (3,1)$, *then we have* $F_{n-1}(A) \leq 9$ *and equality holds if and only if* $A = (3,3,3)$. *In all other cases we have*

$$F_{n-1}(A) \;\leq\; \frac{2\,t_{g,n-1}^2}{g}.$$

Equality holds if and only if A is one of the following unit fraction partitions:

$$(6,6,6), \qquad (2,6,6,6), \qquad (s_{g,1},\ldots,s_{g,n-2},2t_{g,n-1},2t_{g,n-1}).$$

(iii) *Assume* $k = n-2$. *If* $n = 4$ *and* $g \in \{1,2\}$, *then we have* $F_{n-2}(A) \leq 16g^2$ *and equality holds if and only if* $A = (4g,4g,4g,4g)$. *If* $(n,g) = (5,1)$, *then we have* $F_{n-2}(A) \leq 128$ *and equality holds if and only if* $A = (2,8,8,8,8)$. *In all other cases we have*

$$F_{n-2}(A) \;\leq\; \frac{3\,t_{g,n-2}^2}{g}.$$

Equality holds if and only if A is one of the following unit fraction partitions:

$$(12,12,12,12), \qquad (s_{g,1},\ldots,s_{g,n-3},3t_{g,n-2},3t_{g,n-2},3t_{g,n-2}).$$

Remark 1.4.3. In the literature the sequence $S_1 = (s_{1,1}, s_{1,2}, \ldots)$ is known as *Sylvester's sequence*, see for instance [67]. Our naming for the sequences S_g and T_g is derived from that. We list some properties of the sequences S_g and T_g that we will use frequently.

(i) For any $n \geq 1$ we have

$$\frac{1}{g} \;=\; \frac{1}{s_{g,1}} + \cdots + \frac{1}{s_{g,n-1}} + \frac{1}{t_{g,n}}.$$

(ii) For any $n \geq 1$ we have

$$\frac{g}{t_{g,n}} \;=\; \frac{1}{s_{g,1}} \cdots \frac{1}{s_{g,n-1}}.$$

(iii) For any $g, n \geq 1$ we have $s_{g,n+1} > s_{g,n}$ and $s_{g+1,n} > s_{g,n}$.

(iv) For $i \neq j$ we have $\gcd(s_{g,i}, s_{g,j}) = 1$.

The strategy for the proof of Theorem 1.4.2 is as follows: For given g and n we define a certain compact subset $A_g^n \subseteq \mathbb{R}^n$, which has the property that for any unit fraction partition $A = (\alpha_1, \ldots, \alpha_n)$ of g with $\alpha_1 \leq \cdots \leq \alpha_n$, the point $(1/\alpha_1, \ldots, 1/\alpha_n)$ is contained in A_g^n. For $k \in \{n-2, n-1, n\}$ we minimize the function $f_k(x) := x_1 \cdots x_k$ on A_g^n and show that it attains its minimum precisely at the points corresponding to the unit fraction partitions listed in Theorem 1.4.2. This strategy for minimizing functions on unit fraction partitions was first used by Izhboldin and Kurliandchik in [54], see also [4, 65] for generalizations. In [4] the authors call this type of optimization problems *Izhboldin-Kurliandchik problems*. In the following we adopt their naming convention.

Definition 1.4.4. Let $g, n \geq 1$. We denote by $A_g^n \subseteq \mathbb{R}^n$ the compact set of all points $(x_1, \ldots, x_n) \in \mathbb{R}^n$ that satisfy the following conditions:

(A1) $x_1 \geq \cdots \geq x_n \geq 0$.
(A2) $x_1 + \cdots + x_n = 1/g$.
(A3) $x_1 \cdots x_k \leq g(x_{k+1} + \cdots + x_n)$ for all $k = 1, \ldots, n-1$.
For $x \in \mathbb{R}^n$ we denote by $\mathrm{SUM}(g)$ the equality $x_1 + \cdots + x_n = 1/g$, by $\mathrm{ORD}(k)$ we denote the inequality $x_k \geq x_{k+1}$ and by $\mathrm{PS}(g,k)$ the inequality $x_1 \cdots x_k \leq g(x_{k+1} + \ldots + x_n)$. Thus the set A_g^n consists of the points $(x_1, \ldots, x_n) \in \mathbb{R}^n_{\geq 0}$ that satisfy the equality $\mathrm{SUM}(g)$ and the inequalities $\mathrm{ORD}(k)$ and $\mathrm{PS}(g,k)$ for all $k = 1, \ldots, n-1$.

Lemma 1.4.5. *Let $g \geq 1$ and $n \geq 1$. For any unit fraction partition $A = (\alpha_1, \ldots, \alpha_n)$ of g with $\alpha_1 \leq \cdots \leq \alpha_n$ the point $(1/\alpha_1, \ldots, 1/\alpha_n)$ is contained in A_g^n.*

Proof. The tuple $(1/\alpha_1, \ldots, 1/\alpha_n)$ fulfills conditions (A1) and (A2). For the third condition let $1 \leq k \leq n-1$. Then we have

$$g\left(\frac{1}{\alpha_{k+1}} + \cdots + \frac{1}{\alpha_n}\right) \;=\; 1 - g\left(\frac{1}{\alpha_1} + \cdots + \frac{1}{\alpha_k}\right) \;=\; \frac{\alpha_1 \cdots \alpha_k - g\left(\sum_{j=1}^{k} \prod_{i \neq j} \alpha_i\right)}{\alpha_1 \cdots \alpha_k}.$$

The numerator on the right hand side is at least one by Lemma 1.3.13. This yields the desired inequality. □

In the following Proposition we gather important properties of the set A_g^n. It is a generalization of [4, Lemma 4.1].

Proposition 1.4.6. *Let $g \geq 1$ and $n \geq 3$. For any point $x \in A_g^n$ the following hold:*
(i) *We have $1/g > x_1 \geq x_n > 0$.*
(ii) *The inequalities $\mathrm{ORD}(k)$ and $\mathrm{PS}(g,k)$ cannot simultaneously be fulfilled with equality.*
(iii) *If for some $1 \leq k \leq n-1$ the inequality $\mathrm{PS}(g,i)$ is fulfilled with equality for all $i \leq k$, then $x_i = 1/s_{g,i}$ holds for all $i = 1, \ldots, k$.*

Proof. We prove (i). Since the x_i are all non-negative, the equality $\mathrm{SUM}(g)$ implies that $x_1 \leq 1/g$ holds. Assume $x_n = 0$. Then by the inequality $\mathrm{PS}(g, n-1)$ we have $x_1 \cdots x_{n-1} = 0$. Thus $x_i = 0$ holds for some $i = 1, \ldots, n-1$. The inequality $\mathrm{ORD}(j)$ thus implies that $x_j = 0$ holds for all $j = i, \ldots, n$. We can repeat this argument to obtain $x_1 = \cdots = x_n = 0$, which contradicts the equality $\mathrm{SUM}(g)$. Thus $x_n > 0$ holds. We prove (ii). Assume that for some k the inequalities $\mathrm{ORD}(k)$ and $\mathrm{PS}(g,k)$ hold simultaneously with equality. Using $x_{k+1} = x_k$, we may then write

$$0 \;=\; g(x_{k+1} + \cdots + x_n) - x_1 \cdots x_k \;=\; g(x_{k+2} + \cdots + x_n) + x_k(g - x_1 \cdots x_{k-1}).$$

The first summand on the right hand side is non-negative, the second summand is positive. This means that they cannot add to zero, a contradiction. Thus $\mathrm{ORD}(k)$ and $\mathrm{PS}(g,k)$ cannot simultaneously be fulfilled with equality. We prove (iii) by induction on i. In case $i = 1$, if $\mathrm{PS}(g,1)$ holds with equality, we get

$$x_1 \;=\; g(x_2 + \cdots + x_n) \;=\; g\left(\frac{1}{g} - x_1\right).$$

Solving this equation for x_1 yields $x_1 = 1/s_{g,1}$. Now assume $i > 1$ and $x_j = 1/s_{g,j}$ holds for all $j \le i$. If $\mathrm{PS}(g,i)$ holds with equality, we obtain

$$\frac{g}{t_{g,i}} x_i \;=\; x_1 \cdots x_i \;=\; g(x_{i+1} + \cdots + x_n) \;=\; g\left(\frac{1}{g} - x_1 - \cdots - x_i\right) \;=\; g\left(\frac{1}{t_{g,i}} - x_i\right).$$

Solving this equation for x_i yields $x_i = 1/s_{g,i}$. This completes the proof. $\square$

Definition 1.4.7. Let $g \ge 1$ and $n \ge 3$. For $k = 1, \ldots, n$ we define a function f_k by

$$f_k \colon A_g^n \;\to\; \mathbb{R}, \qquad x \;\mapsto\; x_1 \cdots x_k.$$

Moreover, for g, n and k as above, we set

$$y(g,n,k) \;:=\; \left(\frac{1}{s_{g,1}}\,,\;\ldots\;,\;\frac{1}{s_{g,k-1}}\,,\;\frac{1}{(n-k+1)t_{g,k}}\;,\;\ldots\;,\;\frac{1}{(n-k+1)t_{g,k}} \right).$$

Note that the point $y(g,n,k)$ belongs to A_g^n.

Proposition 1.4.8. *Let $g \ge 1$, $n \ge 3$ and $k \in \{1, \ldots, n\}$. Let $y = (y_1, \ldots, y_n) \in A_g^n$ such that f_k attains its minimum at y. Then $y = y(g, n, i_0)$ holds for some $i_0 \le k$.*

The major part of the proof of Proposition 1.4.8 is governed by the following Lemma.

Lemma 1.4.9. *Let $n \ge 3$ and $k \in \{1, \ldots, n\}$. Let $y = (y_1, \ldots, y_n) \in A_g^n$ such that f_k attains its minimum at y. Denote by i_0 the minimal index such that $y_{i_0} = y_n$ holds. Then the following hold:*

(i) $i_0 \le k$.

(ii) *The inequality* $\mathrm{ORD}(i)$ *is strict for all* $1 \le i \le i_0 - 1$.

(iii) *The inequality* $\mathrm{PS}(g,i)$ *holds with equality for all* $1 \le i \le i_0 - 1$.

Proof. The strategy for proving (i)-(iii) is the same in each of the three cases. We will assume that the assertion is false and this will allow us to construct a point $y' \in A_g^n$ with $f_k(y') < f_k(y)$, which contradicts the choice of y. Thus the assertion must be true. We prove (i). For $k = n$ there is nothing to prove. Let $k < n$. Assume that $i_0 > k$ holds. Let j_0 maximal with $y_k = y_{j_0}$. By assumption $j_0 < i_0$ holds. The entries of y satisfy

$$y_1 \;\ge\; \ldots \;\ge\; y_k \;=\; \ldots \;=\; y_{j_0} \;>\; y_{j_0+1} \;\ge\; \ldots \;\ge\; y_{i_0-1} \;>\; y_{i_0} \;=\; \ldots \;=\; y_n.$$

We first consider the case $i_0 = n$. In that case we have $y_{n-1} > y_n$. Let

$$0 \;<\; \epsilon \;<\; \min\left(\frac{y_{j_0+1} - y_{j_0}}{2}, \frac{y_{n-1} - y_n}{2(j_0 - k + 1)}\right),$$

$$y' \;=\; (y_1, \ldots, y_{k-1}, y_k - \epsilon, \ldots, y_{j_0} - \epsilon, y_{j_0+1}, \ldots, y_{n-1}, y_n + \tilde{\epsilon}),$$

where $\tilde{\epsilon} = (j_0 - k + 1)\,\epsilon$. We show that y' lies in A_g^n. By the choice of ϵ and $\tilde{\epsilon}$, the equality $\mathrm{SUM}(g)$ holds for y' and the inequality $\mathrm{ORD}(i)$ holds for y' for all i. Moreover, for all $i \le n-1$ we have

$$y'_1 \cdots y'_i \;\le\; y_1 \cdots y_i \;\le\; g(y_{i+1} + \cdots + y_n) \;\le\; g(y'_{i+1} + \cdots + y'_n),$$

thus $\mathrm{PS}(g,i)$ holds as well. This shows that y' lies in A_g^n. Evaluating f_k on y' we obtain

$$f_k(y') \;=\; y'_1 \cdots y'_k \;=\; y_1 \cdots y_{k-1} \cdot (y_k - \epsilon) \;<\; f_k(y),$$

which contradicts the choice of y. Now assume that $i_0 < n$ holds. Note that since $\mathrm{ORD}(i)$ holds with equality for all $i \geq i_0$, by Proposition 1.4.6 (ii) the inequality $\mathrm{PS}(g,i)$ is strict for y. Thus for each $i \geq i_0$ we can find $\delta_i > 0$, such that

$$y_1 \cdots y_{i_0-1} \cdot (y_{i_0} + \delta_i) \cdots (y_i + \delta_i) \;<\; g(y_{i+1} + \cdots + y_n) - g(n-i)\delta_i.$$

We denote by δ the minimum of all the δ_i. Let

$$0 \;<\; \epsilon \;<\; \min\left(\frac{y_{j_0+1} - y_{j_0}}{2}, \frac{y_{i_0-1} - y_{i_0}}{2}, \frac{n - i_0 + 1}{j_0 - k + 1}\delta\right),$$

$$y' \;=\; (y_1, \ldots, y_{k-1}, y_k - \epsilon, \ldots, y_{j_0} - \epsilon, y_{j_0+1}, \ldots, y_{i_0-1}, y_{i_0} + \tilde{\epsilon}, \ldots, y_n + \tilde{\epsilon}),$$

where $\tilde{\epsilon} = \frac{j_0-k+1}{n-i_0+1}\,\epsilon$. Again, ϵ and $\tilde{\epsilon}$ are chosen such that y' satisfies equality $\mathrm{SUM}(g)$ and the inequality $\mathrm{ORD}(i)$ for all i. We show that $\mathrm{PS}(g,i)$ holds for y'. This is clear for $i < i_0$. For $i \geq i_0$ note that $\tilde{\epsilon} < \delta \leq \delta_i$ holds. Thus we have

$$y'_1 \cdots y'_i \;<\; y_1 \cdots y_{i_0-1} \cdot (y_{i_0} + \delta_i) \cdots (y_i + \delta_i) \;<\; g(y'_{i+1} + \cdots + y'_n).$$

This shows that y' belongs to A_g^n. Evaluating f_k on y' we again obtain the inequality $f_k(y') < f_k(y)$, which contradicts the choice of y. Thus $i_0 \leq k$ holds, which proves (i).

We prove (ii). Note that by definition of i_0 we have $y_{i_0-1} < y_{i_0}$, so for $i_0 \leq 2$ there is nothing to show. Assume $i_0 \geq 3$ holds. This also means we have $n \geq k \geq 3$. We show that $\mathrm{ORD}(i)$ is strict for all $i \leq i_0 - 2$. Assume on the contrary that $y_i = y_{i+1}$ holds for some $i \leq i_0 - 2$. Let $j_0 \leq i$ maximal with $y_{j_0} = y_i$ and $j_1 \geq i$ minimal with $y_i = y_{j_1}$. We have $1 \leq j_0 \leq i < j_1 \leq i_0 - 1$. For the entries of y we have:

$$y_{j_0-1} \;>\; y_{j_0} \;=\; \ldots \;=\; y_i \;=\; \ldots \;=\; y_{j_1} \;>\; y_{j_1+1} \;\geq\; \ldots \;\geq\; y_{i_0-1} \;>\; y_{i_0}.$$

By Proposition 1.4.6 The inequality $\mathrm{PS}(g,l)$ is strict for all $j_0 \leq l < j_1$. There is thus $\delta > 0$ such that the following inequality holds for all $j_0 \leq l < j_1$:

$$y_1 \cdots y_{j_0-1} \cdot (y_{j_0} + \delta) \cdot y_{j_0+1} \cdots y_l \;<\; g(y_{l+1} + \cdots + y_n - \delta)$$

With this value δ we may choose an $\epsilon > 0$ as follows and define a point y' depending on this ϵ:

$$0 \;<\; \epsilon \;<\; \min\left(\frac{y_{j_0-1} - y_{j_0}}{2}, \frac{y_{j_1} - y_{j_1-1}}{2}, \delta\right),$$

$$y' \;=\; (y_1, \ldots, y_{j_0} + \epsilon, \ldots, y_{j_1} - \epsilon, \ldots y_n)\,.$$

We show that y' lies in A_g^n. Clearly y' satisfies $\mathrm{SUM}(g)$ and $\mathrm{ORD}(l)$ for all l. By the choice of ϵ, the inequality $\mathrm{PS}(g,l)$ holds for y' for $l < j_1$. For $l \geq j_1$ note that

$$y'_{j_0} \cdot y'_{j_1} \;=\; (y_{j_0} + \epsilon)(y_{j_1} - \epsilon) \;=\; y_{j_0} y_{j_1} - \epsilon^2 \;<\; y_{j_0} y_{j_1}.$$

Thus in this case $\mathrm{PS}(g,l)$ holds for y' as well. This shows that y' belongs to A_g^n. As before, we have $f_k(y') < f_k(y)$, which contradicts the choice of y. Thus $\mathrm{ORD}(i)$ is strict, which proves (ii).

We prove (iii). First we assume there is $i \le i_0 - 2$ such that $\mathrm{PS}(g,i)$ is strict. Then there is $\delta > 0$ such that the following inequality holds:

$$y_1 \cdots y_{i-1} \cdot (y_i + \delta) \ < \ g(y_{i+1} + \cdots + y_n - \delta).$$

With this value δ we may choose an $\epsilon > 0$ as follows and define a point y' depending on this ϵ:

$$0 \ < \ \epsilon \ < \ \min\left(\frac{y_{i-1} - y_i}{2}, \frac{y_i - y_{i+1}}{2}, \delta\right),$$

$$y' \ = \ (y_1, \ldots, y_{i-1}, y_i + \epsilon, y_i - \epsilon, y_{i+1}, \ldots, y_n).$$

With the same arguments as in (i) and (ii) we see that y' lies in A_g^n and again $f_k(y') < f_k(y)$ holds, contradicting the choice of y. Thus $\mathrm{PS}(g,i)$ holds with equality. Now assume that $\mathrm{PS}(g, i_0 - 1)$ is strict. For $t \in \mathbb{R}$ consider the points

$$\tilde{y}(t) \ := \ (y_1, \ldots, y_{i_0-2}, y_{i_0-1} + t, y_{i_0} - \tilde{t}, \ldots, y_n - \tilde{t}),$$

where $\tilde{t} = \frac{t}{n-i_0+1}$. Note that $\tilde{y}(0) = y$ holds. We define a function $f \colon \mathbb{R} \to \mathbb{R}$ by

$$f(t) \ := \ f_k(\tilde{y}(t)) \ = \ y_1 \cdots y_{i_0-2}(y_{i_0-1} + t)(y_{i_0} - \tilde{t})^{k-i_0+1}.$$

The derivative of f is given by

$$f'(t) \ = \ y_1 \cdots y_{i_0-2}(y_{i_0-1} - \tilde{t})^{k-i_0} \left[\left(y_{i_0} - \frac{k-i_0+1}{n-i_0+1} y_{i_0-1}\right) - (k-i_0+2)\tilde{t}\right].$$

Note that for t close to zero, the factor before the square brackets is positive. The behaviour of f close to $t = 0$ is thus governed by the term

$$\delta \ = \ y_{i_0} - \frac{k-i_0+1}{n-i_0+1} y_{i_0-1}.$$

If δ is negative, then f is monotone decreasing in a neighborhood of $t = 0$. We can thus find $t > 0$ with $\tilde{y}(t) \in A_g^n$ such that $f(t) < f(0)$ holds. On the other hand, if δ is positive, then f is monotone decreasing in a neighborhood of $t = 0$ and we can find $t < 0$ with $\tilde{y}(t) \in A_g^n$ and $f(t) < f(0)$. If $\delta = 0$, then f has a local maximum at $t = 0$. There is thus a neighborhood of $t = 0$ with $\tilde{y}(t) \in A_g^n$ and we have $f(t) < f(0)$ for all $t \neq 0$ in that neighborhood. In all cases there is a point $\tilde{y}(t) \in A_g^n$ with $f_k(\tilde{y}(t)) < f_k(y)$. A contradiction to the choice of y, thus $\mathrm{PS}(g, i_0 - 1)$ holds with equality for y. ☐

Proof of Proposition 1.4.8. Let $y = (y_1, \ldots, y_n) \in A_g^n$ such that f_k attains its minimum at y. Let i_0 minimal such that $y_{i_0} = y_n$ holds. By Lemma 1.4.9 (i) we have $i_0 \le k$. We show that $y = y(g, n, i_0)$ holds. By Lemma 1.4.9 (ii) the inequality $\mathrm{PS}(g,i)$ holds with equality for all $i = 1 \dots i_0 - 1$. Proposition 1.4.6 (iii) then tells us that $y_i = 1/s_{g,i}$ holds. Now using $y_{i_0} = \cdots = y_n$ and the fact that $\mathrm{PS}(g, i_0 - 1)$ holds with equality, we obtain

$$\frac{g}{t_{g,i_0}} \;=\; \frac{1}{s_{g,1}} \cdots \frac{1}{s_{g,i_0-1}} \;=\; g(y_{i_0} + \cdots + y_n) \;=\; g(n - i_0 + 1)y_{i_0}.$$

This shows that $y = y(g, n, i_0)$ holds, which completes the proof. $\square$

Proposition 1.4.10. *Let $g \geq 1$, $n \geq 3$ and let $k \in \{1, \dots, n\}$. Let $y \in A_g^n$ such that f_k attains its minimum at y.*

(i) *Assume $k = n$. Then we have $f_n(y) = g/t_{g,n}^2$ and $y = y(g, n, n)$ holds.*

(ii) *Assume $k = n - 1$. If $(n, g) = (3, 1)$, then we have $f_2(y) = 1/9$ and y is the point $y(1, 3, 1)$. In all other cases we have*

$$f_{n-1}(y) \;=\; \frac{g}{2t_{g,n-1}^2}$$

and the point y is one of the following:

$$y(2, 3, 1), \qquad y(1, 4, 2), \qquad y(g, n, n - 1).$$

(iii) *Assume $k = n - 2$. If $n = 4$ and $g \in \{1, 2\}$, then we have $f_2(y) = 1/(16g^2)$ and $y = y(g, 4, 1)$ holds. If $(n, g) = (5, 1)$, then we have $f_3(y) = 1/128$ and $y = y(1, 5, 2)$ holds. In all other cases we have*

$$f_{n-2}(y) \;=\; \frac{g}{3t_{g,n-2}^2}$$

and either $y = y(3, 4, 1)$ or $y = y(g, n, n - 2)$.

For the proof of Proposition 1.4.10 we need the following Lemma.

Lemma 1.4.11. *Let $g \geq 1$ and $n \geq 3$. Then the following hold:*

(i) *For all $1 \leq r \leq n$ we have*

$$r^r t_{g,n-r+1}^{r+1} \;\leq\; t_{g,n}^2.$$

Equality holds if and only if $r = 1$.

(ii) *Assume $(n, g) \neq (3, 1)$. Then for all $1 \leq r \leq n - 1$ we have*

$$(r + 1)^r t_{g,n-r}^{r+1} \;\leq\; 2t_{g,n-1}^2.$$

Equality holds if and only if $r = 1$ or (g, r, n) equals $(1, 2, 4)$ or $(2, 2, 3)$.

(iii) *Assume $(n, g) \notin \{(4, 1), (4, 2), (5, 1)\}$. Then for all $1 \leq r \leq n - 2$ we have*

$$(r + 2)^r t_{g,n-r-1}^{r+1} \;\leq\; 3t_{g,n-2}^2.$$

Equality holds if and only if $r = 1$ or $(g, r, n) = (3, 2, 4)$.

Proof. We prove the assertions (i)–(iii) by induction on r and n. Note that for $r = 1$ the inequalities in (i)–(iii) even hold with equality for any $n \geq 3$. We may thus assume $r \geq 2$. Moreover, we will use the following, which can be verified by direct computation:

(a) $(r+1)/r \leq s_{g,n}$ holds for all values of g, n and r.

(b) If $n \geq 3$, then $(r+1)^2/r \leq s_{g,n}$ holds for all g and all $1 \leq r \leq n$.

(c) If $n \geq 4$, or $n \geq 3$ and $g \geq 2$, then $(r+1)^2/r \leq s_{g,n-1}$ holds for all $1 \leq r \leq n$.

(d) If $n \geq 6$, or $n \geq 4$ and $g \geq 2$, then $(r+1)^2/r \leq s_{g,n-2}$ holds for all $1 \leq r \leq n$.

We prove (i). The cases $(r, n) = (2, 3)$ and $(r, n) = (3, 3)$ are verified by direct computation. In these two cases, the inequality is strict. Assume the assertion is true for a fixed pair (r, n). Then we have:

$$
\begin{aligned}
(r+1)^{r+1} t_{g,(n+1)-(r+1)+1}^{(r+1)+1} &= r^r t_{g,n-r+1}^{r+1} (r+1) \left(\frac{r+1}{r}\right)^r t_{g,n-r+1} \\
&\leq t_{g,n}^2 (r+1) \left(\frac{r+1}{r}\right)^r t_{g,n-r+1} \\
&\leq t_{g,n}^2 s_{g,n} s_{g,n-1} \cdots s_{g,n-r+1} t_{g,n-r+1} \\
&= t_{g,n}^2 t_{g,n+1} \\
&< t_{g,n+1}^2.
\end{aligned}
$$

In the second step we used the induction hypothesis for the pair (r, n) and in the third step we used (a) and (b). Thus the inequality (i) holds for the pair $(r + 1, n + 1)$ and it is strict in this case.

We prove (ii). The cases $(g, r, n) = (1, 2, 4)$ and $(g, r, n) = (1, 3, 4)$ as well as the cases $(g, r, n) = (g, 2, 3)$ for all $g \geq 2$ are verified by direct computation. Here (ii) holds with equality for $(g, r, n) = (1, 2, 4)$ and for $(g, r, n) = (2, 2, 3)$ and is strict otherwise. Assume the assertion is true for a fixed pair (r, n). Then we have:

$$
\begin{aligned}
((r+1)+1)^{r+1} t_{g,n-r}^{(r+1)+1} &= (r+1)^r t_{g,n-r}^{r+1} (r+2) \left(\frac{r+2}{r+1}\right)^r t_{g,(n+1)-(r+1)} \\
&\leq 2t_{g,n-1}^2 \frac{(r+2)^2}{r+1} \left(\frac{r+2}{r+1}\right)^{r-1} t_{g,n-r} \\
&\leq 2t_{g,n-1}^2 s_{g,n-1} s_{g,n-2} \cdots s_{g,n-r} t_{g,n-r} \\
&= 2t_{g,n-1}^2 t_{g,n} \\
&< 2t_{g,n}^2 \\
&= 2t_{g,(n-1)+1}^2.
\end{aligned}
$$

In the second step we used the induction hypothesis for the pair (r, n) and in the third step we used (a) and (c). Thus the inequality (ii) holds for the pair $(r + 1, n + 1)$ and it is strict in this case.

We prove (iii). For $n = 3$ there is nothing to prove. The cases $(g, r, n) = (g, 2, 4)$ for $g \geq 3$, as well as $(g, r, n) = (2, r, 5)$ and $(g, r, n) = (1, r, 6)$ for $2 \leq r \leq n - 2$ are verified by direct computation. Here (iii) holds with equality for $(g, r, n) = (3, 2, 4)$ and is strict otherwise. Assume the assertion is true for a fixed pair (r, n). We may assume that $g \geq 2$ and $n \geq 4$, or $g = 1$ and $n \geq 6$. Then we have:

$$\begin{aligned}
((r+1)+2)^{r+1} t_{g,(n+1)-(r+1)-1}^{(r+1)+1} &= (r+2)^r t_{g,n-r-1}^{r+1} (r+3) \left(\frac{r+3}{r+2}\right)^r t_{g,n-r-1} \\
&\leq 3t_{g,n-2}^2 \frac{(r+3)^2}{r+2} \left(\frac{r+3}{r+2}\right)^{r-1} t_{g,n-r-1} \\
&\leq 3t_{g,n-2}^2 s_{g,n-2} s_{g,n-3} \cdots s_{g,n-r-1} t_{g,n-r-1} \\
&= 3t_{g,n-2}^2 t_{g,n-1} \\
&< 3t_{g,n-1}^2 \\
&= 3t_{g,(n+1)-2}^2.
\end{aligned}$$

In the second step we used the induction hypothesis for the pair (r, n) and in the third step we used (a) and (d), thus (iii) holds for the pair $(r+1, n+1)$ and it is strict in this case. $\square$

Proof of Proposition 1.4.10. We compare the values of the function f_k on the points $y(g, n, k)$ and $y(g, n, l)$ for $1 \leq l \leq k$. On $y(g, n, l)$, the value of f_k is given by

$$f_k(y(g,n,l)) = \frac{g}{(n-l+1)^{k-l+1} t_{g,l}^{k-l+2}}.$$

We prove (i). Let $1 \leq l \leq n$ and set $r := n - l + 1$. Then we have

$$\frac{f_n(y(g,n,l))}{f_n(y(g,n,n))} = \frac{f_n(y(g,n,n-r+1))}{f_n(y(g,n,n))} = \frac{t_{g,n}^2}{r^r t_{g,n-r+1}^{r+1}}.$$

By Lemma 1.4.11 (i) this ratio is at least one for all $1 \leq r \leq n-1$ and equality holds if and only if $r = 1$, ie. if and only if $l = n$. This proves (i).

We prove (ii). Let $k = n - 1$. Let $1 \leq l \leq n - 1$ and set $r := n - l$. Then we have

$$\frac{f_{n-1}(y(g,n,l))}{f_{n-1}(y(g,n,n-1))} = \frac{f_{n-1}(y(g,n,n-r))}{f_{n-1}(y(g,n,n-1))} = \frac{2t_{g,n-1}^2}{(r+1)^r t_{g,n-r}^{r+1}}.$$

Assume $(n, g) \neq (3, 1)$. Then this ratio is at least one for all $1 \leq r \leq n-1$ by Lemma 1.4.11 (ii). Moreover it is equal to one if and only if $r = 1$ or $(g, r, n) = (1, 2, 4)$ or $(g, r, n) = (2, 2, 3)$. This means that f_{n-1} attains its minimum on $y(g, n, l)$ if and only if $(g, l, n) = (1, 2, 4)$ or $(g, l, n) = (2, 1, 3)$ or $l = n - 1$. In the case $(n, g) = (3, 1)$ we have

$$f_2(y(1,3,1)) > f_2(y(1,3,2)).$$

Thus in this case f_2 attains its minimum at $y = y(1, 3, 1)$ and we have $f_2(y) = 1/9$.

We prove (iii). Let $k = n - 2$. Let $1 \leq l \leq n - 2$ and set $r := n - l - 1$. Then we have

$$\frac{f_{n-2}(y(g,n,l))}{f_{n-2}(y(g,n,n-2))} = \frac{f_{n-2}(y(g,n,n-r-1))}{f_{n-2}(y(g,n,n-2))} = \frac{3t_{g,n-2}^2}{(r+2)^r t_{g,n-r-1}^{r+1}}.$$

Assume $(n, g) \notin \{(4,1), (4,2), (5,1)\}$. Then by Lemma 1.4.11 (iii) this ratio is at least one for all $1 \le r \le n-2$ and it is equal to one if and only if $r = 1$ or (g, r, n) equals $(3, 2, 4)$, ie. if and only if $l = n-2$ or $(g, l, n) = (3, 1, 4)$ holds. For the three cases that were excluded, plugging in the actual values, we obtain

$$
\begin{aligned}
f_2(y(1,4,1)) &< f_2(y(1,4,2)),\\
f_2(y(2,4,1)) &< f_2(y(2,4,2)),\\
f_3(y(1,5,2)) &< f_3(y(1,5,1)) \ < \ f_3(y(1,5,3)).
\end{aligned}
$$

This completes the proof of Proposition 1.4.10. $\square$

Proof of Theorem 1.4.2. Let $A = (\alpha_1, \ldots, \alpha_n)$ a unit fraction partition of g and assume $\alpha_1 \le \cdots \le \alpha_n$ holds. Let $y(A) := (1/\alpha_1, \ldots, 1/\alpha_n)$. This point belongs to A_g^n by Lemma 1.4.5. For $1 \le k \le n$ we have

$$F_k(A) \ = \ \frac{\alpha_1 \cdots \alpha_n}{\operatorname{lcm}(\alpha_1, \ldots, \alpha_n)^{n-k}} \ \le \ \alpha_1 \cdots \alpha_k \ = \ f_k^{-1}(y(A)).$$

We prove (i). Using Proposition 1.4.10 (i) for the point $y(A)$ we obtain

$$F_n(A) \ \le \ \frac{1}{f_n(y(A))} \ \le \ \frac{t_{g,n}^2}{g}.$$

Equality holds if and only if $y(A) = y(g, n, n)$, ie. if and only if A is the unit fraction partition $(s_{g,1}, \ldots, s_{g,n-1}, t_{g,n})$. We prove (ii). We use Proposition 1.4.10 (ii) for the point $y(A)$. If $(n, g) = (3, 1)$ holds, then we have

$$F_2(A) \ \le \ \frac{1}{f_2(y(A))} \ \le \ 9$$

and equality holds if and only if $y(A) = y(1, 3, 1)$, ie. if and only if $A = (3, 3, 3)$. If $(n, g) \ne (3, 1)$, then we have

$$F_{n-1}(A) \ \le \ \frac{1}{f_{n-1}(y(A))} \ \le \ \frac{2t_{g,n-1}^2}{g}.$$

Checking the points where f_{n-1} attains its minimium, we see that equality holds if and only if $A = (6, 6, 6)$ or $A = (2, 6, 6, 6)$ or $A = (s_{g,1}, \ldots, s_{g,n-2}, 2t_{g,n-1}, 2t_{g,n-1})$. We prove (iii). We use Proposition 1.4.10 (iii) for the point $y(A)$ and distinguish three cases:

(a) If $n = 4$ and $g \in \{1, 2\}$, then we have

$$F_2(A) \ \le \ \frac{1}{f_2(y(A))} \ \le \ 16g^2$$

and equality holds if and only if $y(A) = y(g, 4, 1)$, ie. $A = (4g, 4g, 4g, 4g)$.

(b) If $(n, g) = (5, 1)$, then we have

$$F_3(A) \ \le \ \frac{1}{f_3(y(A))} \ \le \ 128$$

and equality holds if and only if $y(A) = y(1, 5, 2)$, ie. $A = (2, 8, 8, 8, 8)$.

(c) If $(n,g) \notin \{(4,1),(4,2),(5,1)\}$, then

$$F_{n-2}(A) \ \le \ \frac{1}{f_{n-2}(y(A))} \ \le \ \frac{3t_{g,n-2}^2}{g}.$$

Equality holds if and only if $y(A) = y(3,4,1)$ or $y(A) = y(g,n,n-2)$, ie. if and only if A is one of

$$(12,12,12,12), \qquad (s_{g,1},\dots,s_{g,n-3},3t_{g,n-2},3t_{g,n-2},3t_{g,n-2}).$$

□

1.5 Proofs of Theorems 1.1.1 – 1.1.4

Proof of Theorem 1.1.1. Let Z a d-dimensional fake weighted projective space of Gorenstein index g. Let $\Delta \subseteq N_{\mathbb{Q}}$ a d-dimensional Fano simplex with $Z(\Delta) \cong Z$. Then Δ has Gorenstein index g. Let $A := A(\Delta) = (\alpha_0,\dots,\alpha_d)$ the unit fraction partition of g associated to Δ. We may assume that A is ordered non-decreasingly. By Lemma 1.2.6 and Proposition 1.3.3 we have

$$(-\mathcal{K}_Z)^d \ = \ \mathrm{Vol}(\Delta^*) \ \le \lambda(\Delta)\mathrm{Vol}(\Delta^*) \ = \ \frac{1}{g^d}\frac{\alpha_0\cdots\alpha_d}{\mathrm{lcm}(\alpha_0,\dots,\alpha_d)}.$$

For $d = 1$ there is only one fake weighted projective space, namely $\mathbb{P}^1$, which has anticanonical degree $-\mathcal{K}_{\mathbb{P}^1} = 2$. Let $d \ge 2$. In case $g = 1$ and $d = 2$ the right hand side of the inequality is bounded from above by 9 and $\mathbb{P}^2$ is the only Gorenstein fake weighted projective plane whose degree attains that value, see [65, Ex. 4.7]. If $(d,g) \ne (2,1)$, then Theorem 1.4.2 (ii) provides the upper bound

$$(-\mathcal{K}_Z)^d \ \le \ \frac{1}{g^d}\frac{\alpha_0\cdots\alpha_d}{\mathrm{lcm}(\alpha_0,\dots,\alpha_d)} \ \le \ \frac{2t_{g,d}^2}{g^{d+1}}.$$

Equality in the first case holds if and only if Z is a weighted projective space, see Corollary 1.2.14. By Theorem 1.4.2 (ii) equality in the second case holds if and only if one of the following holds:

(i) $(d,g) = (2,2)$ and $A = (6,6,6)$.
(ii) $(d,g) = (3,1)$ and $A = (2,6,6,6)$.
(iii) $A = (s_{g,1},\dots,s_{g,d-1},2t_{g,d},2t_{g,d})$.

Note that the unit fraction partition in (i) is not reduced. In particular, there is no weighted projective plane $Z(\Delta)$ of Gorenstein index 2 with $A(\Delta) = (6,6,6)$. The unit fraction partitions in (ii) and (iii) are reduced and well-formed. By Corollary 1.2.13 and Proposition 1.3.10 the unit fraction partition $A = (2,6,6,6)$ corresponds to the three-dimensional Gorenstein weighted projective space $X = \mathbb{P}(3,1,1,1)$ and the unit fraction partition $A = (s_{g,1},\dots,s_{g,d-1},2t_{g,d},2t_{g,d})$ corresponds to the d-dimensional weighted projective space $Z = \mathbb{P}(Q_{d,g})$. □

The following Lemmas will be used in the proof of Theorem 1.1.2.

Lemma 1.5.1. *For any d-dimensional Fano simplex Δ of Gorenstein index g the product $g^{d-1}\lambda(\Delta^*)$ is an integer.*

Proof. Let Δ a d-dimensional Fano simplex of Gorenstein index g. For the weight system of its dual we write $Q_{\Delta^*} = (q_0^*, \ldots, q_d^*)$. We show that $g^{d-1}q_i^*$ is an integer for all $i = 0, \ldots, d$. Denote by $v_0, \ldots, v_d$ the vertices of Δ and by $u_0, \ldots, u_d$ the vertices of Δ^*, ordered in such a way that $\langle u_i, v_j \rangle = -1$ holds whenever $i \neq j$. We have

$$q_i^* \;=\; |\det(u_0, \ldots, \hat{u}_i, \ldots, u_d)|.$$

Let $i \in \{0, \ldots, d\}$ and extend (v_i) to a basis $(v_i = b_1, b_2, \ldots, b_d)$ of $\mathbb{Z}^d$. Denote by $C = (c_1, \ldots, c_d)$ the dual basis. For $j \neq i$ we write u_j as a linear combination of the basis C with coefficients $\mu_{j1}, \ldots, \mu_{jd} \in \frac{1}{g}\mathbb{Z}$. We have

$$\mu_{j1} \;=\; \sum_{k=1}^{d} \mu_{jk}\langle c_k, b_1 \rangle \;=\; \langle u_j, v_i \rangle \;=\; -1.$$

Using this presentation of u_j with respect to the basis $C = (c_1, \ldots, c_d)$, we obtain for q_i^*:

$$q_i^* \;=\; |\det(u_0, \ldots, \hat{u}_i, \ldots, u_d)| \;=\; |\det \begin{bmatrix} -1 & \ldots & -1 \\ \mu_{02} & \cdots & \mu_{d2} \\ \vdots & & \vdots \\ \mu_{0d} & \cdots & \mu_{dd} \end{bmatrix}| \;=\; \frac{K}{g^{d-1}}$$

for some $K \in \mathbb{Z}_{>0}$. Thus $g^{d-1}Q(\Delta^*)$ is an integral weight system, which shows that $g^{d-1}\lambda(\Delta^*)$ is an integer. $\square$

For a simplex $\Delta \in \mathbb{Q}^d$ we write $\Delta = \Delta(P)$ with the $d \times (d+1)$ matrix P having the vertices of Δ as its columns.

Lemma 1.5.2. *Let Δ a Fano simplex of dimension $d \geq 2$ and Gorenstein index g. Let $A(\Delta) = (\alpha_0, \ldots, \alpha_d)$ the associated unit fraction partition. Write $\Delta = \Delta(P)$, where*

$$P \;:=\; \begin{bmatrix} 1 & a_{12} & \cdots & a_{1d} & -b_1 \\ 0 & a_{22} & \cdots & a_{2d} & -b_2 \\ \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & \cdots & 0 & a_{dd} & -b_d \end{bmatrix}$$

is in Hermite normal form. Then the following hold:

(i) *a_{kk} divides α_{k-1} for all $k = 2, \ldots, d$.*

(ii) *a_{22} divides α_0.*

(iii) *If $\gcd(\alpha_i, \alpha_{k-1}) = 1$ holds for all $i = 0, \ldots, k-2$, then we have $a_{kk} = 1$.*

Proof. For item (i) we refer to the proof of Proposition 1.6.2. We prove (ii). Denote the columns of P by $v_0, \ldots, v_d$. For the first and the last Gorenstein form of Δ we have

$$u_0 \;=\; \left(\frac{\alpha_0}{g} - 1, \frac{a_{12}-1}{a_{22}} - \frac{a_{12}\alpha_0}{a_{22}g}, u_{03}, \ldots, u_{0d}\right) \;\in\; \frac{1}{g}\mathbb{Z},$$

$$u_d \;=\; \left(-1, \frac{a_{12}-1}{a_{22}}, u_{d3}, \ldots, u_{dd}\right) \;\in\; \frac{1}{g}\mathbb{Z}.$$

Taking their difference, we see that $a_{12}\alpha_0/a_{22}$ must be an integer. Since a_{12} and a_{22} are coprime, this means that a_{22} divides α_0. We prove (iii). Assume that $\gcd(\alpha_0, \ldots, \alpha_k) = 1$ holds. We show by induction on l that $a_{ll} = 1$ holds for all $l \leq k$. For $l = 1$ there is nothing to prove. Let $l = 2$. By item (i), a_{22} divides α_1 and by item (ii), a_{22} divides α_0. As they are coprime, we obtain $a_{22} = 1$. Now assume $l > 2$ and $a_{ii} = 1$ for all $i < l$. Then the ith Gorenstein form for $i < l$ and the last Gorenstein form of Δ are given by

$$u_i \;=\; \left(-1, \ldots, \frac{\alpha_{i-1}}{g} - 1, \ldots, -1, u_{il}, \ldots, u_{id}\right) \;\in\; \frac{1}{g}\mathbb{Z},$$

$$u_d \;=\; (-1, \ldots, -1, u_{dl}, \ldots, u_{dd}) \;\in\; \frac{1}{g}\mathbb{Z},$$

where the entry $\alpha_{i-1}/g - 1$ of u_i is at the ith position. Evaluating their difference on the vector $v_{l-1} = (a_{1l}, \ldots, a_{ll}, 0, \ldots, 0)$ shows that a_{ll} divides $\alpha_{i-1}a_{il}$. Since a_{ll} divides α_{k-1} by item (i), it is coprime to α_{i-1}. Thus a_{ll} divides a_{il}. This is only possible if $a_{il} = 0$. Now, the column v_{l-1} is a primitive point in $\mathbb{Z}^d$. This yields $a_{ll} = 1$. □

Proposition 1.5.3. *Let Δ a Fano triangle of Gorenstein index g. If $Q_\Delta^{\mathrm{red}} = (1,1,1)$ holds then g is odd.*

Proof. Let Δ a Fano triangle with even Gorenstein index g and assume that $Q_\Delta^{\mathrm{red}} = (1,1,1)$ holds. The unit fraction partition of g associated with Δ is

$$A(\Delta) \;=\; (\alpha_0, \alpha_1, \alpha_2) \;=\; (3g, 3g, 3g).$$

Let $P \in \mathrm{Mat}(2,3;\mathbb{Z})$ such that $\Delta(P) \cong \Delta$ holds. We may write

$$P \;=\; \begin{bmatrix} 1 & a & -(a+1) \\ 0 & b & -b \end{bmatrix}$$

for some non-negative $a, b \in \mathbb{Z}$. Note that for the columns of P to all be primitive, b must be odd. The Gorenstein forms u_0, u_1, u_2 of Δ are given by

$$u_0 \;=\; \left(2, -\frac{2a+1}{b}\right), \qquad u_1 \;=\; \left(-1, \frac{a+2}{b}\right), \qquad u_2 \;=\; \left(-1, \frac{a-1}{b}\right).$$

Thus the local Gorenstein indices g_0, g_1, g_2 of Δ all divide b. In particular, the Gorenstein index $g = \mathrm{lcm}(g_0, g_1, g_2)$ divides b. Since g is even, this contradicts the fact that b is odd. Thus Q_Δ^{red} cannot be equal to $(1,1,1)$. □

Proof of Theorem 1.1.2. Let Δ a d-dimensional IP lattice simplex of Gorenstein index g and associated unit fraction partition $A(\Delta) = (\alpha_0, \ldots, \alpha_d)$. We may assume that $\alpha_0 \le \ldots \le \alpha_d$ holds. By Lemma 1.5.1, the product $g^{d-1}\lambda(\Delta^*)$ is an integer. In particular $g^{d-1}\lambda(\Delta^*) \ge 1$ holds. With Proposition 1.3.3 (ii) we obtain the following volume bound for Δ:

$$\mathrm{Vol}(\Delta) \;\le\; g^{d-1}\lambda(\Delta^*)\mathrm{Vol}(\Delta) \;=\; \frac{1}{g}\frac{\alpha_0\cdots\alpha_d}{\mathrm{lcm}(\alpha_0,\ldots,\alpha_d)}. \tag{1.5.3.1}$$

Equality holds if and only if $\lambda(\Delta^*) = 1/g^{d-1}$, which by Proposition 1.3.3 (iv) is equivalent to

$$\lambda(\Delta) \;=\; \frac{\alpha_0\cdots\alpha_d}{\mathrm{lcm}(\alpha_0,\ldots,\alpha_d)^2}.$$

We use Theorem 1.4.2 (ii) to bound the right hand side of Equation 1.5.3.1 from above. In case $(d, g) = (2, 1)$ we have

$$\mathrm{Vol}(\Delta) \;\le\; \frac{1}{g}\frac{\alpha_0\cdots\alpha_d}{\mathrm{lcm}(\alpha_0,\ldots,\alpha_d)} \;=\; \frac{\alpha_0\cdots\alpha_d}{\mathrm{lcm}(\alpha_0,\ldots,\alpha_d)} \;\le\; 9.$$

If equality holds, then we have $A(\Delta) = (3, 3, 3)$, ie. $Q^{\mathrm{red}}_\Delta = (1, 1, 1)$. Thus Δ is isomorphic to $H\Delta(1, 1, 1)$ for some 2×2 integer matrix H with $\det(H) = \lambda(\Delta) = 3$. We may assume that H is in Hermite normal form. Thus we have $\Delta \cong \Delta(P)$ with

$$P \;=\; \begin{bmatrix} 1 & a & -(a+1) \\ 0 & 3 & -3 \end{bmatrix},$$

for $a \in \{1, 2\}$. The two choices of a lead to isomorphic simplices. We may choose $a = 1$, which yields

$$P \;=\; \begin{bmatrix} 1 & 1 & -2 \\ 0 & 3 & -3 \end{bmatrix}.$$

Now assume $(d, g) \ne (2, 1)$ holds. Then by Theorem 1.4.2 (ii) we have

$$\mathrm{Vol}(\Delta) \;\le\; \frac{1}{g}\frac{\alpha_0\cdots\alpha_d}{\mathrm{lcm}(\alpha_0,\ldots,\alpha_d)} \;\le\; \frac{2t_{g,d}^2}{g^2}.$$

If equality holds, then we have $A(\Delta) = A$, where A is one of the following:

$$A \;=\; (6, 6, 6), \qquad A \;=\; (2, 6, 6, 6), \qquad A \;=\; (s_{g,1}, \ldots, s_{g,d-1}, 2t_{g,d}, 2t_{g,d}).$$

Note that by Proposition 1.5.3 there is no Fano simplex Δ with associated unit fraction partition $A(\Delta) = (6, 6, 6)$. The other two cases give the following reduced weight systems

$$Q \;=\; (3, 1, 1, 1), \qquad Q \;=\; \left(\frac{2t_{g,d}}{s_{g,1}}, \ldots, \frac{2t_{g,d}}{s_{g,d-1}}, 1, 1\right)$$

and Δ is isomorphic to $H\Delta(Q)$, where Q is one of the reduced weight systems above and H is a square integer matrix in Hermite normal form with $\det(H) = \lambda(\Delta)$. We first consider the case

$$A(\Delta) \;=\; (2,6,6,6), \qquad g(\Delta) \;=\; 1, \qquad Q_\Delta^{\mathrm{red}} = (3,1,1,1), \qquad \lambda(\Delta) \;=\; 12.$$

We consider the diagonal entries (a_{11}, a_{22}, a_{33}). Since Δ has primitive vertices, we have $a_{11} = 1$. Moreover, Propositions 1.6.1 and 1.6.2 tell us that a_{22} and a_{33} are both divisors of 6. As we have

$$a_{22} \cdot a_{33} \;=\; \det(H) \;=\; \lambda(\Delta) \;=\; 12,$$

this leaves for the diagonal of H only the two possibilities $(a_{11}, a_{22}, a_{33}) = (1,2,6)$ and $(a_{11}, a_{22}, a_{33}) = (1,6,2)$. The second case can be transformed into the first by switching the second and third column of H and bringing it in Hermite normal form again. Thus there are $0 \le a, b < 5$ such that $\Delta \cong \Delta(P)$, where

$$P \;=\; \begin{bmatrix} 1 & 1 & a & -(4+a) \\ 0 & 2 & b & -(2+b) \\ 0 & 0 & 6 & -6 \end{bmatrix}.$$

The Gorenstein forms of Δ are then given by

$$u_0 \;=\; \left(1, -1, \frac{b-a-1}{6}\right), \qquad u_1 \;=\; \left(-1, 3, \frac{a-3b-1}{6}\right),$$
$$u_2 \;=\; \left(-1, 0, \frac{5+a}{6}\right), \qquad u_3 \;=\; \left(-1, 0, \frac{a-1}{6}\right).$$

Since Δ is of Gorenstein index 1, all its Gorenstein forms are integral. The last entry of u_3 thus dictates $a = 1$. Plugging this into P, we obtain $u_1 = (-1, 3, b/2)$. We obtain $b = 2$ and P is the first matrix from Theorem 1.1.2 (ii). Now consider the case

$$A(\Delta) \;=\; (s_{g,1}, \ldots, s_{g,d-1}, 2t_{g,d}, 2t_{g,d}), \qquad g(\Delta) \;=\; g,$$
$$Q_\Delta^{\mathrm{red}} \;=\; \left(\frac{2t_{g,d}}{s_{g,1}}, \ldots, \frac{2t_{g,d}}{s_{g,d-1}}, 1, 1\right), \qquad \lambda(\Delta) \;=\; \frac{t_{g,d}}{g}.$$

We have $\Delta \cong H \cdot \Delta(Q_\Delta^{\mathrm{red}}) = \Delta(P)$, where the matrices H and P are given by

$$H \;=\; \begin{bmatrix} 1 & a_{12} & \cdots & a_{1d} \\ 0 & a_{22} & \cdots & a_{2d} \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & a_{dd} \end{bmatrix}, \qquad P \;=\; \begin{bmatrix} 1 & a_{12} & \cdots & a_{1d} & -b_1 \\ 0 & a_{22} & \cdots & a_{2d} & -b_2 \\ \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & \cdots & 0 & a_{dd} & -b_d \end{bmatrix}.$$

The entries $b_1, \ldots, b_d$ in the last column of P can be expressed via the a_{ij} by solving the linear system $P \cdot Q_\Delta^{\mathrm{red}} = 0$. The determinant of H satisfies

$$\det(H) \;=\; \lambda(\Delta) \;=\; \frac{t_{g,d}}{g} \;=\; s_{g,1} \cdots s_{g,d-1}.$$

We first consider the case $d = 2$. Then $a_{22} = s_{g,1} = g+1$ holds. The last Gorenstein form of Δ reads

$$u_2 \;=\; \left(-1, \frac{a_{12}-1}{g+1}\right)$$

and we have $0 \le a_{12} < g+1$. The entries of u_2 are in $\frac{1}{g}\mathbb{Z}$. Thus $a_{12}-1$ is a multiple of $g+1$. This is only possible for $a_{12} = 1$, which yields

$$P \;=\; \begin{bmatrix} 1 & 1 & -(2g+1) \\ 0 & g+1 & -(g+1) \end{bmatrix} \;=\; \begin{bmatrix} 1 & \frac{(s_{g,1}-g)}{s_{g,1}}\frac{t_{g,2}}{g} & -\frac{(s_{g,1}+g)}{s_{g,1}}\frac{t_{g,2}}{g} \\ 0 & \frac{t_{g,2}}{g} & -\frac{t_{g,2}}{g} \end{bmatrix}.$$

Now assume $d > 2$. Note that the entries $\alpha_0, \ldots, \alpha_{d-2}$ of the ufp $A(\Delta)$ are pairwise coprime. By Lemma 1.5.2 (iii) we have $a_{kk} = 1$ for all $k = 2, \ldots, d-1$. Moreover we obtain $a_{dd} = \det(H) = s_{g,1} \cdots s_{g,d-1}$. We now show that $a_{kd} = \frac{(s_{g,k}-g)}{s_{g,k}}\frac{t_{g,d}}{g}$ holds for all $1 \le k \le d-1$. We set $m := a_{1d} + \cdots + a_{(d-1)d} - 1$. The Gorenstein forms of Δ are given by

$$u_{k-1} \;=\; \left(-1, \ldots, \frac{s_{g,k}}{g} - 1, \ldots, -1, \frac{m}{a_{dd}} - \frac{s_{g,k}\, a_{kd}}{g\, a_{dd}}\right) \;\in\; \frac{1}{g}\mathbb{Z},$$

$$u_d \;=\; \left(-1, \ldots, -1, \frac{m}{a_{dd}}\right) \;\in\; \frac{1}{g}\mathbb{Z},$$

where $k = 1, \ldots, d$ and the entry $s_{g,k}/g - 1$ of u_{k-1} is at the kth position. Note that a_{dd} is coprime to g. The last entry of u_d thus dictates that a_{dd} divides m. Moreover, by the last entry of u_{k-1}, the integer a_{kd} is a multiple of $s_{g,1} \cdots \hat{s}_{g,k} \cdots s_{g,d-1}$, where $\hat{s}_{g,k}$ means, that $s_{g,k}$ is omitted in the product. There is thus $\Lambda_k \in \mathbb{Z}$ with

$$a_{kd} \;=\; \Lambda_k\, s_{g,1} \cdots \hat{s}_{g,k} \cdots s_{g,d-1}.$$

Using these Λ_k, we can write the integer m as

$$m \;=\; \frac{a_{1d} + \cdots + a_{(d-1)d} - 1}{a_{dd}} \;=\; \frac{\Lambda_1}{s_{g,1}} + \cdots + \frac{\Lambda_{d-1}}{s_{g,d-1}} - \frac{g}{t_{g,d}}.$$

We now treat the $\Lambda_1, \ldots, \Lambda_{d-1}$ as indeterminates. Note that they only appear in the last entry of the Gorenstein forms $u_0, \ldots, u_d$, whereas they appear only in the first $d-1$ entries of the last column v_d of P. Evaluating $u_0, \ldots, u_{d-2}$ on v_d thus gives a system of $d-1$ linear equations in the $d-1$ variables $\mu_1, \ldots, \mu_{d-1}$, which are independent since the Gorenstein forms $u_0, \ldots, u_{d-2}$ are linearly independent. This system thus has at most one solution. A direct computation shows that the choice $\Lambda_k = s_{g,k} - g$ is a solution for that system. This shows that P is the second matrix in Theorem 1.1.2 (ii), which completes the proof of the Theorem. □

Proof of Theorem 1.1.3. Let Δ a d-dimensional Fano simplex of Gorenstein index g and associated unit fraction partition $A(\Delta) = (\alpha_0, \ldots, \alpha_d)$. We may assume that the entries of $A(\Delta)$ satisfy $\alpha_0 \le \cdots \le \alpha_d$. By Proposition 1.3.3 (iv) and Lemma 1.5.1 we have

$$\lambda(\Delta) \;\le\; g^{d-1}\lambda(\Delta)\lambda(\Delta^*) \;=\; \frac{\alpha_0 \cdots \alpha_d}{\operatorname{lcm}(\alpha_0, \ldots, \alpha_d)^2}. \tag{1.5.3.2}$$

We prove (i). Let $d = 3$ and $g \in \{1, 2\}$. By Theorem 1.4.2 (iii) the right hand side of Equation 1.5.3.2 is bounded by $16g^2$. Assume $\lambda(\Delta) = 16g^2$ holds. Then we have

$$A(\Delta) \;=\; (4g, 4g, 4g, 4g), \qquad Q_\Delta^{\mathrm{red}} \;=\; (1, 1, 1, 1)$$

and there is a 3×3 integer matrix H in Hermite normal form with determinant equal to $\lambda(\Delta) = 16g^2$, such that $\Delta \cong H \cdot \Delta(1, 1, 1, 1)$ holds. Thus we can write $\Delta \cong \Delta(P)$ with

$$P \;=\; \begin{bmatrix} 1 & a_{12} & a_{13} & -(a_{12} + a_{13} + 1) \\ 0 & a_{22} & a_{23} & -(a_{22} + a_{23}) \\ 0 & 0 & a_{33} & -a_{33} \end{bmatrix}$$

in Hermite normal form. By Lemma 1.5.2 (i), a_{22} and a_{33} each divide $4g$. Moreover we have $a_{22} \cdot a_{33} = \det(H) = 16g^2$. Thus $a_{22} = a_{33} = 4g$ holds. The difference of the Gorenstein forms u_1 and u_2 of Δ is given by

$$u_1 - u_2 \;=\; \left(0, \frac{1}{g}, -\frac{a_{23} + 4g}{4g^2}\right) \;\in\; \frac{1}{g}\mathbb{Z}.$$

Thus $4g$ divides a_{23}. This is only possible for $a_{23} = 0$. The last Gorenstein form of Δ then reads

$$u_3 \;=\; \left(-1, \frac{a_{12} - 1}{4g}, \frac{a_{13} - 1}{4g}\right),$$

which yields $a_{12} = 4k + 1$ and $a_{13} = 4l + 1$ for some $k, l \in \mathbb{Z}$. Taking the restrictions on a_{12} and a_{13} into account we obtain $0 \leq k, l \leq g - 1$. In case $g = 1$ we have $k = 0$ and $l = 0$. In case $g = 2$ the different choices for k and l lead to isomorphic simplices. We may thus assume $k = l = 0$ and P is of the form stated in Theorem 1.1.3 (i).

We prove (ii). Let $(d, g) = (4, 1)$. By Equation 1.5.3.2 and Theorem 1.4.2 (iii) we have $\lambda(\Delta) \leq 128$. If equality holds, then we have

$$A(\Delta) \;=\; (2, 8, 8, 8, 8), \qquad Q_\Delta^{\mathrm{red}} \;=\; (4, 1, 1, 1, 1), \qquad \lambda(\Delta) \;=\; 128.$$

There is a 4×4 integer matrix H in Hermite normal form with determinant equal to $\lambda(\Delta) = 128$, such that $\Delta \cong H \cdot \Delta(4, 1, 1, 1, 1)$ holds. Thus $\Delta \cong \Delta(P)$ holds with

$$P \;=\; \begin{bmatrix} 1 & a_{12} & a_{13} & a_{14} & -(4 + a_{12} + a_{13} + a_{14}) \\ 0 & a_{22} & a_{23} & a_{24} & -(a_{22} + a_{23} + a_{24}) \\ 0 & 0 & a_{33} & a_{34} & -(a_{33} + a_{34}) \\ 0 & 0 & 0 & a_{44} & -a_{44} \end{bmatrix}$$

in Hermite normal form. By Lemma 1.5.2 we have $a_{22} \mid 2$ and $a_{33}, a_{44} \mid 8$. Moreover the product of the diagonal entries is the determinant of H. The only possibility for the diagonal is thus $(a_{22}, a_{33}, a_{44}) = (2, 8, 8)$. Calculating the Gorenstein forms of Δ and using the fact that Δ is of Gorenstein index 1, we obtain

$$a_{12} \;=\; a_{13} \;=\; a_{14} \;=\; 1, \qquad a_{23} \;=\; a_{24} \;=\; 2, \qquad a_{34} \;=\; 8.$$

This shows that P is the matrix from Theorem 1.1.3 (ii).

We prove (iii). Assume that (d,g) is neither of $(3,1)$, $(3,2)$, $(4,1)$. By Equation 1.5.3.2 and Theorem 1.4.2 (iii) we have

$$\lambda(\Delta) \ \leq \ \frac{3t_{g,d-1}^2}{g}.$$

If equality holds, then we have $A(\Delta) = A$, where A is one of the following unit fraction partitions:

$$A \ = \ (12,12,12,12), \qquad A \ = \ (s_{g,1},\dots,s_{g,d-2},3t_{g,d-1},3t_{g,d-1},3t_{g,d-1}).$$

In the first case we are in the situation $(d,g) = (3,3)$ and we have

$$A(\Delta) \ = \ (12,12,12,12), \qquad Q_\Delta^{\mathrm{red}} \ = \ (1,1,1,1), \qquad \lambda(\Delta) \ = \ 144.$$

Again, we have $\Delta \cong \Delta(P)$ with

$$P \ = \ \begin{bmatrix} 1 & a_{12} & a_{13} & -(1+a_{12}+a_{13}) \\ 0 & a_{22} & a_{23} & -(a_{22}+a_{23}) \\ 0 & 0 & a_{33} & -a_{33} \end{bmatrix}$$

in Hermite normal form. By Lemma 1.5.2 both a_{22} and a_{33} are divisors of 12. Moreover we have $a_{22} \cdot a_{33} = \lambda(\Delta) = 144$. Thus $a_{22} = a_{33} = 12$ holds. Calculating the Gorenstein forms of Δ and using the fact that Δ is of Gorenstein index 3, we obtain

$$a_{23} \ = \ 0, \qquad a_{12} \ = \ 4k+1, \qquad a_{13} \ = \ 4l+1,$$

where $0 \leq k,l \leq 2$. The cases $k = 2$ and $l = 2$, as well as $(k,l) = (0,0)$ lead to a non-primitive column of P. Thus these cases are excluded. All other choices for k,l lead to isomorphic matrices. We may thus choose $(k,l) = (0,1)$ and P is the first matrix from Theorem 1.1.3 (iii). We now consider the second possible unit fraction partition for Δ, ie. we have $(d,g) \neq (3,3)$ and

$$A(\Delta) \ = \ (s_{g,1},\dots,s_{g,d-2},3t_{g,d-1},3t_{g,d-1},3t_{g,d-1}),$$

$$Q_\Delta^{\mathrm{red}} \ = \ \left(\frac{3t_{g,d-1}}{s_{g,1}},\dots,\frac{3t_{g,d-1}}{s_{g,1}},1,1,1\right), \qquad \lambda(\Delta) \ = \ \frac{3t_{g,d-1}^2}{g}.$$

As before, we can write $\Delta \cong \Delta(P)$, where

$$P \ = \ \begin{bmatrix} 1 & a_{12} & \cdots & a_{1d} & -b_1 \\ 0 & a_{22} & \cdots & a_{2d} & -b_2 \\ \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & \cdots & 0 & a_{dd} & -b_d \end{bmatrix}$$

is in Hermite normal form. The entries b_k of the last column of P can be computed from the entries $a_{k,j}$ by solving the linear system $P \cdot Q_\Delta^{\mathrm{red}} = 0$. Moreover we have

$$a_{22} \cdots a_{dd} \;=\; \lambda(\Delta) \;=\; \frac{3t_{g,d-1}^2}{g} \;=\; 3t_{g,d-1} \cdot s_{g,1} \cdots s_{g,d-2}.$$

Note that the entries $\alpha_0, \ldots, \alpha_{d-3}$ of $A(\Delta)$ are pairwise coprime. Thus for the diagonal entries of P Lemma 1.5.2 (iii) yields $a_{22} = \cdots = a_{(d-2)(d-2)} = 1$. Moreover both $a_{(d-1)(d-1)}$ and a_{dd} are divisors of $3t_{g,d-1}$ by Lemma 1.5.2 (i). Comparing this to the product of the diagonal entries we obtain that both $a_{(d-1)(d-1)}$ and a_{dd} are multiples of the product $s_{g,1} \cdots s_{g,d-2}$ and that

$$a_{(d-1)(d-1)} \cdot a_{dd} \;=\; s_{g,1} \cdots s_{g,d-2} \cdot 3g \cdot s_{g,1} \cdots s_{g,d-2}$$

holds. We can thus write $a_{(d-1)(d-1)} = \Lambda s_{g,1} \cdots s_{g,d-2}$ for some divisor Δ of $3g$. We show that $\Lambda = 1$ holds. Set $m := a_{1(d-1)} + \cdots + a_{(d-2)(d-1)} - 1$. The Gorenstein forms u_d as well as u_k for $k = 0, \ldots, d-3$ of Δ are given by

$$u_k \;=\; \left(-1, \ldots, \frac{\alpha_k}{g} - 1, \ldots, -1, \frac{m}{a_{(d-1)(d-1)}} - \frac{\alpha_k a_{(k+1)(d-1)}}{g a_{(d-1)(d-1)}}, u_{kd}\right) \;\in\; \frac{1}{g}\mathbb{Z},$$

$$u_d \;=\; \left(-1, \ldots, -1, \frac{m}{a_{(d-1)(d-1)}}, u_{dd}\right) \;\in\; \frac{1}{g}\mathbb{Z}.$$

Here the entry $\alpha_k/g - 1$ is at the position $k+1$ of u_k. Let $1 \le k \le d-2$. Comparing the second to last entries of u_{k-1} and u_d and using the fact that they are in $\frac{1}{g}\mathbb{Z}$, we must have that $a_{(d-1)(d-1)}$ divides $\alpha_{k-1} a_{k(d-1)} = s_{g,k} a_{k(d-1)}$. Thus we can write

$$a_{k(d-1)} \;=\; \Lambda_k s_{g,1} \cdots \hat{s}_{g,k} \cdots s_{g,d-2}$$

for some $\Lambda_k \in \mathbb{Z}_{\ge 1}$. Here $\hat{s}_{g,k}$ means that $s_{g,k}$ is omitted. Note that since $s_{g,k} a_{k(d-1)}$ is a multiple of $a_{(d-1)(d-1)}$, the number Λ_k is a multiple of Λ. So in order for the column v_{d-2} to be primitive, Λ must be equal to 1. We thus have

$$a_{(d-1)(d-1)} \;=\; s_{g,1} \cdots s_{g,d-2}, \qquad a_{dd} \;=\; 3t_{g,d-1}.$$

It remains show that $\Lambda_k = (s_{g,k} - g)$ holds for all $k = 1, \ldots, d-2$. The situation is very similar to the last part of the proof of Theorem 1.1.2. However, as we do not have information about the entries $a_{1d}, \ldots, a_{(d-1)d}$ of P, we need to employ a different strategy. Note that $a_{(d-1)(d-1)}$ is coprime to g. Considering again the last Gorenstein form u_d of Δ, its entry

$$u_{d(d-1)} \;=\; \frac{m}{a_{(d-1)(d-1)}} \;=\; \frac{a_{1(d-1)} + \cdots + a_{(d-2)(d-1)} - 1}{a_{(d-1)(d-1)}}$$

must be an integer. In particular, $s_{g,k}$ divides m for all $k = 1, \ldots, d-2$. Since $a_{l(d-1)}$ is a multiple of $s_{g,k}$ for $l \ne k$, this means that we have

$$s_{g,k} \;\mid\; \Lambda_k s_{g,1} \cdots \hat{s}_{g,k} \cdots s_{g,d-2} - 1.$$

As $s_{g,l} = t_{g,l} + 1$ and $s_{g,k} \mid t_{g,l}$ holds for $l > k$, this implies that we have

$$s_{g,k} \ \mid \ s_{g,1} \cdots s_{g,k-1} \Lambda_k - 1.$$

Thus there is $B \in \mathbb{Z}$ with $Bs_{g,k} \ = \ s_{g,1} \cdots s_{g,k-1} \Lambda_k - 1$. Since P is in Hermite normal form, Λ_k is in the range $0 \leq \Lambda_k < s_{g,k}$. Thus B is at least one, but less than $s_{g,1} \cdots s_{g,k-1}$. Moreover, we obtain the identity

$$s_{g,1} \cdots s_{g,k-1} \Lambda_k \ = \ Bs_{g,k} + 1 \ = \ B(t_{g,k} + 1) + 1,$$

and since $t_{g,k}$ is a multiple of $s_{g,1}, \ldots, s_{g,k-1}$, this equation is only fulfilled if $s_{g,1} \cdots s_{g,k-1}$ divides $B+1$. Comparing this to the possible values of B, we obtain $B = s_{g,1} \cdots s_{g,k-1} - 1$. Plugging this in for B and solving for Λ_k, we obtain $\Lambda_k = s_{g,k} - g$. This shows that P is the second matrix from Theorem 1.1.3. Finally assume g is odd. We plug in the values for $a_{1d}, \ldots, a_{(d-1)d}$ provided in Theorem 1.1.3 (iii) and check that the resulting matrix has primitive columns. This shows that this is a valid choice for $a_{1d}, \ldots, a_{(d-1)d}$, which completes the proof. $\square$

Proof of Theorem 1.1.4. Let Δ a d-dimensional IP simplex of Gorenstein index g and associated unit fraction partition $A(\Delta) = (\alpha_0, \ldots, \alpha_d)$. We may assume that A is ordered, ie. that $\alpha_0 \leq \cdots \leq \alpha_d$ holds. By Proposition 1.3.3 (i) we have

$$\mathrm{Vol}(\Delta)\mathrm{Vol}(\Delta^*) \ = \ \frac{\alpha_0 \cdots \alpha_d}{g^{d+1}}.$$

By Theorem 1.4.2 (i) the numerator of the right hand side is bounded by $t_{g,d+1}^2/g$. Thus we obtain

$$\mathrm{Vol}(\Delta)\mathrm{Vol}(\Delta^*) \ \leq \ \frac{t_{g,d+1}^2}{g^{d+2}}.$$

If equality holds, then by Theorem 1.4.2 (i) we have $A(\Delta) = (s_{g,1}, \ldots, s_{g,d}, t_{g,d+1})$, which is equivalent to

$$Q_\Delta^{\mathrm{red}} \ = \ Q(A(\Delta)) \ = \ \left(\frac{t_{g,d+1}}{s_{g,1}}, \ldots, \frac{t_{g,d+1}}{s_{g,d}}, 1 \right).$$

On the other hand, assume that Q_Δ^{red} is of this form. Let $\Delta' = \Delta(Q_\Delta^{\mathrm{red}})$. There is $H \in \mathrm{GL}(d+1, \mathbb{Q})$ such that $\Delta \cong H \cdot \Delta'$ holds. For the Mahler volume of Δ we thus obtain

$$\mathrm{Vol}(\Delta)\mathrm{Vol}(\Delta^*) \ = \ \mathrm{Vol}(H\Delta')\,\mathrm{Vol}((H^*)^{-1}(\Delta')^*) \ = \ \mathrm{Vol}(\Delta')\mathrm{Vol}((\Delta')^*) \ = \ \frac{t_{g,d+1}^2}{g^{d+2}}.$$

$\square$

1.6 A classification procedure for IP lattice simplices

Throughout this section we develop a procedure for the classification of all IP lattice simplices of given dimension and Gorenstein index, see Algorithm 1.6.7. It is easily adapted to only classify Fano simplices, see Remark 1.6.8.

Proposition 1.6.1. *Fix an integer $d \geq 2$ and $d+2$ positive integers $g, g_0, \ldots, g_d$ with $g = \operatorname{lcm}(g_0, \ldots, g_d)$. Let $A = (\alpha_0, \ldots, \alpha_d) \in \mathbb{Z}_{\geq 1}^{d+1}$ a unit fraction partition of g, ie.*

$$\frac{1}{g} \;=\; \frac{1}{\alpha_0} + \cdots + \frac{1}{\alpha_d}.$$

Denote by $w = (w_0, \ldots, w_d) = Q(A)$ the weight system associated with A. Consider the $d \times (d+1)$ integer matrices of the form

$$P \;:=\; [\, v_0 \;\ldots\; v_d \,] \;:=\; \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1d} & -b_1 \\ 0 & a_{22} & \cdots & a_{2d} & -b_2 \\ \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & \cdots & 0 & a_{dd} & -b_d \end{bmatrix}$$

such that for all $k = 1, \ldots, d$ the entries of P satisfy

(i) $a_{kk} \in \mathbb{Z}_{\geq 1},\ a_{kk} \mid \alpha_{k-1}$,

(ii) $0 \leq a_{ik} < a_{kk}$ *for all* $1 \leq i < k$,

(iii) $b_k w_d = a_{kk} w_{k-1} + \cdots + a_{kd} w_{d-1}$.

Let $\Delta := \Delta(P)$ the convex hull of the columns of P. Then Δ is a d-dimensional IP lattice simplex whose associated weight system satisfies $Q_\Delta^{\mathrm{red}} = (w_0, \ldots, w_d)$. The k-th Gorenstein form $u_k = (u_{k1}, \ldots, u_{kd})$ of Δ is explicitly given by

$$u_{kj} \;=\; \begin{cases} \frac{|w| - w_k}{a_{jj} w_k} - \frac{\sum_{l=1}^{j-1} a_{lj} u_{kl}}{a_{jj}}, & \text{if } j = k+1, \\ \frac{-1 - \sum_{l=1}^{j-1} a_{lj} u_{kl}}{a_{jj}}, & \text{otherwise.} \end{cases} \qquad (1.6.1.1)$$

If each of $g_k u_k$ is a primitive vector in $\mathbb{Z}^d$, then Δ is of Gorenstein index g with local Gorenstein indices g_k, where $k = 0, \ldots, d$.

Proof. As P is of rank d, the polytope Δ is full-dimensional. Its vertices are precisely the columns $v_0, \ldots, v_d$. It is thus a lattice simplex. By condition (iii) we have $P \cdot w = 0$. With $\beta_k := w_k/|w|$ we can write

$$\mathbf{0} \;=\; \beta_0 v_0 + \cdots + \beta_d v_d,$$

which is a convex combination of $v_0, \ldots, v_d$ with non-vanishing coefficients. Thus the origin is contained in the interior of Δ, making it an IP lattice simplex. By Remark 1.2.10 (iii) we have $Q_\Delta^{\mathrm{red}} = w$. Let $u_k = (u_{k1}, \ldots, u_{kd})$ the k-th Gorenstein form of Δ. Let $1 \leq j \leq d$. If $j \neq k+1$, then $\langle u_k, v_j \rangle = -1$ holds and for $j = k+1$ we have $\langle u_k, v_j \rangle = |w|/w_k - 1$. Solving these equations for u_{kj} produces the identities in Equation 1.6.1.1. The last assertion is just the definition of the Gorenstein index and the local Gorenstein indices of Δ. $\square$

Proposition 1.6.2. *Let Δ a d-dimensional IP lattice simplex of Gorenstein index g. Then $\Delta \cong \Delta(P)$ holds with a matrix P as provided by Proposition 1.6.1.*

Proof. Write $w := Q_\Delta^{\mathrm{red}} = (w_0, \ldots, w_d)$ and let $A = A(\Delta) = (\alpha_0, \ldots, \alpha_d)$ the unit fraction partition of g associated with Δ. By Proposition 1.3.10 the reduced weight systems w and $Q(A(\Delta))$ coincide. Let P the $d \times (d+1)$ integer matrix whose columns are the vertices of Δ. By bringing P in Hermite normal form, we may assume

$$P \;=\; [\, v_0 \;\ldots\; v_d \,] \;=\; \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1d} & -b_1 \\ 0 & a_{22} & \cdots & a_{2d} & -b_2 \\ \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & \cdots & 0 & a_{dd} & -b_d \end{bmatrix},$$

where $a_{kk} \in \mathbb{Z}_{\geq 1}$ holds for all $k = 1, \ldots, d$ as well as $0 \leq a_{ik} < a_{kk}$ for all $1 \leq i < k$. Solving $P \cdot w = 0$ for the entries b_k, we obtain the identity

$$b_k w_d \;=\; a_{kk} w_{k-1} + \cdots + a_{kd} w_{d-1}$$

It thus remains to show that for all k the diagonal entry a_{kk} divides α_{k-1}. Consider the following sequence of rational numbers

$$q_1 \;:=\; -\frac{1}{a_{11}}, \qquad q_j \;:=\; -\frac{1 + a_{1j} q_1 + \cdots + a_{j-1,j} q_{j-1}}{a_{jj}}.$$

Let $k \geq 1$ and let u_k the k-th Gorenstein form of Δ. For each $1 \leq j \leq k$ we have $\langle u_k, v_j \rangle = -1$. Solving this for u_{kj} we get $u_{kj} = q_j$. In particular $g\, q_k$ is an integer. Evaluating u_{k-1} on v_k, we obtain

$$a_{1k} q_1 + \cdots + a_{(k-1)k} q_{k-1} + a_{kk} u_{(k-1)k} \;=\; \langle u_{k-1}, v_k \rangle \;=\; \frac{|w|}{w_{k-1}} - 1.$$

With the definition of q_k, we can rewrite this equation as

$$a_{kk}(u_{(k-1)k} - q_k) \;=\; -\frac{|w|}{w_{k-1}}.$$

Note that the g-fold of both $u_{(k-1)k}$ and q_k is an integer. Multiplying both sides by $g\, w_{k-1}$ thus shows that $a_{kk} w_{k-1}$ is a divisor of $g|w| = \alpha_{k-1} w_{k-1}$. Clearing w_{k-1} on both sides, we see that a_{kk} divides α_{k-1}. $\square$

Propositions 1.6.1 and 1.6.2 provide us with a procedure to enumerate up to isomorphy all IP lattice simplices Δ with a given constellation of local Gorenstein indices $(g_0, \ldots, g_d)$ and given reduced weight system w. The list produced may contain redundancies, ie. matrices P and P' that give isomorphic simplices $\Delta(P)$ and $\Delta(P')$. In practice, we want the list to be redundancy free without having to check each pair of matrices for isomorphy. The solution to this problem is to define a normal form $\mathrm{NF}(P)$ for these matrices P, which has the property that two matrices P and P' give isomorphic simplices if and only if their normal forms coincide. The normal form we present in Definition 1.6.3 is similar to the PALP normal form for lattice polytopes described in [61], see also [38]. To fix some notation, if B is a $m \times n$ integer matrix with columns $b_1, \dots, b_n$ and $\sigma \in S_n$ is a permutation of $\{1, \dots, n\}$, then we denote by B_σ the matrix with columns $b_{\sigma(1)}, \dots, b_{\sigma(n)}$. Moreover, by $\mathrm{HNF}(B)$ we denote the hermite normal form of B.

Definition 1.6.3. Let P a $d \times (d+1)$ integer matrix whose columns generate $\mathbb{Q}^d$ as a convex cone. Let $w = (w_0, \dots, w_d)$ the reduced weight system and $(g_0, \dots, g_d)$ the local Gorenstein indices of the IP lattice simplex $\Delta(P)$. We denote by S_P the subset of S_{d+1} consisting of all permutations $\sigma \in S_{d+1}$ with the following properties:
(i) If $\sigma(i) \leq \sigma(j)$ holds, then $w_{\sigma(i)} \geq w_{\sigma(j)}$.
(ii) If $\sigma(i) \leq \sigma(j)$ and $w_{\sigma(i)} = w_{\sigma(j)}$ holds, then $g_{\sigma(i)} \geq g_{\sigma(j)}$.
We define the *normal form* of P as

$$\mathrm{NF}(P) \ := \ \min\{\mathrm{HNF}(P_\sigma);\ \sigma \in S_P\},$$

where the minimum is taken lexicographically, ie. we write the entries of the matrix $\mathrm{HNF}(P_\sigma) = (h_{ij})_{ij}$ as a list of integers $(h_{11}, \dots, h_{1d}, h_{21}, \dots, h_{(d+1)d})$ and take the lexicographic minimum among those lists.

Proposition 1.6.4. *For $d \times (d+1)$ integer matrices P and P', whose columns generate $\mathbb{Q}^d$ as a convex cone, we have $\Delta(P) \cong \Delta(P')$ if and only if their normal forms* $\mathrm{NF}(P)$ *and* $\mathrm{NF}(P')$ *coincide.*

Proof. Assume $\Delta(P) \cong \Delta(P')$ holds. Then there is a permutation $\sigma \in S_{d+1}$ and a $d \times d$ unimodular matrix S such that $S \cdot P_\sigma = P'$ holds. A quick comparison shows $S_P = \sigma S_{P'}$. Thus the sets of hermite normal forms, among which the lexicographic minimum is chosen, coincide. We obtain $\mathrm{NF}(P) = \mathrm{NF}(P')$. On the other hand, if $\mathrm{NF}(P) = \mathrm{NF}(P')$ holds, then there are $\sigma, \sigma' \in S_{d+1}$ and unimodular $d \times d$ matrices S and S' with $S \cdot P_\sigma = S' \cdot P'_{\sigma'}$. Thus $\Delta(P)$ and $\Delta(P')$ are isomorphic. □

We translate Proposition 1.6.1 into a classification procedure realized in Algorithm 1.6.5. As input it takes a unit fraction partition $A = (\alpha_0, \dots, \alpha_d)$ of g and a tuple $(g_0, \dots, g_d)$ of positive integers with $g = \mathrm{lcm}(g_0, \dots, g_d)$. It then produces a list of matrices P corresponding to IP lattice simplices $\Delta(P)$ with associated unit fraction partition A and local Gorenstein indices $(g_0, \dots, g_d)$. This list is complete, ie. every IP lattice simplex Δ with associated unit fraction partition A and local Gorenstein indices $(g_0, \dots, g_d)$ is isomorphic to some $\Delta(P)$ with P from that list, and the list is redundancy free, ie. two different matrices P and P' from the list give non-isomorphic simplices $\Delta(P)$ and $\Delta(P')$.

Algorithm 1.6.5. ClassifySimp(A, $[g_0, \dots, g_d]$)

Input: – A unit fraction partition $A = [\alpha_0, \dots, \alpha_d] \in \mathbb{Z}_{\geq 1}^{n+1}$ of g,
– A list of positive integers $[g_0, \dots, g_d] \in \mathbb{Z}_{\geq 1}^{n+1}$ with $g = \mathrm{lcm}(g_0, \dots, g_d)$

1: $L \leftarrow [\ ]$

$w \leftarrow Q(A)$

$\mathrm{div}_k \leftarrow \{\text{Divisors of } \alpha_k\}$ for $k = 0, \ldots, d-1$

for all $(a_{11}, a_{12}, a_{22}, \ldots, a_{1d}, \ldots, a_{dd})$ with $a_{kk} \in \mathrm{div}_{k-1}$ and $0 \le a_{ik} < a_{kk}$ **do**

  $b_k \leftarrow (a_{kk}w_{k-1} + \cdots + a_{kd}w_{d-1})/w_d$ for $k = 1, \ldots, d$

  $$P \leftarrow \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1d} & -b_1 \\ 0 & a_{22} & \cdots & a_{2d} & -b_2 \\ \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & \cdots & 0 & a_{dd} & -b_d \end{bmatrix}$$

  $u_k \leftarrow k$-th linear form as in Equation 1.6.1.1 for $k = 0, \ldots, d$

  if $b_k \in \mathbb{Z}$ **and** $g_k u_k$ is a primitive point in $\mathbb{Z}^d$ **and** $\mathrm{NF}(P) \notin L$ **then**

    add $\mathrm{NF}(P)$ to L

  end if

end for

return L

With Algorithm 1.6.5 we can classify the d-dimensional IP lattice simplices with a fixed constellation of local Gorenstein indices and fixed unit fraction partition. To obtain the classification of all d-dimensional IP lattice simplices of Gorenstein index g, we thus need a list of all length $d+1$ unit fraction partitions of g. This is done by the following Algorithm, which takes as input a reduced positive rational number p/q and a natural number $n \ge 1$ and produces a list of all ordered unit fraction partitions $\alpha_1 \le \cdots \le \alpha_n$ of p/q of length n. For two unit fraction partitions A and A' we write $A \sim A'$, if they only differ by order.

Algorithm 1.6.6. UFP(p/q, n)

Input: – A reduced positive rational $p/q \in \mathbb{Q}_{>0}$
– A positive integer $n \in \mathbb{Z}_{\ge 1}$

if $n = 1$ **and** $p = 1$ **then**

  return $[(p/q)]$

end if

$L \leftarrow [\,]$

for $k = \lceil q/p \rceil, \ldots, \lfloor nq/p \rfloor$ **do**

  $L_2 \leftarrow \mathrm{UFP}(p/q - 1/k, n-1)$

  for all $(1/\alpha_2, \ldots, 1/\alpha_n) \in L_2$ **do**

    if $(1/k, 1/\alpha_1, \ldots, 1/a_n) \not\sim A'$ for all $A' \in L$ **then**

      sort $(1/k, 1/\alpha_1, \ldots, 1/a_n)$ decreasingly and add it to L

    end if

  end for

end for

return L

The following Algorithm takes as input integers $d \ge 2$ and $g \ge 1$ and performs the classification of all d-dimensional IP lattice simplices of Gorenstein index g. As in the case of Algorithm 1.6.5, the output list of matrices P is complete and redundancy free.

Algorithm 1.6.7. ClassifyAllSimp(d, g)

Input: – An integer $d \geq 2$
– An integer $g \geq 1$

1: $L \leftarrow [\ ]$
2: **for all** $A \in \mathrm{UFP}(1/g, d+1)$ **do**
3: $\quad L_2 \leftarrow [\ ]$
4: $\quad$ **for all** $(g_0, \ldots, g_d)$ with $g_k \mid g$ such that $g = \mathrm{lcm}(g_0, \ldots, g_d)$ **do**
5: $\quad\quad$ **for all** $P \in \mathrm{ClassifySimp}(A, (g_0, \ldots, g_d))$ **do**
6: $\quad\quad\quad$ **if** $P \notin L_2$ **then**
7: $\quad\quad\quad\quad$ add P to L_2
8: $\quad\quad\quad$ **end if**
9: $\quad\quad$ **end for**
10: $\quad$ **end for**
11: $\quad$ Append L_2 to L
12: **end for**
13: **return** L

Remark 1.6.8. To classify only the Fano ones among the IP lattice simplices of given dimension d and Gorenstein index g, we perform the following two modifications:

(i) In Algorithm 1.6.5 line 6 we consider those matrices P whose columns are all primitive vectors in $\mathbb{Z}^d$.

(ii) In Algorithm 1.6.7 line 2 we only loop over well-formed unit fraction partitions of g, see Remark 1.2.10 (ii) and Proposition 1.3.10.

1.7 Classification results

We discuss our classification results for Fano simplices; the complete classification data, as well as the Julia code [22] to produce these results can be found at [13]. We start in dimension two.

Theorem 1.7.1. *See Theorem 1.1.5. Up to isomorphy there are* 2,992,229 *Fano triangles of Gorenstein index* $g \leq 1000$.

The following table contains for each $g \leq 1000$ the number $N(g)$ of Fano triangles of Gorenstein index g. The sequence $(N(g))_{g \geq 1}$ is OEIS sequence A145582, available at [68].

g	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24
N	5	7	18	13	33	26	45	27	51	51	67	53	69	74	133	48	89	81	102	110	178	105	124	109
g	25	26	27	28	29	30	31	32	33	34	35	36	37	38	39	40	41	42	43	44	45	46	47	48
N	161	119	164	135	142	187	140	105	274	159	383	169	145	166	329	221	177	266	180	230	404	189	220	213
g	49	50	51	52	53	54	55	56	57	58	59	60	61	62	63	64	65	66	67	68	69	70	71	72
N	315	264	384	233	225	260	573	298	420	241	276	393	216	252	593	202	607	394	247	321	540	560	310	353
g	73	74	75	76	77	78	79	80	81	82	83	84	85	86	87	88	89	90	91	92	93	94	95	96
N	249	283	701	336	783	458	316	439	464	318	341	557	764	307	638	464	363	612	816	389	639	368	914	432
g	97	98	99	100	101	102	103	104	105	106	107	108	109	110	111	112	113	114	115	116	117	118	119	120
N	341	551	893	549	352	583	385	539	1377	383	409	536	377	840	756	580	377	642	1058	512	1010	462	1191	807
g	121	122	123	124	125	126	127	128	129	130	131	132	133	134	135	136	137	138	139	140	141	142	143	144
N	702	402	811	478	888	876	416	406	869	946	480	868	1202	483	1321	680	450	772	505	1172	931	522	1395	707
g	145	146	147	148	149	150	151	152	153	154	155	156	157	158	159	160								
N	1204	482	1319	540	518	997	499	745	1261	1204	1308	965	493	543	1088	919								

g	161	162	163	164	165	166	167	168	169	170	171	172	173	174	175	176	177	178	179	180
N	1477	748	517	670	2128	590	635	1160	895	1211	1395	613	562	962	2017	907	1156	646	689	1285
g	181	182	183	184	185	186	187	188	189	190	191	192	193	194	195	196	197	198	199	200
N	554	1338	1119	864	1442	963	1710	762	1864	1307	655	865	579	661	2507	1025	647	1319	651	1169
g	201	202	203	204	205	206	207	208	209	210	211	212	213	214	215	216	217	218	219	220
N	1206	665	1781	1236	1642	690	1756	1009	1971	1937	599	787	1350	736	1799	1137	1736	710	1353	1845
g	221	222	223	224	225	226	227	228	229	230	231	232	233	234	235	236	237	238	239	240
N	1812	1127	743	1266	2152	709	727	1348	721	1578	2935	1088	733	1510	1915	926	1528	1862	886	1632
g	241	242	243	244	245	246	247	248	249	250	251	252	253	254	255	256	257	258	259	260
N	685	1175	1443	815	2812	1221	2046	1056	1580	1431	894	1818	2264	805	3043	802	753	1291	2020	2038
g	261	262	263	264	265	266	267	268	269	270	271	272	273	274	275	276	277	278	279	280
N	1989	856	885	1782	2056	1897	1625	929	912	1947	838	1363	3391	879	3055	1642	818	917	2210	2424
g	281	282	283	284	285	286	287	288	289	290	291	292	293	294	295	296	297	298	299	300
N	790	1377	856	1071	3460	2076	2391	1453	1484	1877	1681	948	904	2012	2383	1249	2850	976	2588	2103
g	301	302	303	304	305	306	307	308	309	310	311	312	313	314	315	316	317	318	319	320
N	2271	927	1793	1426	2250	1972	938	2473	1860	2026	1005	2087	926	1025	4488	1081	928	1569	2796	1812
g	321	322	323	324	325	326	327	328	329	330	331	332	333	334	335	336	337	338	339	340
N	1927	2271	2713	1586	3203	973	1919	1446	2760	2926	936	1200	2361	1107	2620	2310	963	1589	1995	2600
g	341	342	343	344	345	346	347	348	349	350	351	352	353	354	355	356	357	358	359	360
N	2811	2142	2250	1409	4063	1052	1037	2026	1083	3030	3339	1922	981	1748	2795	1338	4351	1206	1286	2648
g	361	362	363	364	365	366	367	368	369	370	371	372	373	374	375	376	377	378	379	380
N	1840	1106	3122	2561	2555	1646	1068	1677	2638	2301	2894	2000	1091	2691	3684	1651	3009	2783	1105	2855
g	381	382	383	384	385	386	387	388	389	390	391	392	393	394	395	396	397	398	399	400
N	2155	1200	1203	1768	6332	1141	2799	1285	1219	3465	3270	2295	2265	1239	3054	2866	1105	1249	4731	2305
g	401	402	403	404	405	406	407	408	409	410	411	412	413	414	415	416	417	418	419	420
N	1116	1885	2957	1390	3915	2826	3158	2565	1115	2555	2287	1358	3398	2556	3205	2165	2492	2949	1431	4108
g	421	422	423	424	425	426	427	428	429	430	431	432	433	434	435	436	437	438	439	440
N	1094	1217	3078	1760	4164	2023	3040	1500	5493	2644	1378	2278	1219	2933	4870	1425	3631	2018	1358	3754
g	441	442	443	444	445	446	447	448	449	450	451	452	453	454	455	456	457	458	459	460
N	4423	3008	1237	2359	3292	1353	2650	2466	1379	3274	3514	1447	2560	1414	7089	2848	1253	1398	4114	3378
g	461	462	463	464	465	466	467	468	469	470	471	472	473	474	475	476	477	478	479	480
N	1373	4169	1293	2138	5160	1406	1421	3172	3335	2915	2590	2064	3658	2278	4860	3652	3369	1547	1588	3338
g	481	482	483	484	485	486	487	488	489	490	491	492	493	494	495	496	497	498	499	500
N	3188	1401	5700	2548	3406	2281	1344	1830	2754	4192	1404	2668	3902	3328	6896	2038	3814	2330	1416	3011
g	501	502	503	504	505	506	507	508	509	510	511	512	513	514	515	516	517	518	519	520
N	2913	1613	1657	3799	3520	3272	4384	1576	1494	4305	3575	1640	4517	1503	3701	2737	4273	3403	2996	4267
g	521	522	523	524	525	526	527	528	529	530	531	532	533	534	535	536	537	538	539	540
N	1478	3104	1466	1802	7509	1634	4037	3564	2789	3212	3868	3678	3737	2476	3887	2106	3155	1718	6181	4065
g	541	542	543	544	545	546	547	548	549	550	551	552	553	554	555	556	557	558	559	560
N	1420	1575	3027	2885	3867	4836	1465	1770	3682	4529	4392	3504	4015	1618	6008	1820	1620	3368	4072	4936
g	561	562	563	564	565	566	567	568	569	570	571	572	573	574	575	576	577	578	579	580
N	6390	1592	1552	2965	3889	1689	5544	2333	1652	4795	1602	4434	3271	3852	5863	2908	1549	2706	3169	4176
g	581	582	583	584	585	586	587	588	589	590	591	592	593	594	595	596	597	598	599	600
N	4269	2612	4576	2233	7696	1736	1722	4184	4322	3590	3298	2372	1718	4234	9257	1951	3366	3875	1881	4389
g	601	602	603	604	605	606	607	608	609	610	611	612	613	614	615	616	617	618	619	620
N	1574	3728	3972	1866	6423	2696	1722	3098	6950	3522	4622	4139	1644	1836	6694	5131	1649	2773	1736	4341
g	621	622	623	624	625	626	627	628	629	630	631	632	633	634	635	636	637	638	639	640
N	5618	1870	4809	4156	4346	1802	7274	1959	4514	6352	1612	2433	3333	1823	4371	3354	6323	4188	4520	3635
g	641	642	643	644	645	646	647	648	649	650	651	652	653	654	655	656	657	658	659	660
N	1715	2916	1823	4506	7074	4319	1933	3365	5292	5106	6968	1946	1741	2920	4675	2804	4320	4422	2027	6340
g	661	662	663	664	665	666	667	668	669	670	671	672	673	674	675	676	677	678	679	680
N	1778	1907	7743	2702	10245	3736	5199	2225	3779	4070	5059	4770	1785	1975	7115	3138	1800	3010	4765	5302
g	681	682	683	684	685	686	687	688	689	690	691	692	693	694	695	696	697	698	699	700
N	3703	4377	1932	4516	4614	3880	3765	2694	4884	5691	1879	2204	9817	2016	4935	4245	4979	2058	3967	6113
g	701	702	703	704	705	706	707	708	709	710	711	712	713	714	715	716	717	718	719	720
N	2006	4880	4827	3846	7448	1979	4874	3689	1899	4258	4962	2968	5426	6400	10949	2430	4126	2273	2346	5396
g	721	722	723	724	725	726	727	728	729	730	731	732	733	734	735	736	737	738	739	740
N	4971	3142	3797	2211	6642	4539	2031	5772	4310	4074	5131	3557	1995	2170	10340	3594	5315	4129	2017	4953
g	741	742	743	744	745	746	747	748	749	750	751	752	753	754	755	756	757	758	759	760
N	8510	4641	2029	4253	5087	2140	5199	5744	5462	5214	2016	3193	4254	4889	5371	5749	1994	2193	8978	5904
g	761	762	763	764	765	766	767	768	769	770	771	772	773	774	775	776	777	778	779	780
N	1980	3312	5242	2458	9606	2233	5766	3489	2119	8985	4011	2268	2107	4317	7042	2956	8393	2360	5875	7332
g	781	782	783	784	785	786	787	788	789	790	791	792	793	794	795	796	797	798	799	800
N	6100	5006	6667	4496	5100	3398	2075	2481	4397	4630	5611	5867	4901	2220	8460	2466	2260	6880	6078	4896
g	801	802	803	804	805	806	807	808	809	810	811	812	813	814	815	816	817	818	819	820
N	5314	2253	5600	3984	12112	4939	4437	3058	2309	5814	2171	5646	4392	5038	5581	5234	5700	2350	11102	5643
g	821	822	823	824	825	826	827	828	829	830	831	832	833	834	835	836	837	838	839	840
N	2153	3545	2238	3159	11531	5398	2283	5397	2187	4884	4469	4285	8861	3765	6056	6464	7017	2620	2694	8441
g	841	842	843	844	845	846	847	848	849	850	851	852	853	854	855	856	857	858	859	860
N	3965	2269	4384	2431	8326	4714	8833	3382	4553	6596	5856	4319	2288	5164	10993	3328	2356	7478	2312	5801
g	861	862	863	864	865	866	867	868	869	870	871	872	873	874	875	876	877	878	879	880
N	9136	2547	2497	4697	5718	2446	6730	5484	6534	6917	5540	3252	5557	5321	10713	4270	2334	2542	4902	7522
g	881	882	883	884	885	886	887	888	889	890	891	892	893	894	895	896	897	898	899	900
N	2460	6670	2344	6160	9235	2471	2423	5080	5915	5214	8441	2714	6693	4007	6512	4990	10773	2695	6610	6814
g	901	902	903	904	905	906	907	908	909	910	911	912	913	914	915	916	917	918	919	920
N	6272	5506	9442	3258	5816	3838	2399	2809	5908	10177	2550	5709	6829	2558	9287	2764	6498	6340	2496	7013
g	921	922	923	924	925	926	927	928	929	930	931	932	933	934	935	936	937	938	939	940
N	4939	2718	6713	9004	7724	2584	6153	4575	2529	7322	9182	2842	5078	2779	13868	6796	2430	5740	4960	6454
g	941	942	943	944	945	946	947	948	949	950	951	952	953	954	955	956	957	958	959	960
N	2449	4071	6991	4101	14187	5486	2486	4802	5801	7129	5081	7792	2524	5155	6491	3133	10846	2883	6745	6675
g	961	962	963	964	965	966	967	968	969	970	971	972	973	974	975	976	977	978	979	980
N	4205	5672	6246	2775	6352	8171	2564	5300	10330	5464	2671	4823	6707	2774	13411	3500	2540	4188	7462	8759
g	981	982	983	984	985	986	987	988	989	990	991	992	993	994	995	996	997	998	999	1000
N	6201	2733	2678	5591	6671	6360	10574	6643	7241	9723	2559	4449	5145	6171	6583	5017	2633	2858	8000	6407

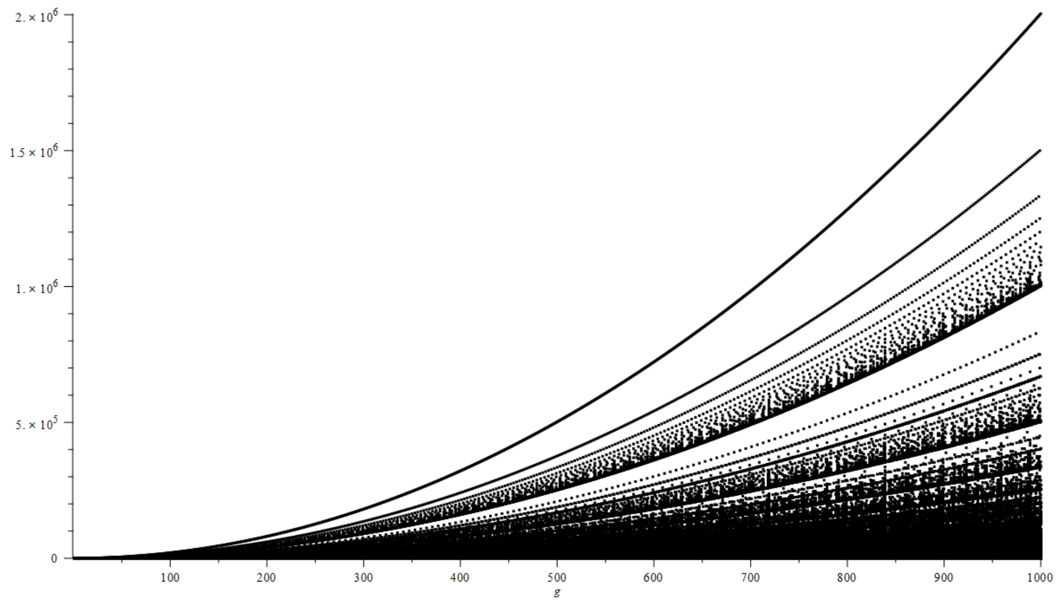


Figure 1.1: Volume of Fano triangles plotted against their Gorenstein index.

In Figure 1.1 we plot the volume of all Fano triangles against their Gorenstein index. The bounding curve on top is the curve $y = 2(x+1)^2$. The one right below is the curve $y = 3/2(x+1)^2$. In fact, all the points in Figure 1.1 that lie above the first limiting curve, i.e. the first curve where they seem to accumulate, lie on a curve of the form

$$y \; = \; \frac{k+1}{k}(x+1)^2.$$

These are precisely the Fano triangles described by the following Construction.

Construction 1.7.2. For $g, k \in \mathbb{Z}_{\geq 1}$ with $k \mid (g+1)$ let

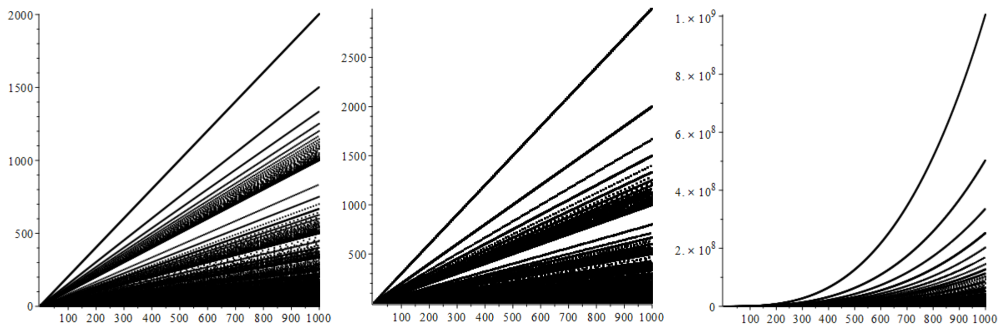
$$P_V(g,k) \; := \; \begin{bmatrix} 1 & 1 & 1-(k+1)(g+1) \\ 0 & \frac{g+1}{k} & -(g+1) \end{bmatrix}, \qquad \Delta_V(g,k) \; := \; \Delta(P_V(g,k)).$$

Then $\Delta = \Delta_V(g,k)$ has Gorenstein index $g(\Delta) = g$ and we have

$$Q_\Delta^{\mathrm{red}} \; = \; ((k+1)g, k, 1), \qquad \lambda(\Delta) \; = \; \frac{g+1}{k}, \qquad \mathrm{Vol}(\Delta) \; = \; \frac{k+1}{k}(g+1)^2.$$

Remark 1.7.3. In [57, Example 4.2] the authors describe a family of Fano triangles of Gorenstein index g whose volume grows as $O(g^{2/3})$. For the family $\Delta_V(g,1)$ from 1.7.2 the volume grows as $O(g^2)$. This shows that the volume of Fano polygons grows at least as $O(g^2)$.

Plotting the normalized volume of the dual, the multiplicity or the Mahler volume of all Fano triangles against their Gorenstein index, we obtain very similar pictures, see



(a) Volume of dual: $\mathrm{Vol}(\Delta^*)$ (b) Multiplicity: $\lambda(\Delta)$ (c) Mahler volume: $\mathrm{MV}(\Delta)$

Figure 1.2: Volume of dual (a), multiplicity (b) and Mahler volume (c) of Fano triangles plotted against their Gorenstein index.

Figure 1.2. Construction 1.7.4 is the analog of Construction 1.7.2 for the volume of the dual, providing the Fano triangles that describe the curves in the upper half of Figure 1.2 (a).

Construction 1.7.4. For $g, k \in \mathbb{Z}_{\geq 1}$ with $k \mid (g+1)$ let

$$P_{V^*}(g,k) \;:=\; \begin{bmatrix} 1 & 0 & -(k+1)g \\ 0 & 1 & -k \end{bmatrix}, \qquad \Delta_{V^*}(g,k) \;:=\; \Delta(P_{V^*}(g,k)).$$

Then $\Delta = \Delta_{V^*}(g,k)$ has Gorenstein index $g(\Delta) = g$ and we have

$$Q_\Delta^{\mathrm{red}} \;=\; ((k+1)g, k, 1), \qquad \lambda(\Delta) \;=\; 1, \qquad \mathrm{Vol}(\Delta^*) \;=\; \frac{k+1}{k}\frac{(g+1)^2}{g}.$$

Our final observation in dimension two is that there are no Fano triangles Δ with even Gorenstein index and reduced weight system $Q_\Delta^{\mathrm{red}} = (1,1,1)$. This observation is proved in Proposition 1.5.3. We restate the classification results in dimensions three and four.

Theorem 1.7.5. *See Theorem 1.1.6. Up to isomorphy there are* 9,368,501 *Fano simplices of dimension three and Gorenstein index $g \leq 30$. The number of simplices $N(g)$ for given Gorenstein index g develops as follows:*

g	1	2	3	4	5	6	7	8
$N(g)$	48	435	1,703	3,042	7,506	14,527	16,627	21,789
g	9	10	11	12	13	14	15	16
$N(g)$	39,288	61,295	54,404	100,670	59,500	157,071	269,037	121,530

g	17	18	19	20	21	22	23	24
$N(g)$	133,559	319,176	173,707	473,732	523,939	401,328	332,612	695,989
g	25	26	27	28	29	30		
$N(g)$	515,042	565,225	824,950	1,007,089	513,356	1,960,325		

Theorem 1.7.6. *See Theorem 1.1.6. Up to isomorphy there are* 87,532 *Fano simplices of dimension four and Gorenstein index* $g \leq 2$*. Of those,* 1,561 *are of Gorenstein index* $g = 1$*. The remaining* 85,971 *simplices are of Gorenstein index* $g = 2$*.*

Remark 1.7.7. By the correspondence between Fano simplices and fake weighted projective spaces, Theorems 1.1.5 - 1.1.7 are also classifications of fake weighted projective spaces of corresponding dimension and Gorenstein index.

Let us compare our results to existing classifications. In dimension two, Theorem 1.1.5 encompasses in particular the classification by Dais [32] of fake weighted projective planes of Gorenstein index at most three and the toric part of the classification in [39]. In dimension three we mention [56], where Kasprzyk classifies the three-dimensional *canonical* Fano polytopes, ie. those with a single interior lattice point. The overlap with Theorem 1.1.6 consists of precisely 204 canonical Fano simplices of Gorenstein index at most 30. There are only 21 three-dimensional canonical Fano simplices that have Gorenstein index larger than 30. The largest Gorenstein index among those is $g = 420$. This data has been taken from [25]. In dimension four we have the classification of the 1561 reflexive simplices by Kreuzer and Skarke [60], which correspond to the 1561 Fano simplices of Gorenstein index one from Theorem 1.1.7. Note that there is no overlap between Theorem 1.1.7 and the classification of empty 4-simplices by Iglesias-Valino and Santos [50] as our simplices have at least one interior lattice point. Let us also mention Balletti's recent extensive classification of lattice polytopes by their volume [8], where the polytopes are classified up to *affine unimodular equivalence*, ie. also allowing for translations. As this does not leave the Gorenstein index invariant, their results are not immediately comparable to Theorems 1.1.5 - 1.1.7.

Remark 1.7.8. Whereas the bounds in Theorems 1.1.2 and 1.1.4 are all sharp, we obtain sharpness in Theorem 1.1.3 (iii) for odd Gorenstein indices only. Indeed, for even Gorenstein index g, the values provided for $a_1, \ldots, a_{d-1}$ in Theorem 1.1.3 (iii) result in a matrix P with the last column being non-primitive. In fact our classification results suggest, that for even Gorenstein index the multiplicity bound in Theorem 1.1.3 (iii) is too high. We conjecture that in this case, apart from $(d, g) = (3, 2), (3, 4)$, the multiplicity is bounded by

$$\mathrm{mult}(\Delta) \ \leq \ \frac{2t_{g,d-1}^2}{g}$$

and this bound is sharp, ie. there is a Fano simplex of dimension d and Gorenstein index g that attains this bound.

CHAPTER
TWO

GORENSTEIN FANO THREEFOLDS OF PICARD NUMBER ONE

We classify the non-toric, $\mathbb{Q}$-factorial, Gorenstein, log terminal Fano threefolds of Picard number one that admit an effective action of a two-dimensional algebraic torus. The chapter is organized as follows. In Section 2.1 we present our classification results. Section 2.2 serves to provide the necessary background on the approach to rational projective varieties X with a torus action of complexity one via the Cox ring based on [41, 46]. In Picard number one, this approach represents any family of $\mathbb{Q}$-factorial varieties X in terms of an integral matrix P. Very first constraints arise from log terminality: Proposition 2.2.24, originally due to [21], shows that log terminality leaves us with eight types of matrices P to consider. These eight cases are treated in Sections 2.4 to 2.11. The classification tables are presented in Section 2.12. In Section 2.13, we compute the Hilbert–Poincaré series of our varieties. The results of this chapter have been achieved under the supervision of Jürgen Hausen and are published in [16].

2.1 Classification results

We work over an algebraically closed field $\mathbb{K}$ of characteristic zero. By a Fano variety we mean a normal projective variety X over $\mathbb{K}$ admitting an ample anticanonical divisor $-\mathcal{K}_X$. We classify the non-toric, $\mathbb{Q}$-factorial, log terminal, Gorenstein, Fano threefolds X of Picard number one that come with an effective action of a two-dimensional torus. Here, log terminal means discrepancies greater than -1 and Gorenstein means that $-\mathcal{K}_X$ is Cartier. We use the Cox ring based approach to rational varieties with a torus action of complexity one developed in [41, 46]. The Cox ring of a normal projective variety X with finitely generated divisor class group $\mathrm{Cl}(X)$ is defined as

$$\mathcal{R}(X) = \bigoplus_{\mathrm{Cl}(X)} \Gamma(X, \mathcal{O}_X(D)),$$

where we refer to [6] for the details. For our Fano threefolds X of Picard number one acted on by a two-dimensional torus, the divisor class group $\mathrm{Cl}(X)$ is of the form $\mathbb{Z} \oplus \Gamma$

with a finite abelian torsion part Γ and the Cox ring $\mathcal{R}(X)$ is a finitely generated complete intersection ring with a very specific system of trinomial relations. Moreover, the variety X can be reconstructed from the list of generator degrees in $\mathrm{Cl}(X)$ and the defining relations of the Cox ring $\mathcal{R}(X)$ which allows us to encode X via these Cox ring data in a compact manner.

Classification 2.1.1. *We obtain* 538 *families of non-toric,* $\mathbb{Q}$*-factorial, Gorenstein, log terminal Fano threefolds of Picard number one acted on effectively by a two-dimensional torus. Listed according to the possible divisor class groups, we have:*

Divisor class group	*Sporadic varieties*	*True families*
$\mathbb{Z}$	*242*	*3 one-dimensional*
$\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$	*163*	*4 one-dimensional*
$\mathbb{Z} \times (\mathbb{Z}/2\mathbb{Z})^2$	*46*	*5 one-dimensional,* *1 two-dimensional*
$\mathbb{Z} \times (\mathbb{Z}/2\mathbb{Z})^3$	*6*	*1 one-dimensional*
$\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z}$	*4*	*1 one-dimensional*
$\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/6\mathbb{Z}$	*1*	*0*
$\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$	*26*	*1 one-dimensional*
$\mathbb{Z} \times (\mathbb{Z}/3\mathbb{Z})^2$	*1*	*0*
$\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z}$	*18*	*1 one-dimensional*
$\mathbb{Z} \times \mathbb{Z}/5\mathbb{Z}$	*4*	*0*
$\mathbb{Z} \times \mathbb{Z}/6\mathbb{Z}$	*8*	*0*
$\mathbb{Z} \times \mathbb{Z}/8\mathbb{Z}$	*2*	*0*

Moreover, every non-toric, $\mathbb{Q}$*-factorial, Gorenstein, log terminal Fano threefold of Picard number one with an effective action of a two-dimensional torus is isomorphic to precisely one member of these* 538 *families.*

The defining data of each of our 538 families are stored in the file [15]. Moreover, we store in this file geometric invariants such as genus, codimension, anticanonical self intersection, Hilbert series, etc., which allows to extract varieties with given properties.

Note that being Gorenstein and log terminal, all varieties from Classification 2.1.1 are canonical. The overlap with the classification of non-toric Mori–Fano threefolds coming with an action of a two-dimensisonal torus given in [21] consists precisely of the smooth quadric in $\mathbb{P}_4$. As mentioned before, the main tool of [21], which settles the terminal case, is the anticanonical complex $\mathcal{A}_X^c$ associated with X, a polyhedral complex extending directly the features of the Fano polytope from toric geometry: X is terminal if and only if $\mathcal{A}_X^c$ has only the origin as an interior lattice point. This allows to bound the possible Cox ring data via the volumes of suitable lattice polytopes constructed out of the complex. In the log terminal Gorenstein case, one could think of proceeding analogously by using canonicity, which, however, appears to end in overflowing computations, even when building on the classification of canonical threefold singularities with action of a two-dimensional torus provided in [24]. Instead we can benefit in a completely different way and much more directly from the Gorenstein property: It gives rise to unit fraction identities involving the Cox ring data that admit only a finite number of integral solutions; see Proposition 2.9.3 (i) for an example. Moreover, the computation of these integral solutions turns out to be easily feasible, which in the end makes the classification possible.

Remark 2.1.2. The following figure shows how the 538 families from the classification 2.1.1 are distributed over the genus-codimension landscape of Fano threefolds presented in [26, Figure 1]. Here the genus of a Fano threefold X is $h^0(X, -\mathcal{K}_X) - 2$ and the codimension is taken with respect to embedding into a weighted projective space by means of a minimal system of homogeneous generators of the anticanonical ring

$$A_X = \bigoplus_{n \in \mathbb{Z}_{\geq 0}} \Gamma(X, -n\mathcal{K}_X).$$

Codimension \ Genus	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28
1	21	5																									
2	15	54	31																								
3				22																							
4			37		17																						
5						62																					
6				71																							
7								51																			
8									41																		
9										10																	
10											1																
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25																											
26																											3

Distribution of the 538 families over the genus-codimension landscape.

2.2 Torus actions of complexity one

We recall the necessary background on rational varieties with a torus action of complexity one and fix our notation. The reader is assumed to be familiar with the very basics of toric geometry, in particular the correspondence between fans and toric varieties; see [30, 33, 37]. We restrict ourselves to spending just a few words on Cox's quotient presentation [29] of a toric variety arising from a fan.

Construction 2.2.1. Let Z be the toric variety defined by a fan Σ in a lattice N such that the primitive generators $v_1, \ldots, v_r$ of the rays of Σ span the rational vector space $N_{\mathbb{Q}} = N \otimes_{\mathbb{Z}} \mathbb{Q}$. We have a linear map

$$P\colon \mathbb{Z}^r \to N, \qquad e_i \mapsto v_i.$$

In case $N = \mathbb{Z}^n$, we also speak of the *generator matrix* $P = [v_1, \ldots, v_r]$ of Σ. The divisor class group and the Cox ring of Z are

$$\mathrm{Cl}(Z) \ = \ K \ := \ \mathbb{Z}^r/\mathrm{im}(P^*), \qquad \mathcal{R}(Z) \ = \ \mathbb{K}[T_1, \ldots, T_r], \qquad \deg(T_i) \ = \ Q(e_i),$$

where P^* denotes the dual map of P and $Q\colon \mathbb{Z}^r \to K$ the projection. Now, one defines a fan $\hat{\Sigma}$ in $\mathbb{Z}^r$ consisting of faces of the positive orthant of $\mathbb{Q}^r$ by

$$\hat{\Sigma} \ := \ \{\delta_0 \preceq \mathbb{Q}^r_{\geq 0};\ P(\delta_0) \subseteq \sigma \text{ for some } \sigma \in \Sigma\}.$$

The toric variety $\hat{Z}$ associated with $\hat{\Sigma}$ is an open toric subset in $\bar{Z} := \mathbb{K}^r$. As P is a map of the fans $\hat{\Sigma}$ and Σ, it defines a toric morphism $p\colon \hat{Z} \to Z$. The quasitorus

$$H \ = \ \mathrm{Spec}\,\mathbb{K}[K] \ = \ \ker(p) \ \subseteq \ \mathbb{T}^r \ = \ (\mathbb{K}^*)^r$$

acts as a subgroup of the torus $\mathbb{T}^r$ on $\hat{Z}$ and the morphism $p\colon \hat{Z} \to Z$ turns out to be a good quotient with respect to the H-action.

The quotient presentation of toric varieties is a central piece in the Cox ring based approach to rational varieties with a torus action of complexity one provided by [41, 46]; see also [6, Section 3.4]. The first step, however, is the following purely algebraic construction of a certain class of graded algebras; see [6, Construction 3.4.2.1] and more generally [42, Constructions 3.5 and 6.3].

Construction 2.2.2. Fix $r \in \mathbb{Z}_{\geq 1}$, a sequence $n_0, \ldots, n_r \in \mathbb{Z}_{\geq 1}$, set $n := n_0 + \cdots + n_r$, and fix integers $m \in \mathbb{Z}_{\geq 0}$ and $0 < s < n + m - r$. The input data are matrices

$$A \ = \ [a_0, \ldots, a_r] \ \in \ \mathrm{Mat}(2, r+1; \mathbb{K}), \qquad P \ = \ \begin{bmatrix} L & 0 \\ d & d' \end{bmatrix} \ \in \ \mathrm{Mat}(r+s, n+m; \mathbb{Z}),$$

where A has pairwise linearly independent columns and P is built from an $(s \times n)$-block d, an $(s \times m)$-block d' and an $(r \times n)$-block L of the shape

$$L \ = \ \begin{bmatrix} -l_0 & l_1 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ -l_0 & 0 & \ldots & l_r \end{bmatrix}, \qquad l_i \ = \ (l_{i1}, \ldots, l_{in_i}) \in \mathbb{Z}^{n_i}_{\geq 1}$$

such that the columns v_{ij}, v_k of P are pairwise different primitive vectors generating $\mathbb{Q}^{r+s}$ as a cone. Consider the polynomial algebra

$$\mathbb{K}[T_{ij}, S_k] \ := \ \mathbb{K}[T_{ij}, S_k;\ 0 \le i \le r,\ 1 \le j \le n_i, 1 \le k \le m].$$

Denote by $\mathfrak{I}$ the set of all triples $I = (i_1, i_2, i_3)$ with $0 \le i_1 < i_2 < i_3 \le r$ and define for any $I \in \mathfrak{I}$ a trinomial

$$g_I \ := \ g_{i_1,i_2,i_3} \ := \ \det \begin{bmatrix} T_{i_1}^{l_{i_1}} & T_{i_2}^{l_{i_2}} & T_{i_3}^{l_{i_3}} \\ a_{i_1} & a_{i_2} & a_{i_3} \end{bmatrix}, \qquad T_i^{l_i} \ := \ T_{i1}^{l_{i1}} \cdots T_{in_i}^{l_{in_i}}.$$

Consider the factor group $K := \mathbb{Z}^{n+m}/\mathrm{im(P^*)}$ and the projection $Q\colon \mathbb{Z}^{n+m} \to K$. We define a K-grading on $\mathbb{K}[T_{ij}, S_k]$ by setting

$$\deg(T_{ij}) \ := \ \omega_{ij} \ := \ Q(e_{ij}), \qquad \deg(S_k) \ := \ \omega_k \ := \ Q(e_k).$$

Then the trinomials g_I just introduced are K-homogeneous and they all share the same K-degree. In particular, we obtain a K-graded factor algebra

$$R(A, P) \ := \ \mathbb{K}[T_{ij}, S_k]/\langle g_I;\ I \in \mathfrak{I}\rangle.$$

Example 2.2.3. We choose $r = 2$, moreover $n_0 = 2$, $n_1 = n_2 = 1$ and $m = 1$ and, finally $s = 2$. In this setting, consider the defining matrices

$$A \ := \ \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}, \qquad P \ := \ \begin{bmatrix} -1 & -1 & 4 & 0 & 0 \\ -1 & -1 & 0 & 2 & 0 \\ 0 & 0 & -3 & 1 & 1 \\ 0 & -2 & 4 & 0 & 0 \end{bmatrix}.$$

The algebra $R(A, P)$ arising from these matrices comes due to $r = 2$ with a single trinomial relation and is explicitly given by

$$R(A, P) \ = \ \mathbb{K}[T_{01}, T_{02}, T_{11}, T_{21}, S_1]/\langle T_{01}T_{02} + T_{11}^4 + T_{21}^2\rangle.$$

We have $K = \mathbb{Z}^5/\mathrm{im}(P^*) = \mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$ and the degrees of the T_{ij} and S_1 are the columns of the *degree matrix*

$$Q \ = \ \begin{bmatrix} 2 & 2 & 1 & 2 & 1 \\ \bar{1} & \bar{1} & \bar{0} & \bar{0} & \bar{0} \\ \bar{1} & \bar{1} & \bar{1} & \bar{1} & \bar{0} \end{bmatrix}.$$

Theorem 2.2.4. *See* [6, Theorem 3.4.2.3], *also* [42, Theorems 3.10 and 6.5]. *The ring $R(A, P)$ produced by Construction 2.2.2 is a normal complete intersection ring and its ideal of relations is generated by the trinomials $g_i = g_{i,i+1,i+2}$, where $i = 0, \dots, r-2$.*

Remark 2.2.5. We call a defining matrix P *irredundant* if we have $l_{i1}n_i \ge 2$ for all $i = 0, \dots, r$. Each $R(A, P)$ is isomorphic as a graded algebra to some $R(A', P')$ with P' irredundant. Note that for $r \ge 2$ and an irredundant P, the ring $R(A, P)$ is not a polynomial ring.

Remark 2.2.6. Consider a defining matrix P as in Construction 2.2.2. By an *admissible operation* on the matrix P we mean one of the following:

(i) adding a multiple of one of the upper r rows to one of the lower s rows,
(ii) applying a unimodular matrix from the left to the (d, d') block,
(iii) swapping two columns v_{ij_1} and v_{ij_2} inside a leaf $v_{i1}, \ldots, v_{in_i}$,
(iv) swapping two leafs $v_{i1}, \ldots, v_{in_i}$ and $v_{j1}, \ldots, v_{jn_j}$ and rearranging the L-block by elementary row operations into its required shape,
(v) swapping two columns v_{k_1} and v_{k_2} of the d'-block.

If P' arises from P via admissible operations, then with a suitable A', the graded rings $R(A, P)$ and $R(A', P')$ are isomorphic.

Remark 2.2.7. The matrix A of a ring $R(A, P)$ is responsible for the coefficients of the defining trinomials $g_i = g_{i,i+1,i+2}$. By rescaling variables we can always reduce to defining relations of the shape

$$T_0^{l_0} + T_1^{l_1} + T_2^{l_2}, \qquad \lambda_1 T_1^{l_1} + T_2^{l_2} + T_3^{l_3}, \qquad \ldots \qquad \lambda_{r-2} T_{r-2}^{l_{r-2}} + T_{r-1}^{l_{r-1}} + T_r^{l_r}$$

with pairwise distinct $1 \neq \lambda_i \in \mathbb{C}^*$. In particular, in case of a single defining relation, there is no need to care about the coefficients. The matrix A is motivated by the geometry behind $R(A, P)$, see Remark 2.2.12.

We enter the second step, producing rational normal varieties X with a torus action $\mathbb{T}^s \times X \to X$ of complexity one. Each of the resulting X comes embedded in a toric variety Z, is defined in homogeneous coordinates by the above trinomials $g_0, \ldots, g_{r-1}$ and the torus $\mathbb{T}^s$ acting on X is a subtorus of the acting torus $\mathbb{T}^{r+s}$ of Z. The original references are again [41, 46]; see also [6, Construction 3.4.3.6] as well as the more general [42, Constructions 3.5 and 6.13].

Construction 2.2.8. In the situation of Construction 2.2.2, assume $r \geq 2$ and that P is irredundant. Consider the common zero set of the defining relations of $R(A, P)$:

$$\bar{X} \ := \ V(g_I;\ I \in \mathfrak{I}) \ \subseteq \ \bar{Z} \ := \ \mathbb{K}^{n+m}.$$

Let Σ be any fan in $N = \mathbb{Z}^{r+s}$ having the columns of P as the primitive generators of its rays. Then $\hat{X} := \bar{X} \cap \hat{Z}$ and Construction 2.2.1 yield a commutative diagram

$$\begin{tikzcd}
\bar{X} \arrow[r, phantom, "\subseteq"] & \bar{Z} \\
\hat{X} \arrow[u, phantom, "\cup|"] \arrow[r] \arrow[d, "p", "/\!\!/ H"'] & \hat{Z} \arrow[u, phantom, "\cup|"] \arrow[d, "/\!\!/ H", "p"'] \\
X \arrow[r] & Z,
\end{tikzcd}$$

where $X := X(A, P, \Sigma) := p(\hat{X})$ is a non-toric, closed subvariety of the toric variety Z arising from Σ. Dimension, divisor class group and Cox ring of X are

$$\dim(X) \ = \ s + 1, \qquad \mathrm{Cl}(X) \ \cong \ K, \qquad \mathcal{R}(X) \ \cong \ R(A, P).$$

The subtorus $\mathbb{T}^s \subseteq \mathbb{T}^{r+s}$ of the acting torus of Z associated with the sublattice $\mathbb{Z}^s \subseteq \mathbb{Z}^{r+s}$ leaves X invariant and the induced T-action on X is of complexity one.

Example 2.2.9. We continue Example 2.2.3. Let Σ be the fan in $\mathbb{Z}^4$ having P as its generator matrix and the maximal cones

$$\begin{array}{lll} \mathrm{cone}(v_{02}, v_{11}, v_{21}, v_1), & \mathrm{cone}(v_{01}, v_{11}, v_{21}, v_1), & \mathrm{cone}(v_{01}, v_{02}, v_{21}, v_1), \\ \mathrm{cone}(v_{01}, v_{02}, v_{11}, v_1), & \mathrm{cone}(v_{01}, v_{02}, v_{11}, v_{21}). & \end{array}$$

The associated toric variety Z is a four-dimensional fake weighted projective space with divisor class group

$$\mathrm{Cl}(Z) = K = \mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}.$$

Moreover, $H = \mathbb{K}^* \times \{\pm 1\} \times \{\pm 1\}$ acts on $\bar{Z} = \mathbb{K}^5$ via the weights given by the columns of the degree matrix Q and Construction 2.2.8 becomes

$$\begin{tikzcd}[row sep=small] V(T_{01}T_{02} + T_{11}^4 + T_{21}^2) \arrow[r, phantom, "="] & \bar{X} \arrow[r, phantom, "\subseteq"] & \bar{Z} \arrow[r, phantom, "="] & \mathbb{K}^5 \\ \cup\!\mathsf{I} & \cup\!\mathsf{I} & \cup\!\mathsf{I} & \cup\!\mathsf{I} \\ \bar{X} \setminus \{0\} \arrow[r, phantom, "="] & \hat{X} \arrow[r] \arrow[d, "p", "/\!\!/H"'] & \hat{Z} \arrow[d, "/\!\!/H", "p"'] \arrow[r, phantom, "="] & \mathbb{K}^5 \setminus \{0\}. \\ & X \arrow[r] & Z & \end{tikzcd}$$

Theorem 2.2.10. *See* [6, Theorem 4.4.1.6] *and* [42, Theorems 3.10 and 6.18]. *Every non-toric rational normal projective variety with a torus action of complexity one is equivariantly isomorphic to some $X(A, P, \Sigma)$ arising from Construction* 2.2.2.

Any variety $X = X(A, P, \Sigma)$ inherits many geometric properties from its ambient toric variety Z. A first observation concerns the restriction of the invariant divisors from Z to X; see [6, Proposition 3.2.4.5].

Remark 2.2.11. Consider $X = X(A, P, \Sigma)$ as in Construction 2.2.8. The columns v_{ij} and v_k of P define prime divisors $D_{ij} = V_Z(T_{ij})$ and $D_k = V_Z(T_k)$ on Z. The restrictions of them $D_{ij}^X = V_X(T_{ij})$ and $D_k^X = V_X(S_k)$ are prime divisors on X and in the class group $\mathrm{Cl}(Z) = K = \mathrm{Cl}(X)$, we have

$$[D_{ij}] \;=\; \deg(T_{ij}) \;=\; [D_{ij}^X], \qquad [D_k] \;=\; \deg(T_k) \;=\; [D_k^X].$$

We recover the divisors D_{ij}^X as the components of the critical values $c_i \in \mathbb{P}_1$ of a certain quotient map; see [42, Proposition 3.16] for a general treatment.

Remark 2.2.12. Consider $X = X(A, P, \Sigma)$ as in Construction 2.2.8. Consider the open sets of points having finite isotropy groups with respect to the $\mathbb{T}^s$-action:

$$Z_0 \;=\; \{z \in Z;\; \mathbb{T}^s_z \text{ is finite}\}, \qquad X_0 \;=\; X \cap Z_0 \;=\; \{x \in X;\; \mathbb{T}^s_x \text{ is finite}\}.$$

Then $Z_0 \subseteq Z$ is invariant under the torus $\mathbb{T}^{r+s}$ and $X_0 \subseteq X$ is invariant under $\mathbb{T}^s$. Moreover, we have a commutative diagram

$$\begin{array}{ccc} X & \subseteq & Z \\ \cup\!\vert & & \cup\!\vert \\ X_0 & \longrightarrow & Z_0 \\ {\scriptstyle /\!\!/\mathbb{T}^s}\big\downarrow{\scriptstyle \pi_X} & & {\scriptstyle \pi_Z}\big\downarrow{\scriptstyle /\!\!/\mathbb{T}^s} \\ \mathbb{P}_1 & \longrightarrow & \mathbb{P}_r, \end{array}$$

where π_X and π_Z are categorical quotients with respect to the actions of $\mathbb{T}^s$ on X and Z respectively and π_Z is a toric morphism. Moreover, we obtain

$$\overline{\pi_X^{-1}(c_i)} \;=\; \bigcup_{j=1}^{n_i} D_{ij}^X \subseteq X, \qquad \overline{\pi_Z^{-1}(C_i)} \;=\; \bigcup_{j=1}^{n_i} D_{ij} \subseteq Z$$

with the toric divisors $C_0, \ldots, C_r \subseteq \mathbb{P}_r$ and the points $c_i \in \mathbb{P}_1$ having the i-th column of A as its homogeneous coordinates. Finally,

$$|\mathbb{T}^s_{x_{ij}}| \;=\; l_{ij}$$

holds for the order of the isotropy group $\mathbb{T}^s_{x_{ij}}$ of the action of the torus $\mathbb{T}^s$ at any general point $x_{ij} \in D_{ij}^X$.

The divisors from Remark 2.2.12 also allow an explicit presentation of an anticanonical divisor; see [6, Proposition 3.4.4.1].

Remark 2.2.13. Let $X = X(A, P, \Sigma)$ arise from Construction 2.2.8. Then the anticanonical divisor class of X is given as

$$-\mathcal{K}_X \;=\; \sum_{i,j} \deg(T_{ij}) \,+\, \sum_k \deg(S_k) \,-\, (r-1)\sum_{i=1}^{n_0} l_{0j}\deg(T_{0j}) \;\in\; K \;=\; \mathrm{Cl}(X).$$

In particular, due to $\deg(T_{ij}) = [D_{ij}^X]$ and $\deg(T_k) = [D_k^X]$, we have the following anticanonical divisor on X:

$$D_0^X \;:=\; \sum_{i,j} D_{ij}^X \,+\, \sum_k D_k^X \,-\, (r-1)\sum_{j=1}^{n_0} l_{0j} D_{0j}^X.$$

Example 2.2.14. For the variety X from Example 2.2.9, we can compute the anticanonical class as

$$-\mathcal{K}_X \;=\; \deg(T_{11}) + \deg(T_{21}) + \deg(S_1) \;=\; (4, \bar{0}, \bar{0}) \;\in\; \mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \;=\; \mathrm{Cl}(X).$$

In particular, we see that the anticanonical class is ample and, consequently, X is a Fano variety.

Remark 2.2.15. Let $X = X(A, P, \Sigma)$ arise from Construction 2.2.8. We call $\sigma \in \Sigma$ an *X-cone* if the corresponding toric orbit $\mathbb{T}^{r+s} \cdot z_\sigma \subseteq Z$ meets $X \subseteq Z$. A cone $\sigma \in \Sigma$ is an X-cone if and only if one of the following holds:

(i) σ is a *big cone*, that means $v_{0j_0}, \dots, v_{rj_r} \in \sigma$ for some $j_0, \dots, j_r$,

(ii) σ is a *leaf cone*, that means $\sigma \subseteq \mathrm{cone}(v_{i1}, \dots, v_{in_i}, v_1, \dots, v_m)$ for some i.

Every X-cone $\sigma \in \Sigma$ defines an affine open subvariety $X_\sigma =: X \cap Z_\sigma$ in X by cutting down the corresponding affine toric chart $Z_\sigma \subseteq Z$. Note that X is covered by the X_σ, where σ runs through the X-cones of Σ.

Example 2.2.16. Consider again the variety X from Example 2.2.9. Then the fan Σ has exactly four maximal X-cones, namely

$$\begin{array}{ll} \mathrm{cone}(v_{02}, v_{11}, v_{21}, v_1), & \mathrm{cone}(v_{01}, v_{11}, v_{21}, v_1), \\ \mathrm{cone}(v_{01}, v_{02}, v_{11}, v_{21}), & \mathrm{cone}(v_{01}, v_{02}, v_1). \end{array}$$

The first three are big cones, whereas the fourth one is a leaf cone. Thus, X is covered by four open affine subvarieties, given by the maximal X-cones of Σ.

Let us see how to detect Cartier divisors, that means locally principal Weil divisors, on a variety $X = X(A, P, \Sigma)$ in terms of the defining data.

Proposition 2.2.17. *Let $X = X(A, P, \Sigma)$ arise from Construction 2.2.8. Consider on Z and X the Weil divisors*

$$D \ = \ \sum a_{ij} D_{ij} + \sum a_k D_k, \qquad D^X \ = \ \sum a_{ij} D_{ij}^X + \sum a_k D_k^X.$$

In $K = \mathrm{Cl}(Z) = \mathrm{Cl}(X)$ consider the classes $\omega = [D] = [D^X]$, $\omega_{ij} = [D_{ij}] = [D_{ij}^X]$ and $\omega_k = [D_k] = [D_k^X]$ and let $\sigma \in \Sigma$ an X-cone. Then the following statements are equivalent:

(i) *The divisor D^X is Cartier on X_σ.*

(ii) *The divisor D is Cartier on Z_σ.*

(iii) *We have $D = \mathrm{div}(\chi^u)$ on Z_σ with a character χ^u of $\mathbb{T}^{r+s}$.*

(iv) *There is $u \in \mathbb{Z}^{r+s}$ with $\langle u, v_{ij} \rangle = a_{ij}$ and $\langle u, v_k \rangle = a_k$ for all $v_{ij}, v_k \in \sigma$.*

(v) *We have $\omega \in \langle \omega_{ij}, \omega_k;\ v_{ij}, v_k \notin \sigma \rangle$ in $K = \mathrm{Cl}(X)$.*

In particular, D is a Cartier divisor on X if and only if one of these conditions holds for all maximal X-cones $\sigma \in \Sigma$.

Proof. The equivalence of (i), (ii) and (v) follows from Proposition [6, Proposition 3.3.1.5]. The rest is basic toric geometry. □

A normal variety X is *$\mathbb{Q}$-factorial* if every Weil divisor D on X admits a Cartier multiple nD with $n \in \mathbb{Z}_{\geq 1}$.

Corollary 2.2.18. *A variety $X = X(A, P, \Sigma)$ as in Construction* 2.2.8 *is $\mathbb{Q}$-factorial if and only if each X-cone $\sigma \in \Sigma$ is simplicial.*

Now, recall that a variety is *Gorenstein* if its canonical class is Cartier. Combining Remark 2.2.13 and Proposition 2.2.17, we obtain the following characterization.

Corollary 2.2.19. *Consider $X = X(A, P, \Sigma)$ and let $D^X = \sum a_{ij} D^X_{ij} + \sum a_k D^X_k$ be an anticanonical divisor on X. Then X is Gorenstein if and only if for every maximal X-cone σ, there is a linear form $u \in \mathbb{Z}^{r+s}$ with*

$$\langle u, v_{ij} \rangle \ = \ a_{ij}, \qquad \langle u, v_k \rangle \ = \ a_k \qquad \textit{for all} \quad v_{ij}, v_k \ \in \ \sigma.$$

Example 2.2.20. Consider again the variety X from Example 2.2.9 and the four maximal X-cones given in Example 2.2.16. Listed accordingly, we have linear forms

$$(2, 0, 1, -1), \qquad (0, 0, 1, 1), \qquad (-2, 2, -3, 0), \qquad (-1, 1, 1, 0)$$

representing the anticanonical divisor D^X_0 on the corresponding affine open subvarieties of X. In particular, X is Gorenstein.

If X is a $\mathbb{Q}$-factorial Fano variety of Picard number one, then the divisor class group $\mathrm{Cl}(X)$ allows a positive splitting into a free cyclic part and its torsion part Γ, that means that we have an isomorphism

$$\mathrm{Cl}(X) \ \cong \ \mathbb{Z} \oplus \Gamma$$

such that for the anticanonical class $\omega_X = (w_X, \eta_X)$, the $\mathbb{Z}$-part w_X is positive. Note that in this setting the $\mathbb{Z}$-part of any divisor class $\omega = (w, \eta)$ does not depend on the particular choice of the splitting.

Corollary 2.2.21. *Let $X = X(A, P, \Sigma)$ be $\mathbb{Q}$-factorial, Gorenstein, Fano and of Picard number one. Then, for every maximal X-cone σ, the $\mathbb{Z}$-parts w_{ij}, w_k of the generator degrees and w_X of the anticanonical class satisfy*

$$\gcd(w_{ij}, w_k; \ v_{ij} \notin \sigma, v_k \notin \sigma) \ \mid \ w_X.$$

Proof. As X is Gorenstein, the canonical class ω_X represents a Cartier divisor. Proposition 2.2.17 tells us that for every maximal X-cone σ, the ω_X lies in the subgroup of $\mathrm{Cl}(X) = K$ generated by the classes ω_{ij}, ω_k, where $v_{ij} \notin \sigma$, $v_k \notin \sigma$. Thus, the $\mathbb{Z}$-part w_X lies in the ideal of $\mathbb{Z}$ generated by the $\mathbb{Z}$-parts w_{ij}, w_k, where $v_{ij} \notin \sigma$, $v_k \notin \sigma$. The assertion follows. $\square$

Finally, we discuss log terminality. Recall that given any resolution of singularities $\pi \colon X' \to X$ of a normal variety, we have the ramification formula

$$\mathcal{K}_{X'} - \pi^* \mathcal{K}_X \ = \ \sum_{i=1}^{r} a_i E_i$$

with canonical divisors on X' and X and the exceptional divisors $E_i, \ldots, E_r$. Then X is called *log terminal* if we have $a_i > -1$ for $i = 1, \ldots, r$. We characterize log terminality of a given $\mathbb{Q}$-factorial Fano variety $X = X(A, P, \Sigma)$. A *platonic tuple* is a tuple $(l_0, \dots, l_r)$ of positive integers such that after re-ordering the l_i decreasingly, we obtain a tuple $(a, b, c, 1, \dots, 1)$ with (a, b, c) one of

$$(x, y, 1), \qquad (y, 2, 2), \qquad (5, 3, 2), \qquad (4, 3, 2), \qquad (3, 3, 2).$$

Proposition 2.2.22. *See* [5, Theorem 3.13]. *A $\mathbb{Q}$-factorial Fano variety $X = X(A, P, \Sigma)$ has at most log terminal singularities if and only if for any X-cone $\sigma = \operatorname{cone}(v_{0j_0}, \dots, v_{rj_r})$ the exponents $l_{0j_0}, \dots, l_{rj_r}$ form a platonic tuple.*

Example 2.2.23. For the variety $X = X(A, P, \Sigma)$ from Example 2.2.9 we have to consider the X-cones

$$\operatorname{cone}(v_{02}, v_{11}, v_{21}), \qquad \operatorname{cone}(v_{01}, v_{11}, v_{21}).$$

Both of them yield the exponent tuple $(1, 4, 2)$ which is platonic. Consequently, X is log terminal.

Log terminality leads to the following first constraints on the defining matrix P of our Fano varieties $X = X(A, P, \Sigma)$.

Proposition 2.2.24. *See* [21, Lemma 5.2]. *Let $X = X(A, P, \Sigma)$ a non-toric, $\mathbb{Q}$-factorial, log terminal Fano threefold of Picard number one, where P is irredundant. Then, after suitable admissible operations, P fits into one of the following cases:*

(i) *$m = 0$, $r = 2$ and $n = 5$, where $n_0 = n_1 = 2$, $n_2 = 1$,*
(ii) *$m = 0$, $r = 3$ and $n = 6$, where $n_0 = n_1 = 2$, $n_2 = n_3 = 1$,*
(iii) *$m = 0$, $r = 4$ and $n = 7$, where $n_0 = n_1 = 2$, $n_2 = n_3 = n_4 = 1$,*
(iv) *$m = 0$, $r = 2$ and $n = 5$, where $n_0 = 3$, $n_1 = n_2 = 1$,*
(v) *$m = 0$, $r = 3$ and $n = 6$, where $n_0 = 3$, $n_1 = n_2 = n_3 = 1$,*
(vi) *$m = 1$, $r = 2$ and $n = 4$, where $n_0 = 2$, $n_1 = n_2 = 1$,*
(vii) *$m = 1$, $r = 3$ and $n = 5$, where $n_0 = 2$, $n_1 = n_2 = n_3 = 1$,*
(viii) *$m = 2$, $r = 2$ and $n = 3$, where $n_0 = n_1 = n_2 = 1$.*

Remark 2.2.25. Every rational Gorenstein del Pezzo surface has at most canonical singularities and thus is in particular log terminal; see [47]. For Gorenstein Fano varieties of higher dimension even the latter property need not hold. For instance,

$$P = \begin{bmatrix} -3 & -1 & 3 & 1 & 0 \\ -3 & -1 & 0 & 0 & k \\ -4 & -1 & 1 & 0 & k \\ 1 & 0 & 0 & 0 & -1 \end{bmatrix}$$

defines for each $k \geq 4$ a $\mathbb{Q}$-factorial Fano threefold $X = X(A, P, \Sigma)$ of Picard number one, which is not log terminal by Proposition 2.2.22. More explicitly, Σ consists of all pointed cones generated by columns of P and we have

$$X = V(T_{01}^3 T_{02} + T_{11}^3 T_{12} + T_{21}^k) \subseteq \mathbb{P}_{1,\,k-3,\,1,\,k-3,\,1},$$

$$[-\mathcal{K}_X] = k - 3 \in \mathbb{Z} = \operatorname{Cl}(X).$$

This example series shows moreover that the Gorenstein and Fano condition together are even in the specific setting of threefolds with an action of a two-dimensional torus not enough to guarantee finiteness in fixed dimension and Picard number.

2.3 Proof of Classification 2.1.1: Preparation

Proposition 2.2.24 divides the proof of the classification theorem into cases (i) to (viii). These cases are treated in Sections 2.4 to 2.11. The strategy in each of these eight cases is very formulaic. The pattern is as follows.

(i) Using log terminality and the Gorenstein property, we obtain constraints on the entries of the defining matrix P, such that each row of P admits at most one entry which is not bounded by other entries of P. See 2.9.2 for an example.

(ii) We establish unit fraction identities involving the Cox ring data, which bound the entries of the $\mathbb{Z}$-part of the degree matrix Q^0. See 2.9.3 for an example.

(iii) Combining items (i) and (ii), and using the fact that P annihilates the transpose of Q^0, we determine the remaining entries of P. This produces a finite list of candidates for defining matrices P.

(iv) From the resulting list of explicit matrices P we remove those not defining a Gorenstein Fano variety and remove redundancies, ie. matrices defining isomorphic varieties.

For item (iv) we need criteria to decide computationally whether or not given defining data lead to isomorphic varieties. For this, we say that a defining matrix P as in Construction 2.2.8 has *ordered exponents* if we have

(i) $n_0 \geq \cdots \geq n_r$,

(ii) $l_{i1} \geq \cdots \geq l_{in_i}$ for each $i = 0, \ldots, r$ and

(iii) if $n_i = n_{i+1}$ then $l_{i1} \geq l_{i+1,1}$.

If P has ordered exponents, then we call the data $(n_0, \ldots, n_r, m)$ the *format* of P. Note that via admissible column operations, we can always assume that P has ordered exponents.

Proposition 2.3.1. *Let* (A, P, Σ) *and* (A', P', Σ') *be as in Construction* 2.2.8 *such that the associated varieties* X *and* X' *are isomorphic to each other.*

(i) *There is an isomorphism* $\varphi\colon X \to X'$ *which is equivariant with respect to the torus actions.*

(ii) If P *and* P' *have ordered exponents, then they share the same format and for each* i *there is an* i' *with* $n_{i'} = n_i$ *and* $(l_{i1}, \ldots, l_{in_i}) = (l'_{i'1}, \ldots, l'_{i'n_i})$ *such that*

$$\langle \deg(T_{ij});\, j = 1, \ldots, n_i \rangle \;\cong\; \langle \deg(T_{i'j});\, j = 1, \ldots, n_i \rangle$$

holds for the subgroups in $\mathrm{Cl}(X)$ *and* $\mathrm{Cl}(X')$*, respectively, generated by the corresponding degrees.*

Proof. For the first assertion, observe that for any isomorphism $\varphi\colon X \to X'$ of varieties, we can install a torus action on X' making φ equivariant. Now, any torus action of complexity one on the non-toric X' corresponds to a maximal torus in the affine algebraic group $\mathrm{Aut}(X')$; see for instance [7, Theorem 2.1]. Thus, the assertion follows from the fact that any two maximal tori in an affine algebraic group are conjugate. The second assertion follows from the first one and the fact that any equivariant isomorphism respects the data described in Remark 2.2.12. □

The following Lemma is a combination of Proposition 2.2.17 and Corollary 2.2.21. Its specific formulation will be used numerous times throughout Sections 2.4 – 2.11.

Lemma 2.3.2. *Let $X = X(A, P, \Sigma)$ as in Construction 2.2.8. Assume X is non-toric, $\mathbb{Q}$-factorial, Gorenstein, Fano and of Picard number one.*

(i) *If there is $0 \le i_0 \le r$ with $n_{i_0} > 1$, then for every $1 \le j_0 \le n_{i_0}$ the cone*

$$\sigma_{i_0 j_0} \ := \ \mathrm{cone}(\, v_{ij}, v_k\, ;\, (i,j) \neq (i_0, j_0),\ k = 1, \ldots, m\,)$$

is a maximal X-cone. In this case the $\mathbb{Z}$-part $w_{i_0 j_0}$ is a divisor of w_X. Moreover, if $-\mathcal{K} = \sum a_{ij} D_{ij}^X + \sum a_k D_k^X$ is an anticanonical divisor on X, then there is a linear form $u \in \mathbb{Z}^{r+s}$ with

$$\langle u, v_{i_0 j_0} \rangle \ = \ a_{i_0 j_0} - \frac{w_X}{w_{i_0 j_0}}; \qquad \langle u, v_{ij} \rangle \ = \ a_{ij}, \qquad \langle u, v_k \rangle \ = \ a_k$$

for all $(i,j) \neq (i_0, j_0)$ and all $k = 1, \ldots, m$.

(ii) *If $m \geq 1$ holds, then for every $1 \le k_0 \le m$ the cone*

$$\sigma_{k_0} \ := \ \mathrm{cone}(\, v_{ij}, v_k\, ;\, 0 \le i \le r,\ 1 \le j \le n_i,\ k \neq k_0\,)$$

is a maximal X-cone. In this case the $\mathbb{Z}$-part w_{k_0} is a divisor of w_X. Moreover, if $-\mathcal{K} = \sum a_{ij} D_{ij}^X + \sum a_k D_k^X$ is an anticanonical divisor on X, then there is a linear form $u \in \mathbb{Z}^{r+s}$ with

$$\langle u, v_{k_0} \rangle \ = \ a_{k_0} - \frac{w_X}{w_{k_0}}; \qquad \langle u, v_{ij} \rangle \ = \ a_{ij}, \qquad \langle u, v_k \rangle \ = \ a_k$$

for all $0 \le i \le r$, $1 \le j \le n_i$ and all $k \neq k_0$.

Proof. We prove (i), Item (ii) is proved similarly. The assumption $n_{i_0} > 1$ guarantees that $\sigma_{i_0 j_0}$ is a big cones, see Remark 2.2.15. Its dimension equals that of Σ, thus it is a maximal X-cone. We use Corollary 2.2.21 to see that w_X is divisible by $w_{i_0 j_0}$. Let $-\mathcal{K} = \sum a_{ij} D_{ij}^X + \sum a_k D_k^X$ an anitcanonical divisor on X. Applying Proposition 2.2.17 to the X-cone $\sigma_{i_0 j_0}$, we obtain a linear form u with $\langle u, v_{ij} \rangle = a_{ij}$ for all $(i,j) \neq (i_0, j_0)$ and $\langle u, v_k \rangle = a_k$ for all $k = 1, \ldots, m$. Note that the defining matrix P annihilates the transpose of the $\mathbb{Z}$-part Q^0 of the degree matrix. Thus we have

$$0 \ = \ u\, P\, (Q^0)^* \ = \ \sum_{i,j} \langle u, v_{ij} \rangle w_{ij} + \sum_k \langle u, v_k \rangle w_k$$

We solve this equation for $\langle u, v_{i_0 j_0} \rangle$ to obtain the assertion. □

The second tool that we will use extensively throughout Sections 2.4 – 2.11 is an upper bound on sums of unit fractions. It involves Sylvester's sequence, which we already encountered in Chapter 1, see Definition 1.4.1. We state the relevant part of that definition, the sequences for $g = 1$.

Definition 2.3.3. See Definition 1.4.1. We define two integer sequences, *Sylvester's sequence* $S = (s_1, s_2, s_3, \dots)$ and the *truncated Sylvester sequence* $T = (t_1, t_2, t_3 \dots)$ via

$$s_1 \ := \ 2, \qquad s_k \ := \ s_k(s_k - 1) + 1 \qquad t_k \ := \ s_k - 1.$$

Lemma 2.3.4. *See* [31, Thm. I] *and* [54, Thm. 1]. *For any positive integers $a_1, \dots, a_n$ the following hold.*

(i) *If $\sum_{i=0}^{n} \frac{1}{a_i} < 1$ holds, then we have*

$$\sum_{i=1}^{n} \frac{1}{a_i} \ \leq \ \sum_{i=0}^{n} \frac{1}{s_i} \ = \ 1 - \frac{1}{t_{n+1}}.$$

(ii) *If $\sum_{i=0}^{n} \frac{1}{a_i} = 1$ holds, then we have*

$$a_i \ \leq \ t_n \quad \text{for all } i \ = \ 1 \dots, n.$$

Example 2.3.5. We describe how we use Lemma 2.3.4 to obtain effective bounds on the denominators of unit fractions. The first six terms of Sylvester's sequence are

$$s_1 = 2, \quad s_2 = 3, \quad s_3 = 7, \quad s_4 = 43, \quad s_5 = 1807, \quad s_6 = 3,263,443.$$

Assume we have four positive integers a_1, a_2, a_3, a_4 satisfying $1/a_1 + \dots + 1/a_4 = 1$. Then Lemma 2.3.4 (ii) provides the bound $a_k \leq 42$ for all k and we can use the computer to easily enumerate all possible solutions $(a_1, \dots, a_4)$. Looking at the members of Sylvester's sequence shows that the effectiveness of this strategy deteriorates quickly with the number of summands. Already for six summands, there are in the order of 10^{38} possible constellations for $(a_1, \dots, a_6)$. Nevertheless, this strategy is still effective, if we have more information about the summands. Assume for example that we are interested in the positive solutions for the equation

$$\frac{1}{a_1} + \frac{1}{a_2} + \frac{1}{a_3} + \frac{3}{a_4} \ = \ 1.$$

Naively, splitting the last summand into a sum of three unit fractions, we run into the problem of computational complexity described above. Instead, we can split off the last summand, to obtain the inequality

$$\frac{1}{a_1} + \frac{1}{a_2} + \frac{1}{a_3} \ < \ 1.$$

Lemma 2.3.4 (i) tells us, that the sum is at most 41/42. We can apply this bound to the summand that was split off, to get the bound $a_4 \leq 126$. In this way we are able to obtain bounds on the denominators which allow us to enumerate all possible solutions via computer in a reasonable time frame.

2.4 Proof of Classification 2.1.1: Case 1 - format $(2,2,1,0)$

Proposition 2.2.24 divides the proof of the classification theorem 2.1.1 into cases (i) to (viii). In this section we treat case (i). The setting is as follows.

Setting 2.4.1. Let $X = X(A,P,\Sigma)$ a $\mathbb{Q}$-factorial threefold of Picard number one of format $(2,2,1,0)$. Then

$$P = [v_{01}, v_{02}, v_{11}, v_{12}, v_{21}] = \begin{bmatrix} -l_{01} & -l_{02} & l_{11} & l_{12} & 0 \\ -l_{01} & -l_{02} & 0 & 0 & l_{21} \\ d_{011} & d_{021} & d_{111} & d_{121} & d_{211} \\ d_{012} & d_{022} & d_{112} & d_{122} & d_{212} \end{bmatrix}$$

holds with pairwise different primitive columns $v_{01}, v_{02}, v_{11}, v_{12}$ and v_{21} generating $\mathbb{Q}^4$ as a cone. We assume P to have ordered exponents. The maximal X-cones of the fan Σ of Z are given by

$$\sigma_{01} = \mathrm{cone}(v_{02}, v_{11}, v_{12}, v_{21}), \qquad \sigma_{02} = \mathrm{cone}(v_{01}, v_{11}, v_{12}, v_{21}),$$

$$\sigma_{11} = \mathrm{cone}(v_{01}, v_{02}, v_{12}, v_{21}), \qquad \sigma_{12} = \mathrm{cone}(v_{01}, v_{02}, v_{11}, v_{21}).$$

We have $K = \mathbb{Z} \oplus \Gamma$ with the torsion part Γ and denote $\deg(T_{ij}) = (w_{ij}, \eta_{ij})$ as well as $\deg(T_k) = (w_k, \eta_k)$ accordingly. In particular, we write

$$Q^0 = [w_{01}, w_{02}, w_{11}, w_{12}, w_{21}]$$

for the free part of the degree matrix Q. Note that the vector $(w_{01}, w_{02}, w_{11}, w_{12}, w_{21})$ is primitive in $\mathbb{Z}^5$ and generates $\ker(P)$.

Very first constraints on the exponents of the defining relation g come from log terminality of X.

Proposition 2.4.2. *Consider $X = X(A,P,\Sigma)$ as in Setting 2.4.1. Assume that X is non-toric, Fano and log-terminal. Then the tuple (l_{01}, l_{11}, l_{21}) fits into precisely one of the following constellations:*

$$\begin{array}{llll} (x,1,y), & x \geq 1,\ y \geq 2; & (3,2,z), & 3 \leq z \leq 5; \\ (2,2,y), & y \geq 2; & (z,2,3), & 4 \leq z \leq 5; \\ (y,2,2), & y \geq 3; & (z,3,2), & 3 \leq z \leq 5. \end{array}$$

Proof. We apply Proposition 2.2.22 to the X-cone $\mathrm{cone}(v_{01}, v_{11}, v_{21})$ to see that (l_{01}, l_{11}, l_{21}) is a platonic tuple. As P has ordered exponents, $l_{01} \geq l_{11}$ holds. Moreover, since X is non-toric, we have $l_{21} \geq 2$. This leaves us with the six constellations for (l_{01}, l_{11}, l_{21}) stated in the Proposition. □

The next series of constraints arises from log terminality and the Gorenstein property and directly aims for the entries of the defining matrix P.

Proposition 2.4.3. *Consider $X = X(A, P, \Sigma)$ as in Setting 2.4.1. Assume that X is non-toric, Fano, log-terminal and Gorenstein. Then the following hold:*

(i) *An anticanonical divisor on X is explicitly given by*

$$-\mathcal{K} \;=\; D_{01}^X + D_{02}^X + D_{11}^X + D_{12}^X + (1 - l_{21})D_{21}^X.$$

(ii) *The weights w_{01}, w_{02}, w_{11} and w_{12} are divisors of w_X.*

(iii) *We have $1 \le l_{11} \le 3$ and $l_{12} \le l_{11}$.*

(iv) *Admissible row operations turn the defining matrix P into one of the following forms, according to the possible values of l_{11} and l_{12}:*

$$P \;=\; \begin{bmatrix} -l_{01} & -l_{02} & l_{11} & 1 & 0 \\ -l_{01} & -l_{02} & 0 & 0 & l_{21} \\ d_{011} & d_{021} & d_{111} & 0 & d_{211} \\ d_{012} & d_{022} & 0 & 0 & d_{212} \end{bmatrix}, \quad \begin{array}{l} 1 \le l_{11} \le 3, \\ 0 \le d_{011} \le l_{21}\frac{w_X}{w_{01}},\ 0 \le d_{012} \le \frac{w_X}{w_{01}}, \\ 1 \le d_{111} \le l_{11} - 1 + \frac{w_X}{w_{11}}, \\ 0 \le d_{211}, d_{212} < l_{21}, \end{array}$$

$$P \;=\; \begin{bmatrix} -l_{01} & -l_{02} & l & l & 0 \\ -l_{01} & -l_{02} & 0 & 0 & l_{21} \\ d_{011} & d_{021} & 1 & 1 & d_{211} \\ d_{012} & d_{022} & 0 & d_{122} & d_{212} \end{bmatrix}, \quad \begin{array}{l} 2 \le l \le 3, \\ 1 - l_{01} - \frac{w_X}{w_{01}} \le d_{011} \le l_{01} - 1 + \frac{w_X}{w_{01}}, \\ 0 \le d_{012} < l\left(l_{01} + l_{02}\left(\frac{w_X}{w_{01}} - 1\right)\right), \\ 0 \le d_{021}, d_{022} < l_{02},\ 1 \le d_{122} \le \frac{w_X}{w_{12}}, \end{array}$$

$$P \;=\; \begin{bmatrix} -l_{01} & -l_{02} & 3 & 2 & 0 \\ -l_{01} & -l_{02} & 0 & 0 & 2 \\ 1 & 1 - \frac{w_X}{w_{02}} & 1 & 1 & -1 \\ d_{012} & d_{022} & 0 & 0 & d_{212} \end{bmatrix}, \quad \begin{array}{l} -l_{01} - \frac{w_X}{w_{01}} < d_{012} \le \frac{w_X}{w_{01}}, \\ 0 \le d_{212} \le 1. \end{array}$$

Proof. Item (i) follows immediately from Remark 2.2.13. Item (ii) follows from applying Corollary 2.2.21 to the four X-cones $\sigma_{01}, \sigma_{02}, \sigma_{11}$ and σ_{12}. Item (iii) is a consequence of Proposition 2.4.2. We prove (iv). There are six possible constellations for (l_{11}, l_{12}), which we group into three cases as follows:

(a) $l_{12} = 1$. The possible constellations are $(l_{11}, l_{12}) = (1, 1), (2, 1), (3, 1)$.
(b) $l_{11} = l_{21} > 1$. The possible constellations are $(l_{11}, l_{12}) = (2, 2), (3, 3)$.
(c) $(l_{11}, l_{12}) = (3, 2)$.

We start with case (a). Assume $l_{12} = 1$. By adding multiples of the first row to the third and fourth row of P we achieve $d_{121} = d_{122} = 0$. Moreover, applying a suitable unimodular 2×2 matrix to the d-block, we may assume $d_{111} > 0$ and $d_{112} = 0$. Multiplying the last row of P by -1 if necessary, we may assume that $d_{012} \ge 0$ holds. We add multiples of the second row of P to the third and fourth to guarantee $0 \le d_{121}, d_{122} < l_{21}$. The matrix P is now of the form

$$P \;=\; \begin{bmatrix} -l_{01} & -l_{02} & l_{11} & 1 & 0 \\ -l_{01} & -l_{02} & 0 & 0 & l_{21} \\ d_{011} & d_{021} & d_{111} & 0 & d_{211} \\ d_{012} & d_{022} & 0 & 0 & d_{212} \end{bmatrix},$$

with $d_{012} \ge 0$, $d_{111} > 0$ and $0 \le d_{211}, d_{212} < l_{21}$. We make use of the Gorenstein property. Consider the X-cone $\sigma_{11} = \mathrm{cone}(v_{01}, v_{02}, v_{12}, v_{21})$. By Lemma 2.3.2 there is a linear form $u \in \mathbb{Z}^4$ with

$$\langle u, v_{11} \rangle \;=\; 1 - \frac{w_X}{w_{11}}, \qquad \langle u, v_{12} \rangle \;=\; 1.$$

The second equation yields $u_1 = 1$. Plugging this into the first equation and solving for d_{111} we obtain

$$1 - l_{11} - \frac{w_X}{w_{11}} \;=\; d_{111} u_3.$$

Note that the left hand side of this equation is strictly negative. Thus $u_3 \neq 0$ holds and d_{111} is a divisor of $l_{11} - 1 + \frac{w_X}{w_{11}}$. In particular, we get the bound

$$1 \;\leq\; d_{111} \;\leq\; l_{11} - 1 + \frac{w_X}{w_{11}}.$$

Now consider the X-cones $\sigma_{01} = \operatorname{cone}(v_{02}, v_{11}, v_{12}, v_{21})$ and $\sigma_{02} = \operatorname{cone}(v_{01}, v_{11}, v_{12}, v_{21})$. By Lemma 2.3.2 there are linear forms $u', u'' \in \mathbb{Z}^4$ with

$$\begin{array}{llll} \langle u', v_{01} \rangle \;=\; 1 - \frac{w_X}{w_{01}}, & \langle u', v_{02} \rangle \;=\; 1, & \langle u'', v_{01} \rangle \;=\; 1, & \langle u'', v_{02} \rangle \;=\; 1 - \frac{w_X}{w_{02}}, \\ \langle u', v_{11} \rangle \;=\; 1, & \langle u', v_{12} \rangle \;=\; 1, & \langle u'', v_{11} \rangle \;=\; 1, & \langle u'', v_{12} \rangle \;=\; 1, \\ \langle u', v_{21} \rangle \;=\; 1 - l_{21}. & & \langle u'', v_{21} \rangle \;=\; 1 - l_{21}. & \end{array}$$

Consider their difference $u := u' - u''$. Evaluating u on the columns of P yields

$$\begin{array}{ll} \langle u, v_{01} \rangle \;=\; -\frac{w_X}{w_{01}}, & \langle u, v_{02} \rangle \;=\; \frac{w_X}{w_{02}}, \\ \langle u, v_{11} \rangle \;=\; 0, & \langle u, v_{12} \rangle \;=\; 0, \\ \langle u, v_{21} \rangle \;=\; 0. & \end{array} \tag{2.4.3.1}$$

Combining the third and fourth equation of 2.4.3.1 we obtain $u_1 = u_3 = 0$. Plugging this into the first equation and multiplying by l_{21}, we obtain

$$-l_{21} \frac{w_X}{w_{01}} \;=\; -l_{01} l_{21} u_2 + u_4 l_{21} d_{012} \;=\; u_4 (l_{01} d_{212} + l_{21} d_{012}). \tag{2.4.3.2}$$

In the second step we used the identity $u_2 l_{21} = -u_4 d_{212}$, which we obtain from the last equation in 2.4.3.1. Note that the left hand side of Equation 2.4.3.2 is strictly negative. Thus $u_4 \neq 0$ and $l_{01} d_{212} + l_{21} d_{012} \neq 0$ holds and $l_{01} d_{212} + l_{21} d_{012}$ is a divisor of $l_{21} \frac{w_X}{w_{01}}$. Using the bounds on d_{212}, we obtain

$$0 \;\leq\; d_{012} \;\leq\; \frac{w_X}{w_{01}}.$$

Finally, to get bounds on d_{011}, consider the following 4×4 integer matrix:

$$S \;=\; \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -d_{212} & 1 & l_{21} \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

It leaves the first two rows of P unchanged and has determinant $\det(S) = 1$. It thus consists of admissible row operations of P. Multiplying P from the left by multiples of S, we achieve

$$0 \ \leq \ d_{011} \ \leq \ l_{01}d_{212} + l_{21}d_{012} \ \leq \ l_{21}\frac{w_X}{w_{01}}.$$

Thus the matrix P is of the first form described in Proposition 2.4.3 (iv).

We treat case (b). Assume $l_{11} = l_{12} > 1$ holds. Let $l := l_{11}$, this is either 2 or 3. Applying a suitable unimodular 2×2 matrix to the d-block we achieve $d_{112} = 0$. Primitivity of v_{11} ensures that $d_{111} \neq 0$. Adding multiples of the first row of P to the third row and multiplying by -1 if necessary, we achieve $d_{111} = 1$. Multiplying the last row by -1 if necessary, we may assume that $d_{122} \geq 0$ holds. The matrix P is now of the form

$$P \ = \ \begin{bmatrix} -l_{01} & -l_{02} & l & l & 0 \\ -l_{01} & -l_{02} & 0 & 0 & l_{21} \\ d_{011} & d_{021} & 1 & d_{121} & d_{211} \\ d_{012} & d_{022} & 0 & d_{122} & d_{212} \end{bmatrix},$$

with $d_{122} \geq 0$. We make use of the Gorenstein property. Consider the X-cone $\mathrm{cone}(v_{11}, v_{12})$. By Lemma 2.3.2 there is a linear form $u \in \mathbb{Z}^4$ with

$$\langle u, v_{11} \rangle \ = \ 1, \qquad \langle u, v_{12} \rangle \ = \ 1. \tag{2.4.3.3}$$

The first equation ensures that u_3 is coprime to l. In particular $u_3 \neq 0$ holds. Linear independence of v_{11} and v_{12} guarantees that $d_{122} > 0$ holds. We show that via admissible row operations we can achieve $d_{121} = 1$. We write $d_{122} = l^e d_2$, where $e \in \mathbb{Z}_{\geq 0}$ is chosen such that d_2 is not divisible by l. Combining the two equations from 2.4.3.3 we obtain

$$u_3(d_{121} - 1) \ = \ -l^e u_4 d_2.$$

As u_3 is not divisible by l, and l is prime, there is $d_1 \in \mathbb{Z}$ such that $d_{121} = l^e d_1 + 1$ holds. Let $c = \gcd(ld_1, d_2)$. There are $\alpha, \beta, \gamma, \delta \in \mathbb{Z}$ with

$$\begin{array}{lcllcl} c & = & \alpha l d_1 + \beta d_2, & 1 & = & \alpha\gamma + \beta\delta, \\ \gamma c & = & l d_1, & \delta c & = & d_2, \end{array}$$

As d_2 is not divisible by l, neither are δ and c. Thus γ is divisible by l. We write $\gamma = l\gamma'$ and $\delta = l\delta' + f$, where $f = \pm 1$. Consider the 4×4 integer matrix

$$S \ = \ \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -f\delta' & 0 & f\delta & -f\gamma' \\ -\alpha & 0 & l\alpha & \beta \end{bmatrix}.$$

The matrix S leaves the first two rows of P unchanged and it has determinant

$$\det(S) \ = \ f(\alpha\gamma + \beta\delta) \ = \ \pm 1.$$

It thus consists of admissible row operations of P. Multiplying P from the left by S transforms it into the matrix

$$P \;=\; \begin{bmatrix} -l_{01} & -l_{02} & l & l & 0 \\ -l_{01} & -l_{02} & 0 & 0 & l_{21} \\ d_{011} & d_{021} & 1 & 1 & d_{211} \\ d_{012} & d_{022} & 0 & l^e c & d_{212} \end{bmatrix},$$

which we again call P. We also write again d_{122} for $l^e c$. The entries d_{ijk} are understood to be indeterminates. Transforming P by S changes their actual values. Consider the X-cone $\sigma_{12} = \mathrm{cone}(v_{01}, v_{02}, v_{11}, v_{21})$. By Lemma 2.3.2 there is a linear form $u \in \mathbb{Z}^4$ with

$$\begin{array}{rclrcl} \langle u, v_{01}\rangle &=& 1, & \langle u, v_{02}\rangle &=& 1, \\ \langle u, v_{11}\rangle &=& 1, & \langle u, v_{12}\rangle &=& 1 - \frac{w_X}{w_{12}}, \\ \langle u, v_{21}\rangle &=& 1 - l_{21}. \end{array} \tag{2.4.3.4}$$

Combining the third and fourth equation of 2.4.3.4 we obtain

$$-\frac{w_X}{w_{12}} \;=\; u_4 d_{122}.$$

The left hand side is strictly negative. Thus $u_4 \neq 0$ holds and d_{122} is a divisor of $\frac{w_X}{w_{12}}$. In particular we get the bounds

$$1 \;\leq\; d_{122} \;\leq\; \frac{w_X}{w_{12}}.$$

We treat the remaining entries of the d-block. We add multiples of the second row of P to the third and fourth row to achieve $0 \leq d_{021}, d_{022} < l_{02}$. Consider the X-cone $\sigma_{01} = \mathrm{cone}(v_{02}, v_{11}, v_{12}, v_{21})$. By Lemma 2.3.2 there is a linear form $u \in \mathbb{Z}^4$ with

$$\begin{array}{rclrcl} \langle u, v_{01}\rangle &=& 1 - \frac{w_X}{w_{01}}, & \langle u, v_{02}\rangle &=& 1, \\ \langle u, v_{11}\rangle &=& 1, & \langle u, v_{12}\rangle &=& 1, \\ \langle u, v_{21}\rangle &=& 1 \end{array} \tag{2.4.3.5}$$

Combining the third and fourth equation of 2.4.3.5 we obtain $u_4 = 0$ plugging this into the first equation and multiplying by l_{02} yields

$$l_{02}\left(1 - \frac{w_X}{w_{01}}\right) \;=\; -l_{01}l_{02}(u_1+u_2) + u_3 l_{02} d_{011} \;=\; l_{01} + u_3(l_{02}d_{011} - l_{01}d_{021}). \tag{2.4.3.6}$$

In the second step we used the identity $l_{02}(u_1+u_2) = u_3 d_{021} - 1$, which we obtain from the second equation in 2.4.3.5. Not that both u_3 and $l_{02}d_{011} - l_{01}d_{021}$ are non-trivial. Subtracting l_{01} on both sides in Equation 2.4.3.6 and using the bounds on d_{021} we obtain

$$1 - l_{01} - \frac{w_X}{w_{01}} \;\leq\; d_{011} \;\leq\; l_{01} - 1 + \frac{w_X}{w_{01}}.$$

To get bounds on d_{012} consider the 4×4 integer matrix

$$S \;=\; \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -l_{02} & l d_{021} + l_{02} & l\, l_{02} & 1 \end{bmatrix}.$$

It leaves the first two rows of P unchanged and has determinant $\det(S) = 1$. It thus consists of admissible row operations of P. Note that S also leaves the columns v_{02}, v_{11} andv_{12} unchanged. Multiplying P from the left by multiples of S we achieve

$$0 \;\leq\; d_{012} \;\leq\; |l(l_{02}d_{011} - l_{01}d_{021})| \;<\; l\left(l_{01} + l_{02}\left(\frac{w_X}{w_{01}} - 1\right)\right).$$

This shows that the matrix P is of the second form described in Proposition 2.4.3 (iv).

We treat case (c). Assume $(l_{11}, l_{12}) = (3, 2)$ holds. Note that by Proposition 2.4.2 we then also have $l_{21} = 2$. Consider the X-cone $\sigma_{02} = \operatorname{cone}(v_{01}, v_{11}, v_{12}, v_{21})$. By Lemma 2.3.2 there is a linear form $u \in \mathbb{Z}^4$ with

$$\begin{array}{rclrcl} \langle u, v_{01}\rangle & = & 1, & \langle u, v_{02}\rangle & = & 1 - \frac{w_X}{w_{02}}, \\ \langle u, v_{11}\rangle & = & 1, & \langle u, v_{12}\rangle & = & 1, \\ \langle u, v_{21}\rangle & = & -1. & & & \end{array} \tag{2.4.3.7}$$

Consider the 4×4 integer matrix

$$S \;=\; \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ u_1 & u_2 & u_3 & u_4 \\ d_{111}d_{122} - d_{112}d_{121} & 0 & 2d_{112} - 3d_{122} & 3d_{121} - 2d_{111} \end{bmatrix}.$$

It leaves the first two rows of P unchanged and has determinant

$$\det(S) \;=\; (3d_{121} - 2d_{111})u_3 + (3d_{122} + 2d_{112})u_4 \;=\; 3\langle u, v_{12}\rangle - 2\langle u, v_{11}\rangle \;=\; 1.$$

The matrix S thus consists of admissible row operations of P. Multiplying P from the left by S transforms it into the matrix

$$P \;=\; \begin{bmatrix} -l_{01} & -l_{02} & 3 & 2 & 0 \\ -l_{01} & -l_{02} & 0 & 0 & 2 \\ 1 & 1 - \frac{w_X}{w_{02}} & 1 & 1 & -1 \\ d_{012} & d_{022} & 0 & 0 & d_{212} \end{bmatrix},$$

which we again call P. The entries d_{ijk} are understood to be indeterminates. Transforming P by S changes their actual values. We add multiples of the second row of P to the fourth row to achieve

$$0 \;\leq\; d_{212} \;\leq\; 1.$$

Consider now the X-cone $\sigma_{01} = \operatorname{cone}(v_{02}, v_{11}, v_{12}, v_{21})$. By Lemma 2.3.2 there is a linear form $u \in \mathbb{Z}^4$ with

$$\begin{array}{rclrcl} \langle u, v_{01}\rangle & = & 1 - \frac{w_X}{w_{01}}, & \langle u, v_{02}\rangle & = & 1, \\ \langle u, v_{11}\rangle & = & 1, & \langle u, v_{12}\rangle & = & 1, \\ \langle u, v_{21}\rangle & = & -1. & & & \end{array} \tag{2.4.3.8}$$

Combining the third and fourth equation yields $u_1 = 0$ and $u_3 = 1$. Plugging this into the first equation and multiplying by 2, we obtain

$$2 - 2\frac{w_X}{w_{01}} \;=\; 2 - 2l_{01}u_2 + 2u_4d_{012} \;=\; 2 + u_4(2d_{012} + l_{01}d_{212}). \tag{2.4.3.9}$$

In the second step we used the identity $2u_2 = -u_4 d_{212}$, which we obtain from the last equation of 2.4.3.8. Note that both u_4 and $2d_{012} + l_{01}d_{212}$ are non-trivial. Subtracting 2 on both sides of Equation 2.4.3.9 and using the bounds on d_{212} we obtain

$$-l_{01} - \frac{w_X}{w_{01}} \;<\; d_{012} \;\leq\; \frac{w_X}{w_{01}}.$$

This shows that the matrix P is of the third form described in Proposition 2.4.3 (iv), which completes the proof. ☐

The final series of constraints shows that all entries of the $\mathbb{Z}$-part of the degree matrix $Q^0 = [w_{01}, w_{02}, w_{11}, w_{12}, w_{21}]$ are bounded.

Proposition 2.4.4. *Consider $X = X(A, P, \Sigma)$ as in Setting 2.4.1. Assume that X is non-toric, Fano, log-terminal and Gorenstein.*

(i) *For any four positive integers $\alpha_{01}, \alpha_{02}, \alpha_{11}$ and α_{12} consider the 6×5 matrix*

$$G \;:=\; \begin{bmatrix} 1-\alpha_{01} & 1 & 1-l_{11} & 1-l_{12} & 1 \\ 1 & 1-\alpha_{02} & 1-l_{11} & 1-l_{12} & 1 \\ 1 & 1 & 1-l_{11}-\alpha_{11} & 1-l_{12} & 1 \\ 1 & 1 & 1-l_{11} & 1-l_{12}-\alpha_{12} & 1 \\ -l_{01} & -l_{02} & l_{11} & l_{12} & 0 \\ -l_{01} & -l_{02} & 0 & 0 & l_{21} \end{bmatrix}.$$

The matrix G is of rank at least four. Moreover, $\operatorname{rank}(G) = 4$ *holds if and only if $\alpha_{01}, \alpha_{02}, \alpha_{11}, \alpha_{12}$ and $l_{01}, l_{02}, l_{11}, l_{12}, l_{21}$ satisfy the identities*

$$\begin{aligned} \left(\frac{l_{01}}{\alpha_{01}} + \frac{l_{02}}{\alpha_{02}}\right) - \left(\frac{l_{11}}{\alpha_{11}} + \frac{l_{12}}{\alpha_{12}}\right) &= 0, \\ \frac{1}{\alpha_{01}} + \frac{1}{\alpha_{02}} + \frac{1}{\alpha_{11}} + \frac{1}{\alpha_{12}} + \left(\frac{l_{11}}{\alpha_{11}} + \frac{l_{12}}{\alpha_{12}}\right)\left(\frac{1}{l_{21}} - 1\right) &= 1. \end{aligned}$$

(ii) *There are unique $\alpha_{01}, \alpha_{02}, \alpha_{11}, \alpha_{12} \in \mathbb{Z}_{\geq 1}$ with $\alpha_{ij} w_{ij} = w_X$ for all $0 \leq i \leq 1$ and all $1 \leq j \leq 2$ and the corresponding matrix G from* (i) *satisfies*

$$\ker(G) \;=\; \ker(P) \;=\; \mathbb{Z} \cdot (w_{01}, w_{02}, w_{11}, w_{12}, w_{21}).$$

(iii) *According to the possible constellations of (l_{01}, l_{11}, l_{21}) from Proposition 2.4.2 we have the following upper bounds on the entries of the matrix G from* (ii)*:*

	l_{01}	l_{02}	l_{11}	l_{12}	l_{21}	α_{01}	α_{02}	α_{11}	α_{12}
$(x,1,y)$	83	42	1	1	12	21	21	42	6
$(2,2,y)$	2	2	2	2	24	42	42	42	42
$(y,2,2)$	23	12	2	2	2	6	6	12	3
$(3,2,z)$	3	3	2	2	5	21	21	2	21
$(z,2,3)$	5	5	2	2	3	21	21	4	21
$(z,3,2)$	5	5	3	3	2	903	35	15	28

Proof. We prove (i). In order to see that G is of rank at least four, we just compute the minor obtained by deleting rows 5 and 6 and column 1:

$$G_{(1,2,3,4),(2,3,4,5)} \;=\; \det \begin{bmatrix} 1 & 1-l_{11} & 1-l_{12} & 1 \\ 1-\alpha_{02} & 1-l_{11} & 1-l_{12} & 1 \\ 1 & 1-l_{11}-\alpha_{11} & 1-l_{12} & 1 \\ 1 & 1-l_{11} & 1-l_{12}-\alpha_{12} & 1 \end{bmatrix} \;=\; \alpha_{02}\alpha_{11}\alpha_{12} \;\neq\; 0.$$

Moreover, G is of rank exactly four if and only if all its 5-minors vanish. Rearranging these six equations and removing redundancies, we arrive at the identities in $\alpha_{01}, \alpha_{02}, \alpha_{11}, \alpha_{12}$ and $l_{01}, l_{02}, l_{11}, l_{12}, l_{21}$.

We prove (ii). By Proposition 2.4.3 (ii) there are positive integers α_{ij} for $0 \le i \le 1$ and $1 \le j \le 2$ with

$$\alpha_{ij}w_{ij} \;=\; w_{01} + w_{02} + (1-l_{11})w_{11} + (1-l_{12})w_{12} + w_{21}.$$

Moreover, by homogeneity of the defining relation g we have

$$l_{01}w_{01} + l_{02}w_{02} \;=\; l_{11}w_{11} + l_{12}w_{12} \;=\; l_{21}w_{21}.$$

The matrix G from (i) is the coefficient matrix of the corresponding system of linear equations. In particular, the integral matrix G has kernel generated by the primitive vector $(w_{01}, w_{02}, w_{11}, w_{12}, w_{21}) \in \mathbb{Z}^5$.

We prove (iii). We treat the configuration $(l_{01}, l_{11}, l_{21}) = (x, 1, y)$. In this case the identities from (i) read

$$\frac{x}{\alpha_{01}} + \frac{l_{02}}{\alpha_{02}} - \left(\frac{1}{\alpha_{11}} + \frac{1}{\alpha_{12}}\right) \;=\; 0, \tag{2.4.4.1}$$

$$\frac{1}{y}\left(\frac{1}{\alpha_{11}} + \frac{1}{\alpha_{12}}\right) + \frac{1}{\alpha_{01}} + \frac{1}{\alpha_{02}} \;=\; 1. \tag{2.4.4.2}$$

The first summand on the left hand side of Equation 2.4.4.2 is positive. The rest of the sum is thus strictly smaller than one. We can thus apply Lemma 2.3.4 (i) to obtain

$$\frac{1}{\alpha_{01}} + \frac{1}{\alpha_{02}} \;\le\; \frac{5}{6}, \qquad\qquad \frac{1}{y}\left(\frac{1}{\alpha_{11}} + \frac{1}{\alpha_{12}}\right) \;\ge\; \frac{1}{6}.$$

Since l_{11} and l_{12} are equal, we may assume $\alpha_{11} \ge \alpha_{12}$. Moreover we have $x \ge l_{02} \ge 1$ and $y \ge 2$. From this we get the bounds $\alpha_{12} \le 6$ and $y \le 12$. On the other hand, we can apply Lemma 2.3.4 (ii) to Equation 2.4.4.2 directly, to obtain $y\alpha_{01}, y\alpha_{02}, \alpha_{11} \le 42$. As y is at least two, this gives the bounds $\alpha_{01}, \alpha_{02} \le 21$. With these bounds on α_{ij}, we can rewrite Equation 2.4.4.1 to obtain

$$2 \;\ge\; \frac{1}{\alpha_{11}} + \frac{1}{\alpha_{12}} \;=\; \frac{x}{\alpha_{01}} + \frac{l_{02}}{\alpha_{02}} \;\ge\; \frac{1}{42}(x + l_{02}).$$

Since l_{02} is at least one and bounded from above by x, we obtain the bounds $x \le 83$ and $l_{02} \le 42$.

We treat the configuration $(l_{01}, l_{11}, l_{21}) = (2, 2, y)$. In this case the identities from (i) read

$$\left(\frac{2}{\alpha_{01}} + \frac{l_{02}}{\alpha_{02}}\right) - \left(\frac{2}{\alpha_{11}} + \frac{l_{12}}{\alpha_{12}}\right) = 0, \tag{2.4.4.3}$$

$$\frac{1}{\alpha_{01}} + \frac{1}{\alpha_{02}} + \frac{1}{\alpha_{11}} + \frac{1}{\alpha_{12}} + \left(\frac{2}{\alpha_{01}} + \frac{l_{02}}{\alpha_{02}}\right)\left(\frac{1}{y} - 1\right) = 1. \tag{2.4.4.4}$$

Combining these two equations we obtain the following identity

$$1 = \frac{2 - l_{02}}{2\alpha_{02}} + \frac{2 - l_{12}}{2\alpha_{12}} + \frac{1}{y}\left(\frac{2}{\alpha_{01}} + \frac{l_{02}}{\alpha_{02}}\right). \tag{2.4.4.5}$$

As the last summand on the right hand side is positive, the rest of the sum is strictly smaller than one. Note that the numerators of the first two summands are either 0 or 1, since we have $1 \le l_{02}, l_{12} \le 2$. Thus each one either vanishes or is a unit fraction. Applying Lemma 2.3.4 (i) to the first two summands, we thus obtain

$$\frac{2 - l_{02}}{2\alpha_{02}} + \frac{2 - l_{12}}{2\alpha_{12}} \le \frac{5}{6}, \qquad \frac{1}{y}\left(\frac{2}{\alpha_{01}} + \frac{l_{02}}{\alpha_{02}}\right) \ge \frac{1}{6}.$$

With the second inequality we can give an upper bound on y by

$$y \le 6\left(\frac{2}{\alpha_{01}} + \frac{l_{02}}{\alpha_{02}}\right) \le 24.$$

In order to obtain bounds on α_{01} and α_{02} we rearrange Equation 2.4.4.5. Combining the fractions that contain α_{02} in the denominator, we get

$$1 = \frac{(2 - l_{02})y + 2l_{02}}{2y\alpha_{02}} + \frac{2 - l_{12}}{2\alpha_{12}} + \frac{2}{y\alpha_{01}}. \tag{2.4.4.6}$$

The first summand on the right hand side is positive, so the remaining sum is strictly smaller than one. As it consists of at most three unit fractions, we apply Lemma 2.3.4 (i) to obtain

$$\frac{2 - l_{12}}{2\alpha_{12}} + \frac{1}{y\alpha_{01}} + \frac{1}{y\alpha_{01}} \le \frac{41}{42}, \qquad \frac{(2 - l_{02})y + 2l_{02}}{2y\alpha_{02}} \ge \frac{1}{42}.$$

We solve the second inequality for α_{02}, using $y \ge 2$ to obtain $\alpha_{02} \le 42$. Instead of splitting off the first summand in Equation 2.4.4.6, we can split off the last summand and again invoke Lemma 2.3.4 (i) to obtain

$$\frac{(2 - l_{02})y + 2l_{02}}{2y\alpha_{02}} + \frac{2 - l_{12}}{2\alpha_{12}} \le \frac{41}{42}, \qquad \frac{2}{y\alpha_{01}} \ge \frac{1}{42}.$$

The second inequality gives the bound $\alpha_{01} \le 42$. Note that the equations 2.4.4.3 and 2.4.4.4 are invariant under switching α_{01} with α_{11} and α_{02} with α_{12}. We thus get the same bounds $\alpha_{11} \le 42$ and $\alpha_{12} \le 42$.

We treat the configuration $(l_{01}, l_{11}, l_{21}) = (y, 2, 2)$. In this case the identities from (i) read

$$\begin{aligned} \left(\frac{y}{\alpha_{01}} + \frac{l_{02}}{\alpha_{02}}\right) - \left(\frac{2}{\alpha_{11}} + \frac{l_{12}}{\alpha_{12}}\right) &= 0, && (2.4.4.7)\\ \frac{1}{\alpha_{01}} + \frac{1}{\alpha_{02}} + \frac{2 - l_{12}}{2\alpha_{12}} &= 1. && (2.4.4.8) \end{aligned}$$

Note that the numerator of the last summand in Equation 2.4.4.8 is either 0 or 1. It is thus a sum of at most three unit fractions. Applying Lemma 2.3.4 (ii) we get the bounds $\alpha_{01}, \alpha_{02} \leq 6$ and $\alpha_{12} \leq 3$. Using these bounds on α_{01} and α_{02}, we rearrange Equation 2.4.4.7 to obtain

$$4 \ \geq \ \frac{2}{\alpha_{11}} + \frac{l_{12}}{\alpha_{12}} \ = \ \frac{y}{\alpha_{01}} + \frac{l_{02}}{\alpha_{02}} \ \geq \ \frac{1}{6}(y + l_{02}).$$

Since l_{02} is at least one and bounded from above by y, we get the bounds $y \leq 23$ and $l_{02} \leq 12$. Now solving Equation 2.4.4.7 for α_{11} and applying the bounds that we already established, we obtain $\alpha_{11} \leq 12$.

We treat the configuration $(l_{01}, l_{11}, l_{21}) = (3, 2, z)$. In this case the identities from (i) read

$$\begin{aligned} \left(\frac{3}{\alpha_{01}} + \frac{l_{02}}{\alpha_{02}}\right) - \left(\frac{2}{\alpha_{11}} + \frac{l_{12}}{\alpha_{12}}\right) &= 0, && (2.4.4.9)\\ \frac{1}{\alpha_{01}} + \frac{1}{\alpha_{02}} + \frac{1}{\alpha_{11}} + \frac{1}{\alpha_{12}} + \left(\frac{3}{\alpha_{01}} + \frac{l_{02}}{\alpha_{02}}\right)\left(\frac{1}{z} - 1\right) &= 1. && (2.4.4.10) \end{aligned}$$

We combine these two equations to obtain the following identity:

$$1 \ = \ \frac{6 - z}{2z\alpha_{01}} + \frac{(2 - l_{02})z + 2l_{02}}{2z\alpha_{02}} + \frac{2 - l_{12}}{2\alpha_{12}}.$$

We determine the possible values of the numerators of the three summands on the right. We have $3 \leq z \leq 5$. In this range $2z$ is divisible by $6 - z$. The first summand is thus a unit fraction. The exponent l_{02} can be either $1, 2$ or 3. The numerator of the second summand thus evaluates to $z + 2$, 4 or $6 - z$. In all cases the second summand can be written as a sum of at most two unit fractions. The numerator of the last summand is either 0 or 1. Thus the right hand side is a sum of at most four unit fractions and their denominators are each at least $2\alpha_{ij}$. We apply Lemma 2.3.4 (ii) to obtain the bounds $\alpha_{01}, \alpha_{02}, \alpha_{12} \leq 21$. Now solving Equation 2.4.4.9 for α_{11} and applying the bounds already established, we obtain $\alpha_{11} \leq 2$.

We treat the configuration $(l_{01}, l_{11}, l_{21}) = (z, 2, 3)$. In this case the identities from (i) read

$$\begin{aligned} \left(\frac{z}{\alpha_{01}} + \frac{l_{02}}{\alpha_{02}}\right) - \left(\frac{2}{\alpha_{11}} + \frac{l_{12}}{\alpha_{12}}\right) &= 0, && (2.4.4.11)\\ \frac{1}{\alpha_{01}} + \frac{1}{\alpha_{02}} + \frac{1}{\alpha_{11}} + \frac{1}{\alpha_{12}} - \frac{2}{3}\left(\frac{2}{\alpha_{11}} + \frac{l_{12}}{\alpha_{12}}\right) &= 1. && (2.4.4.12) \end{aligned}$$

We combine these two equations to obtain the following identity:

$$1 \;=\; \frac{6-z}{6\alpha_{01}} + \frac{6-l_{02}}{6\alpha_{02}} + \frac{2-l_{12}}{2\alpha_{12}}.$$

Just as in the previous case, the right hand side is a sum of at most four unit fractions with the denominators at least $2\alpha_{ij}$. We apply Lemma 2.3.4 (ii) to obtain the bounds $\alpha_{01}, \alpha_{02}, \alpha_{12} \leq 21$. Solving Equation 2.4.4.11 for α_{11}, using the bounds we already obtained, we get $\alpha_{11} \leq 4$.

We treat the configuration $(l_{01}, l_{11}, l_{21}) = (z, 3, 2)$. In this case the identities from (i) read

$$\left(\frac{z}{\alpha_{01}} + \frac{l_{02}}{\alpha_{02}}\right) - \left(\frac{3}{\alpha_{11}} + \frac{l_{12}}{\alpha_{12}}\right) \;=\; 0, \tag{2.4.4.13}$$

$$\frac{1}{\alpha_{01}} + \frac{1}{\alpha_{02}} + \frac{1}{\alpha_{11}} + \frac{1}{\alpha_{12}} - \frac{1}{2}\left(\frac{z}{\alpha_{01}} + \frac{l_{02}}{\alpha_{02}}\right) \;=\; 1. \tag{2.4.4.14}$$

We combine these two equations to obtain the following identity:

$$1 \;=\; \frac{6-z}{6\alpha_{01}} + \frac{6-l_{02}}{6\alpha_{02}} + \frac{3-l_{12}}{3\alpha_{12}}. \tag{2.4.4.15}$$

Note that the denominator of the second summand can take the values 1 through 5. Thus, in contrast to the previous two cases, in this case the bounds on α_{ij} we obtain from directly applying Lemma 2.3.4 (ii) are too large to be useful: The right hand side is a sum of at most 5 unit fractions. Their denominators are thus bounded from above by $s_8 - 1$, which is of order 10^7. Instead we adapt the strategy from the earlier cases. Note that each summand is non-negative. Splitting off the second summand, the rest is a sum of at most three unit fractions. Lemma 2.3.4 (i) yields

$$\frac{6-z}{6\alpha_{01}} + \frac{3-l_{12}}{3\alpha_{12}} \;\leq\; \frac{41}{42}, \qquad\qquad \frac{6-l_{02}}{6\alpha_{02}} \geq \frac{1}{42}.$$

From the second inequality we get the bound $\alpha_{02} \leq 35$. Splitting off the first summand in Equation 2.4.4.15 instead of the second, the remainder is a sum of at most four unit fractions and we obtain $\alpha_{01} \leq 903$. Splitting off the last summand, we obtain $\alpha_{12} \leq 28$. Now we solve Equation 2.4.4.13 for α_{11} and use the bounds on the α_{ij} we obtained to get $\alpha_{11} \leq 15$. □

Corollary 2.4.5. *There is a list of* 262 *explicitly given matrices P of format* $(2,2,1,0)$, *each of them defining a non-toric $\mathbb{Q}$-factorial, Gorenstein, log terminal Fano threefold $X(A,P,\Sigma)$ of Picard number one.*

	$\mathbb{Z}$	$\mathbb{Z}+\mathbb{Z}_2$	$\mathbb{Z}+\mathbb{Z}_3$	$\mathbb{Z}+\mathbb{Z}_4$	$\mathbb{Z}+\mathbb{Z}_5$	$\mathbb{Z}+\mathbb{Z}_6$	$\mathbb{Z}+\mathbb{Z}_2^2$	$\mathbb{Z}+\mathbb{Z}_2+\mathbb{Z}_4$	$\mathbb{Z}+\mathbb{Z}_3^2$	$\mathbb{Z}+\mathbb{Z}_2^3$	sum
$(x,1,y)$	94	24	8	3	1	1	2	1	1		135
$(2,2,y)$	9	9		1			3			1	23
$(y,2,2)$	31	25		2			4				62
$(3,2,z)$	6	1									7
$(z,2,3)$	5										5
$(z,3,2)$	21	8	1								30
sum	166	67	9	6	1	1	9	1	1	1	262

Number of members P of the list according to divisor class group and exponent configuration

Distinct matrices from the list yield non-isomorphic varieties and every non-toric, $\mathbb{Q}$-factorial, log terminal and Gorenstein Fano threefold of Picard number one of format $(2,2,1,0)$ is isomorphic to an $X = X(A,P,\Sigma)$ with P from the list.

Proof. Proposition 2.4.4 allows us to write down explicitly all possible matrices G and hence to determine all possible $Q^0 = [w_{01}, w_{02}, w_{11}, w_{12}, w_{21}]$ by computer. Now, recall that P annihilates the transpose of Q^0. This enables us to determine in the matrix P, adjusted according to Proposition 2.4.3 (iv), all the remaining variables. So, we are left with a finite list of explicitly given possible defining matrices P. Checking for the necessary properties by means of [43] and reducing via Proposition 2.3.1 to data defining pairwise non-isomorphic varieties, we obtain the list presented in the assertion. $\square$

2.5 Proof of Classification 2.1.1: Case 2 - format $(2,2,1,1,0)$

Proposition 2.2.24 divides the proof of the classification theorem 2.1.1 into cases (i) to (viii). In this section we treat case (ii). The setting is as follows.

Setting 2.5.1. Let $X = X(A,P,\Sigma)$ a $\mathbb{Q}$-factorial threefold of Picard number one of format $(2,2,1,1,0)$. Then

$$P = [v_{01}, v_{02}, v_{11}, v_{12}, v_{21}, v_{31}] = \begin{bmatrix} -l_{01} & -l_{02} & l_{11} & l_{12} & 0 & 0 \\ -l_{01} & -l_{02} & 0 & 0 & l_{21} & 0 \\ -l_{01} & -l_{02} & 0 & 0 & 0 & l_{31} \\ d_{011} & d_{021} & d_{111} & d_{121} & d_{211} & d_{311} \\ d_{012} & d_{022} & d_{112} & d_{122} & d_{212} & d_{312} \end{bmatrix}$$

holds with pairwise different primitive columns $v_{01}, v_{02}, v_{11}, v_{12}, v_{21}$ and v_{31} generating $\mathbb{Q}^5$ as a cone. We assume P to have ordered exponents. The maximal X-cones of the fan Σ of Z are given by

$$\sigma_{01} = \operatorname{cone}(v_{02}, v_{11}, v_{12}, v_{21}, v_{31}), \qquad \sigma_{02} = \operatorname{cone}(v_{01}, v_{11}, v_{12}, v_{21}, v_{31}),$$

$$\sigma_{11} = \operatorname{cone}(v_{01}, v_{02}, v_{12}, v_{21}, v_{31}), \qquad \sigma_{12} = \operatorname{cone}(v_{01}, v_{02}, v_{11}, v_{21}, v_{31}).$$

We have $K = \mathbb{Z} \oplus \Gamma$ with the torsion part Γ and denote $\deg(T_{ij}) = (w_{ij}, \eta_{ij})$ as well as $\deg(T_k) = (w_k, \eta_k)$ accordingly. In particular, we write

$$Q^0 \ = \ [w_{01}, w_{02}, w_{11}, w_{12}, w_{21}, w_{31}]$$

for the free part of the degree matrix Q. Note that the vector $(w_{01}, w_{02}, w_{11}, w_{12}, w_{21}, w_{31})$ is primitive in $\mathbb{Z}^6$ and generates $\ker(P)$.

Our first series of constraints arising from the log terminality and the Gorenstein property directly aims for entries of the defining matrix P.

Proposition 2.5.2. *Consider $X = X(A, P, \Sigma)$ as in Setting 2.5.1. Assume that X is non-toric, Fano, log-terminal and Gorenstein.*

(i) *We have $l_{11} = l_{12} = 1$ and the tuple $(l_{01}, l_{11}, l_{21}, l_{31})$ fits into precisely one of the following constellations:*

$$\begin{array}{llll} (1,1,x,y), & x \ge y \ge 2; & (2,1,z,3), & 3 \le z \le 5; \\ (2,1,y,2), & y \ge 2; & (3,1,z,2), & 3 \le z \le 5; \\ (y,1,2,2), & y \ge 3; & (z,1,3,2), & 4 \le z \le 5. \end{array}$$

(ii) *$-\mathcal{K} = (1-l_{01})D^X_{01} + (1-l_{02})D^X_{02} + D^X_{11} + D^X_{21}$ is an anitcanonical divisor on X. In particular, the free part of the anticanonical class is given by*

$$w_X \ = \ (1-l_{01})w_{01} + (1-l_{02})w_{02} + w_{21} + w_{31}.$$

(iii) *Admissible row operations turn the defining matrix P into the form*

$$P \ = \ \begin{bmatrix} -l_{01} & -l_{02} & 1 & 1 & 0 & 0 \\ -l_{01} & -l_{02} & 0 & 0 & l_{21} & 0 \\ -l_{01} & -l_{02} & 0 & 0 & 0 & l_{31} \\ d_{011} & d_{021} & 0 & d_{121} & d_{211} & d_{311} \\ d_{012} & d_{022} & 0 & 0 & d_{212} & d_{312} \end{bmatrix}, \qquad \begin{array}{l} l_{01} \ge l_{02} \ge 1, \ l_{21} \ge l_{31} \ge 2, \\ 0 \le d_{011}, d_{012} < l_{01}, \\ 0 \le d_{021} < l_{01}\frac{w_X}{w_{02}}, \\ -\frac{w_X}{w_{02}} \le d_{022} < l_{02} + \frac{w_X}{w_{02}}, \\ 1 \le d_{121} \le \frac{w_X}{w_{12}}, \ 0 \le d_{311}, d_{312} < l_{31}, \end{array}$$

where $w_{02} \mid w_X$ and $w_{12} \mid w_X$.

Proof. We prove (i). We apply Proposition 2.2.22 to the X-cone $\operatorname{cone}(v_{01}, v_{11}, v_{21}, v_{31})$ to see that $(l_{01}, l_{11}, l_{21}, l_{31})$ is a platonic tuple. As P has ordered exponents, $l_{01} \ge l_{11}$ and $l_{21} \ge l_{31}$ holds. Moreover, since X is non-toric, we have $l_{31} \ge 2$. Thus we have $l_{11} = 1$ and consequently $l_{12} = 1$. This leaves us with the six constellations for $(l_{01}, l_{11}, l_{21}, l_{31})$ stated in the assertion. Item (ii) follows immediately from Remark 2.2.13 and homogeneity of the defining relations g_0 and g_1.

We prove (iii). Adding multiples of the first row of P to the fourth and fifth row, we achieve $d_{111} = d_{112} = 0$. We apply a suitable unimodular 2×2 matrix to the d-block to ensure $d_{122} = 0$ and $d_{121} \ge 0$. By linear independence of v_{11} and v_{12} the entry d_{121} is positive. By adding multiples of the second row to the fourth and fifth row, we may assume $0 \le d_{011}, d_{012} < l_{01}$. We make use of the Gorenstein property. Consider the X-cone $\sigma_{12} = \operatorname{cone}(v_{01}, v_{02}, v_{11}, v_{21}, v_{31})$. By Lemma 2.3.2 we have $w_{12} \mid w_X$ and there is a linear form $u \in \mathbb{Z}^5$ with

$$\begin{array}{rclrcl} \langle u, v_{01} \rangle &=& 1 - l_{01}, & \langle u, v_{02} \rangle &=& 1 - l_{02}, \\ \langle u, v_{11} \rangle &=& 0, & \langle u, v_{12} \rangle &=& -\frac{w_X}{w_{12}}, \\ \langle u, v_{21} \rangle &=& 1, & \langle u, v_{31} \rangle &=& 1. \end{array}$$

By the third equation $u_1 = 0$ holds. Plugging this into the fourth equation, we obtain

$$-\frac{w_X}{w_{12}} \; = \; u_4 d_{121}.$$

Thus d_{121} divides $\frac{w_X}{w_{12}}$. In particular, we get the bounds $1 \le d_{121} \le \frac{w_X}{w_{12}}$. Now consider the X-cone $\sigma_{02} = \operatorname{cone}(v_{01}, v_{11}, v_{12}, v_{21}, v_{31})$. By Lemma 2.3.2 we have $w_{02} \mid w_X$ and there is a linear form $u \in \mathbb{Z}^5$ with

$$\begin{array}{rclrcl} \langle u, v_{01} \rangle &=& 1 - l_{01}, & \langle u, v_{02} \rangle &=& 1 - l_{02} - \frac{w_X}{w_{02}}, \\ \langle u, v_{11} \rangle &=& 0, & \langle u, v_{12} \rangle &=& 0, \\ \langle u, v_{21} \rangle &=& 1, & \langle u, v_{31} \rangle &=& 1. \end{array} \tag{2.5.2.1}$$

Again, by the third equation, $u_1 = 0$ holds. Plugging this into the fourth equation now yields $u_4 = 0$. Plugging this into the second equation and multiplying by l_{01}, we obtain

$$l_{01}\left(1 - l_{02} - \frac{w_X}{w_{02}}\right) \; = \; -l_{02} l_{01}(u_2 + u_3) + u_5 l_{01} d_{022} \; = \; u_5(l_{01} d_{022} - l_{02} d_{012}) + l_{02}(1 - l_{01}).$$

In the second step we used the identity $l_{01}(u_2 + u_3) = l_{01} + u_5 d_{012} - 1$, which we obtain from the firs equation in 2.5.2.1. We subtract $l_{02}(1 - l_{01})$ on both sides to obtain

$$u_5(l_{01} d_{022} - l_{02} d_{012}) \; = \; l_{01}\left(1 - \frac{w_X}{w_{02}}\right) - l_{02}.$$

As the right hand side is negative, the left hand side thus not vanish. Thus we have $u_5 \neq 0$ as well as $(l_{01} d_{022} - l_{02} d_{012}) \neq 0$ and $(l_{01} d_{022} - l_{02} d_{012})$ is a divisor of $l_{01}\left(1 - \frac{w_X}{w_{02}}\right) - l_{02}$. Solving for d_{022} und using the bounds on d_{012}, we obtain

$$-\frac{w_X}{w_{02}} \; \le \; d_{022} \; < \; l_{02} + \frac{w_X}{w_{02}}.$$

Adding the d_{012}-fold of the third row and the l_{01}-fold of the fifth row of P to the fourth row leaves the first, second and third entry unchanged. We repeat this to achieve

$$0 \; \le \; d_{021} \; < \; |l_{01} d_{022} - l_{02} d_{012}| \; \le \; l_{01}\left(\frac{w_X}{w_{02}} - 1\right) + l_{02} \; \le \; l_{01} \frac{w_X}{w_{02}}.$$

Finally we add multiples of the difference of the second and third row of P to the fourth and fifth row to obtain $0 \le d_{311}, d_{312} < l_{31}$. □

Our second series of constraints shows that all entries of the $\mathbb{Z}$-part of the degree matrix $Q^0 = [w_{01}, w_{02}, w_{11}, w_{12}, w_{21}, w_{31}]$ are bounded.

Proposition 2.5.3. *Consider $X = X(A,P,\Sigma)$ as in Setting 2.5.1. Assume that X is non-toric, Fano, log-terminal and Gorenstein.*

(i) *For any four positive integers $\alpha_{01}, \alpha_{02}, \alpha_{11}$ and α_{12} consider the 7×6 matrix*

$$G := \begin{bmatrix} 1-l_{01}-\alpha_{01} & 1-l_{02} & 0 & 0 & 1 & 1 \\ 1-l_{01} & 1-l_{02}-\alpha_{02} & 0 & 0 & 1 & 1 \\ 1-l_{01} & 1-l_{02} & -\alpha_{11} & 0 & 1 & 1 \\ 1-l_{01} & 1-l_{02} & 0 & -\alpha_{12} & 1 & 1 \\ -l_{01} & -l_{02} & 1 & 1 & 0 & 0 \\ -l_{01} & -l_{02} & 0 & 0 & l_{21} & 0 \\ -l_{01} & -l_{02} & 0 & 0 & 0 & l_{31} \end{bmatrix}.$$

The matrix G is of rank at least 5. Moreover, $\operatorname{rank}(G) = 5$ holds if and only if $\alpha_{01}, \alpha_{02}, \alpha_{11}, \alpha_{12}$ and $l_{01}, l_{02}, l_{21}, l_{31}$ satisfy the identities

$$\begin{aligned} \frac{l_{01}}{\alpha_{01}} + \frac{l_{02}}{\alpha_{02}} - \frac{1}{\alpha_{11}} - \frac{1}{\alpha_{12}} &= 0, \\ \frac{1}{\alpha_{01}} + \frac{1}{\alpha_{02}} + \left(\frac{l_{01}}{\alpha_{01}} + \frac{l_{02}}{\alpha_{02}}\right)\left(\frac{1}{l_{21}} + \frac{1}{l_{31}} - 1\right) &= 1. \end{aligned}$$

(ii) *There are unique $\alpha_{01}, \alpha_{02}, \alpha_{11}, \alpha_{12} \in \mathbb{Z}_{\geq 1}$ with $\alpha_{ij} w_{ij} = w_X$ for all $0 \leq i \leq 1$ and all $1 \leq j \leq 2$ and the corresponding matrix G from (i) satisfies*

$$\ker(G) = \ker(P) = \mathbb{Z} \cdot (w_{01}, w_{02}, w_{11}, w_{12}, w_{21}, w_{31}).$$

(iii) *According to the possible constellations of $(l_{01}, l_{11}, l_{21}, l_{31})$ from Proposition 2.5.2 (i) we have the following upper bounds on the entries of the matrix G from (ii). An empty line indicates that this exponent configuration does not occur.*

	l_{01}	l_{02}	l_{11}	l_{12}	l_{21}	l_{31}	α_{01}	α_{02}	α_{11}	α_{12}
$(1,1,x,y)$	1	1	1	1	42	4	21	2	21	2
$(2,1,y,2)$	2	2	1	1	6	2	12	3	3	3
$(y,1,2,2)$	3	3	1	1	2	2	2	2	1	1
$(2,1,z,3)$										
$(3,1,z,2)$	3	1	1	1	5	2	3	1	1	1
$(z,1,3,2)$										

Proof. We prove (i). In order to see that G is of rank at least five, we just compute the minor obtained by deleting rows 5 and 7 and column 1:

$$\det \begin{bmatrix} 1-l_{02} & 0 & 0 & 1 & 1 \\ 1-l_{02}-\alpha_{02} & 0 & 0 & 1 & 1 \\ 1-l_{02} & -\alpha_{11} & 0 & 1 & 1 \\ 1-l_{02} & 0 & -\alpha_{12} & 1 & 1 \\ -l_{02} & 0 & 0 & l_{21} & 0 \end{bmatrix} = -\alpha_{02}\alpha_{11}\alpha_{12}l_{21} \neq 0.$$

Moreover, G is of rank exactly five if and only if all its 6-minors vanish. Rearranging these seven equations and removing redundancies, we arrive at the identities in $\alpha_{01}, \alpha_{02}, \alpha_{11}, \alpha_{12}$ and $l_{01}, l_{02}, l_{21}, l_{31}$.

We prove (ii). Applying Corollary 2.2.21 to the four maximal X-cones σ_{01}, σ_{02}, σ_{11}, and σ_{12} we see that each of w_{01}, w_{02}, w_{11}, and w_{12} is a divisor of w_X and hence we obtain positive integers α_{ij} for $0 \le i \le 1$ and $1 \le j \le 2$ with

$$\alpha_{ij} w_{ij} \;=\; (1 - l_{01}) w_{01} + (1 - l_{02}) w_{02} + w_{21} + w_{31}.$$

Moreover, by homogeneity of the defining relations g_0 and g_1 we have

$$l_{01} w_{01} + l_{02} w_{02} \;=\; w_{11} + w_{12} \;=\; l_{21} w_{21} \;=\; l_{31} w_{31}.$$

The matrix G is the coefficient matrix of the corresponding system of linear equations. In particular, $\ker(G)$ is generated by the primitive vector $(w_{01}, w_{02}, w_{11}, w_{12}, w_{21}, w_{31}) \in \mathbb{Z}^6$.

We prove (iii). We treat the configuration $(l_{01}, l_{11}, l_{21}, l_{31}) = (1, 1, x, y)$. In this case the identities from (i) read

$$\frac{1}{\alpha_{01}} + \frac{1}{\alpha_{02}} - \frac{1}{\alpha_{11}} - \frac{1}{\alpha_{12}} \;=\; 0, \qquad (2.5.3.1)$$

$$\left(\frac{1}{\alpha_{01}} + \frac{1}{\alpha_{02}}\right)\left(\frac{1}{x} + \frac{1}{y}\right) \;=\; 1. \qquad (2.5.3.2)$$

Since $l_{01} = l_{02}$ and $l_{11} = l_{12}$ we may assume $\alpha_{01} \ge \alpha_{02}$ and $\alpha_{11} \ge \alpha_{12}$. Moreover we have $x \ge y$. With these assumptions, Equation 2.5.3.2 immediately gives the bounds $\alpha_{02} \le 2$ and $y \le 4$. Moreover, we may expand Equation 2.5.3.2 into a sum of four unit fractions. Lemma 2.3.4 (ii) then gives the bounds $\alpha_{01} \le 21$ and $x \le 42$. We use Equation 2.5.3.1 and the bounds on α_{01} and α_{02} to obtain $\alpha_{11} \le 21$ and $\alpha_{12} \le 2$.

We treat the configuration $(l_{01}, l_{11}, l_{21}, l_{31}) = (2, 1, y, 2)$. In this case the identities from (i) read

$$\frac{2}{\alpha_{01}} + \frac{l_{02}}{\alpha_{02}} - \frac{1}{\alpha_{11}} - \frac{1}{\alpha_{12}} \;=\; 0, \qquad (2.5.3.3)$$

$$\frac{2 - l_{02}}{2\alpha_{02}} + \frac{1}{y}\left(\frac{2}{\alpha_{01}} + \frac{l_{02}}{\alpha_{02}}\right) \;=\; 1. \qquad (2.5.3.4)$$

We use Equation 2.5.3.3 to replace the term in the brackets in 2.5.3.4 and obtain

$$\frac{2 - l_{02}}{2\alpha_{02}} + \frac{1}{y\alpha_{11}} + \frac{1}{y\alpha_{12}} \;=\; 1,$$

which is a sum of at most 3 unit fractions. We can thus apply Lemma 2.3.4 (ii) to get the bounds $\alpha_{11}, \alpha_{12} \le 3$ and $y \le 6$. Combining this equation with 2.5.3.3 and considering the two cases $l_{02} = 1$ and $l_{02} = 2$, we obtain the bounds $\alpha_{01} \le 12$ and $\alpha_{02} \le 3$.

We treat the configuration $(l_{01}, l_{11}, l_{21}, l_{31}) = (y, 1, 2, 2)$. In this case the identities from (i) read

$$\frac{y}{\alpha_{01}} + \frac{l_{02}}{\alpha_{02}} - \frac{1}{\alpha_{11}} - \frac{1}{\alpha_{12}} \;=\; 0,$$

$$\frac{1}{\alpha_{01}} + \frac{1}{\alpha_{02}} \;=\; 1.$$

The second equation immediately yields $\alpha_{01} = \alpha_{02} = 2$. Plugging this into the first equation yields $y = 3$, $l_{02} = 1$ and $\alpha_{11} = \alpha_{12} = 1$.

We treat the configuration $(l_{01}, l_{11}, l_{21}, l_{31}) = (2, 1, z, 3)$. In this case the identities from (i) read

$$\frac{2}{\alpha_{01}} + \frac{l_{02}}{\alpha_{02}} - \frac{1}{\alpha_{11}} - \frac{1}{\alpha_{12}} = 0, \tag{2.5.3.5}$$

$$\frac{6-z}{3z\alpha_{01}} + \frac{(3-2l_{02})z + 3l_{02}}{3z\alpha_{02}} = 1. \tag{2.5.3.6}$$

Since $l_{11} = l_{12}$ we may assume $\alpha_{11} \geq \alpha_{12}$. We have $3 \leq z \leq 5$ In this range, $6 - z$ is a divisor of z. Thus the first summand of Equation 2.5.3.6 is at most $1/3$. For the second summand this means

$$\frac{(3-2l_{02})z + 3l_{02}}{3z\alpha_{02}} \geq \frac{2}{3}.$$

This inequality is only fulfilled for $z = 3$, $l_{02} = 1$ and $\alpha_{02} = 1$ and in this case equality holds. Thus we also have $\alpha_{01} = 1$. Plugging these values into Equation 2.5.3.5, we obtain

$$\frac{1}{\alpha_{11}} + \frac{1}{\alpha_{12}} = 3,$$

which is a contradiction. Thus, the exponent configuration $(l_{01}, l_{11}, l_{21}, l_{31}) = (2, 1, z, 3)$ does not occur.

We treat the configuration $(l_{01}, l_{11}, l_{21}, l_{31}) = (3, 1, z, 2)$. In this case the identities from (i) read

$$\frac{3}{\alpha_{01}} + \frac{l_{02}}{\alpha_{02}} - \frac{1}{\alpha_{11}} - \frac{1}{\alpha_{12}} = 0, \tag{2.5.3.7}$$

$$\frac{6-z}{2z\alpha_{01}} + \frac{(2-l_{02})z + 2l_{02}}{2z\alpha_{02}} = 1. \tag{2.5.3.8}$$

As before, $6 - z$ is a divisor of z for $3 \leq z \leq 5$, thus the first summand in Equation 2.5.3.8 is at most $1/2$. For the second summand this means.

$$\frac{(2-l_{02})z + 2l_{02}}{2z\alpha_{02}} \geq \frac{1}{2}.$$

This inequality is only fulfilled for $\alpha_{02} = 1$. The second summand in Equation 2.5.3.8 is a sum of at most two unit fractions. Applying Lemma 2.3.2 (i), we obtain

$$\frac{6-z}{2z\alpha_{01}} \geq \frac{1}{6},$$

which gives the bound $\alpha_{01} \leq 3$. Plugging these into Equation 2.5.3.7, we get the bounds $l_{02} = 1$, $\alpha_{11} = 1$ and $\alpha_{12} = 1$.

We treat the configuration $(l_{01}, l_{11}, l_{21}, l_{31}) = (z, 1, 3, 2)$, where $z \geq 4$. In this case the identities from (i) read

$$\frac{z}{\alpha_{01}} + \frac{l_{02}}{\alpha_{02}} - \frac{1}{\alpha_{11}} - \frac{1}{\alpha_{12}} = 0, \tag{2.5.3.9}$$

$$\frac{6-z}{6\alpha_{01}} + \frac{6-l_{02}}{6\alpha_{02}} = 1. \tag{2.5.3.10}$$

The left hand side of Equation 2.5.3.10 is a sum of at most three unit fractions, each with denominator at least $2\alpha_{ij}$. We apply Lemma 2.3.2 (ii) to obtain the bounds $\alpha_{01}, \alpha_{02} \leq 3$. Combining Equations 2.5.3.9 and 2.5.3.10, we get the identity

$$\frac{1}{\alpha_{11}} + \frac{1}{\alpha_{12}} = 6\left(\frac{1}{\alpha_{01}} + \frac{1}{\alpha_{02}} - 1\right).$$

This equation is only fulfilled if exactly one of α_{01} and α_{02} is equal to one , the other one is equal to three and $\alpha_{11} = \alpha_{12} = 1$. Plugging this into Equation 2.5.3.9, we obtain

$$2 = \frac{z}{\alpha_{01}} + \frac{l_{02}}{\alpha_{02}} \geq \frac{4}{\alpha_{01}} + \frac{1}{\alpha_{02}} \geq \frac{7}{3},$$

which is a contradiction. Thus, the exponent configuration $(l_{01}, l_{11}, l_{21}, l_{31}) = (z, 1, 3, 2)$ does not occur. This completes the proof. $\square$

Corollary 2.5.4. *There is a list of* 10 *explicitly given matrices P of format* $(2, 2, 1, 1, 0)$, *each of them defining a non-toric $\mathbb{Q}$-factorial, Gorenstein, log terminal Fano threefold $X(A, P, \Sigma)$ of Picard number one.*

	$\mathbb{Z}$	$\mathbb{Z}+\mathbb{Z}_2$	$\mathbb{Z}+\mathbb{Z}_3$	$\mathbb{Z}+\mathbb{Z}_4$	$\mathbb{Z}+\mathbb{Z}_2^2$	$\mathbb{Z}+\mathbb{Z}_2+\mathbb{Z}_4$	sum
$(1, 1, x, y)$	1	1	1	1		1	5
$(2, 1, y, 2)$	1	2			1		4
$(y, 1, 2, 2)$							0
$(3, 1, z, 2)$	1						1
sum	3	3	1	1	1	1	10

Number of members P of the list according to divisor class group and exponent configuration

Distinct matrices from the list yield non-isomorphic varieties and every non-toric, $\mathbb{Q}$-factorial, log terminal and Gorenstein Fano threefold of Picard number one of format $(2, 2, 1, 1, 0)$ *is isomorphic to an $X = X(A, P, \Sigma)$ with P from the list.*

Proof. Proposition 2.5.3 allows us to write down explicitly all possible matrices G and hence to determine all possible $Q^0 = [w_{01}, w_{02}, w_{11}, w_{12}, w_{21}, w_{31}]$ by computer. Recall that P annihilates the transpose of Q^0. This enables us to determine in the matrix P, adjusted according to Proposition 2.5.2 (iii), all the remaining variables. So, we are left with a finite list of explicitly given possible defining matrices P. Checking for the necessary properties by means of [43] and reducing via Proposition 2.3.1 to data defining pairwise non-isomorphic varieties, we obtain the list presented in the assertion. $\square$

2.6 Proof of Classification 2.1.1: Case 3 - format $(2,2,1,1,1,0)$

Proposition 2.2.24 divides the proof of the classification theorem 2.1.1 into cases (i) to (viii). In this section we treat case (iii). The setting is as follows.

Setting 2.6.1. Let $X = X(A,P,\Sigma)$ a $\mathbb{Q}$-factorial threefold of Picard number one of format $(2,2,1,1,1,0)$. Then

$$P = [v_{01}, v_{02}, v_{11}, v_{12}, v_{21}, v_{31}, v_{41}] = \begin{bmatrix} -l_{01} & -l_{02} & l_{11} & l_{12} & 0 & 0 & 0 \\ -l_{01} & -l_{02} & 0 & 0 & l_{21} & 0 & 0 \\ -l_{01} & -l_{02} & 0 & 0 & 0 & l_{31} & 0 \\ -l_{01} & -l_{02} & 0 & 0 & 0 & 0 & l_{41} \\ d_{011} & d_{021} & d_{111} & d_{121} & d_{211} & d_{311} & d_{411} \\ d_{012} & d_{022} & d_{112} & d_{122} & d_{212} & d_{312} & d_{412} \end{bmatrix}$$

holds with pairwise different primitive columns $v_{01}, v_{02}, v_{11}, v_{12}, v_{21}, v_{31}$ and v_{41} generating $\mathbb{Q}^6$ as a cone. We assume P to have ordered exponents. The maximal X-cones of the fan Σ of Z are given by

$$\sigma_{01} = \operatorname{cone}(v_{02}, v_{11}, v_{12}, v_{21}, v_{31}, v_{41}), \qquad \sigma_{02} = \operatorname{cone}(v_{01}, v_{11}, v_{12}, v_{21}, v_{31}, v_{41}),$$

$$\sigma_{11} = \operatorname{cone}(v_{01}, v_{02}, v_{12}, v_{21}, v_{31}, v_{41}), \qquad \sigma_{12} = \operatorname{cone}(v_{01}, v_{02}, v_{11}, v_{21}, v_{31}, v_{41}).$$

We have $K = \mathbb{Z} \oplus \Gamma$ with the torsion part Γ and denote $\deg(T_{ij}) = (w_{ij}, \eta_{ij})$ as well as $\deg(T_k) = (w_k, \eta_k)$ accordingly. In particular, we write

$$Q^0 = [w_{01}, w_{02}, w_{11}, w_{12}, w_{21}, w_{31}, w_{41}]$$

for the free part of the degree matrix Q. Note that the vector $(w_{01}, w_{02}, w_{11}, w_{12}, w_{21}, w_{31}, w_{41})$ is primitive in $\mathbb{Z}^7$ and generates $\ker(P)$.

Our first series of constraints arising from the log terminality and the Gorenstein property directly aims for entries of the defining matrix P.

Proposition 2.6.2. *Consider $X = X(A,P,\Sigma)$ as in Setting 2.6.1. Assume that X is non-toric, Fano, log-terminal and Gorenstein.*

(i) *We have $l_{01} = l_{02} = 1$ as well as $l_{11} = l_{12} = 1$. Moreover, the tuple of exponents $(l_{01}, l_{11}, l_{21}, l_{31}, l_{41})$ fits into precisely one of the following constellations:*

$$(1,1,y,2,2), \quad y \geq 2; \qquad\qquad (1,1,z,3,2), \quad 3 \leq z \leq 5.$$

(ii) *$-\mathcal{K} = (1-l_{21})D_{21}^X + D_{31}^X + D_{41}^X$ is an anticanonical divisor on X. In particular, the free part of the anticanonical divisor class of X is given by*

$$w_X = (1-l_{21})w_{21} + w_{31} + w_{41}.$$

(iii) *Admissible row operations turn the defining matrix P into the form*

$$P = \begin{bmatrix} -1 & -1 & 1 & 1 & 0 & 0 & 0 \\ -1 & -1 & 0 & 0 & l_{21} & 0 & 0 \\ -1 & -1 & 0 & 0 & 0 & l_{31} & 0 \\ -1 & -1 & 0 & 0 & 0 & 0 & l_{41} \\ 0 & d_{021} & 0 & d_{121} & d_{211} & d_{311} & d_{411} \\ 0 & 0 & 0 & d_{122} & d_{212} & d_{312} & d_{412} \end{bmatrix}, \qquad \begin{array}{l} l_{21} \ge l_{31} \ge l_{41} > 1, \\ 1 \le d_{021} \le \frac{w_X}{w_{02}}, \\ 0 \le d_{121} < d_{122} \le \frac{w_X}{w_{12}}, \\ 0 \le d_{311}, d_{312} < l_{31}, \\ 0 \le d_{411}, d_{412} < l_{41}, \end{array}$$

where $w_{02} \mid w_X$ and $w_{12} \mid w_X$.

Proof. We prove (i). We apply Proposition 2.2.22 to the X-cone $\operatorname{cone}(v_{01}, v_{11}, v_{21}, v_{31}, v_{41})$ to see that $(l_{01}, l_{11}, l_{21}, l_{31}, l_{41})$ is a platonic tuple. As P has ordered exponents, we have $l_{01} \ge l_{11}$, $l_{21} \ge l_{31}$ and $l_{31} \ge l_{41}$. Moreover, since X is non-toric, $l_{41} \ge 2$ holds. Thus we have $l_{01} = 1$ as well as $l_{11} = 1$ and consequently $l_{02} = l_{12} = 1$. This leaves us with the two constellations for $(l_{01}, l_{11}, l_{21}, l_{31}, l_{41})$ stated in the assertion. Item (ii) follows immediately from Remark 2.2.13 and homogeneity of the defining relations g_0, g_1 and g_2.

We prove (iii). We add multiples of the first row of P to the fifth and sixth row to achieve $d_{111} = d_{112} = 0$ and we add multiples of the second row to the fifth and sixth row to achieve $d_{011} = d_{012} = 0$. Multiplying the d-block by a suitable 2×2 unimodular matrix, we may assume that $d_{022} = 0$ and $d_{021} \ge 0$ holds. Linear independence of v_{01} and v_{02} ensures that d_{021} is positive. Multiplying the last row by -1 if necessary, we may assume $d_{122} \ge 0$. We make use of the Gorenstein property. Consider the X-cone $\sigma_{02} = \operatorname{cone}(v_{01}, v_{11}, v_{12}, v_{21}, v_{31}, v_{41})$. By Lemma 2.3.2 we have $w_{02} \mid w_X$ and there is a linear form $u \in \mathbb{Z}^6$ with

$$\begin{array}{rclrcl} \langle u, v_{01} \rangle & = & 0, & \langle u, v_{02} \rangle & = & -\frac{w_X}{w_{02}}, \\ \langle u, v_{11} \rangle & = & 0, & \langle u, v_{12} \rangle & = & 0, \\ \langle u, v_{21} \rangle & = & 1 - l_{21}, & \langle u, v_{31} \rangle & = & 1, \\ \langle u, v_{41} \rangle & = & 1. & & & \end{array}$$

Combining the first two equations shows that d_{021} is a divisor of wX/w_{02}. In particular, we get the bound

$$1 \le d_{021} \le \frac{w_X}{w_{02}}.$$

Applying Lemma 2.3.2 to the X-cone $\sigma_{12} = \operatorname{cone}(v_{01}, v_{02}, v_{11}, v_{21}, v_{31}, v_{41})$, we see that $w_{12} \mid w_X$ holds and we obtain a linear form $u \in \mathbb{Z}^6$ with

$$\begin{array}{rclrcl} \langle u, v_{01} \rangle & = & 0, & \langle u, v_{02} \rangle & = & 0, \\ \langle u, v_{11} \rangle & = & 0, & \langle u, v_{12} \rangle & = & -\frac{w_X}{w_{12}}, \\ \langle u, v_{21} \rangle & = & 1 - l_{21}, & \langle u, v_{31} \rangle & = & 1, \\ \langle u, v_{41} \rangle & = & 1. & & & \end{array}$$

Combining the first three equations yields $u_1 = 0$ and $u_5 = 0$. Plugging this into the fourth equation shows that d_{122} is a divisor of w_X/w_{12}. We add multiples of the sixth row of P to the fifth row to obtain the bounds

$$0 \le d_{121} < d_{122} \le \frac{w_X}{w_{12}}.$$

Finally we add multiples of the difference of the second and third row to the fifth and sixth row, to achieve $0 \leq d_{311}, d_{312} < l_{31}$ and we add multiples of the difference of rows two and four to rows five and six to achieve $0 \leq d_{411}, d_{412} < l_{41}$. □

Our second series of constraints shows that all entries of the $\mathbb{Z}$-part of the degree matrix $Q^0 = [w_{01}, w_{02}, w_{11}, w_{12}, w_{21}, w_{31}, w_{41}]$ are bounded.

Proposition 2.6.3. *Consider $X = X(A, P, \Sigma)$ as in Setting 2.6.1. Assume that X is non-toric, Fano, log-terminal and Gorenstein.*

(i) *For any four positive integers $\alpha_{01}, \alpha_{02}, \alpha_{11}$ and α_{12} consider the 8×7 matrix*

$$G := \begin{bmatrix} -\alpha_{01} & 0 & 0 & 0 & 1-l_{21} & 1 & 1 \\ 0 & -\alpha_{02} & 0 & 0 & 1-l_{21} & 1 & 1 \\ 0 & 0 & -\alpha_{11} & 0 & 1-l_{21} & 1 & 1 \\ 0 & 0 & 0 & -\alpha_{12} & 1-l_{21} & 1 & 1 \\ -1 & -1 & 1 & 1 & 0 & 0 & 0 \\ -1 & -1 & 0 & 0 & l_{21} & 0 & 0 \\ -1 & -1 & 0 & 0 & 0 & l_{31} & 0 \\ -1 & -1 & 0 & 0 & 0 & 0 & l_{41} \end{bmatrix}.$$

The matrix G is of rank at least 6. Moreover, $\operatorname{rank}(G) = 6$ holds if and only if $\alpha_{01}, \alpha_{02}, \alpha_{11}, \alpha_{12}$ and l_{21}, l_{31}, l_{41} satisfy the identities

$$\begin{aligned} \frac{1}{\alpha_{01}} + \frac{1}{\alpha_{02}} - \frac{1}{\alpha_{11}} - \frac{1}{\alpha_{12}} &= 0, \\ \left(\frac{1}{\alpha_{01}} + \frac{1}{\alpha_{02}}\right)\left(\frac{1}{l_{21}} + \frac{1}{l_{31}} + \frac{1}{l_{41}} - 1\right) &= 1. \end{aligned}$$

(ii) *There are unique $\alpha_{01}, \alpha_{02}, \alpha_{11}, \alpha_{12} \in \mathbb{Z}_{\geq 1}$ with $\alpha_{ij} w_{ij} = w_X$ for all $0 \leq i \leq 1$ and all $1 \leq j \leq 2$ and the corresponding matrix G from (i) satisfies*

$$\ker(G) = \ker(P) = \mathbb{Z} \cdot (w_{01}, w_{02}, w_{11}, w_{12}, w_{21}, w_{31}, w_{41}).$$

(iii) *According to the possible constellations of the exponents $(l_{01}, l_{11}, l_{21}, l_{31}, l_{41})$ from Proposition 2.6.2 (i) we have the following upper bounds on the entries of the matrix G from (ii). An empty line indicates that this exponent configuration does not occur.*

	l_{01}	l_{02}	l_{11}	l_{12}	l_{21}	l_{31}	l_{41}	α_{01}	α_{02}	α_{11}	α_{12}
$(1,1,y,2,2)$	1	1	1	1	2	2	2	1	1	1	1
$(1,1,z,3,2)$											

Proof. We verify (i). In order to see that G is of rank at least six, we just compute the minor obtained by deleting rows 5 and 8 and column 1:

$$\det \begin{bmatrix} 0 & 0 & 0 & 1-l_{21} & 1 & 1 \\ -\alpha_{02} & 0 & 0 & 1-l_{21} & 1 & 1 \\ 0 & -\alpha_{11} & 0 & 1-l_{21} & 1 & 1 \\ 0 & 0 & -\alpha_{12} & 1-l_{21} & 1 & 1 \\ -1 & 0 & 0 & l_{21} & 0 & 0 \\ -1 & 0 & 0 & 0 & l_{31} & 0 \end{bmatrix} = \alpha_{02}\alpha_{11}\alpha_{12}l_{21}l_{31} \neq 0.$$

Moreover, G is of rank exactly six if and only if all its 7-minors vanish. Rearranging these eight equations and removing redundancies, we arrive at the two identities in $\alpha_{01}, \alpha_{02}, \alpha_{11}, \alpha_{12}$ and l_{21}, l_{31}, l_{41}.

We prove (ii). Applying Corollary 2.2.21 to the four maximal X-cones σ_{01}, σ_{02}, σ_{11}, and σ_{12} we see that each of w_{01}, w_{02}, w_{11}, and w_{12} is a divisor of w_X and hence we obtain positive integers α_{ij} for $0 \le i \le 1$ and $1 \le j \le 2$ with

$$\alpha_{ij}w_{ij} = (1-l_{21})w_{21} + w_{31} + w_{41}.$$

Moreover, by homogeneity of the defining relations g_0, g_1 and g_2 we have

$$w_{01} + w_{02} = w_{11} + w_{12} = l_{21}w_{21} = l_{31}w_{31} = l_{41}w_{41}.$$

The matrix G is the coefficient matrix of the corresponding system of linear equations. In particular, the kernel of the matrix G is generated by the primitive vector $(w_{01}, w_{02}, w_{11}, w_{12}, w_{21}, w_{31}, w_{41}) \in \mathbb{Z}^7$.

We prove (iii). We treat the configuration $(l_{01}, l_{11}, l_{21}, l_{31}, l_{41}) = (1, 1, y, 2, 2)$. In this case the identities from (i) read

$$\begin{aligned} \frac{1}{\alpha_{01}} + \frac{1}{\alpha_{02}} - \frac{1}{\alpha_{11}} - \frac{1}{\alpha_{12}} &= 0, \\ \frac{1}{y}\left(\frac{1}{\alpha_{01}} + \frac{1}{\alpha_{02}}\right) &= 1. \end{aligned}$$

By the second equation we immediately get $y = 2$ and $\alpha_{01} = \alpha_{02} = 1$. Plugging this into the first equation, we obtain $\alpha_{11} = \alpha_{12} = 1$.

We treat the configuration $(l_{01}, l_{11}, l_{21}, l_{31}, l_{41}) = (1, 1, z, 3, 2)$. In this case the identities from (i) read

$$\begin{aligned} \frac{1}{\alpha_{01}} + \frac{1}{\alpha_{02}} - \frac{1}{\alpha_{11}} - \frac{1}{\alpha_{12}} &= 0, \\ \frac{6-z}{6z}\left(\frac{1}{\alpha_{01}} + \frac{1}{\alpha_{02}}\right) &= 1. \end{aligned}$$

The second equation implies $2 \ge \frac{6z}{6-z}$, which is only possible for $z = 1$. This is a contradiction to the assumption $z \ge 3$. Thus this exponent configuration does not occur. □

Corollary 2.6.4. *For every choice $\lambda_1 \in \mathbb{K}^*$ and $\lambda_2 \in \mathbb{K}^*\backslash\{\lambda_1\}$ the matrix P of format $(2,2,1,1,1,0)$ given by*

$$P = \begin{bmatrix} -1 & -1 & 1 & 1 & 0 & 0 & 0 \\ -1 & -1 & 0 & 0 & 2 & 0 & 0 \\ -1 & -1 & 0 & 0 & 0 & 2 & 0 \\ -1 & -1 & 0 & 0 & 0 & 0 & 2 \\ 0 & 1 & 0 & 0 & -3 & 1 & 1 \\ 0 & 0 & 0 & 1 & -3 & 1 & 1 \end{bmatrix}$$

defines a non-toric $\mathbb{Q}$-factorial, Gorenstein, log terminal Fano threefold $X = X(A, P, \Sigma)$ of Picard number one with divisor class group and Cox ring

$$\mathcal{R}(X) \cong \mathbb{K}[T_{01}, T_{02}, T_{11}, T_{12}, T_{21}, T_{31}, T_{41}]/\langle g_0, g_1, g_2 \rangle,$$

$$g_0 = T_{01}T_{02} + T_{11}T_{12} + T_{21}^2, \quad g_1 = \lambda_1 T_{11}T_{12} + T_{21}^2 + T_{31}^2, \quad g_2 = \lambda_2 T_{21}^2 + T_{31}^2 + T_{41}^2,$$

$$\mathrm{Cl}(X) \cong \mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}, \qquad Q = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ \bar{0} & \bar{0} & \bar{0} & \bar{0} & \bar{1} & \bar{1} & \bar{0} \\ \bar{1} & \bar{1} & \bar{1} & \bar{1} & \bar{1} & \bar{0} & \bar{0} \end{bmatrix}.$$

Every non-toric, $\mathbb{Q}$-factorial, log terminal and Gorenstein Fano threefold of Picard number one of format $(2,2,1,1,1,0)$ is isomorphic to $X = X(A, P, \Sigma)$ for a choice of λ_1 and λ_2 as above with that matrix P.

Proof. Proposition 2.6.3 provides a single matrix G, namely

$$G = \begin{bmatrix} -1 & 0 & 0 & 0 & -1 & 1 & 1 \\ 0 & -1 & 0 & 0 & -1 & 1 & 1 \\ 0 & 0 & -1 & 0 & -1 & 1 & 1 \\ 0 & 0 & 0 & -1 & -1 & 1 & 1 \\ -1 & -1 & 1 & 1 & 0 & 0 & 0 \\ -1 & -1 & 0 & 0 & 2 & 0 & 0 \\ -1 & -1 & 0 & 0 & 0 & 2 & 0 \\ -1 & -1 & 0 & 0 & 0 & 0 & 2 \end{bmatrix}.$$

Its kernel is generated by the primitive vector

$$Q^0 = [w_{01}, w_{02}, w_{11}, w_{12}, w_{21}, w_{31}] = [1,1,1,1,1,1,1].$$

Recall that P annihilates the transpose of Q^0. This enables us to determine in the matrix P, adjusted according to Proposition 2.6.2 (iii), all the remaining variables. Checking the list of possible matrices P for the necessary properties by means of [43] and reducing via Proposition 2.3.1 to data defining pairwise non-isomorphic varieties, we end up with the single matrix P presented in the assertion. The description of the Cox ring and the class group follow from Construction 2.2.2. $\square$

2.7 Proof of Classification 2.1.1: Case 4 - format $(3, 1, 1, 0)$

Proposition 2.2.24 divides the proof of the classification theorem 2.1.1 into cases (i) to (viii). In this section we treat case (iv). The setting is as follows.

Setting 2.7.1. Let $X = X(A, P, \Sigma)$ a $\mathbb{Q}$-factorial threefold of Picard number one of format $(3, 1, 1, 0)$. Then

$$P = [v_{01}, v_{02}, v_{03}, v_{11}, v_{21}] = \begin{bmatrix} -l_{01} & -l_{02} & -l_{03} & l_{11} & 0 \\ -l_{01} & -l_{02} & -l_{03} & 0 & l_{21} \\ d_{011} & d_{021} & d_{031} & d_{111} & d_{211} \\ d_{012} & d_{022} & d_{032} & d_{112} & d_{212} \end{bmatrix}$$

holds with pairwise different primitive columns $v_{01}, v_{02}, v_{03}, v_{11}$ and v_{21} generating $\mathbb{Q}^4$ as a cone. We assume P to have ordered exponents. The maximal X-cones of the fan Σ of Z are given by

$$\sigma_{01} = \operatorname{cone}(v_{02}, v_{03}, v_{11}, v_{21}), \qquad \sigma_{02} = \operatorname{cone}(v_{01}, v_{03}, v_{11}, v_{21}),$$

$$\sigma_{03} = \operatorname{cone}(v_{01}, v_{02}, v_{11}, v_{21}), \qquad \tau_0 = \operatorname{cone}(v_{01}, v_{02}, v_{03}).$$

We have $K = \mathbb{Z} \oplus \Gamma$ with the torsion part Γ and denote $\deg(T_{ij}) = (w_{ij}, \eta_{ij})$ as well as $\deg(T_k) = (w_k, \eta_k)$ accordingly. In particular, we write

$$Q^0 = [w_{01}, w_{02}, w_{03}, w_{11}, w_{21}]$$

for the free part of the degree matrix Q. Note that the vector $(w_{01}, w_{02}, w_{03}, w_{11}, w_{21})$ is primitive in $\mathbb{Z}^5$ and generates $\ker(P)$.

Very first constraints on the exponents of the defining relation g come from log terminality of X.

Proposition 2.7.2. *Consider $X = X(A, P, \Sigma)$ as in Setting 2.7.1. Assume that X is non-toric, Fano and log-terminal. Then the tuple (l_{01}, l_{11}, l_{21}) fits into precisely one of the following constellations:*

$$\begin{array}{llll} (1, x, y), & x \geq y \geq 2; & (2, z, 3), & 3 \leq z \leq 5; \\ (y, 2, 2), & y \geq 2; & (3, z, 2), & 3 \leq z \leq 5; \\ (2, y, 2), & y \geq 3; & (z, 3, 2), & 4 \leq z \leq 5. \end{array}$$

The following Lemma treats the exponents (l_{01}, l_{02}, l_{03}) of the first monomial of the defining relation g.

Lemma 2.7.3. *Consider $X = X(A, P, \Sigma)$ as in Setting 2.7.1. Assume that X is Gorenstein. If the exponents l_{01}, l_{02} and l_{03} coincide, then they must all be equal one.*

Proof. Let $l := l_{01} = l_{02} = l_{03}$. Assume $l > 1$ holds. We apply a suitable unimodular 2×2 matrix to the d-block and add multiples of the first row of P to the third row, so that P is of the shape

$$P = \begin{bmatrix} -l & -l & -l & l_{11} & 0 \\ -l & -l & -l & 0 & l_{21} \\ d_{011} & d_{021} & d_{031} & d_{111} & d_{211} \\ 0 & d_{022} & d_{032} & d_{112} & d_{212} \end{bmatrix},$$

where $0 < d_{011} < l$. Consider the X-cone $\tau_0 = \mathrm{cone}(v_{01}, v_{02}, v_{03})$. An anticanonical divisor on X is given by

$$-\mathcal{K} = D_{01}^X + D_{02}^X + D_{03}^X + D_{11}^X + (1 - l_{21}) D_{21}^X.$$

By Lemma 2.3.2 there is thus a linear form $u \in \mathbb{Z}^4$ with

$$\langle u, v_{01} \rangle = 1, \qquad \langle u, v_{02} \rangle = 1, \qquad \langle u, v_{03} \rangle = 1.$$

This implies $d_{022} \neq 0$, since otherwise we had $v_{01} = v_{02}$. We expand the first equation

$$1 = \langle u, v_{01} \rangle = -l(u_1 + u_2) + d_{011} u_3.$$

Thus l and u_3 are coprime. In particular, since $l > 0$, we have $u_3 \neq 0$. We combine this with the second and third equation to obtain

$$\begin{aligned} u_3(d_{021} - d_{011}) + u_4 d_{022} &= 0, \\ u_3(d_{031} - d_{011}) + u_4 d_{032} &= 0. \end{aligned}$$

Combining these two together, we obtain the following identity

$$u_3(d_{022}(d_{031} - d_{011}) - d_{032}(d_{021} - d_{011})) = 0.$$

As $u_3 \neq 0$, the second factor must vanish. This contradicts the fact that the first three columns v_{01}, v_{02}, v_{03} of P are linearly independent. Thus we have $l = 1$, which completes the proof. $\square$

We deviate from the formula established in the other parts of the proof by first treating the $\mathbb{Z}$-part of the degree matrix $Q^0 = [w_{01}, w_{02}, w_{03}, w_{11}, w_{21}]$. This is our second series of constraints.

Proposition 2.7.4. *Consider $X = X(A, P, \Sigma)$ as in Setting 2.7.1. Assume that X is non-toric, Fano, log-terminal and Gorenstein.*

(i) $-\mathcal{K} = (1 - l_{01})D_{01}^X + (1 - l_{02})D_{02}^X + (1 - l_{03})D_{03}^X + D_{11}^X + D_{21}^X$ *is an anticanonical divisor on X. In particular, the free part of the anticanonical class is given by*

$$w_X \;=\; (1 - l_{01})w_{01} + (1 - l_{02})w_{02} + (1 - l_{03})w_{03} + w_{11} + w_{21}.$$

(ii) *For any three positive integers α_{01}, α_{02} and α_{03} consider the 5×5 matrix*

$$G \;:=\; \begin{bmatrix} 1 - l_{01} - \alpha_{01} & 1 - l_{02} & 1 - l_{03} & 1 & 1 \\ 1 - l_{01} & 1 - l_{02} - \alpha_{02} & 1 - l_{03} & 1 & 1 \\ 1 - l_{01} & 1 - l_{02} & 1 - l_{03} - \alpha_{03} & 1 & 1 \\ -l_{01} & -l_{02} & -l_{03} & l_{11} & 0 \\ -l_{01} & -l_{02} & -l_{03} & 0 & l_{21} \end{bmatrix}.$$

The matrix G is of rank at least four. Moreover, we have $\det(G) = 0$ *if and only if $\alpha_{01}, \alpha_{02}, \alpha_{03}$ and $l_{01}, l_{02}, l_{03}, l_{11}, l_{21}$ satisfy the identity*

$$\frac{1}{\alpha_{01}} + \frac{1}{\alpha_{02}} + \frac{1}{\alpha_{03}} + \left(\frac{l_{01}}{\alpha_{01}} + \frac{l_{02}}{\alpha_{02}} + \frac{l_{03}}{\alpha_{03}}\right)\left(\frac{1}{l_{11}} + \frac{1}{l_{21}} - 1\right) \;=\; 1.$$

(iii) *There are unique $\alpha_{01}, \alpha_{02}, \alpha_{03} \in \mathbb{Z}_{\geq 1}$ with $\alpha_{01}w_{01} = \alpha_{02}w_{02} = \alpha_{03}w_{03} = w_X$, and the corresponding matrix G from (iii) satisfies*

$$\ker(G) \;=\; \ker(P) \;=\; \mathbb{Z} \cdot (w_{01}, w_{02}, w_{03}, w_{11}, w_{21}).$$

(iv) *According to the possible constellations of (l_{01}, l_{11}, l_{21}) from Proposition 2.7.2 we have the following upper bounds on the entries of the matrix G from* (ii)*:*

	l_{01}	l_{02}	l_{03}	l_{11}	l_{21}	α_{01}	α_{02}	α_{03}
$(1, x, y)$	1	1	1	126	6	30	10	3
$(y, 2, 2)$	10	5	1	2	2	6	6	6
$(2, y, 2)$	2	2	1	30	2	1806	1806	1806
$(2, z, 3)$	2	2	1	5	3	602	28	28
$(3, z, 2)$	3	3	2	5	2	903	35	35
$(z, 3, 2)$	5	5	4	3	2	602	35	35

Proof. We prove (i). We have $r = 1$ and the defining relation g of the Cox ring is given by

$$g \;=\; T_{01}^{l_{01}} T_{02}^{l_{02}} T_{03}^{l_{03}} + T_{11}^{l_{11}} + T_{21}^{l_{21}}.$$

Thus, $\deg(g) = l_{01}\deg(T_{01}) + l_{02}\deg(T_{02}) + l_{03}\deg(T_{03})$ holds and Remark 2.2.13 shows that the anticanonical divisor $-\mathcal{K}$ is as claimed.

We prove (ii). In order to see that G is of rank at least four, we just compute the minor obtained by deleting row 5 and column 1:

$$G_{5,1} \;=\; \det \begin{bmatrix} 1-l_{02} & 1-l_{03} & 1 & 1 \\ 1-l_{02}-\alpha_{02} & 1-l_{03} & 1 & 1 \\ 1-l_{02} & 1-l_{03}-\alpha_{03} & 1 & 1 \\ -l_{02} & -l_{03} & l_{11} & 0 \end{bmatrix} \;=\; -\alpha_{02}\alpha_{03}l_{11} \;\neq\; 0.$$

Moreover, suitably rearranging the equation $\det(G) = 0$, we arrive at the displayed identity in $\alpha_{01}, \alpha_{02}, \alpha_{03}$ and $l_{01}, l_{02}, l_{03}, l_{11}, l_{21}$.

We prove (iii). Applying Corollary 2.2.21 to the three maximal X-cones σ_{01}, σ_{02} and σ_{03} we see that each of w_{01}, w_{02} and w_{03} divides w_X and hence we obtain positive integers α_{01}, α_{02} and α_{03} with

$$\alpha_{01}w_{01} \;=\; \alpha_{02}w_{02} \;=\; \alpha_{03}w_{03} \;=\; (1-l_{01})w_{01} + (1-l_{02})w_{02} + (1-l_{03})w_{03} + w_{11} + w_{21}.$$

Moreover, by homogeneity of the defining relation g we have

$$l_{01}w_{01} + l_{02}w_{02} + l_{03}w_{03} \;=\; l_{11}w_{11} \;=\; l_{21}w_{21}.$$

The matrix G from (i) is the coefficient matrix of the corresponding system of linear equations. In particular, the integral matrix G has kernel generated by the primitive vector $(w_{01}, w_{02}, w_{11}, w_{12}, w_{21}) \in \mathbb{Z}^5$.

We prove (iv). We treat the configuration $(l_{01}, l_{11}, l_{21}) = (1, x, y)$. In this case the identity from (ii) reads

$$\left(\frac{1}{\alpha_{01}} + \frac{1}{\alpha_{02}} + \frac{1}{\alpha_{03}}\right)\left(\frac{1}{x} + \frac{1}{y}\right) \;=\; 1. \tag{2.7.4.1}$$

Since $l_{01} = l_{02} = l_{03}$ holds, we may assume $\alpha_{01} \geq \alpha_{02} \geq \alpha_{03}$. We can then directly infer that $\alpha_{03} \leq 3$ and $y \leq 6$ holds. Note that we have $x \geq y \geq 2$. We distinguish two cases. If $x = y = 2$, then Equation 2.7.4.1 reduces to

$$\frac{1}{\alpha_{01}} + \frac{1}{\alpha_{02}} + \frac{1}{\alpha_{03}} \;=\; 1.$$

Applying Lemma 2.3.4 (ii) yields $\alpha_{01}, \alpha_{02} \leq 6$. If $x \geq 3$, then the second factor in Equation 2.7.4.1 is strictly smaller than one. Lemma 2.3.4 (i) says that the second factor is at most $5/6$ and we get the inequality

$$\frac{6}{5} \;\leq\; \frac{1}{\alpha_{01}} + \frac{1}{\alpha_{02}} + \frac{1}{\alpha_{03}} \;\leq\; \frac{2}{\alpha_{02}} + 1.$$

This gives the bound $\alpha_{02} \leq 10$. To obtain an upper bound for x, we rearrange Equation 2.7.4.1 to obtain

$$x \;=\; \frac{\frac{1}{\alpha_{01}} + \frac{1}{\alpha_{02}} + \frac{1}{\alpha_{03}}}{1 - \left(\frac{1}{y\alpha_{01}} + \frac{1}{y\alpha_{02}} + \frac{1}{y\alpha_{03}}\right)}.$$

As we are looking for positive solutions, the denominator on the right hand side must be positive. We can thus invoke Lemma 2.3.4 (i) to obtain

$$\frac{1}{y\alpha_{01}} + \frac{1}{y\alpha_{02}} + \frac{1}{y\alpha_{03}} \ \leq\ \frac{41}{42}.$$

Plugging this into the equation for x gives the upper bound $x \leq 126$. Finally we solve Equation 2.7.4.1 for α_{01} and use the established bounds on x, y and α_{02}, α_{03} to obtain $\alpha_{01} \leq 30$.

We treat the configuration $(l_{01}, l_{11}, l_{21}) = (y, 2, 2)$. In this case the identity from (ii) reads

$$\frac{1}{\alpha_{01}} + \frac{1}{\alpha_{02}} + \frac{1}{\alpha_{03}} \ =\ 1. \tag{2.7.4.2}$$

This yields $\alpha_{01}, \alpha_{02}, \alpha_{03} \leq 6$ according to Lemma 2.3.4 (ii). Additionally, since $l_{01} = l_{02}$ holds, we have $w_{11} = w_{21}$. We apply Corollary 2.2.21 to the fourth maximal X-cone $\tau_0 = \operatorname{cone}(v_{01}, v_{02}, v_{03})$ to see that there is $\gamma \in \mathbb{Z}_{\geq 1}$ with $w_X = \gamma w_{11} = \gamma w_{21}$. Homogeneity of the defining relation g yields the identity

$$\frac{y}{\alpha_{01}} + \frac{l_{02}}{\alpha_{02}} + \frac{l_{03}}{\alpha_{03}} \ =\ \frac{2}{\gamma}. \tag{2.7.4.3}$$

With the bounds that we obtained for $\alpha_{01}, \alpha_{02}, \alpha_{03}$ we get the chain of inequalities

$$2 \ \geq\ \frac{2}{\gamma} \ =\ \frac{y}{\alpha_{01}} + \frac{l_{02}}{\alpha_{02}} + \frac{l_{03}}{\alpha_{03}} \ \geq\ \frac{1}{6}(y + l_{02} + l_{03}).$$

Using $y \geq l_{02}$ and $l_{02} \geq l_{03}$, this inequality gives the bounds $y \leq 10$ and $l_{02} \leq 5$. We also obtain the inequality

$$2 \ \geq\ \frac{y}{\alpha_{01}} + \frac{l_{02}}{\alpha_{02}} + \frac{l_{03}}{\alpha_{03}} \ \geq\ l_{03}\left(\frac{1}{\alpha_{01}} + \frac{1}{\alpha_{02}} + \frac{1}{\alpha_{03}}\right) \ =\ l_{03},$$

which shows that $l_{03} \leq 2$ holds. If $l_{03} = 2$ holds, then these must all be equalities, thus in this case $\gamma = 1$ holds. Plugging $\gamma = 1$ and $l_{03} = 2$ into Equation 2.7.4.3, we obtain

$$\frac{y-2}{\alpha_{01}} + \frac{l_{02}-2}{\alpha_{02}} \ =\ 0.$$

Note that both summands on the left are non-negative. This is only possible if we have $y = l_{02} = 2$. This is a contradiction to Lemma 2.7.3, thus $l_{03} = 1$ holds.

We treat the configuration $(l_{01}, l_{11}, l_{21}) = (2, y, 2)$. Note that by Lemma 2.7.3 we immediately get $l_{03} = 1$. We rearrange the identity from (ii) to obtain

$$1 \ =\ \left(\frac{2}{\alpha_{01}y} + \frac{l_{02}}{\alpha_{02}y} + \frac{1}{\alpha_{03}y}\right) + \left(\frac{2-l_{02}}{2\alpha_{02}} + \frac{1}{2\alpha_{03}}\right). \tag{2.7.4.4}$$

The first bracket is positive and the second bracket is a sum of at most two unit fractions. Lemma 2.3.4 (ii) yields

$$\left(\frac{2-l_{02}}{2\alpha_{02}} + \frac{1}{2\alpha_{03}}\right) \ \leq\ \frac{5}{6}, \qquad \left(\frac{2}{\alpha_{01}y} + \frac{l_{02}}{\alpha_{02}y} + \frac{1}{\alpha_{03}y}\right) \ \geq\ \frac{1}{6}.$$

The inequality on the right gives the bound $y \leq 30$. To obtain bounds on α_{01}, α_{02} and α_{03} we rearrange Equation 2.7.4.4 in three different ways, each time separating the term involving a different α_{ij}:

$$\begin{aligned} 1 &= \frac{2}{\alpha_{01}y} + \left(\frac{l_{02}}{\alpha_{02}y} + \frac{2-l_{02}}{2\alpha_{02}} + \frac{1}{\alpha_{03}y} + \frac{1}{2\alpha_{03}}\right) \\ &= \frac{2l_{02} + (2-l_{02})y}{2y\alpha_{02}} + \left(\frac{2}{\alpha_{01}y} + \frac{1}{\alpha_{03}y} + \frac{1}{2\alpha_{03}}\right) \\ &= \frac{2+y}{2y\alpha_{03}} + \left(\frac{2}{\alpha_{01}y} + \frac{l_{02}}{\alpha_{02}y} + \frac{2-l_{02}}{2\alpha_{02}}\right). \end{aligned}$$

Note that in each of the three cases, the first summand is positive and the second summand is a sum of at most four unit fractions. Applying Lemma 2.3.4 (i) we obtain

$$\frac{2}{\alpha_{01}y} \geq \frac{1}{1806}, \qquad \frac{2l_{02} + (2-l_{02})y}{2y\alpha_{02}} \geq \frac{1}{1806}, \qquad \frac{2+y}{2y\alpha_{03}} \geq \frac{1}{1806}.$$

Solving these inequalities for α_{01}, α_{02} and α_{03} we obtain the bounds

$$\alpha_{01} \leq 1806, \qquad \alpha_{02} \leq 1806, \qquad \alpha_{03} \leq 1806.$$

We treat the configuration $(l_{01}, l_{11}, l_{21}) = (2, z, 3)$. Note that by Lemma 2.7.3 we immediately get $l_{03} = 1$. The identity from (ii) reads

$$\frac{6-z}{3z\alpha_{01}} + \frac{3z + (3-2z)l_{02}}{3z\alpha_{02}} + \frac{3+z}{3z\alpha_{03}} = 1.$$

Note that the numerator of the second summand is positive for all permitted values of z and l_{02}. We have $3 \leq z \leq 5$. In this range $3z$ is divisible by $6-z$. The first summand is thus a unit fraction. Similarly we see that the second and third summand are each a sum of at most two unit fractions. Moreover, each of the three summands is positive. We can thus apply Lemma 2.3.4 (i) for each of the three summands to obtain

$$\frac{6-z}{3z\alpha_{01}} \geq \frac{1}{1806}, \qquad \frac{3z + (3-2z)l_{02}}{3z\alpha_{02}} \geq \frac{1}{42}, \qquad \frac{3+z}{3z\alpha_{03}} \geq \frac{1}{42}.$$

Solving these inequalities for α_{01}, α_{02} and α_{03}, we get the bounds $\alpha_{01} \leq 602$, $\alpha_{02} \leq 28$ and $\alpha_{03} \leq 28$.

We treat the configuration $(l_{01}, l_{11}, l_{21}) = (3, z, 2)$. Using Lemma 2.7.3 we get the bound $l_{03} \leq 2$. The identity from (ii) reads

$$\frac{6-z}{2z\alpha_{01}} + \frac{2z + (2-z)l_{02}}{2z\alpha_{02}} + \frac{2z + (2-z)l_{03}}{2z\alpha_{03}} = 1.$$

Note that the numerators are all positive. Moreover, the first summand is a unit fraction, as $6-z$ divides $2z$, and the second and third summand are each a sum of at most two unit fractions. Each of the three summands is positive, so we can apply Lemma 2.3.4 (i) to obtain

$$\frac{6-z}{2z\alpha_{01}} \geq \frac{1}{1806}, \qquad \frac{2z+(2-z)l_{02}}{2z\alpha_{02}} \geq \frac{1}{42}, \qquad \frac{2z+(2-z)l_{03}}{2z\alpha_{03}} \geq \frac{1}{42}.$$

Solving these inequalities for α_{01}, α_{02} and α_{03}, we get the bounds $\alpha_{01} \leq 903$, $\alpha_{02} \leq 35$ and $\alpha_{03} \leq 35$.

We treat the configuration $(l_{01}, l_{11}, l_{21}) = (z, 3, 2)$. Using Lemma 2.7.3 we get the bound $l_{03} \leq 4$. The identity from (ii) reads

$$\frac{6-z}{6\alpha_{01}} + \frac{6-l_{02}}{6\alpha_{02}} + \frac{6-l_{03}}{6\alpha_{03}} \;=\; 1.$$

The first summand is a unit fraction, the second and third summand are each a sum of at most two unit fractions. Moreover, each of the three summands is positive. We invoke Lemma 2.3.4 (i) to obtain

$$\frac{6-z}{6\alpha_{01}} \geq \frac{1}{1806}, \qquad \frac{6-l_{02}}{6\alpha_{02}} \geq \frac{1}{42}, \qquad \frac{6-l_{03}}{6\alpha_{03}} \geq \frac{1}{42}.$$

Solving these inequalities for α_{01}, α_{02} and α_{03}, we get the bounds $\alpha_{01} \leq 602$, $\alpha_{02} \leq 35$ and $\alpha_{03} \leq 35$. □

Proposition 2.7.5. *Consider $X = X(A, P, \Sigma)$ as in Setting 2.7.1. Assume that X is non-toric, Fano, log-terminal and Gorenstein. Then the following hold:*

(i) *An anticanonical divisor on X is given by*

$$-\mathcal{K} \;=\; D^X_{01} + D^X_{02} + D^X_{03} + D^X_{11} + (1-l_{21})D^X_{21}.$$

(ii) *The weights w_{01}, w_{02} and w_{03} are divisors of w_X.*

(iii) *The exponents l_{01}, l_{02} and l_{03} fit into precisely one of the following cases:*

  (a) $l_{03} = 1$,
  (b) $l_{03} > 1$ *and* $l_{02} = l_{03} + 1$,
  (c) $(l_{02}, l_{03}) = (2, 2)$ *or* $(l_{02}, l_{03}) = (3, 3)$,
  (d) $(l_{02}, l_{03}) = (4, 2)$,
  (e) $(l_{01}, l_{02}, l_{03}) = (5, 5, 2)$.

(iv) *Admissible row operations turn the defining matrix P into one of the following* *forms, according to the cases (a) through (e) from item (ii):*

$$P \;=\; \begin{bmatrix} -l_{01} & -l_{02} & -1 & l_{11} & 0 \\ -l_{01} & -l_{02} & -1 & 0 & l_{21} \\ d_{011} & d_{021} & 0 & d_{111} & d_{211} \\ d_{012} & 0 & 0 & d_{112} & d_{212} \end{bmatrix}, \qquad \begin{array}{l} l_{01}\geq l_{02}\geq 1,\ l_{11}\geq l_{21}\geq 2, \\ 0\leq d_{011}<\frac{w_X}{w_{01}},\ 0<d_{012}\leq\frac{w_X}{w_{01}}, \\ 0<d_{021}\leq\frac{w_X}{w_{02}}, \\ 0\leq d_{211},d_{212}<l_{21}, \end{array}$$

$$P \;=\; \begin{bmatrix} -l_{01} & -(l_{03}+1) & -l_{03} & l_{11} & 0 \\ -l_{01} & -(l_{03}+1) & -l_{03} & 0 & 2 \\ 1 & 1 & 1 & d_{111} & d_{211} \\ d_{012} & 0 & 0 & d_{112} & d_{212} \end{bmatrix}, \qquad \begin{array}{l} l_{01}\geq l_{03}+1,\ l_{11}\geq 3, \\ 1\leq d_{012}\leq\frac{w_X}{w_{01}}, \\ 0\leq d_{211},d_{212}\leq 1, \end{array}$$

$$P \;=\; \begin{bmatrix} -l_{01} & -l & -l & l_{11} & 0 \\ -l_{01} & -l & -l & 0 & 2 \\ d_{011} & 1 & 1 & d_{111} & d_{211} \\ d_{012} & 0 & d_{032} & d_{112} & d_{212} \end{bmatrix}, \qquad \begin{array}{l} 2\leq l\leq 3,\ l_{11}\geq 3, \\ 1-\frac{w_X}{w_{01}}\leq d_{011}\leq\frac{w_X}{w_{01}}+4, \\ 0\leq d_{012}\leq l\left(\frac{w_X}{w_{01}}-1\right)+l_{01}, \\ 1\leq d_{032}\leq\frac{w_X}{w_{03}},\ 0\leq d_{211},d_{212}\leq 1, \end{array}$$

$$P \;=\; \begin{bmatrix} -l_{01} & -4 & -2 & 3 & 0 \\ -l_{01} & -4 & -2 & 0 & 2 \\ 1-\frac{w_X}{w_{01}} & 1 & 1 & 1 & -1 \\ d_{012} & d_{022} & 0 & d_{112} & d_{212} \end{bmatrix}, \qquad \begin{array}{l} 4\leq l_{01}\leq 5, \\ 1-\frac{w_X}{w_{01}}\leq d_{012}\leq\frac{w_X}{w_{01}}+1, \\ 0\leq d_{022}\leq 1, \\ 0\leq d_{212}\leq 1, \end{array}$$

$$P \;=\; \begin{bmatrix} -5 & -5 & -2 & 3 & 0 \\ -5 & -5 & -2 & 0 & 2 \\ d & d & d_{031} & d_{111} & d_{211} \\ 0 & d_{022} & d_{032} & d_{112} & d_{212} \end{bmatrix}, \qquad \begin{array}{l} 1\leq d\leq 2,\ 0<d_{022}\leq\frac{w_X}{w_{02}}, \\ 1-\frac{w_X}{w_{03}}\leq d_{031}\leq\frac{w_X}{w_{03}}, \\ 0\leq d_{032}<5\frac{w_X}{w_{03}}-3, \\ 0\leq d_{211},d_{212}\leq 1. \end{array}$$

Proof. Item (i) follows from Proposition 2.7.4 (i) and homogeneity of the defining relation g. Item (ii) is part of Proposition 2.7.4 (iii). We prove (iii). The cases (a) to (e) are mutually exclusive. We consider the bounds on l_{03} that we obtained in Proposition 2.7.4. For the first four constellations of (l_{01}, l_{11}, l_{21}) we have $l_{03} = 1$. Thus, they all fall under case (a). Assume $l_{03} > 1$ holds. Then either $(l_{01}, l_{11}, l_{21}) = (3, z, 2)$ with $3 \leq z \leq 5$ or $(l_{01}, l_{11}, l_{21}) = (z, 3, 2)$ with $4 \leq z \leq 5$. For the constellations $(l_{01}, l_{02}, l_{03}) = (5, 4, 4)$, $(5, 5, 3)$ and $(5, 5, 4)$ the identity from Proposition 2.7.4 (ii) is never fulfilled. The possible constellations for (l_{01}, l_{02}, l_{03}) are thus

$$\begin{array}{llllll} (3,2,2), & (3,3,2), & & & & \\ (4,2,2), & (4,3,2), & (4,3,3), & (4,4,2), & (4,4,3), & \\ (5,2,2), & (5,3,2), & (5,3,3), & (5,4,2), & (5,4,3), & (5,5,2). \end{array}$$

In this arrangement, columns one and three fall under case (c), columns two and five fall under case (b), column four is case (d) and column six is case (e).

We prove (iv). We start with case (a). Assume $l_{03} = 1$ holds. We add multiples of the first row of P to the third and fourth row to achieve $d_{031} = d_{032} = 0$. Applying a suitable unimodular 2×2 matrix to the d block, we may assume that $d_{022} = 0$ and $d_{021} \geq 0$

holds. Linear independence of v_{02} and v_{03} ensures that d_{021} is positive. Furthermore, by linear independence of the first three columns of P, we have $d_{012} \neq 0$ and we may assume $d_{012} > 0$ by multiplying the last row of P by -1 if necessary. We add multiples of the fourth row of P to the third row to achieve $0 \leq d_{011} < d_{012}$. Now P is of the form

$$P \;=\; \begin{bmatrix} -l_{01} & -l_{02} & -1 & l_{11} & 0 \\ -l_{01} & -l_{02} & -1 & 0 & l_{21} \\ d_{011} & d_{021} & 0 & d_{111} & d_{211} \\ d_{012} & 0 & 0 & d_{112} & d_{212} \end{bmatrix}$$

with $d_{012}, d_{021} > 0$ and $0 \leq d_{011} < d_{012}$. Let $u', u'' \in \mathbb{Z}^4$ the linear forms that Lemma 2.3.2 provides for the X-cones $\sigma_{01} = \operatorname{cone}(v_{02}, v_{03}, v_{11}, v_{21})$ and $\sigma_{02} = \operatorname{cone}(v_{01}, v_{03}, v_{11}, v_{21})$. For their difference $u := u' - u''$ we have

$$\begin{array}{rclrcl} \langle u, v_{01}\rangle & = & -\frac{w_X}{w_{01}}, & \langle u, v_{02}\rangle & = & \frac{w_X}{w_{02}}, \\ \langle u, v_{03}\rangle & = & 0, & \langle u, v_{11}\rangle & = & 0, \\ \langle u, v_{21}\rangle & = & 0. & & & \end{array}$$

Combining the second and third equation, we see that d_{021} is a divisor of w_X/w_{02}. In particular we obtain

$$0 \;<\; d_{021} \;\leq\; \frac{w_X}{w_{02}}.$$

Let $u''' \in \mathbb{Z}^4$ the linear form provided by Lemma 2.3.2 for the X-cone $\tau_0 = \operatorname{cone}(v_{01}, v_{02}, v_{03})$. It evaluates to one on each of v_{01}, v_{02} and v_{03}. For the difference $u := u' - u'''$, where as before u' is the linear form for the X-cone σ_{01}, we have

$$\langle u, v_{01}\rangle \;=\; -\frac{w_{01}}{w_X}, \qquad \langle u, v_{02}\rangle \;=\; 0, \qquad \langle u, v_{03}\rangle \;=\; 0.$$

By the second and third equation we have $u_1 + u_2 = u_3 = 0$. The first equation then tells us that d_{012} is a divisor of w_X/w_{01} and in particular we obtain

$$0 \;<\; d_{012} \;\leq\; \frac{w_X}{w_{01}}.$$

Finally we add multiples of the difference of the first two rows of P to the third and fourth row to achieve $0 \leq d_{211}, d_{212} < l_{21}$. This shows that P is of the first form described in Proposition 2.4.3 (iv).

We consider case (b). Assume $l_{03} > 1$ and $l_{02} = l_{03} + 1$. Note that we have $l_{21} = 2$. Consider the X-cone $\tau_0 = \operatorname{cone}(v_{01}, v_{02}, v_{03})$. Lemma 2.3.2 provides us with a linear form $u \in \mathbb{Z}^4$ that evaluates to 1 on each of v_{01}, v_{02} and v_{03}. Consider the 4×4 integer matrix

$$S \;=\; \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ u_1 & u_2 & u_3 & u_4 \\ d_{022}d_{031} - d_{021}d_{032} & 0 & l_{03}d_{022} - (l_{03}+1)d_{032} & (l_{03}+1)d_{031} - l_{03}d_{021} \end{bmatrix}.$$

It leaves the first two rows of P unchanged and has determinant

$$\det(S) \;=\; (l_{03}+1)\langle u, v_{03}\rangle - l_{03}\langle u, v_{02}\rangle \;=\; 1.$$

Thus S consists of admissible row operations on P and multiplying P from the left by S we obtain the matrix

$$P \;=\; \begin{bmatrix} -l_{01} & -(l_{03}+1) & -l_{03} & l_{11} & 0 \\ -l_{01} & -(l_{03}+1) & -l_{03} & 0 & 2 \\ 1 & 1 & 1 & d_{111} & d_{211} \\ d_{012} & 0 & 0 & d_{112} & d_{212} \end{bmatrix},$$

which we again call P. Here the entries d_{ijk} are understood to be indeterminates. Their actual values are affected by transforming P by S. Multiplying the last row of P by -1 if necessary we may assume that d_{012} is non-negative. Moreover we add multiples of the difference of the first two rows of P to the third and fourth row to ensure $0 \le d_{211}, d_{212} \le 1$. Consider the X-cone $\sigma_{01} = \operatorname{cone}(v_{02}, v_{03}, v_{11}, v_{21})$. Lemma 2.3.2 provides us with a linear form $u \in \mathbb{Z}^4$ with

$$\begin{array}{rclrcl} \langle u, v_{01}\rangle & = & 1 - \frac{w_X}{w_{01}}, & \langle u, v_{02}\rangle & = & 1, \\ \langle u, v_{03}\rangle & = & 1, & \langle u, v_{11}\rangle & = & 1, \\ \langle u, v_{21}\rangle & = & -1. & & & \end{array}$$

The second and third equation yield $u_1 + u_2 = 0$ and $u_3 = 1$. Plugging this into the first equation, we see that d_{012} is a divisor of w_X/w_{01}. In particular we obtain the bounds

$$0 \;<\; d_{012} \;\le\; \frac{w_X}{w_{01}}.$$

This shows that P is of the second form described in Proposition 2.4.3 (iv).

We consider case (c). Assume $l_{02} = l_{03} = l$ where $l = 2$ or $l = 3$. Note that we have $l_{21} = 2$. Applying a suitable unimodular 2×2 matrix to the d-block, we may assume $d_{022} = 0$ and $d_{021} \ge 0$. Primitivity of v_{02} ensures that $d_{021} > 0$ holds. By adding multiples of the first row of P to the third row and multiplying by -1 if necessary, we achieve $d_{021} = 1$. We write $d_{032} = l^e d_2$, where we choose $e \in \mathbb{Z}_{\ge 0}$ such that d_2 is not divisible by l. Applying Corollary 2.2.19 to the X-cone $\operatorname{cone}(v_{02}, v_{03})$, we obtain a linear form $u \in \mathbb{Z}^4$ with $\langle u, v_{02}\rangle = 1$ and $\langle u, v_{03}\rangle = 1$. The first equation ensures that u_3 is coprime to l. In particular, $u_3 \ne 0$ holds. Combining the two equations we get the identity

$$u_3(d_{031} - 1) \;=\; -u_4 l^e d_2.$$

As u_3 is not divisible by l, and l is prime, there is $d_1 \in \mathbb{Z}$ such that $d_{031} = l_{03}^e d_1 + 1$ holds. Let $c = \gcd(ld_1, d_2)$. There are $\alpha, \beta, \gamma, \delta \in \mathbb{Z}$ with

$$\begin{array}{rclrcl} c & = & \alpha l d_1 + \beta d_2, & 1 & = & \alpha\gamma + \beta\delta, \\ \gamma c & = & l d_1, & \delta c & = & d_2, \end{array}$$

As d_2 is not divisible by l, neither are δ and c. Thus γ is divisible by l. We write $\gamma = l\gamma'$ and $\delta = l\delta' + f$, where $f = \pm 1$. Consider the 4×4 integer matrix

$$S \;=\; \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -f\delta' & 0 & f\delta & -f\gamma' \\ -\alpha & 0 & l\alpha & \beta \end{bmatrix}.$$

The matrix S leaves the first two rows of P unchanged and it has determinant

$$\det(S) \;=\; f(\alpha\gamma + \beta\delta) \;=\; \pm 1.$$

It thus consists of admissible row operations of P. Multiplying P from the left by S transforms it into the matrix

$$P \;=\; \begin{bmatrix} -l_{01} & -l & -l & l_{11} & 0 \\ -l_{01} & -l & -l & 0 & 2 \\ d_{011} & 1 & 1 & d_{111} & d_{211} \\ d_{012} & 0 & l^e c & d_{112} & d_{212} \end{bmatrix},$$

which we again call P. We also write again d_{032} for the entry $l^e c$. Here the entries d_{ijk} are understood to be indeterminates. Their actual values are affected by transforming P by S. Note that linear independence of v_{02} and v_{03} ensures $d_{032} \neq 0$ and by multiplying the last row of P by -1 if necessary, we may assume that $d_{032} > 0$ holds. Consider the X-cone $\sigma_{03} = \mathrm{cone}(v_{01}, v_{02}, v_{11}, v_{21})$. By Lemma 2.3.2 there is a linear form $u \in \mathbb{Z}^4$ with

$$\begin{array}{rclrcl} \langle u, v_{01}\rangle & = & 1, & \langle u, v_{02}\rangle & = & 1, \\ \langle u, v_{03}\rangle & = & 1 - \frac{w_X}{w_{03}}, & \langle u, v_{11}\rangle & = & 1, \\ \langle u, v_{21}\rangle & = & -1. & & & \end{array}$$

Combining equations two and three we see that d_{032} is a divisor of w_X/w_{03}. In particular, we obtain the bounds

$$1 \;\leq\; d_{032} \;\leq\; \frac{w_X}{w_{03}}.$$

We apply Lemma 2.3.2 to the X-cone $\sigma_{01} = \mathrm{cone}(v_{02}, v_{03}, v_{11}, v_{21})$ to obtain a linear form $u \in \mathbb{Z}^4$ with

$$\begin{array}{rclrcl} \langle u, v_{01}\rangle & = & 1 - \frac{w_X}{w_{01}}, & \langle u, v_{02}\rangle & = & 1, \\ \langle u, v_{03}\rangle & = & 1, & \langle u, v_{11}\rangle & = & 1, \\ \langle u, v_{21}\rangle & = & -1. & & & \end{array}$$

Equations two and three yield $u_4 = 0$. Plugging this into the first equation and multiplying by l, we obtain

$$l\left(1 - \frac{w_X}{w_{01}}\right) \;=\; -l_{01}l(u_1 + u_2) + u_3 l d_{011} \;=\; l_{01} + u_3(l d_{011} - l_{01}).$$

Note that this implies $u_3 \neq 0$ and $(l d_{011} - l_{01}) \neq 0$. Subtracting l_{01} on both sides we see that $(l d_{011} - l_{01})$ is a divisor of $l(1 - \frac{w_X}{w_{01}}) - l_{01}$. Using the bounds on l_{01}, we obtain

$$1 - \frac{w_X}{w_{01}} \;\leq\; d_{011} \;\leq\; \frac{w_X}{w_{01}} + 4.$$

Now consider the 4×4 integer matrix

$$S = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & l & 1 \end{bmatrix}.$$

It leaves the first two rows of P unchanged and it has determinant one. It thus consists of admissible row operations on P. Multiplying P from the left by multiples of S leaves the columns v_{02} and v_{03} unchanged and we can achieve

$$0 \leq d_{012} < |ld_{011} - l_{01}| \leq l\left(\frac{w_X}{w_{01}} - 1\right) + l_{01}.$$

Finally, we add multiples of the difference of the first two rows of P to the fourth and fifth row to ensure $0 \leq d_{211}, d_{212} \leq 1$. This shows that P is of the third form described in Proposition 2.4.3 (iv).

We consider case (d). Assume $(l_{02}, l_{03}) = (4, 2)$. Note that we have $l_{11} = 3$ and $l_{21} = 2$. Lemma 2.3.2 applied to the X-cone $\sigma_{01} = \mathrm{cone}(v_{02}, v_{03}, v_{11}, v_{21})$ provides us with a linear form $u \in \mathbb{Z}^4$ with

$$\begin{array}{rclrcl} \langle u, v_{01}\rangle & = & 1 - \frac{w_X}{w_{01}}, & \langle u, v_{02}\rangle & = & 1, \\ \langle u, v_{03}\rangle & = & 1, & \langle u, v_{11}\rangle & = & 1, \\ \langle u, v_{21}\rangle & = & -1. & & & \end{array}$$

Let $d = d_{022}d_{031} - d_{021}d_{032}$ and consider the 4×4 integer matrix

$$S = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ u_1 & u_2 & u_3 & u_4 \\ -du_1 & -du_2 & d_{022} - 2d_{032} - du_3 & 2d_{031} - d_{021} - du_4 \end{bmatrix}.$$

It leaves the first two rows of P unchanged and it has determinant

$$\begin{aligned} \det(S) &= u_3(2d_{031} - d_{021}) + u_4(2d_{032} - d_{022}) \\ &= 2\langle u, v_{03}\rangle - \langle u, v_{02}\rangle \\ &= 1. \end{aligned}$$

Multiplying P from the left by S transforms it into the matrix

$$P = \begin{bmatrix} -l_{01} & -4 & -2 & 3 & 0 \\ -l_{01} & -4 & -2 & 0 & 2 \\ 1 - \frac{w_X}{w_{01}} & 1 & 1 & 1 & -1 \\ d_{012} & d & 0 & d_{112} & d_{212} \end{bmatrix},$$

which we again call P. We also write again d_{022} for the entry d. Here the entries d_{ijk} are understood to be indeterminates. Their actual values are affected by transforming P by S. Adding multiples of the second and the two-fold of the third row of P to the fourth row, we achieve $0 \le d_{022} \le 1$. Consider the X-cone $\sigma_{01} = \mathrm{cone}(v_{02}, v_{03}, v_{11}, v_{21})$. By Lemma 2.3.2 there is a linear form $u \in \mathbb{Z}^4$ with

$$\begin{array}{rclrcl} \langle u, v_{01} \rangle & = & 1 - \frac{w_X}{w_{01}}, & \langle u, v_{02} \rangle & = & 1, \\ \langle u, v_{03} \rangle & = & 1, & \langle u, v_{11} \rangle & = & 1, \\ \langle u, v_{21} \rangle & = & -1. & & & \end{array}$$

Combining equations two and three, we obtain $u_3 = d_{022}u_4$ and $2(u_1 + u_2) = d_{022}u_4$. Note that this implies $u_4 \neq 0$. Plugging this into equation one, we obtain

$$2\left(1 - \frac{w_X}{w_{01}}\right) \;=\; u_4(d_{022}(1 - l_{01}) + 2d_{012}).$$

This shows that $d_{022}(1 - l_{01}) + 2d_{012}$ is a divisor of $2\left(1 - \frac{w_X}{w_{01}}\right)$. Using the bounds on l_{01} and d_{022}, we obtain

$$1 - \frac{w_X}{w_{01}} \;\le\; d_{012} \;\le\; \frac{w_X}{w_{01}} + 1.$$

Finally we add multiples of the difference of the first and second row of P to the fourth row to achieve $0 \le d_{212} \le 1$. This shows that P is of the fourth form described in Proposition 2.4.3 (iv).

We consider case (e). Assume $(l_{01}, l_{02}, l_{03}) = (5, 5, 2)$. We have $l_{11} = 3$ and $l_{21} = 2$. Applying a suitable unimodular 2×2 matrix to the d-block, we achieve $d_{012} = 0$ and $d_{011} \ge 0$. By primitivity of v_{01} we have $m := d_{011} > 0$. We add multiples of the first row of P to the third row to achieve $1 \le m \le 4$. We write $d_{022} = 5^e d_2$, where we choose $e \in \mathbb{Z}_{\ge 0}$ such that d_2 is not divisible by 5. Consider the X-cone $\tau_0 = \mathrm{cone}(v_{01}, v_{02}, v_{03})$. Corollary 2.2.19 provides us with a linear form $u \in \mathbb{Z}^4$ with

$$\langle u, v_{01} \rangle \;=\; 1, \qquad \langle u, v_{02} \rangle \;=\; 1, \qquad \langle u, v_{03} \rangle \;=\; 1.$$

The first equation implies that u_3 is coprime to 5. In particular $u_3 \neq 0$ holds. Combining the first and second equation, we obtain

$$u_3(d_{021} - m) \;=\; u_4 5^e d_2.$$

There is thus $d_1 \in \mathbb{Z}$ with $d_{021} = 5^e d_1 + m$. Let $c = \gcd(5d_1, d_2)$. There are $\alpha, \beta, \gamma, \delta \in \mathbb{Z}$ with

$$\begin{array}{rclrcl} c & = & \alpha 5 d_1 + \beta d_2, & 1 & = & \alpha\gamma + \beta\delta, \\ \gamma c & = & 5d_1, & \delta c & = & d_2, \end{array}$$

As d_2 is not divisible by 5, neither are δ and c. Thus γ is divisible by 5. We write $\gamma = 5\gamma'$ and $\delta = 5\delta' + f$, where $1 \le f \le 4$. Consider the 4×4 integer matrix

$$S \;=\; \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \delta' m & 0 & \delta & -\gamma' \\ \alpha m & 0 & 5\alpha & \beta \end{bmatrix}.$$

The matrix S leaves the first two rows of P unchanged and it has determinant $\det(S) = 1$. It thus consists of admissible row operations of P. Multiplying P from the left by S transforms it into the matrix

$$P \;=\; \begin{bmatrix} -5 & -5 & -2 & 3 & 0 \\ -5 & -5 & -2 & 0 & 2 \\ mf & mf & d_{031} & d_{111} & d_{211} \\ 0 & 5^e c & d_{032} & d_{112} & d_{212} \end{bmatrix},$$

which we again call P. Moreover we again write d_{022} for the entry $5^e c$. The entries d_{ijk} are understood to be indeterminates. Their actual values are affected by transforming P by S. We add multiples of the first row to the third row and multiply by -1, if necessary, to replace the entry mf by d, where $1 \le d \le 2$. Consider the maximal X-cone $\sigma_{02} = \operatorname{cone}(v_{01}, v_{03}, v_{11}, v_{21})$. By Lemma 2.3.2 there is a linear form $u \in \mathbb{Z}^4$ with

$$\begin{array}{rclrcl} \langle u, v_{01}\rangle & = & 1, & \langle u, v_{02}\rangle & = & 1 - \frac{w_X}{w_{02}}, \\ \langle u, v_{03}\rangle & = & 1, & \langle u, v_{11}\rangle & = & 1, \\ \langle u, v_{21}\rangle & = & -1. & & & \end{array}$$

Combining the first and second equation we see that d_{022} divides w_X/w_{02}. In particular we obtain the bounds

$$1 \;\le\; d_{022} \;\le\; \frac{w_X}{w_{02}}.$$

Now consider the X-cone $\sigma_{03} = \operatorname{cone}(v_{01}, v_{02}, v_{11}, v_{21})$. Lemma 2.3.2 provides a linear form $u \in \mathbb{Z}^4$ with

$$\begin{array}{rclrcl} \langle u, v_{01}\rangle & = & 1, & \langle u, v_{02}\rangle & = & 1, \\ \langle u, v_{03}\rangle & = & 1 - \frac{w_X}{w_{03}}, & \langle u, v_{11}\rangle & = & 1, \\ \langle u, v_{21}\rangle & = & -1. & & & \end{array}$$

Combining the first and second equation, we obtain $u_4 = 0$ and $5(u_1 + u_2) = du_3 - 1$. Plugging this into the third equation and multiplying by 5, we get

$$5\left(1 - \frac{w_X}{w_{03}}\right) \;=\; -10(u_1 + u_2) + 5u_3 d_{031} \;=\; 2 - u_3(5d_{031} - 2d).$$

Note that this implies $u_3 \neq 0$ as well as $5d_{031} - 2d \neq 0$ and that $(5d_{031} - 2d)$ is a divisor of $5\frac{w_X}{w_{03}} - 3$. For d_{031} we thus obtain the bounds

$$1 - \frac{w_X}{w_{03}} \;\le\; d_{031} \;\le\; \frac{w_X}{w_{03}}.$$

Adding the d-fold of the first row and the 5-fold of the third row of P to the fourth row leaves the first two entries unchanged. Repeating this we achieve

$$0 \;\le\; d_{032} \;<\; |5d_{031} - 2d| \;\le\; 5\frac{w_X}{w_{03}} - 3.$$

Finally we add multiples of the difference of the first and second row of P to the third and fourth row to achieve $0 \le d_{211}, d_{212} \le 1$. This shows that P is of the fifth form described in Proposition 2.4.3 (iv), which completes the proof. $\square$

Corollary 2.7.6. *There is a list of* 87 *explicitly given matrices P of format* $(3, 1, 1, 0)$*, each of them defining a non-toric $\mathbb{Q}$-factorial, Gorenstein, log terminal Fano threefold $X(A, P, \Sigma)$ of Picard number one.*

	$\mathbb{Z}$	$\mathbb{Z}+\mathbb{Z}_2$	$\mathbb{Z}+\mathbb{Z}_3$	$\mathbb{Z}+\mathbb{Z}_4$	$\mathbb{Z}+\mathbb{Z}_5$	$\mathbb{Z}+\mathbb{Z}_6$	$\mathbb{Z}+\mathbb{Z}_8$	$\mathbb{Z}+\mathbb{Z}_2+\mathbb{Z}_2$	$\mathbb{Z}+\mathbb{Z}_2+\mathbb{Z}_4$	$\mathbb{Z}+\mathbb{Z}_2+\mathbb{Z}_6$	sum
$(1, x, y)$	5	4	4	4	2	3	2		1	1	26
$(y, 2, 2)$		3									3
$(2, y, 2)$	9	13		1				3			26
$(2, z, 3)$			1								1
$(3, z, 2)$	7	6	1								14
$(z, 3, 2)$	15	2									17
sum	36	28	6	5	2	3	2	3	1	1	87

Number of members P of the list according to divisor class group and exponent configuration

Distinct matrices from the list yield non-isomorphic varieties and every non-toric, $\mathbb{Q}$-factorial, log terminal and Gorenstein Fano threefold of Picard number one of format $(3, 1, 1, 0)$ *is isomorphic to an $X = X(A, P, \Sigma)$ with P from the list.*

Proof. Proposition 2.7.4 allows us to write down explicitly all possible matrices G and hence to determine all possible $Q^0 = [w_{01}, w_{02}, w_{03}, w_{11}, w_{21}]$ by computer. Recall that P annihilates the transpose of Q^0. This enables us to determine in the matrix P, adjusted according to Proposition 2.7.5 (iv), all the remaining variables. So, we are left with a finite list of explicitly given possible defining matrices P. Checking for the necessary properties by means of [43] and reducing via Proposition 2.3.1 to data defining pairwise non-isomorphic varieties, we obtain the list presented in the assertion. □

2.8 Proof of Classification 2.1.1: Case 5 - format $(3, 1, 1, 1, 0)$

Proposition 2.2.24 divides the proof of the classification theorem 2.1.1 into cases (i) to (viii). In this section we treat case (v). The setting is as follows.

Setting 2.8.1. Let $X = X(A, P, \Sigma)$ a $\mathbb{Q}$-factorial threefold of Picard number one of format $(3, 1, 1, 1, 0)$. Then

$$P \;=\; [v_{01}, v_{02}, v_{03}, v_{11}, v_{21}, v_{31}] \;=\; \begin{bmatrix} -l_{01} & -l_{02} & -l_{03} & l_{11} & 0 & 0 \\ -l_{01} & -l_{02} & -l_{03} & 0 & l_{21} & 0 \\ -l_{01} & -l_{02} & -l_{03} & 0 & 0 & l_{31} \\ d_{011} & d_{021} & d_{031} & d_{111} & d_{211} & d_{311} \\ d_{012} & d_{022} & d_{032} & d_{112} & d_{212} & d_{312} \end{bmatrix}$$

holds with pairwise different primitive columns $v_{01}, v_{02}, v_{03}, v_{11}, v_{21}$ and v_{31} generating $\mathbb{Q}^5$ as a cone. We assume P to have ordered exponents. The maximal X-cones of the fan Σ of Z are given by

$$\sigma_{01} \;=\; \operatorname{cone}(v_{02}, v_{03}, v_{11}, v_{21}, v_{31}), \qquad \sigma_{02} \;=\; \operatorname{cone}(v_{01}, v_{03}, v_{11}, v_{21}, v_{31}),$$

$$\sigma_{03} \ = \ \mathrm{cone}(v_{01}, v_{02}, v_{11}, v_{21}, v_{31}), \qquad \tau_0 \ = \ \mathrm{cone}(v_{01}, v_{02}, v_{03}).$$

We have $K = \mathbb{Z} \oplus \Gamma$ with the torsion part Γ and denote $\deg(T_{ij}) = (w_{ij}, \eta_{ij})$ as well as $\deg(T_k) = (w_k, \eta_k)$ accordingly. In particular, we write

$$Q^0 \ = \ [w_{01}, w_{02}, w_{03}, w_{11}, w_{21}, w_{31}]$$

for the free part of the degree matrix Q. Note that the vector $(w_{01}, w_{02}, w_{03}, w_{11}, w_{21}, w_{31})$ is primitive in $\mathbb{Z}^6$ and generates $\ker(P)$.

Our first series of constraints arising from the log terminality and the Gorenstein property directly aims for entries of the defining matrix P.

Proposition 2.8.2. *Consider $X = X(A, P, \Sigma)$ as in Setting 2.8.1. Assume that X is non-toric, Fano, log-terminal and Gorenstein.*

(i) *We have $l_{01} = l_{02} = l_{03} = 1$. Moreover, the tuple of exponents $(l_{01}, l_{11}, l_{21}, l_{31})$ fits into precisely one of the following constellations:*

$$(1, y, 2, 2), \quad y \geq 2; \qquad\qquad (1, z, 3, 2), \quad 3 \leq z \leq 5.$$

(ii) *$-\mathcal{K} = (1 - l_{11})D^X_{11} + D^X_{21} + D^X_{31}$ is an anticanonical divisor on X. In particular, the free part of the anticanonical class of X is given by*

$$w_X \ = \ (1 - l_{11})w_{11} + w_{21} + w_{31}.$$

(iii) *Admissible row operations turn the defining matrix P into the form*

$$P \ = \ \begin{bmatrix} -1 & -1 & -1 & l_{11} & 0 & 0 \\ -1 & -1 & -1 & 0 & l_{21} & 0 \\ -1 & -1 & -1 & 0 & 0 & l_{31} \\ 0 & d_{021} & d_{031} & d_{111} & d_{211} & d_{311} \\ 0 & 0 & d_{032} & d_{112} & d_{212} & d_{312} \end{bmatrix}, \qquad \begin{array}{l} l_{11} \geq l_{21} \geq l_{31} > 1, \\ 0 < d_{021} \leq \frac{w_X}{w_{02}}, \\ 0 \leq d_{031} < d_{032} \leq \frac{w_X}{w_{03}}, \\ 0 \leq d_{211}, d_{212} < l_{21}, \\ 0 \leq d_{311}, d_{312} < l_{31}, \end{array}$$

where $w_{02} \mid w_X$ and $w_{03} \mid w_X$.

Proof. We prove (i). We apply Proposition 2.2.22 to the X-cone $\mathrm{cone}(v_{01}, v_{11}, v_{21}, v_{31})$ to see that $(l_{01}, l_{11}, l_{21}, l_{31})$ is a platonic tuple. As P has ordered exponents, we have $l_{11} \geq l_{21}$ and $l_{21} \geq l_{31}$. Moreover, since X is non-toric, $l_{31} \geq 2$ holds. Thus we have $l_{01} = 1$ and consequently $l_{02} = l_{03} = 1$. This leaves us with the two constellations for $(l_{01}, l_{11}, l_{21}, l_{31})$ stated in the assertion. Item (ii) follows immediately from Remark 2.2.13 and homogeneity of the defining relations g_0 and g_1.

We prove (iii). Adding multiples of the first row of P to the fourth and fifth row, we achieve $d_{011} = d_{012} = 0$. Multiplying the d-block by a suitable unimodular 2×2 matrix, we may assume $d_{022} = 0$ and $d_{021} \geq 0$. Linear independence of v_{01} and v_{02} ensures that d_{021} is positive. Multiplying the last row of P by -1 if necessary, we may assume that $d_{032} \geq 0$ holds. We make use of the Gorenstein property. Consider the X-cone $\sigma_{02} = \operatorname{cone}(v_{01}, v_{03}, v_{11}, v_{21}, v_{31})$. By Lemma 2.3.2 we have $w_{02} \mid w_X$ and there is a linear form $u \in \mathbb{Z}^5$ with

$$\begin{array}{rclrcl} \langle u, v_{01}\rangle & = & 0, & \langle u, v_{02}\rangle & = & -\frac{w_X}{w_{02}}, \\ \langle u, v_{03}\rangle & = & 0, & \langle u, v_{11}\rangle & = & 1 - l_{11}, \\ \langle u, v_{21}\rangle & = & 1, & \langle u, v_{31}\rangle & = & 1. \end{array}$$

By the first equation, $u_1 + u_2 + u_3 = 0$ holds. Plugging this into the second equation, we see that d_{021} is a divisor of w_X/w_{02}. In particular, we obtain the bound

$$1 \ \le \ d_{021} \ \le \ \frac{w_X}{w_{02}}.$$

Applying Lemma 2.3.2 to the X-cone $\sigma_{03} = \operatorname{cone}(v_{01}, v_{02}, v_{11}, v_{21}, v_{31})$ we see that w_{03} is a divisor of w_X and we obtain a linear form $u \in \mathbb{Z}^5$ with

$$\begin{array}{rclrcl} \langle u, v_{01}\rangle & = & 0, & \langle u, v_{02}\rangle & = & 0, \\ \langle u, v_{03}\rangle & = & -\frac{w_X}{w_{03}}, & \langle u, v_{11}\rangle & = & 1 - l_{11}, \\ \langle u, v_{21}\rangle & = & 1, & \langle u, v_{31}\rangle & = & 1. \end{array}$$

Combining the first two equations, we obtain $u_1 + u_2 + u_3 = 0$ and $u_4 = 0$. Plugging this into the third equation, we see that d_{032} is a divisor of w_X/w_{03}. We add multiples of the last row of P to the fourth row to achieve

$$0 \ \le \ d_{031} \ < \ d_{032} \ \le \ \frac{w_X}{w_{03}}.$$

Finally we add multiples of the difference of the first and second row of P to the fourth and fifth row to get $0 \le d_{211}, d_{212} \le l_{21}$ and we do the same for the first and third row to get $0 \le d_{311}, d_{312} \le l_{31}$. $\square$

Our second series of constraints shows that all entries of the $\mathbb{Z}$-part of the degree matrix $Q^0 = [w_{01}, w_{02}, w_{03}, w_{11}, w_{21}, w_{31}]$ are bounded.

Proposition 2.8.3. *Consider $X = X(A, P, \Sigma)$ as in Setting 2.8.1. Assume that X is non-toric, Fano, log-terminal and Gorenstein.*

(i) *For any three positive integers α_{01}, α_{02} and α_{03} consider the 6×6 matrix*

$$G \ := \ \begin{bmatrix} -\alpha_{01} & 0 & 0 & 1 - l_{11} & 1 & 1 \\ 0 & -\alpha_{02} & 0 & 1 - l_{11} & 1 & 1 \\ 0 & 0 & -\alpha_{03} & 1 - l_{11} & 1 & 1 \\ -1 & -1 & -1 & l_{11} & 0 & 0 \\ -1 & -1 & -1 & 0 & l_{21} & 0 \\ -1 & -1 & -1 & 0 & 0 & l_{31} \end{bmatrix}.$$

The matrix G is of rank at least 5. Moreover, we have $\det(G) = 0$ if and only if $\alpha_{01}, \alpha_{02}, \alpha_{03}$ and l_{11}, l_{21}, l_{31} satisfy the identity

$$\left(\frac{1}{\alpha_{01}} + \frac{1}{\alpha_{02}} + \frac{1}{\alpha_{03}}\right)\left(\frac{1}{l_{11}} + \frac{1}{l_{21}} + \frac{1}{l_{31}} - 1\right) \ = \ 1.$$

(ii) *There are unique $\alpha_{01}, \alpha_{02}, \alpha_{03} \in \mathbb{Z}_{\geq 1}$ with $\alpha_{01}w_{01} = \alpha_{02}w_{02} = \alpha_{03}w_{03} = w_X$ and the corresponding matrix G from (i) satisfies*

$$\ker(G) \;=\; \ker(P) \;=\; \mathbb{Z} \cdot (w_{01}, w_{02}, w_{03}, w_{11}, w_{21}, w_{31}).$$

(iii) *According to the possible constellations of the exponents $(l_{01}, l_{11}, l_{21}, l_{31})$ from Proposition 2.8.2 (i), we have the following upper bounds on the entries of the matrix G from (ii). An empty line indicates that this exponent configuration does not occur.*

	l_{01}	l_{02}	l_{03}	l_{11}	l_{21}	l_{31}	α_{01}	α_{02}	α_{03}
$(1, y, 2, 2)$	1	1	1	3	2	2	2	2	1
$(1, z, 3, 2)$									

Proof. We verify (i). In order to see that G is of rank at least five, we just compute the minor obtained by deleting row 6 and column 1:

$$G_{6,1} \;=\; \det \begin{bmatrix} 0 & 0 & 1-l_{11} & 1 & 1 \\ -\alpha_{02} & 0 & 1-l_{11} & 1 & 1 \\ 0 & -\alpha_{03} & 1-l_{11} & 1 & 1 \\ -1 & -1 & l_{11} & 0 & 0 \\ -1 & -1 & 0 & l_{21} & 0 \end{bmatrix} \;=\; \alpha_{02}\alpha_{03}l_{11}l_{21} \;\neq\; 0.$$

Moreover, suitably rearranging the equation $\det(G) = 0$, we arrive at the displayed identity on $\alpha_{01}, \alpha_{02}, \alpha_{03}$ and l_{11}, l_{21}, l_{31}.

We prove (ii). Applying Corollary 2.2.21 to the three maximal X-cones σ_{01}, σ_{02} and σ_{03} we see that each of w_{01}, w_{02} and w_{03} is a divisor of w_X and hence we obtain positive integers α_{01}, α_{02} and α_{03} with

$$\alpha_{01}w_{01} \;=\; \alpha_{02}w_{02} \;=\; \alpha_{03}w_{03} \;=\; (1-l_{11})w_{11} + w_{21} + w_{31}.$$

Moreover, by homogeneity of the defining relations g_0 and g_1 we have

$$w_{01} + w_{02} + w_{03} \;=\; l_{11}w_{11} \;=\; l_{21}w_{21} \;=\; l_{31}w_{31}.$$

The matrix G is the coefficient matrix of the corresponding system of linear equations. In particular, $\ker(G)$ is generated by the primitive vector $(w_{01}, w_{02}, w_{03}, w_{11}, w_{21}, w_{31}) \in \mathbb{Z}^6$.

We prove (iii). We treat the configuration $(l_{01}, l_{11}, l_{21}, l_{31}) = (1, y, 2, 2)$. In this case the identity from (i) reads

$$\frac{1}{y}\left(\frac{1}{\alpha_{01}} + \frac{1}{\alpha_{02}} + \frac{1}{\alpha_{03}}\right) \;=\; 1.$$

Since we have $l_{01} = l_{02} = l_{03}$, we may assume that $\alpha_{01} \geq \alpha_{02} \geq \alpha_{03}$ holds. We immediately get the bounds $\alpha_{03} = 1$ and $y \leq 3$. Plugging the two possible values for y into the equation yields $\alpha_{01}, \alpha_{02} \leq 2$.

We treat the configuration $(l_{01}, l_{11}, l_{21}, l_{31}) = (1, z, 3, 2)$. In this case the identity from (i) reads

$$\frac{6-z}{6z}\left(\frac{1}{\alpha_{01}} + \frac{1}{\alpha_{02}} + \frac{1}{\alpha_{03}}\right) \;=\; 1.$$

This equation can only be fulfilled for $z \geq 2$, which is a contradiction to the assumption $z \geq 3$. Thus this exponent configuration does not occur. □

Corollary 2.8.4. *For every choice $\lambda_1 \in \mathbb{K}^*$ the matrix P of format $(3, 1, 1, 1, 0)$ given by*

$$P \;=\; \begin{bmatrix} -1 & -1 & -1 & 2 & 0 & 0 \\ -1 & -1 & -1 & 0 & 2 & 0 \\ -1 & -1 & -1 & 0 & 0 & 2 \\ 0 & 2 & 0 & -3 & 1 & 1 \\ 0 & 0 & 1 & -3 & 1 & 1 \end{bmatrix}$$

defines a non-toric $\mathbb{Q}$-factorial, Gorenstein, log terminal Fano threefold $X = X(A, P, \Sigma)$ of Picard number one with divisor class group and Cox ring

$$\mathcal{R}(X) \;\cong\; \mathbb{K}[T_{01}, T_{02}, T_{03}, T_{11}, T_{21}, T_{31}]/\langle g_0, g_1\rangle,$$

$$g_0 \;=\; T_{01}T_{02}T_{03} + T_{11}^2 + T_{21}^2, \qquad g_1 \;=\; \lambda_1 T_{11}^2 + T_{21}^2 + T_{31}^2,$$

$$\mathrm{Cl}(X) \;\cong\; \mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}, \qquad Q \;=\; \begin{bmatrix} 1 & 1 & 2 & 2 & 2 & 2 \\ \bar{0} & \bar{0} & \bar{0} & \bar{1} & \bar{1} & \bar{0} \\ \bar{0} & \bar{0} & \bar{0} & \bar{0} & \bar{1} & \bar{1} \end{bmatrix}.$$

Every non-toric, $\mathbb{Q}$-factorial, log terminal and Gorenstein Fano threefold of Picard number one of format $(3, 1, 1, 1, 0)$ is isomorphic to $X = X(A, P, \Sigma)$ for a choice of λ_1 as above with that matrix P.

Proof. Proposition 2.6.3 allows us to write down explicitly all possible matrices G and hence to determine all possible $Q^0 = [w_{01}, w_{02}, w_{11}, w_{12}, w_{21}, w_{31}]$. Excplicitly, there are the following two possibilities for Q^0:

$$(2, 2, 2, 2, 3, 3), \qquad\qquad (1, 1, 2, 2, 2, 2).$$

The matrix P annihilates the transpose of Q^0. This enables us to determine in the matrix P, adjusted according to Proposition 2.8.2 (iii), all the remaining variables. So, we are left with a finite list of explicitly given possible defining matrices P. We check for the necessary properties by means of [43] and reduce via Proposition 2.3.1 to data defining pairwise non-isomorphic varieties. The first candidate for Q^0 does not produce any valid matrices P. For the second candidate, there is up to admissible operations only one matrix P, namely the one presented in the assertion. The description of the Cox ring and the class group follow from Construction 2.2.2. □

2.9 Proof of Classification 2.1.1: Case 6 - format $(2,1,1,1)$

Proposition 2.2.24 divides the proof of the classification theorem 2.1.1 into cases (i) to (viii). In this section we treat case (vi). The setting is as follows.

Setting 2.9.1. Let $X = X(A,P,\Sigma)$ a $\mathbb{Q}$-factorial threefold of Picard number one of format $(2,1,1,1)$. Then

$$P \;=\; [v_{01}, v_{02}, v_{11}, v_{21}, v_1] \;=\; \begin{bmatrix} -l_{01} & -l_{02} & l_{11} & 0 & 0 \\ -l_{01} & -l_{02} & 0 & l_{21} & 0 \\ d_{011} & d_{021} & d_{111} & d_{211} & d_{11} \\ d_{012} & d_{022} & d_{112} & d_{212} & d_{12} \end{bmatrix}.$$

holds with pairwise different primitive columns $v_{01}, v_{02}, v_{11}, v_{21}$ and v_1 generating $\mathbb{Q}^4$ as a cone. We assume P to have ordered exponents. The maximal X-cones of the fan Σ of Z are given by

$$\sigma_{01} \;=\; \operatorname{cone}(v_{02}, v_{11}, v_{21}, v_1), \qquad \sigma_{02} \;=\; \operatorname{cone}(v_{01}, v_{11}, v_{21}, v_1),$$
$$\sigma_{1} \;=\; \operatorname{cone}(v_{01}, v_{02}, v_{11}, v_{21}), \qquad \tau_{0} \;=\; \operatorname{cone}(v_{01}, v_{02}, v_1).$$

We have $K = \mathbb{Z} \oplus \Gamma$ with the torsion part Γ and denote $\deg(T_{ij}) = (w_{ij}, \eta_{ij})$ as well as $\deg(T_k) = (w_k, \eta_k)$ accordingly. In particular, we write

$$Q^0 \;=\; [w_{01}, w_{02}, w_{11}, w_{21}, w_1]$$

for the free part of the degree matrix Q. Note that the vector $(w_{01}, w_{02}, w_{11}, w_{21}, w_1)$ is primitive in $\mathbb{Z}^5$ and generates $\ker(P)$.

Our first series of constraints arising from the log terminality and the Gorenstein property directly aims for entries of the defining matrix P.

Proposition 2.9.2. *Consider $X = X(A,P,\Sigma)$ as in Setting 2.9.1. Assume that X is non-toric, Fano, log-terminal and Gorenstein.*

(i) *The tuple of exponents (l_{01}, l_{11}, l_{21}) fits into precisely one of the following constellations:*

$$\begin{array}{llll} (1,x,y), & x \ge y > 1; & (2,z,3), & 3 \le z \le 5; \\ (y,2,2), & y \ge 2; & (3,z,2), & 3 \le z \le 5; \\ (2,y,2), & y \ge 3; & (z,3,2), & 4 \le z \le 5. \end{array}$$

(ii) *$-\mathcal{K} = (1-l_{01})D_{01}^X + (1-l_{02})D_{02}^X + D_{11}^X + D_{21}^X + D_1^X$ is an anticanonical divisor on X. In particular, the free part of the anticanonical class of X is given by*

$$w_X \;=\; (1-l_{01})w_{01} + (1-l_{02})w_{02} + w_{11} + w_{21} + w_1.$$

(iii) *Admissible row operations turn the defining matrix P into the shape*

$$P \;=\; \begin{bmatrix} -l_{01} & -l_{02} & l_{11} & 0 & 0 \\ -l_{01} & -l_{02} & 0 & l_{21} & 0 \\ 1-l_{01} & 1-l_{02} & d_{111} & d_{211} & 1 \\ d_{012} & d_{022} & d_{112} & d_{212} & 0 \end{bmatrix}, \qquad \begin{array}{l} l_{01} \ge l_{02} \ge 1, \;\; l_{11} \ge l_{21} \ge 2, \\ 0 \le d_{012} < l_{01}, \\ -\frac{w_X}{w_{02}} \le d_{022} < l_{02} + \frac{w_X}{w_{02}}, \\ 0 \le d_{211}, d_{212} < l_{21}, \end{array}$$

where $w_{02} \mid w_X$.

Proof. We prove (i). We apply Proposition 2.2.22 to the X-cone $\mathrm{cone}(v_{01}, v_{11}, v_{21})$ to see that (l_{01}, l_{11}, l_{21}) is a platonic tuple. As P has ordered exponents, we have $l_{11} \ge l_{21}$. Moreover, since X is non-toric, $l_{21} \ge 2$ holds. This leaves us with the six constellations for (l_{01}, l_{11}, l_{21}) stated in the assertion.

For the second assertion note that we have $r = 2$ and that the defining relation of the Cox ring is given as

$$g \;=\; T_{01}^{l_{01}} T_{02}^{l_{02}} + T_{11}^{l_{11}} + T_{21}^{l_{21}}.$$

Thus $\deg(g) = l_{01}\deg(T_{01}) + l_{02}\deg(T_{02})$ holds and Remark 2.2.13 shows that the anticanonical divisor $-\mathcal{K}$ is as claimed.

We prove (iii). We care about the entries of the (d, d')-block of P. Since $v_1 \in \mathbb{Z}^4$ is primitive, we can apply a suitable unimodular 2×2 matrix from the left to the (d, d') block to ensure

$$d_{11} \;=\; 1, \qquad d_{12} \;=\; 0.$$

We now make use of the assumption that X is Gorenstein. First consider the X-cone $\tau_0 = \mathrm{cone}(v_{01}, v_{02}, v_1)$. Then Corollary 2.2.19 provides a linear form $u \in \mathbb{Z}^4$ such that

$$\langle u, v_{01}\rangle \;=\; 1 - l_{01}, \qquad \langle u, v_{02}\rangle \;=\; 1 - l_{02}, \qquad \langle u, v_1\rangle \;=\; 1.$$

The last equation tells us in particular $u_3 = 1$. Plugging this into the first two equations yields

$$d_{011} \;=\; l_{01}(u_1 + u_2) - u_4 d_{012} + 1 - l_{01}, \qquad d_{021} \;=\; l_{02}(u_1 + u_2) - u_4 d_{022} + 1 - l_{02}.$$

Thus, adding the $(u_1 + u_2)$-fold of the first and the u_4-fold of the fourth row of P to the third one, we obtain

$$d_{011} \;=\; 1 - l_{01}, \qquad d_{021} \;=\; 1 - l_{02}.$$

Moreover, adding an appropriate multiple of the first row of P to the fourth one, we achieve

$$0 \;\le\; d_{012} \;<\; l_{01}.$$

Now consider the maximal X-cone $\sigma_{02} = \mathrm{cone}(v_{01}, v_{11}, v_{21}, v_1)$. Let $u \in \mathbb{Z}^4$ be a linear form representing D_0^X on $X_{\sigma_{02}}$ according to Corollary 2.2.19(*iii*). Then

$$0 \;=\; Q^0 \cdot P^* \cdot u \;=\; \sum \langle u, v_{ij}\rangle w_{ij} + \langle u, v_1\rangle w_1 \;=\; w_X + (u_4 d_{022} - l_{02}(u_1 + u_2))w_{02}.$$

In particular, we see that w_{02} divides w_X. Moreover, we must have $u_3 = 1$. We obtain

$$1 - l_{01} \;=\; \langle u, v_{01}\rangle \;=\; -l_{01}u_1 - l_{01}u_2 + 1 - l_{01} + u_4 d_{012}.$$

This merely means $l_{01}(u_1 + u_2) = u_4 d_{012}$. Plugging this into the previous equation yields

$$-l_{01}\frac{w_X}{w_{02}} \;=\; u_4(d_{022}l_{01} - d_{012}l_{02}).$$

Thus, $(d_{022}l_{01} - d_{012}l_{02})$ divides $l_{01}\frac{w_X}{w_{02}}$. As a consequence, we can estimate d_{022} as follows:

$$l_{02}\frac{d_{012}}{l_{01}} - \frac{w_X}{w_{02}} \;\le\; d_{022} \;\le\; l_{02}\frac{d_{012}}{l_{01}} + \frac{w_X}{w_{02}}.$$

Combining this with $0 \le d_{012} < l_{01}$, we arrive at the desired bounds for d_{022}. Finally, we achieve

$$0 \;\le\; d_{211}, d_{212} \;<\; l_{21}$$

by adding suitable multiples of the difference of the first two rows of P to third and the fourth one. $\square$

The second series of constraints shows that all entries of the $\mathbb{Z}$-part of the degree matrix $Q^0 = [w_{01}, w_{02}, w_{11}, w_{21}, w_1]$ are bounded.

Proposition 2.9.3. *Consider $X = X(A, P, \Sigma)$ as in Setting 2.9.1. Assume that X is non-toric, Fano, log-terminal and Gorenstein.*

(i) *For any three positive integers α_{01}, α_{02} and β_1 consider the 5×5*

$$G \;:=\; \begin{bmatrix} 1-l_{01}-\alpha_{01} & 1-l_{02} & 1 & 1 & 1 \\ 1-l_{01} & 1-l_{02}-\alpha_{02} & 1 & 1 & 1 \\ 1-l_{01} & 1-l_{02} & 1 & 1 & 1-\beta_1 \\ -l_{01} & -l_{02} & l_{11} & 0 & 0 \\ -l_{01} & -l_{02} & 0 & l_{21} & 0 \end{bmatrix}.$$

The matrix G is of rank at least four. Moreover, $\det(G) = 0$ holds if and only if $\alpha_{01}, \alpha_{02}, \beta_1$ and $l_{01}, l_{02}, l_{11}, l_{21}$ satisfy the identity

$$\frac{1}{\beta_1} + \frac{1}{\alpha_{01}} + \frac{1}{\alpha_{02}} + \left(\frac{l_{01}}{\alpha_{01}} + \frac{l_{02}}{\alpha_{02}}\right)\left(\frac{1}{l_{11}} + \frac{1}{l_{21}} - 1\right) \;=\; 1.$$

(ii) *There are unique $\alpha_{01}, \alpha_{02}, \beta_1 \in \mathbb{Z}_{\ge 1}$ with $\alpha_{01}w_{01} = \alpha_{02}w_{02} = \beta_1 w_1 = w_X$ and the corresponding matrix G from (i) satisfies*

$$\ker(G) \;=\; \ker(P) \;=\; \mathbb{Z} \cdot (w_{01}, w_{02}, w_{11}, w_{21}, w_1).$$

(iii) *According to the possible constellations of (l_{01}, l_{11}, l_{21}) from Proposition 2.9.2 (i) we have the following upper bounds on the entries of the matrix G from (ii):*

	l_{01}	l_{02}	l_{11}	l_{21}	α_{01}	α_{02}	β_1
$(1,x,y)$	1	1	84	8	42	4	36
$(y,2,2)$	11	6	2	2	6	6	6
$(2,y,2)$	2	2	24	2	28	35	1806
$(2,z,3)$	2	2	5	3	14	4	42
$(3,z,2)$	3	3	5	2	14	5	42
$(z,3,2)$	5	5	3	2	14	5	42

Proof. We prove (i). In order to see that G is of rank at least four, we just compute the minor obtained by deleting row 3 and column 1:

$$G_{3,1} \;=\; \det \begin{bmatrix} 1-l_{02} & 1 & 1 & 1 \\ 1-l_{02}-\alpha_{02} & 1 & 1 & 1 \\ -l_{02} & l_{11} & 0 & 0 \\ -l_{02} & 0 & l_{21} & 0 \end{bmatrix} \;=\; \alpha_{02} l_{11} l_{21} \;\neq\; 0.$$

Moreover, suitably rearranging the equation $\det(G) = 0$, we arrive at the displayed identity on $\alpha_{01}, \alpha_{02}, \beta_1$ and $l_{01}, l_{02}, l_{11}, l_{21}$.

We prove (ii). Applying Corollary 2.2.21 to the three maximal X-cones σ_{01}, σ_{02} and σ_1 shows that each of w_{01}, w_{02} and w_1 is a multiple of w_X and hence we obtain positive integers α_{01}, α_{02} and β_1 with

$$\alpha_{01}w_{01} \;=\; \alpha_{02}w_{02} \;=\; \beta_1 w_1 \;=\; (1-l_{01})w_{01} + (1-l_{02})w_{02} + w_{11} + w_{21} + w_1.$$

Moreover, by homogeneity of the defining relation g we have

$$l_{01}w_{01} + l_{02}w_{02} \;=\; l_{11}w_{11} \;=\; l_{21}w_{21}.$$

The matrix G is the coefficient matrix of the corresponding system of linear equations. In particular, $\ker(G)$ is generated by the primitive vector $(w_{01}, w_{02}, w_{11}, w_{21}, w_1) \in \mathbb{Z}^5$.

We turn to (iii). We treat the configuration $(l_{01}, l_{11}, l_{21}) = (1, x, y)$. In this case the identity from (i) reads

$$\frac{1}{\beta_1} + \left(\frac{1}{\alpha_{01}} + \frac{1}{\alpha_{02}}\right)\left(\frac{1}{x} + \frac{1}{y}\right) \;=\; 1. \tag{2.9.3.1}$$

Since $l_{01} = l_{02}$ holds, we may assume $\alpha_{01} \geq \alpha_{02}$. We immediately get the bounds

$$\beta_1 \geq 2, \qquad \alpha_{02} \leq 4, \qquad y \leq 8.$$

Restricting Equation 2.9.3.1 to partial sums, we obtain the inequalities

$$\begin{aligned} \frac{1}{\alpha_{02}x} + \frac{1}{\alpha_{02}y} + \frac{1}{\beta_1} &< 1, \\ \frac{1}{\alpha_{01}y} + \frac{1}{\alpha_{02}y} + \frac{1}{\beta_1} &< 1. \end{aligned}$$

In both cases, we can apply Lemma 2.3.4 (i), which tells us the sum on the left is at most $41/42$. This gives lower bounds on the parts of the sum in Equation 2.9.3.1 that were split off, which yields the bounds $\alpha_{01} \leq 42$ and $x \leq 84$. Finally, we solve Equation 2.9.3.1 for β_1 and check the possible values within the established bounds for x,y, α_{01} and α_{02} to obtain $\beta_1 \leq 36$.

We treat the configuration $(l_{01}, l_{11}, l_{21}) = (y, 2, 2)$. In this case the identity from (i) reads

$$\frac{1}{\alpha_{01}} + \frac{1}{\alpha_{02}} + \frac{1}{\beta_1} \;=\; 1.$$

This gives the bounds $\alpha_{01}, \alpha_{02}, \beta_1 \leq 6$. Additionally, since l_{11} equals l_{21} in this case, we have $w_{11} = w_{21}$. Applying Corollary 2.2.19 to the X-cone $\tau_0 = \operatorname{cone}(v_{01}, v_{02}, v_1)$, we see that there is $\gamma \in \mathbb{Z}_{\geq 1}$ with $\gamma w_{11} = \gamma w_{21} = w_X$. Homogeneity of the defining relation g yields

$$\frac{y}{\alpha_{01}} + \frac{l_{02}}{\alpha_{02}} \;=\; \frac{2}{\gamma}.$$

Using the bounds for α_{01} and α_{02}, we obtain $y \leq 11$ and $l_{02} \leq 6$.

We treat the configuration $(l_{01}, l_{11}, l_{21}) = (2, y, 2)$. In this case the identity from (i) reads

$$\frac{1}{\beta_1} + \frac{2 - l_{02}}{2\alpha_{02}} + \frac{1}{y}\left(\frac{2}{\alpha_{01}} + \frac{l_{02}}{\alpha_{02}}\right) \;=\; 1. \tag{2.9.3.2}$$

Note that the second summand is either 0 or a unit fraction. Moreover, each summand si positive. By splitting of the summands that contain y, respectively α_{01} and α_{02} in their denominators, we can apply Lemma 2.3.4 (i) to obtain

$$\frac{1}{y}\left(\frac{2}{\alpha_{01}} + \frac{l_{02}}{\alpha_{02}}\right) \;\geq\; \frac{1}{6}, \qquad \frac{2}{\alpha_{01} y} \;\geq\; \frac{1}{42}, \qquad \frac{(2 - l_{02})y + 2l_{02}}{2y\alpha_{02}} \;\geq\; \frac{1}{42}.$$

The first inequality gives the bound $y \leq 24$, the second gives $\alpha_{01} \leq 28$ and the third gives $\alpha_{02} \leq 35$. To get a bound on β_1, we expand the left hand side of Equation 2.9.3.2. It is a sum of at most five unit fractions. We apply Lemma 2.3.4 (ii) to obtain $\beta_1 \leq 1806$.

We treat the configuration $(l_{01}, l_{11}, l_{21}) = (2, z, 3)$. In this case the identity from (i) reads

$$\frac{1}{\beta_1} + \frac{6 - z}{3\alpha_{01} z} + \frac{3l_{02} + (3 - 2l_{02})z}{3\alpha_{02} z} \;=\; 1. \tag{2.9.3.3}$$

Note that we have $3 \leq z \leq 5$ and in this range $6 - z$ is a divisor of z. Thus the first two summands are unit fractions. We apply Lemma 2.3.4 (i) to obtain

$$\frac{3l_{02} + (3 - 2l_{02})z}{3\alpha_{02} z} \;\geq\; \frac{1}{6}.$$

Solving this for α_{02}, we get the bound $\alpha_{02} \leq 4$. Note that the third summand in Equation 2.9.3.3 is a sum of at most two unit fractions. Applying Lemma 2.3.4 (iii) to that sum thus yields

$$\frac{1}{\beta_1} \;\leq\; \frac{1}{42}, \qquad \frac{6 - z}{3\alpha_{01} z} \;\leq\; \frac{1}{42}.$$

From this we obtain the bounds $\alpha_{01} \leq 14$ and $\beta_1 \leq 42$.

We treat the configuration $(l_{01}, l_{11}, l_{21}) = (3, z, 2)$. In this case the identity from (i) reads

$$\frac{1}{\beta_1} + \frac{6 - z}{2\alpha_{01} z} + \frac{2l_{02} + (2 - l_{02})z}{2\alpha_{02} z} \;=\; 1.$$

Applying exactly the same arguments as for the previous configuration, we now get the bounds $\alpha_{01} \leq 14$, $\alpha_{02} \leq 5$ and $\beta_1 \leq 42$.

We treat the configuration $(l_{01}, l_{11}, l_{21}) = (z, 3, 2)$. In this case the identity from (i) reads

$$\frac{1}{\beta_1} + \frac{6 - z}{6\alpha_{01}} + \frac{6 - l_{02}}{6\alpha_{02}} \;=\; 1.$$

Again, the same arguments apply as for the previous two exponent configurations, now giving the bounds $\alpha_{01} \leq 14$, $\alpha_{02} \leq 5$ and $\beta \leq 42$. $\square$

Corollary 2.9.4. *There is a list of 155 explicitly given generator matrices P of format $(2,1,1,1)$ each of them defining a non-toric $\mathbb{Q}$-factorial, Gorenstein, log terminal Fano threefold $X(A,P,\Sigma)$ of Picard number one.*

	$\mathbb{Z}$	$\mathbb{Z}+\mathbb{Z}_2$	$\mathbb{Z}+\mathbb{Z}_3$	$\mathbb{Z}+\mathbb{Z}_4$	$\mathbb{Z}+\mathbb{Z}_5$	$\mathbb{Z}+\mathbb{Z}_6$	$\mathbb{Z}+\mathbb{Z}_2^2$	$\mathbb{Z}+\mathbb{Z}_2+\mathbb{Z}_4$	$\mathbb{Z}+\mathbb{Z}_2^3$	sum
$(1,x,y)$	13	19	6	4	1	3	4	2		52
$(y,2,2)$		12		3			9		1	25
$(2,y,2)$	13	24					8		1	46
$(z,3,2)$	10	7					1			18
$(2,z,3)$	2		3			1				6
$(3,z,2)$	2	5					1			8
sum	40	67	9	7	1	4	23	2	2	155

Number of members P of the list according to divisor class group and exponent configuration

Distinct matrices from the list yield non-isomorphic varieties and every non-toric, $\mathbb{Q}$-factorial, log-terminal and Gorenstein Fano threefold of Picard number one of format $(2,1,1,1)$ is isomorphic to an $X = X(A,P,\Sigma)$ with P from the list.

Proof. Proposition 2.9.3 allows us to write down explicitly all possible matrices G and hence to determine all possible $Q^0 = [w_{01}, w_{02}, w_{11}, w_{21}, w_1]$ by computer. Now, recall that P annihilates the transpose of Q^0. This enables us to determine in the matrix P, adjusted according to Proposition 2.9.2 (iii), all the remaining variables. So, we are left with a finite list of explicitly given possible defining matrices P. Checking for the necessary properties by means of [43] and reducing via Proposition 2.3.1 to data defining pairwise non-isomorphic varieties, we arrive a the list presented in the assertion. □

2.10 Proof of Classification 2.1.1: Case 7 - format $(2,1,1,1,1)$

Proposition 2.2.24 divides the proof of the classification theorem 2.1.1 into cases (i) to (viii). In this section we treat case (vii). The setting is as follows.

Setting 2.10.1. Let $X = X(A,P,\Sigma)$ a $\mathbb{Q}$-factorial threefold of Picard number one of format $(2,1,1,1,1)$. Then

$$P = [v_{01}, v_{02}, v_{11}, v_{21}, v_{31}, v_1] = \begin{bmatrix} -l_{01} & -l_{02} & l_{11} & 0 & 0 & 0 \\ -l_{01} & -l_{02} & 0 & l_{21} & 0 & 0 \\ -l_{01} & -l_{02} & 0 & 0 & l_{31} & 0 \\ d_{011} & d_{021} & d_{111} & d_{211} & d_{311} & d_{11} \\ d_{012} & d_{022} & d_{112} & d_{212} & d_{312} & d_{12} \end{bmatrix}$$

holds with pairwise different primitive columns $v_{01}, v_{02}, v_{11}, v_{21}, v_{31}$ and v_1 generating $\mathbb{Q}^5$ as a cone. We assume P to have ordered exponents. The maximal X-cones of the fan Σ of Z are given by

$$\sigma_{01} = \operatorname{cone}(v_{02}, v_{11}, v_{21}, v_{31}, v_1), \qquad \sigma_{02} = \operatorname{cone}(v_{01}, v_{11}, v_{21}, v_{31}, v_1),$$

$$\sigma_1 \ = \ \mathrm{cone}(v_{01}, v_{02}, v_{11}, v_{21}, v_{31}), \qquad \tau_0 \ = \ \mathrm{cone}(v_{01}, v_{02}, v_1).$$

We have $K = \mathbb{Z} \oplus \Gamma$ with the torsion part Γ and denote $\deg(T_{ij}) = (w_{ij}, \eta_{ij})$ as well as $\deg(T_k) = (w_k, \eta_k)$ accordingly. In particular, we write

$$Q^0 \ = \ [w_{01}, w_{02}, w_{11}, w_{21}, w_{31}, w_1]$$

for the free part of the degree matrix Q. Note that the vector $(w_{01}, w_{02}, w_{11}, w_{21}, w_{31}, w_1)$ is primitive in $\mathbb{Z}^6$ and generates $\ker(P)$.

Our first series of constraints arising from the log terminality and the Gorenstein property directly aims for entries of the defining matrix P.

Proposition 2.10.2. *Consider $X = X(A, P, \Sigma)$ as in Setting 2.10.1. Assume that X is non-toric, Fano, log-terminal and Gorenstein.*

(i) *We have $l_{01} = l_{02} = 1$. Moreover, the tuple of exponents $(l_{01}, l_{11}, l_{21}, l_{31})$ fits into precisely one of the following constellations:*

$$(1, y, 2, 2), \quad y \geq 2, \qquad\qquad (1, z, 3, 2), \quad 3 \leq z \leq 5.$$

(ii) *$-\mathcal{K} = (1-l_{11})D_{11}^X + D_{21}^X + D_{31}^X + D_1^X$ is an anticanonical divisor on X. In particular, the free part of the anticanonical class of X is given by*

$$w_X \ = \ (1 - l_{11})w_{11} + w_{21} + w_{31} + w_1.$$

(iii) *Admissible row operations turn the defining matrix P into the form*

$$P \ = \ \begin{bmatrix} -1 & -1 & l_{11} & 0 & 0 & 0 \\ -1 & -1 & 0 & l_{21} & 0 & 0 \\ -1 & -1 & 0 & 0 & l_{31} & 0 \\ 0 & 0 & d_{111} & d_{211} & d_{311} & 1 \\ 0 & d_{022} & d_{112} & d_{212} & d_{312} & 0 \end{bmatrix}, \qquad \begin{array}{l} l_{11} \geq l_{21} \geq l_{31} > 1, \\ 0 < d_{022} \leq \frac{w_X}{w_{02}}, \\ 0 \leq d_{211}, d_{212} < l_{21}, \\ 0 \leq d_{311}, d_{312} < l_{31}, \end{array}$$

where $w_{02} \mid w_X$.

Proof. We prove (i). We apply Proposition 2.2.22 to the X-cone $\mathrm{cone}(v_{01}, v_{11}, v_{21}, v_{31})$ to see that $(l_{01}, l_{11}, l_{21}, l_{31})$ is a platonic tuple. As P has ordered exponents, we have $l_{11} \geq l_{21}$ and $l_{21} \geq l_{31}$. Moreover, since X is non-toric, $l_{31} \geq 2$ holds. Thus we have $l_{01} = 1$ and consequently $l_{02} = 1$. This leaves us with the two constellations for $(l_{01}, l_{11}, l_{21}, l_{31})$ stated in the assertion. Item (ii) follows immediately from Remark 2.2.13 and homogeneity of the defining relations g_0 and g_1.

We prove (iii). Multiplying the (d, d')-block by a suitable unimodular 2×2 matrix, we may assume $d_{11} = 1$ and $d_{12} = 0$. Adding multiples of the first row of P to the fourth and fifth, we achieve $d_{011} = d_{012} = 0$. We make use of the Gorenstein property. Consider the X-cone $\tau_0 = \mathrm{cone}(v_{01}, v_{02}, v_1)$. By Corollary 2.2.19 there is a linear form $u \in \mathbb{Z}^5$ with

$$\langle u, v_{01} \rangle \ = \ 0, \qquad \langle u, v_{21} \rangle \ = \ 0, \qquad \langle u, v_1 \rangle \ = \ 1.$$

Equations one and three yield $u_1+u_2+u_3=0$ and $u_4=1$. Plugging this into the second equation, we see that d_{022} is a divisor of d_{021}. Note that linear independence of v_{01} and v_{02} demands that d_{022} is not zero. We add the u_4-fold of the fifth row of P to the fourth row to achieve $d_{021}=0$. By multiplying the last row of P by -1 if necessary, we may assume $d_{022}>0$. We add an appropriate multiple of the difference of the first and second row to the fourth and fifth to achieve $0 \le d_{211}, d_{212} < l_{21}$. Doing the same for the first and third row yields $0 \le d_{311}, d_{312} < l_{31}$. Consider the X-cone $\sigma_{02} = \operatorname{cone}(v_{01}, v_{11}, v_{21}, v_{31}, v_1)$. By Lemma 2.3.2 we have $w_{02} \mid w_X$ and there is a linear form $u \in \mathbb{Z}^5$ with

$$\begin{array}{rclrcl} \langle u, v_{01}\rangle & = & 0, & \langle u, v_{02}\rangle & = & -\frac{w_X}{w_{02}}, \\ \langle u, v_{11}\rangle & = & 1-l_{11}, & \langle u, v_{21}\rangle & = & 1, \\ \langle u, v_{31}\rangle & = & 1, & \langle u, v_{1}\rangle & = & 1. \end{array}$$

The first and second equation show that d_{022} is a divisor of w_X/w_{02}, which established the bound on d_{022}. □

Our second series of constraints shows that all entries of the $\mathbb{Z}$-part of the degree matrix $Q^0 = [w_{01}, w_{02}, w_{11}, w_{21}, w_{31}, w_1]$ are bounded.

Proposition 2.10.3. *Consider $X = X(A, P, \Sigma)$ as in Setting 2.10.1. Assume that X is non-toric, Fano, log-terminal and Gorenstein.*

(i) *For any three positive integers α_{01}, α_{02} and β_1 consider the 6×6 matrix*

$$G := \begin{bmatrix} -\alpha_{01} & 0 & 1-l_{11} & 1 & 1 & 1 \\ 0 & -\alpha_{02} & 1-l_{11} & 1 & 1 & 1 \\ 0 & 0 & 1-l_{11} & 1 & 1 & 1-\beta_1 \\ -1 & -1 & l_{11} & 0 & 0 & 0 \\ -1 & -1 & 0 & l_{21} & 0 & 0 \\ -1 & -1 & 0 & 0 & l_{31} & 0 \end{bmatrix}.$$

The matrix G is of rank at least five. Moreover, we have $\det(G)=0$ if and only if $\alpha_{01}, \alpha_{02}, \beta_1$ and l_{11}, l_{21}, l_{31} satisfy the identity

$$\frac{1}{\beta_1} + \left(\frac{1}{\alpha_{01}} + \frac{1}{\alpha_{02}}\right)\left(\frac{1}{l_{11}} + \frac{1}{l_{21}} + \frac{1}{l_{31}} - 1\right) = 1.$$

(ii) *There are unique $\alpha_{01}, \alpha_{02}, \beta_1 \in \mathbb{Z}_{\ge 1}$ with $\alpha_{01}w_{01} = \alpha_{02}w_{02} = \beta_1 w_1 = w_X$, and the corresponding matrix G from (i) satisfies*

$$\ker(G) = \ker(P) = \mathbb{Z} \cdot (w_{01}, w_{02}, w_{11}, w_{21}, w_{31}, w_1).$$

(iii) *According to the possible constellations of the exponents $(l_{01}, l_{11}, l_{21}, l_{31})$ from Proposition 2.10.2 (i) we have the following upper bounds on the entries of the matrix G from (ii). An empty line indicates that this exponent configuration does not occur.*

	l_{01}	l_{02}	l_{11}	l_{21}	l_{31}	α_{01}	α_{02}	β_1
$(1, y, 2, 2)$	1	1	4	2	2	3	2	4
$(1, z, 3, 2)$								

Proof. We prove (i). In order to see that G is of rank at least five, we just compute the minor obtained by deleting row 3 and column 1:

$$G_{3,1} \;=\; \det \begin{bmatrix} 0 & 1-l_{11} & 1 & 1 & 1 \\ -\alpha_{02} & 1-l_{11} & 1 & 1 & 1 \\ -1 & l_{11} & 0 & 0 & 0 \\ -1 & 0 & l_{21} & 0 & 0 \\ -1 & 0 & 0 & l_{31} & 0 \end{bmatrix} \;=\; \alpha_{02}l_{11}l_{21}l_{31} \;\neq\; 0.$$

Moreover, suitably rearranging the equation $\det(G) = 0$, we arrive at the displayed identity on $\alpha_{01}, \alpha_{02}, \beta_1$ and l_{11}, l_{21}, l_{31}.

We prove (ii). Applying Corollary 2.2.21 to the three maximal X-cones σ_{01}, σ_{02} and σ_1 shows that each of w_{01}, w_{02} and w_1 is a multiple of w_X and hence we obtain positive integers α_{01}, α_{02} and β_1 with

$$\alpha_{01}w_{01} \;=\; \alpha_{02}w_{02} \;=\; \beta_1 w_1 \;=\; (1-l_{11})w_{11} + w_{21} + w_{31} + w_1.$$

Moreover, by homogeneity of the defining relations g_0 and g_1 we have

$$w_{01} + w_{02} \;=\; l_{11}w_{11} \;=\; l_{21}w_{21} \;=\; l_{31}w_{31}.$$

The matrix G is the coefficient matrix of the corresponding system of linear equations. In particular, $\ker(G)$ is generated by the primitive vector $(w_{01}, w_{02}, w_{11}, w_{21}, w_{31}, w_1) \in \mathbb{Z}^6$.

We prove (iii). We treat the configuration $(l_{01}, l_{11}, l_{21}, l_{31}) = (1, y, 2, 2)$. In this case the identity from (i) reads

$$\frac{1}{\beta_1} + \frac{1}{y}\left(\frac{1}{\alpha_{01}} + \frac{1}{\alpha_{02}}\right) \;=\; 1.$$

Since $l_{01} = l_{02}$ holds, we may assume $\alpha_{01} \geq \alpha_{02}$. We immediately get the bounds $\alpha_{02} \leq 2$ and $y \leq 4$. Expanding the left hand side, we see that it is a sum of three unit fractions. Applying 2.3.4 (ii) shows that the denominator of each summand is at most 6. Taking into account $y \geq 2$, this gives the bounds $\alpha_{01} \leq 3$ and $\beta_1 \leq 6$.

We treat the configuration $(l_{01}, l_{11}, l_{21}, l_{31}) = (1, z, 3, 2)$. In this case the identity from (i) reads

$$\frac{1}{\beta_1} + \frac{6-z}{6z}\left(\frac{1}{\alpha_{01}} + \frac{1}{\alpha_{02}}\right) \;=\; 1.$$

Note that $6 - z$ is a divisor of z for $3 \leq z \leq 5$. Thus the left hand side is a sum of three unit fractions, two of them have denominator at least 6. This cannot add up to one. Thus this exponent configuration does not occur. □

Corollary 2.10.4. *There is a list of* 5 *explicitly given generator matrices* P *of format* $(2,1,1,1,1)$ *each of them defining a non-toric* $\mathbb{Q}$*-factorial, Gorenstein, log terminal Fano threefold* $X(A,P,\Sigma)$ *of Picard number one.*

	$\mathbb{Z}+\mathbb{Z}_2$	$\mathbb{Z}+\mathbb{Z}_2^2$	$\mathbb{Z}+\mathbb{Z}_2^3$	sum
$(1,y,2,2)$	1	3	1	5

Number of members P of the list according to divisor class group and exponent configuration

Distinct matrices from the list yield non-isomorphic varieties and every non-toric, $\mathbb{Q}$*-factorial, log-terminal and Gorenstein Fano threefold of Picard number one of format* $(2,1,1,1,1)$ *is isomorphic to an* $X = X(A,P,\Sigma)$ *with* P *from the list.*

Proof. Proposition 2.10.3 allows us to write down explicitly all possible matrices G and hence to determine all possible $Q^0 = [w_{01}, w_{02}, w_{11}, w_{21}, w_{31}, w_1]$ by computer. Now, recall that P annihilates the transpose of Q^0. This enables us to determine in the matrix P, adjusted according to Proposition 2.10.2 (iii), all the remaining variables. So, we are left with a finite list of explicitly given possible defining matrices P. Checking for the necessary properties by means of [43] and reducing via Proposition 2.3.1 to data defining pairwise non-isomorphic varieties, we arrive a the list presented in the assertion. $\square$

2.11 Proof of Classification 2.1.1: Case 8 - format $(1,1,1,2)$

Proposition 2.2.24 divides the proof of the classification theorem 2.1.1 into cases (i) to (viii). In this section we treat case (viii). The setting is as follows.

Setting 2.11.1. Let $X = X(A,P,\Sigma)$ a $\mathbb{Q}$-factorial threefold of Picard number one of format $(1,1,1,2)$. Then

$$P = [v_{01}, v_{11}, v_{21}, v_1, v_2] = \begin{bmatrix} -l_{01} & l_{11} & 0 & 0 & 0 \\ -l_{01} & 0 & l_{21} & 0 & 0 \\ d_{011} & d_{111} & d_{211} & d_{11} & d_{21} \\ d_{012} & d_{112} & d_{212} & d_{12} & d_{22} \end{bmatrix}$$

holds with pairwise different primitive columns $v_{01}, v_{11}, v_{21}, v_1$ and v_2 generating $\mathbb{Q}^4$ as a cone. We assume P to have ordered exponents. The maximal X-cones of the fan Σ of Z are given by

$$\sigma_1 = \operatorname{cone}(v_{01}, v_{11}, v_{21}, v_2), \qquad \sigma_2 = \operatorname{cone}(v_{01}, v_{11}, v_{21}, v_1),$$

$$\tau_0 = \operatorname{cone}(v_{01}, v_1, v_2), \quad \tau_1 = \operatorname{cone}(v_{11}, v_1, v_2), \quad \tau_2 = \operatorname{cone}(v_{21}, v_1, v_2).$$

We have $K = \mathbb{Z} \oplus \Gamma$ with the torsion part Γ and denote $\deg(T_{ij}) = (w_{ij}, \eta_{ij})$ as well as $\deg(T_k) = (w_k, \eta_k)$ accordingly. In particular, we write

$$Q^0 = [w_{01}, w_{11}, w_{21}, w_1, w_2]$$

for the free part of the degree matrix Q. Note that the vector $(w_{01}, w_{11}, w_{21}, w_1, w_2)$ is primitive in $\mathbb{Z}^5$ and generates $\ker(P)$.

Proposition 2.11.2. *Consider $X = X(A, P, \Sigma)$ as in Setting 2.11.1. Assume that X is non-toric, Fano, log-terminal and Gorenstein.*

(i) *The tuple of exponents (l_{01}, l_{11}, l_{21}) fits into precisely one of the following constellations:*

$$(y, 2, 2), \quad y \geq 2, \qquad\qquad (z, 3, 2), \quad 3 \leq z \leq 5.$$

(ii) *$-\mathcal{K} = (1 - l_{01})D_{01}^X + D_{11}^X + D_{21}^X + D_1^X + D_2^X$ is an anticanonical divisor on X. In particular, the free part of the anticanonical class of X is given by*

$$w_X \; = \; (1 - l_{01})w_{01} + w_{11} + w_{21} + w_1 + w_2.$$

(iii) *Admissible row operations turn the defining matrix P into the form*

$$P \; = \; \begin{bmatrix} -l_{01} & l_{11} & 0 & 0 & 0 \\ -l_{01} & 0 & l_{21} & 0 & 0 \\ d_{011} & 1 & 1 & 1 & 1 \\ d_{012} & d_{112} & d_{212} & 0 & d_{22} \end{bmatrix}, \qquad \begin{array}{l} l_{01} \geq l_{11} \geq l_{21} \geq 2, \\ 0 \leq d_{112} < l_{11}, \\ 0 \leq d_{212} < l_{21}, \\ 0 < d_{22} \leq \frac{w_X}{w_2}, \end{array}$$

where $w_2 \mid w_X$.

Proof. We prove (i). We apply Proposition 2.2.22 to the X-cone $\mathrm{cone}(v_{01}, v_{11}, v_{21})$ to see that (l_{01}, l_{11}, l_{21}) is a platonic tuple. As P has ordered exponents, we have $l_{01} \geq l_{11}$ and $l_{11} \geq l_{21}$. Moreover, since X is non-toric, $l_{21} \geq 2$ holds. This leaves us with the two constellations for (l_{01}, l_{11}, l_{21}) stated in the assertion.

For the second assertion note that we have $r = 2$ and that the defining relation of the Cox ring is given as

$$g \; = \; T_{01}^{l_{01}} + T_{11}^{l_{11}} + T_{21}^{l_{21}}.$$

Thus $\deg(g) = l_{01} \deg(T_{01})$ holds and Remark 2.2.13 shows that the anticanonical divisor $-\mathcal{K}$ is as claimed.

We prove (iii). Applying a suitable unimodular 2×2 matrix to the (d, d')-block, we may assume $d_{11} = 1$ and $d_{12} = 0$. We make use of the Gorenstein property. Consider the X-cone $\tau_2 = \mathrm{cone}(v_{21}, v_1, v_2)$. There is a linear form $u \in \mathbb{Z}^4$ with

$$\langle u, v_{21} \rangle \; = \; 1, \qquad \langle u, v_1 \rangle \; = \; 1, \qquad \langle u, v_2 \rangle \; = \; 1.$$

The second equation shows that $u_3 = 1$ holds. The other two equations the read

$$\begin{aligned} 1 \; &= \; u_2 l_{21} + d_{211} + u_4 d_{212}, \\ 1 \; &= \; d_{21} + u_4 d_{22}. \end{aligned}$$

By adding the u_2-fold of the second row of P and the u_4-fold of the fourth row to the third, we achieve $d_{211} = 1$ and $d_{21} = 1$. Note that linear independence of v_1 and v_2 demands that d_{22} is not zero. Multiplying the last row of P by -1 if necessary, we may assum that d_{22} is positive. Now consider the X-cone $\tau_1 = \mathrm{cone}(v_{11}, v_1, v_2)$. Again, there is a linear form $u \in \mathbb{Z}^4$ with

$$\langle u, v_{21} \rangle \; = \; 1, \qquad \langle u, v_1 \rangle \; = \; 1, \qquad \langle u, v_2 \rangle \; = \; 1.$$

The second and third equation yield $u_3 = 1$ and $u_4 = 0$. The first equation thus reads

$$1 \;=\; u_1 l_{11} + d_{111}.$$

Thus, adding the u_1-fold of the first equation of P to the third, we achieve $d_{111} = 1$. Finally consider the X-cone $\sigma_2 = \operatorname{cone}(v_{01}, v_{11}, v_{21}, v_1)$. By Lemma 2.3.2 we have $w_2 \mid w_X$ and there is a linear form $u \in \mathbb{Z}^4$ with

$$\begin{array}{rclrcl} \langle u, v_{01}\rangle & = & 1 - l_{01}, & \langle u, v_{11}\rangle & = & 1, \\ \langle u, v_{21}\rangle & = & 1, & \langle u, v_1\rangle & = & 1, \\ \langle u, v_2\rangle & = & 1 - \frac{w_X}{w_2}. & & & \end{array}$$

Combining the fourth and fifth equation shows that d_{22} is a divisor of w_X/w_2. In particular, we get the bound

$$1 \;\le\; d_{22} \;\le\; \frac{w_X}{w_2}.$$

Finally, we add multiples of the first and the second row of P to the last row to achieve $0 \le d_{112} < l_{11}$ and $0 \le d_{212} < l_{21}$. □

Our second series of constraints shows that all entries of the $\mathbb{Z}$-part of the degree matrix $Q^0 = [w_{01}, w_{11}, w_{21}, w_1, w_2]$ are bounded.

Proposition 2.11.3. *Consider $X = X(A, P, \Sigma)$ as in Setting 2.11.1. Assume that X is non-toric, Fano, log-terminal and Gorenstein.*

(i) *For any three positive integers α_{01}, β_1 and β_2 consider the 5×5 matrix*

$$G \;:=\; \begin{bmatrix} 1 - l_{01} - \alpha_{01} & 1 & 1 & 1 & 1 \\ 1 - l_{01} & 1 & 1 & 1 - \beta_1 & 1 \\ 1 - l_{01} & 1 & 1 & 1 & 1 - \beta_2 \\ -l_{01} & l_{11} & 0 & 0 & 0 \\ -l_{01} & 0 & l_{21} & 0 & 0 \end{bmatrix}.$$

The matrix G is of rank at least five. Moreover, we have $\det(G) = 0$ if and only if $\alpha_{01}, \beta_1, \beta_2$ and l_{01}, l_{11}, l_{21} satisfy the identity

$$\frac{1}{\alpha_{01}} + \frac{1}{\beta_1} + \frac{1}{\beta_2} + \frac{l_{01}}{\alpha_{01}}\left(\frac{1}{l_{11}} + \frac{1}{l_{21}} - 1\right) \;=\; 1.$$

(ii) *There are unique $\alpha_{01}, \beta_1, \beta_2 \in \mathbb{Z}_{\ge 1}$ with $\alpha_{01} w_{01} = \beta_1 w_1 = \beta_2 w_2 = w_X$, and the corresponding matrix G from (i) satisfies*

$$\ker(G) \;=\; \ker(P) \;=\; \mathbb{Z} \cdot (w_{01}, w_{11}, w_{21}, w_1, w_2).$$

(iii) *According to the possible constellations of (l_{01}, l_{11}, l_{21}) from Proposition 2.11.2 (i) we have the following upper bounds on the entries of the matrix G from (ii):*

	l_{01}	l_{11}	l_{21}	α_{01}	β_1	β_2
$(y, 2, 2)$	12	2	2	6	6	6
$(z, 3, 2)$	5	3	2	3	6	6

Proof. We prove (i). In order to see that G is of rank at least four, we just compute the minor obtained by deleting row 3 and column 1:

$$G_{3,1} \;=\; \det \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1-\beta_1 & 1 \\ l_{11} & 0 & 0 & 0 \\ 0 & l_{21} & 0 & 0 \end{bmatrix} \;=\; \beta_1 l_{11} l_{21} \;\neq\; 0.$$

Moreover, suitably rearranging the equation $\det(G) = 0$, we arrive at the displayed identity on $\alpha_{01}, \beta_1, \beta_2$ and l_{01}, l_{11}, l_{21}.

We prove (ii). Applying Corollary 2.2.21 to the maximal X-cones σ_1 and σ_2 shows that each of w_1 and w_2 is a multiple of w_X and hence we obtain positive integers β_1 and β_2 with

$$\beta_1 w_1 \;=\; \beta_2 w_2 \;=\; (1 - l_{01})w_{01} + w_{11} + w_{21} + w_1 + w_2.$$

By applying admissible row operations to the matrix P, we may assume that it is of the form presented in Proposition 2.11.2 (iii). Note that P annihilates the transpose of Q^0. Thus from the third row of P, we obtain the identity

$$d_{011}w_{01} + w_{11} + w_{21} + w_1 + w_2 \;=\; 0.$$

Set $\alpha_{01} := 1 - l_{01} - d_{011}$. Then we obtain the identity

$$\alpha_{01}w_{01} \;=\; (1 - l_{01})w_{01} + w_{11} + w_{21} + w_1 + w_2.$$

Moreover, by homogeneity of the defining relation g we have

$$w_{01} + w_{02} \;=\; l_{11}w_{11} \;=\; l_{21}w_{21} \;=\; l_{31}w_{31}.$$

Now, G from (i) is the coefficient matrix of the corresponding system of linear equations. In particular, for any choice of α_{01}, β_1 and β_2 the integral matrix G has kernel generated by the primitive vector $(w_{01}, w_{11}, w_{21}, w_1, w_2) \in \mathbb{Z}^5$.

We prove (iii). We treat the configuration $(l_{01}, l_{11}, l_{21}) = (y, 2, 2)$. In this case the identity from (i) reads

$$\frac{1}{\alpha_{01}} + \frac{1}{\beta_1} + \frac{1}{\beta_2} \;=\; 1.$$

We apply Lemma 2.3.4 (ii) to get the bounds $\alpha_{01} \leq 6$, $\beta_1 \leq 6$ and $\beta_2 \leq 6$. Additionally, since $l_{11} = l_{21}$ holds, we have $w_{11} = w_{21}$. Thus, applying Corollary 2.2.21 to the X-cone $\tau_0 = \operatorname{cone}(v_{01}, v_1, v_2)$, we see that there is $\gamma \in \mathbb{Z}_{\geq 1}$ with $\gamma w_{11} = \gamma w_{21} = w_X$. Homogeneity of the defining relation g yields

$$\frac{y}{\alpha_{01}} \;=\; \frac{2}{\gamma}.$$

Using the bound on α_{01}, we obtain $y \leq 12$.

We treat the configuration $(l_{01}, l_{11}, l_{21}) = (z, 3, 2)$. In this case the identity from (i) reads

$$\frac{6-z}{6\alpha_{01}} + \frac{1}{\beta_1} + \frac{1}{\beta_2} \;=\; 1.$$

Note that $6 - z$ is a divisor of 6 for $3 \leq z \leq 5$. Thus the left hand side is a sum of three unit fractions. By Lemma 2.3.4 (ii) we obtain the bounds $\alpha_{01} \leq 3$, $\beta_1 \leq 6$ and $\beta_2 \leq 6$. □

Corollary 2.11.4. *There is a list of* 17 *explicitly given generator matrices P of format* $(1,1,1,2)$ *each of them defining a non-toric $\mathbb{Q}$-factorial, Gorenstein, log terminal Fano threefold $X(A, P, \Sigma)$ of Picard number one.*

	$\mathbb{Z}+\mathbb{Z}_2$	$\mathbb{Z}+\mathbb{Z}_3$	$\mathbb{Z}+\mathbb{Z}_2^2$	$\mathbb{Z}+\mathbb{Z}_2^3$	sum
$(y,2,2)$	2		11	3	16
$(z,3,2)$		1			1
sum	2	1	11	3	17

Number of members P of the list according to divisor class group and exponent configuration

Distinct matrices from the list yield non-isomorphic varieties and every non-toric, $\mathbb{Q}$-factorial, log-terminal and Gorenstein Fano threefold of Picard number one of format $(1,1,1,2)$ *is isomorphic to an $X = X(A, P, \Sigma)$ with P from the list.*

Proof. Proposition 2.11.3 allows us to write down explicitly all possible matrices G and hence to determine all possible $Q^0 = [w_{01}, w_{11}, w_{21}, w_1, w_2]$ by computer. Now, recall that P annihilates the transpose of Q^0. This enables us to determine in the matrix P, adjusted according to Proposition 2.11.2 (iii), all the remaining variables. So, we are left with a finite list of explicitly given possible defining matrices P. Checking for the necessary properties by means of [43] and reducing via Proposition 2.3.1 to data defining pairwise non-isomorphic varieties, we arrive a the list presented in the assertion. □

2.12 Classification lists

Here we provide the detailed presentation of our classification result. Let us briefly recall the background. Each non-toric, $\mathbb{Q}$-factorial, Gorenstein, log terminal Fano threefold X of Picard number one coming with an effective action of a two-dimensional torus is uniquely determined by its Cox ring. In particular, X can be encoded by the degree matrix Q, that means the list of degrees of the Cox ring generators in $\mathrm{Cl}(X)$ and the defining trinomial relations $g_0, \ldots, g_{r-1}$. For instance, our example variety X from Examples 2.2.3, 2.2.9, 2.2.16 and 2.2.20 is encoded by

$$Q = \begin{bmatrix} 2 & 2 & 1 & 2 & 1 \\ \bar{1} & \bar{1} & \bar{0} & \bar{0} & \bar{0} \\ \bar{1} & \bar{1} & \bar{1} & \bar{1} & \bar{0} \end{bmatrix}, \qquad g_0 = T_1T_2 + T_3^4 + T_4^2,$$

where the columns of Q live in $\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$. Indeed, the defining matrix P is determined up to admissible operations by Q, the format $(2,1,1,1)$ and the list of exponents of g_0. Alternatively, X is the hypersurface defined by g_0 in the fake weighted projective space $Z = \hat{Z}/H$, where $\hat{Z} = \mathbb{K}^5 \setminus \{0\}$ and the quasitorus H and its action on $\mathbb{K}^5$ are given by

$$H = \mathbb{K}^* \times \{\pm 1\} \times \{\pm 1\}, \qquad (t, \zeta, \eta) \cdot z = (t^2\zeta\eta z_1, t^2\zeta\eta z_2, t\eta z_3, t^2\eta z_4, tz_5).$$

We turn to the classification lists. Every non-toric, $\mathbb{Q}$-factorial, Gorenstein, log terminal Fano threefold X of Picard number one coming with an effective action of a two-dimensional torus is isomorphic to precisely one of the listed varieties. Conversely, each of the listed data defines a non-toric, $\mathbb{Q}$-factorial, Gorenstein, log terminal Fano threefold X of Picard number one coming with an effective action of a two-dimensional torus.

Each of the lists represents a possible divisor class group and format. Each variety in such a list is specified by its matrix Q of generator degrees and its defining trinomial relations; observe that we number the variables of the relation conventionally and not in accordance with the double-indexed enumeration of the columns of the associated defining matrix P. Besides the specifying data, we list the anticanonical self intersection. A file containing also the defining matrices P and further invariants is available at [15].

Classification list 2.12.1. *Non-toric, $\mathbb{Q}$-factorial, Gorenstein, log terminal Fano threefolds of Picard number one with an effective two-torus action: Specifying data for divisor class group $\mathbb{Z}$ and format $(2,2,1,0)$.*

ID	relations	gd-matrix	$-\mathcal{K}^3$
10	$T_1^5T_2+T_3^3T_4^3+T_5^2$	$\begin{bmatrix}1&1&1&1&3\end{bmatrix}$	2
11	$T_1^4T_2^3+T_3^3T_4^2+T_5^2$	$\begin{bmatrix}1&2&2&2&5\end{bmatrix}$	2
14	$T_1^5T_2+T_3^3T_4^2+T_5^2$	$\begin{bmatrix}2&4&4&1&7\end{bmatrix}$	4
15	$T_1^3T_2^2+T_3^3T_4^2+T_5^2$	$\begin{bmatrix}2&4&4&1&7\end{bmatrix}$	4
16	$T_1^3T_2^2+T_3^3T_4^2+T_5^2$	$\begin{bmatrix}2&1&2&1&4\end{bmatrix}$	4
17	$T_1^5T_2^3+T_3^3T_4+T_5^2$	$\begin{bmatrix}1&1&2&2&4\end{bmatrix}$	4
18	$T_1^3T_2+T_3^3T_4^2+T_5^2$	$\begin{bmatrix}4&2&4&1&7\end{bmatrix}$	4
31	$T_1^3T_2+T_3^3T_4+T_5^2$	$\begin{bmatrix}3&1&3&1&5\end{bmatrix}$	6
32	$T_1^4T_2+T_3^2T_4^2+T_5^3$	$\begin{bmatrix}1&2&2&1&2\end{bmatrix}$	6
33	$T_1^5T_2+T_3^2T_4+T_5^3$	$\begin{bmatrix}1&1&2&2&2\end{bmatrix}$	6
34	$T_1^4T_2^2+T_3^2T_4+T_5^3$	$\begin{bmatrix}1&1&2&2&2\end{bmatrix}$	6
35	$T_1^3T_2+T_3^2T_4^2+T_5^3$	$\begin{bmatrix}2&6&3&3&4\end{bmatrix}$	6
36	$T_1^3T_2^2+T_3^2T_4+T_5^3$	$\begin{bmatrix}1&6&6&3&5\end{bmatrix}$	6
37	$T_1^3T_2^2+T_3^2T_4+T_5^3$	$\begin{bmatrix}2&3&3&6&4\end{bmatrix}$	6
38	$T_1^{16}T_2+T_3^2T_4+T_5^2$	$\begin{bmatrix}1&2&6&6&9\end{bmatrix}$	6
39	$T_1^{12}T_2^3+T_3^2T_4+T_5^2$	$\begin{bmatrix}1&2&6&6&9\end{bmatrix}$	6
40	$T_1^8T_2^5+T_3^2T_4+T_5^2$	$\begin{bmatrix}1&2&6&6&9\end{bmatrix}$	6
41	$T_1^7T_2^4+T_3^2T_4+T_5^2$	$\begin{bmatrix}2&1&6&6&9\end{bmatrix}$	6
42	$T_1^4T_2+T_3^2T_4+T_5^2$	$\begin{bmatrix}3&2&6&2&7\end{bmatrix}$	6
52	$T_1^4T_2+T_3^3T_4^3+T_5^2$	$\begin{bmatrix}1&2&1&1&3\end{bmatrix}$	8
53	$T_1^4T_2^3+T_3^3T_4+T_5^2$	$\begin{bmatrix}1&2&2&4&5\end{bmatrix}$	8
54	$T_1^{11}T_2+T_3^2T_4+T_5^2$	$\begin{bmatrix}1&1&4&4&6\end{bmatrix}$	8
55	$T_1^9T_2^3+T_3^2T_4+T_5^2$	$\begin{bmatrix}1&1&4&4&6\end{bmatrix}$	8
56	$T_1^8T_2+T_3^2T_4+T_5^2$	$\begin{bmatrix}1&2&4&2&5\end{bmatrix}$	8
57	$T_1^7T_2^5+T_3^2T_4+T_5^2$	$\begin{bmatrix}1&1&4&4&6\end{bmatrix}$	8
58	$T_1^4T_2^3+T_3^2T_4+T_5^2$	$\begin{bmatrix}1&2&4&2&5\end{bmatrix}$	8
65	$T_1^3T_2+T_3^2T_4+T_5^5$	$\begin{bmatrix}1&2&2&1&1\end{bmatrix}$	10
66	$T_1^2T_2+T_3^2T_4+T_5^5$	$\begin{bmatrix}2&1&2&1&1\end{bmatrix}$	10
67	$T_1^{15}T_2+T_3T_4+T_5^5$	$\begin{bmatrix}1&5&10&10&4\end{bmatrix}$	10
68	$T_1^{10}T_2^2+T_3T_4+T_5^5$	$\begin{bmatrix}1&5&10&10&4\end{bmatrix}$	10
69	$T_1^5T_2^3+T_3T_4+T_5^5$	$\begin{bmatrix}1&5&10&10&4\end{bmatrix}$	10
70	$T_1^5T_2+T_3T_4+T_5^5$	$\begin{bmatrix}2&5&5&10&3\end{bmatrix}$	10

ID	relations	gd-matrix	$-\mathcal{K}^3$
77	$T_1^3T_2+T_3^3T_4^3+T_5^2$	$\begin{bmatrix}2&6&3&1&6\end{bmatrix}$	12
78	$T_1^3T_2+T_3^3T_4^2+T_5^2$	$\begin{bmatrix}2&12&4&3&9\end{bmatrix}$	12
79	$T_1^5T_2+T_3^2T_4+T_5^3$	$\begin{bmatrix}1&4&4&1&3\end{bmatrix}$	12
80	$T_1^{10}T_2+T_3^2T_4+T_5^2$	$\begin{bmatrix}1&2&3&6&6\end{bmatrix}$	12
81	$T_1^7T_2+T_3^2T_4+T_5^2$	$\begin{bmatrix}2&4&3&12&9\end{bmatrix}$	12
82	$T_1^6T_2^3+T_3^2T_4+T_5^2$	$\begin{bmatrix}1&2&3&6&6\end{bmatrix}$	12
83	$T_1^5T_2^2+T_3^2T_4+T_5^2$	$\begin{bmatrix}2&4&3&12&9\end{bmatrix}$	12
84	$T_1^5T_2^2+T_3^2T_4+T_5^2$	$\begin{bmatrix}2&1&3&6&6\end{bmatrix}$	12
85	$T_1^4T_2+T_3^2T_4+T_5^2$	$\begin{bmatrix}4&2&3&12&9\end{bmatrix}$	12
86	$T_1^3T_2^3+T_3^2T_4+T_5^2$	$\begin{bmatrix}2&4&3&12&9\end{bmatrix}$	12
87	$T_1^2T_2+T_3^2T_4+T_5^3$	$\begin{bmatrix}4&1&4&1&3\end{bmatrix}$	12
88	$T_1^2T_2+T_3^2T_4+T_5^2$	$\begin{bmatrix}3&2&3&2&4\end{bmatrix}$	12
89	$T_1^{21}T_2+T_3T_4+T_5^3$	$\begin{bmatrix}1&3&12&12&8\end{bmatrix}$	12
90	$T_1^{18}T_2^2+T_3T_4+T_5^3$	$\begin{bmatrix}1&3&12&12&8\end{bmatrix}$	12
91	$T_1^{12}T_2^4+T_3T_4+T_5^3$	$\begin{bmatrix}1&3&12&12&8\end{bmatrix}$	12
92	$T_1^{10}T_2+T_3T_4+T_5^4$	$\begin{bmatrix}1&2&6&6&3\end{bmatrix}$	12
93	$T_1^9T_2^5+T_3T_4+T_5^3$	$\begin{bmatrix}1&3&12&12&8\end{bmatrix}$	12
94	$T_1^9T_2+T_3T_4+T_5^6$	$\begin{bmatrix}1&3&6&6&2\end{bmatrix}$	12
95	$T_1^7T_2^3+T_3T_4+T_5^3$	$\begin{bmatrix}3&1&12&12&8\end{bmatrix}$	12
96	$T_1^6T_2^3+T_3T_4+T_5^4$	$\begin{bmatrix}1&2&6&6&3\end{bmatrix}$	12
97	$T_1^5T_2^2+T_3T_4+T_5^4$	$\begin{bmatrix}2&1&6&6&3\end{bmatrix}$	12
98	$T_1^3T_2^2+T_3T_4+T_5^{12}$	$\begin{bmatrix}2&3&6&6&1\end{bmatrix}$	12
99	$T_1^3T_2+T_3T_4+T_5^3$	$\begin{bmatrix}4&3&3&12&5\end{bmatrix}$	12
100	$T_1^2T_2+T_3T_4+T_5^2$	$\begin{bmatrix}3&4&4&6&5\end{bmatrix}$	12
106	$T_1^4T_2+T_3^3T_4^2+T_5^2$	$\begin{bmatrix}1&4&2&1&4\end{bmatrix}$	16
107	$T_1^3T_2+T_3^3T_4^2+T_5^2$	$\begin{bmatrix}2&8&4&1&7\end{bmatrix}$	16
108	$T_1^3T_2+T_3^3T_4+T_5^2$	$\begin{bmatrix}1&1&1&1&2\end{bmatrix}$	16
109	$T_1^3T_2+T_3^2T_4+T_5^4$	$\begin{bmatrix}1&1&1&2&1\end{bmatrix}$	16
110	$T_1^7T_2+T_3^2T_4+T_5^2$	$\begin{bmatrix}1&1&2&4&4\end{bmatrix}$	16
111	$T_1^5T_2^3+T_3^2T_4+T_5^2$	$\begin{bmatrix}1&1&2&4&4\end{bmatrix}$	16
112	$T_1^4T_2+T_3^2T_4+T_5^2$	$\begin{bmatrix}1&2&2&2&3\end{bmatrix}$	16
113	$T_1^8T_2+T_3T_4+T_5^4$	$\begin{bmatrix}1&4&4&8&3\end{bmatrix}$	16
114	$T_1^7T_2+T_3T_4+T_5^4$	$\begin{bmatrix}1&1&4&4&2\end{bmatrix}$	16
115	$T_1^6T_2+T_3T_4+T_5^8$	$\begin{bmatrix}1&2&4&4&1\end{bmatrix}$	16
116	$T_1^5T_2^3+T_3T_4+T_5^4$	$\begin{bmatrix}1&1&4&4&2\end{bmatrix}$	16
117	$T_1^3T_2^2+T_3T_4+T_5^8$	$\begin{bmatrix}2&1&4&4&1\end{bmatrix}$	16
126	$T_1^4T_2+T_3^3T_4^2+T_5^2$	$\begin{bmatrix}1&6&2&2&5\end{bmatrix}$	18
127	$T_1^5T_2+T_3^3T_4+T_5^2$	$\begin{bmatrix}1&1&1&3&3\end{bmatrix}$	18
128	$T_1^3T_2+T_3^3T_4^3+T_5^2$	$\begin{bmatrix}1&3&1&1&3\end{bmatrix}$	18
129	$T_1^8T_2+T_3^2T_4+T_5^2$	$\begin{bmatrix}1&2&2&6&5\end{bmatrix}$	18
130	$T_1^4T_2^3+T_3^2T_4+T_5^2$	$\begin{bmatrix}1&2&2&6&5\end{bmatrix}$	18
131	$T_1^{15}T_2+T_3T_4+T_5^3$	$\begin{bmatrix}1&9&6&18&8\end{bmatrix}$	18
132	$T_1^{11}T_2+T_3T_4+T_5^3$	$\begin{bmatrix}1&1&6&6&4\end{bmatrix}$	18
133	$T_1^{10}T_2^2+T_3T_4+T_5^3$	$\begin{bmatrix}1&1&6&6&4\end{bmatrix}$	18
134	$T_1^8T_2^4+T_3T_4+T_5^3$	$\begin{bmatrix}1&1&6&6&4\end{bmatrix}$	18
135	$T_1^7T_2^5+T_3T_4+T_5^3$	$\begin{bmatrix}1&1&6&6&4\end{bmatrix}$	18
136	$T_1^7T_2+T_3T_4+T_5^3$	$\begin{bmatrix}1&2&3&6&3\end{bmatrix}$	18
137	$T_1^6T_2^2+T_3T_4+T_5^3$	$\begin{bmatrix}1&9&6&18&8\end{bmatrix}$	18
138	$T_1^6T_2+T_3T_4+T_5^3$	$\begin{bmatrix}2&9&3&18&7\end{bmatrix}$	18
139	$T_1^5T_2^2+T_3T_4+T_5^3$	$\begin{bmatrix}1&2&3&6&3\end{bmatrix}$	18
140	$T_1^5T_2+T_3T_4+T_5^6$	$\begin{bmatrix}1&1&3&3&1\end{bmatrix}$	18
141	$T_1^4T_2+T_3T_4+T_5^3$	$\begin{bmatrix}2&1&3&6&3\end{bmatrix}$	18
142	$T_1^3T_2+T_3T_4+T_5^9$	$\begin{bmatrix}2&3&3&6&1\end{bmatrix}$	18
143	$T_1^2T_2+T_3T_4+T_5^3$	$\begin{bmatrix}2&2&3&3&2\end{bmatrix}$	18
145	$T_1^3T_2+T_3^3T_4+T_5^2$	$\begin{bmatrix}2&10&5&1&8\end{bmatrix}$	20
146	$T_1^{26}T_2+T_3T_4+T_5^2$	$\begin{bmatrix}1&4&10&20&15\end{bmatrix}$	20
147	$T_1^{18}T_2^3+T_3T_4+T_5^2$	$\begin{bmatrix}1&4&10&20&15\end{bmatrix}$	20
148	$T_1^{10}T_2^5+T_3T_4+T_5^2$	$\begin{bmatrix}1&4&10&20&15\end{bmatrix}$	20
149	$T_1^7T_2^2+T_3T_4+T_5^2$	$\begin{bmatrix}4&1&10&20&15\end{bmatrix}$	20
150	$T_1^3T_2^2+T_3T_4+T_5^2$	$\begin{bmatrix}4&5&2&20&11\end{bmatrix}$	20
151	$T_1^2T_2+T_3T_4+T_5^4$	$\begin{bmatrix}5&2&2&10&3\end{bmatrix}$	20
160	$T_1^4T_2+T_3^3T_4+T_5^2$	$\begin{bmatrix}1&6&3&1&5\end{bmatrix}$	24
161	$T_1^4T_2+T_3^2T_4+T_5^3$	$\begin{bmatrix}1&2&1&4&2\end{bmatrix}$	24
162	$T_1^3T_2+T_3^2T_4+T_5^3$	$\begin{bmatrix}1&12&6&3&5\end{bmatrix}$	24
163	$T_1^2T_2^2+T_3^2T_4+T_5^3$	$\begin{bmatrix}1&2&1&4&2\end{bmatrix}$	24
164	$T_1^2T_2+T_3^2T_4+T_5^6$	$\begin{bmatrix}1&4&2&2&1\end{bmatrix}$	24
165	$T_1^2T_2+T_3^2T_4+T_5^4$	$\begin{bmatrix}3&2&1&6&2\end{bmatrix}$	24
166	$T_1^2T_2+T_3^2T_4+T_5^3$	$\begin{bmatrix}1&1&1&1&1\end{bmatrix}$	24
167	$T_1^{22}T_2+T_3T_4+T_5^2$	$\begin{bmatrix}1&8&6&24&15\end{bmatrix}$	24
168	$T_1^{16}T_2+T_3T_4+T_5^2$	$\begin{bmatrix}1&2&6&12&9\end{bmatrix}$	24

ID	relations	gd-matrix	$-\mathcal{K}^3$
169	$T_1^{13}T_2+T_3T_4+T_5^2$	$\begin{bmatrix}1&3&4&12&8\end{bmatrix}$	24
170	$T_1^{12}T_2^3+T_3T_4+T_5^2$	$\begin{bmatrix}1&2&6&12&9\end{bmatrix}$	24
171	$T_1^9T_2+T_3T_4+T_5^3$	$\begin{bmatrix}1&6&3&12&5\end{bmatrix}$	24
172	$T_1^8T_2^5+T_3T_4+T_5^2$	$\begin{bmatrix}1&2&6&12&9\end{bmatrix}$	24
173	$T_1^7T_2^4+T_3T_4+T_5^2$	$\begin{bmatrix}2&1&6&12&9\end{bmatrix}$	24
174	$T_1^7T_2^3+T_3T_4+T_5^2$	$\begin{bmatrix}1&3&4&12&8\end{bmatrix}$	24
175	$T_1^6T_2^3+T_3T_4+T_5^2$	$\begin{bmatrix}1&8&6&24&15\end{bmatrix}$	24
176	$T_1^6T_2+T_3T_4+T_5^2$	$\begin{bmatrix}3&8&2&24&13\end{bmatrix}$	24
177	$T_1^5T_2+T_3T_4+T_5^4$	$\begin{bmatrix}1&3&2&6&2\end{bmatrix}$	24
178	$T_1^5T_2+T_3T_4+T_5^3$	$\begin{bmatrix}1&1&2&4&2\end{bmatrix}$	24
179	$T_1^5T_2+T_3T_4+T_5^2$	$\begin{bmatrix}3&1&4&12&8\end{bmatrix}$	24
180	$T_1^4T_2^2+T_3T_4+T_5^3$	$\begin{bmatrix}1&1&2&4&2\end{bmatrix}$	24
181	$T_1^4T_2+T_3T_4+T_5^6$	$\begin{bmatrix}1&2&2&4&1\end{bmatrix}$	24
182	$T_1^4T_2+T_3T_4+T_5^2$	$\begin{bmatrix}1&6&4&6&5\end{bmatrix}$	24
183	$T_1^4T_2+T_3T_4+T_5^2$	$\begin{bmatrix}1&2&3&3&3\end{bmatrix}$	24
184	$T_1^4T_2+T_3T_4+T_5^2$	$\begin{bmatrix}3&2&2&12&7\end{bmatrix}$	24
185	$T_1^3T_2^2+T_3T_4+T_5^3$	$\begin{bmatrix}1&6&3&12&5\end{bmatrix}$	24
186	$T_1^3T_2+T_3T_4+T_5^3$	$\begin{bmatrix}1&3&3&3&2\end{bmatrix}$	24
187	$T_1^2T_2+T_3T_4+T_5^8$	$\begin{bmatrix}3&2&2&6&1\end{bmatrix}$	24
190	$T_1^3T_2+T_3^3T_4+T_5^2$	$\begin{bmatrix}1&15&5&3&9\end{bmatrix}$	30
191	$T_1^2T_2+T_3^2T_4+T_5^3$	$\begin{bmatrix}1&10&5&2&4\end{bmatrix}$	30
192	$T_1^{13}T_2+T_3T_4+T_5^2$	$\begin{bmatrix}1&5&3&15&9\end{bmatrix}$	30
193	$T_1^7T_2+T_3T_4+T_5^3$	$\begin{bmatrix}1&5&2&10&4\end{bmatrix}$	30
194	$T_1^3T_2^3+T_3T_4+T_5^2$	$\begin{bmatrix}1&5&3&15&9\end{bmatrix}$	30
195	$T_1^2T_2^2+T_3T_4+T_5^3$	$\begin{bmatrix}1&5&2&10&4\end{bmatrix}$	30
196	$T_1T_2+T_3T_4+T_5^5$	$\begin{bmatrix}2&3&2&3&1\end{bmatrix}$	30
203	$T_1^5T_2+T_3^2T_4+T_5^2$	$\begin{bmatrix}1&1&1&4&3\end{bmatrix}$	32
204	$T_1^4T_2+T_3^2T_4+T_5^2$	$\begin{bmatrix}2&2&1&8&5\end{bmatrix}$	32
205	$T_1^3T_2^3+T_3^2T_4+T_5^2$	$\begin{bmatrix}1&1&1&4&3\end{bmatrix}$	32
206	$T_1^3T_2^2+T_3^2T_4+T_5^2$	$\begin{bmatrix}2&2&1&8&5\end{bmatrix}$	32
207	$T_1^2T_2+T_3^2T_4+T_5^2$	$\begin{bmatrix}1&2&1&2&2\end{bmatrix}$	32
208	$T_1^{11}T_2+T_3T_4+T_5^2$	$\begin{bmatrix}1&1&4&8&6\end{bmatrix}$	32

ID	relations	gd-matrix	$-\mathcal{K}^3$
209	$T_1^9T_2^3+T_3T_4+T_5^2$	$\begin{bmatrix}1&1&4&8&6\end{bmatrix}$	32
210	$T_1^8T_2+T_3T_4+T_5^2$	$\begin{bmatrix}1&2&2&8&5\end{bmatrix}$	32
211	$T_1^7T_2^5+T_3T_4+T_5^2$	$\begin{bmatrix}1&1&4&8&6\end{bmatrix}$	32
212	$T_1^4T_2^3+T_3T_4+T_5^2$	$\begin{bmatrix}1&2&2&8&5\end{bmatrix}$	32
213	$T_1^3T_2+T_3T_4+T_5^2$	$\begin{bmatrix}1&1&2&2&2\end{bmatrix}$	32
214	$T_1^2T_2+T_3T_4+T_5^4$	$\begin{bmatrix}1&2&2&2&1\end{bmatrix}$	32
215	$T_1^2T_2+T_3T_4+T_5^2$	$\begin{bmatrix}1&4&2&4&3\end{bmatrix}$	32
217	$T_1^6T_2+T_3^2T_4+T_5^2$	$\begin{bmatrix}1&2&1&6&4\end{bmatrix}$	36
218	$T_1^5T_2+T_3^2T_4+T_5^2$	$\begin{bmatrix}2&4&1&12&7\end{bmatrix}$	36
219	$T_1^3T_2^2+T_3^2T_4+T_5^2$	$\begin{bmatrix}2&4&1&12&7\end{bmatrix}$	36
220	$T_1^3T_2^2+T_3^2T_4+T_5^2$	$\begin{bmatrix}2&1&1&6&4\end{bmatrix}$	36
221	$T_1^3T_2+T_3^2T_4+T_5^2$	$\begin{bmatrix}4&2&1&12&7\end{bmatrix}$	36
222	$T_1^{10}T_2+T_3T_4+T_5^2$	$\begin{bmatrix}1&4&2&12&7\end{bmatrix}$	36
223	$T_1^7T_2+T_3T_4+T_5^2$	$\begin{bmatrix}1&1&2&6&4\end{bmatrix}$	36
224	$T_1^5T_2^3+T_3T_4+T_5^2$	$\begin{bmatrix}1&1&2&6&4\end{bmatrix}$	36
225	$T_1^3T_2^2+T_3T_4+T_5^2$	$\begin{bmatrix}4&1&2&12&7\end{bmatrix}$	36
226	$T_1^3T_2+T_3T_4+T_5^4$	$\begin{bmatrix}1&1&1&3&1\end{bmatrix}$	36
227	$T_1T_2+T_3T_4+T_5^2$	$\begin{bmatrix}3&1&2&2&2\end{bmatrix}$	36
230	$T_1^3T_2+T_3T_4+T_5^5$	$\begin{bmatrix}1&2&1&4&1\end{bmatrix}$	40
231	$T_1^2T_2+T_3T_4+T_5^5$	$\begin{bmatrix}2&1&1&4&1\end{bmatrix}$	40
232	$T_1^5T_2+T_3T_4+T_5^2$	$\begin{bmatrix}3&7&1&21&11\end{bmatrix}$	42
233	$T_1^4T_2+T_3T_4+T_5^3$	$\begin{bmatrix}2&7&1&14&5\end{bmatrix}$	42
234	$T_1^2T_2+T_3T_4+T_5^7$	$\begin{bmatrix}2&3&1&6&1\end{bmatrix}$	42
235	$T_1^5T_2+T_3T_4+T_5^3$	$\begin{bmatrix}1&4&1&8&3\end{bmatrix}$	48
236	$T_1^2T_2+T_3T_4+T_5^3$	$\begin{bmatrix}4&1&1&8&3\end{bmatrix}$	48
237	$T_1T_2+T_3T_4+T_5^3$	$\begin{bmatrix}1&2&1&2&1\end{bmatrix}$	48
238	$T_1T_2+T_3T_4+T_5^2$	$\begin{bmatrix}1&3&1&3&2\end{bmatrix}$	48
241	$T_1^5T_2+T_3T_4+T_5^2$	$\begin{bmatrix}1&1&1&5&3\end{bmatrix}$	50
242	$T_1^3T_2^3+T_3T_4+T_5^2$	$\begin{bmatrix}1&1&1&5&3\end{bmatrix}$	50
243	$T_1^7T_2+T_3T_4+T_5^2$	$\begin{bmatrix}1&3&1&9&5\end{bmatrix}$	54
244	$T_1^3T_2+T_3T_4+T_5^2$	$\begin{bmatrix}3&1&1&9&5\end{bmatrix}$	54
245	$T_1T_2+T_3T_4+T_5^2$	$\begin{bmatrix}1&1&1&1&1\end{bmatrix}$	54

Classification list 2.12.2. *Non-toric, $\mathbb{Q}$-factorial, Gorenstein, log terminal Fano three-folds of Picard number one with an effective two-torus action: Specifying data for divisor class group $\mathbb{Z}$ and format* $(2,2,1,1,0)$.

ID	relations	gd-matrix	$-\mathcal{K}^3$
30	$T_1^3T_2+T_3T_4+T_5^3,$ $\lambda T_3T_4+T_5^3+T_6^2$	$\begin{bmatrix}1&3&3&3&2&3\end{bmatrix}$	6
76	$T_1^2T_2+T_3T_4+T_5^3,$ $\lambda T_3T_4+T_5^3+T_6^2$	$\begin{bmatrix}1&4&2&4&2&3\end{bmatrix}$	12
189	$T_1T_2+T_3T_4+T_5^3,$ $\lambda T_3T_4+T_5^3+T_6^2$	$\begin{bmatrix}1&5&1&5&2&3\end{bmatrix}$	30

Classification list 2.12.3. *Non-toric, $\mathbb{Q}$-factorial, Gorenstein, log terminal Fano three-folds of Picard number one with an effective two-torus action: Specifying data for divisor class group $\mathbb{Z}$ and format* $(3,1,1,0)$.

ID	relations	gd-matrix	$-\mathcal{K}^3$
3	$T_1^5T_2^3T_3^2+T_4^3+T_5^2$	$\begin{bmatrix}1&1&2&4&6\end{bmatrix}$	2
4	$T_1^4T_2^3T_3^2+T_4^3+T_5^2$	$\begin{bmatrix}1&2&1&4&6\end{bmatrix}$	2
5	$T_1^5T_2^5T_3+T_4^3+T_5^2$	$\begin{bmatrix}1&1&2&4&6\end{bmatrix}$	2
6	$T_1^5T_2^3T_3+T_4^3+T_5^2$	$\begin{bmatrix}1&2&1&4&6\end{bmatrix}$	2
7	$T_1^5T_2T_3+T_4^3+T_5^2$	$\begin{bmatrix}2&1&1&4&6\end{bmatrix}$	2
8	$T_1^4T_2^3T_3+T_4^3+T_5^2$	$\begin{bmatrix}2&1&1&4&6\end{bmatrix}$	2
9	$T_1^3T_2^2T_3+T_4^5+T_5^2$	$\begin{bmatrix}2&1&2&2&5\end{bmatrix}$	2
13	$T_1^2T_2^2T_3+T_4^7+T_5^2$	$\begin{bmatrix}4&1&4&2&7\end{bmatrix}$	4
25	$T_1^5T_2^2T_3+T_4^3+T_5^2$	$\begin{bmatrix}2&1&6&6&9\end{bmatrix}$	6
26	$T_1^4T_2^4T_3+T_4^3+T_5^2$	$\begin{bmatrix}2&1&6&6&9\end{bmatrix}$	6
27	$T_1^4T_2^2T_3+T_4^3+T_5^2$	$\begin{bmatrix}1&6&2&6&9\end{bmatrix}$	6
28	$T_1^3T_2T_3+T_4^3+T_5^2$	$\begin{bmatrix}2&3&3&4&6\end{bmatrix}$	6
29	$T_1^2T_2T_3+T_4^7+T_5^2$	$\begin{bmatrix}3&2&6&2&7\end{bmatrix}$	6
46	$T_1^5T_2^3T_3+T_4^3+T_5^2$	$\begin{bmatrix}1&1&4&4&6\end{bmatrix}$	8
47	$T_1^4T_2^4T_3+T_4^3+T_5^2$	$\begin{bmatrix}1&1&4&4&6\end{bmatrix}$	8
48	$T_1^4T_2T_3+T_4^3+T_5^2$	$\begin{bmatrix}1&1&1&2&3\end{bmatrix}$	8
49	$T_1^3T_2^2T_3+T_4^3+T_5^2$	$\begin{bmatrix}1&4&1&4&6\end{bmatrix}$	8
50	$T_1^3T_2^2T_3+T_4^3+T_5^2$	$\begin{bmatrix}1&1&1&2&3\end{bmatrix}$	8
51	$T_1^2T_2^2T_3+T_4^5+T_5^2$	$\begin{bmatrix}2&1&4&2&5\end{bmatrix}$	8
63	$T_1T_2T_3+T_4^7+T_5^3$	$\begin{bmatrix}1&10&10&3&7\end{bmatrix}$	10
64	$T_1T_2T_3+T_4^3+T_5^2$	$\begin{bmatrix}2&5&5&4&6\end{bmatrix}$	10
75	$T_1^3T_2^3T_3+T_4^3+T_5^2$	$\begin{bmatrix}3&1&12&8&12\end{bmatrix}$	12
104	$T_1^2T_2T_3+T_4^7+T_5^2$	$\begin{bmatrix}1&4&8&2&7\end{bmatrix}$	16
105	$T_1^2T_2T_3+T_4^3+T_5^2$	$\begin{bmatrix}1&2&2&2&3\end{bmatrix}$	16
121	$T_1^5T_2T_3+T_4^3+T_5^2$	$\begin{bmatrix}1&6&1&4&6\end{bmatrix}$	18
122	$T_1^4T_2^2T_3+T_4^3+T_5^2$	$\begin{bmatrix}1&1&6&4&6\end{bmatrix}$	18
123	$T_1^3T_2^3T_3+T_4^3+T_5^2$	$\begin{bmatrix}1&1&6&4&6\end{bmatrix}$	18
124	$T_1^2T_2T_3+T_4^5+T_5^2$	$\begin{bmatrix}1&2&6&2&5\end{bmatrix}$	18
125	$T_1T_2T_3+T_4^7+T_5^2$	$\begin{bmatrix}1&9&18&4&14\end{bmatrix}$	18
144	$T_1^2T_2^2T_3+T_4^3+T_5^2$	$\begin{bmatrix}4&1&20&10&15\end{bmatrix}$	20
158	$T_1^4T_2T_3+T_4^3+T_5^2$	$\begin{bmatrix}1&12&2&6&9\end{bmatrix}$	24
159	$T_1^2T_2^2T_3+T_4^3+T_5^2$	$\begin{bmatrix}2&1&12&6&9\end{bmatrix}$	24
188	$T_1T_2T_3+T_4^3+T_5^2$	$\begin{bmatrix}1&5&30&12&18\end{bmatrix}$	30
201	$T_1^3T_2T_3+T_4^3+T_5^2$	$\begin{bmatrix}1&8&1&4&6\end{bmatrix}$	32
202	$T_1^2T_2^2T_3+T_4^3+T_5^2$	$\begin{bmatrix}1&1&8&4&6\end{bmatrix}$	32
240	$T_1T_2T_3+T_4^3+T_5^2$	$\begin{bmatrix}1&1&10&4&6\end{bmatrix}$	50

Classification list 2.12.4. *Non-toric, $\mathbb{Q}$-factorial, Gorenstein, log terminal Fano three-folds of Picard number one with an effective two-torus action: Specifying data for divisor class group $\mathbb{Z}$ and format* $(2,1,1,1)$.

ID	relations	gd-matrix	$-\mathcal{K}^3$
1	$T_1^5T_2^2+T_3^3+T_4^2$	$\begin{bmatrix}2&1&4&6&1\end{bmatrix}$	2
2	$T_1^3T_2^2+T_3^5+T_4^2$	$\begin{bmatrix}2&2&2&5&1\end{bmatrix}$	2
12	$T_1^2T_2+T_3^4+T_4^3$	$\begin{bmatrix}4&4&3&4&1\end{bmatrix}$	4
19	$T_1^3T_2^2+T_3^3+T_4^2$	$\begin{bmatrix}2&3&4&6&3\end{bmatrix}$	6
20	$T_1^3T_2^2+T_3^3+T_4^2$	$\begin{bmatrix}2&6&6&9&1\end{bmatrix}$	6
21	$T_1^3T_2+T_3^3+T_4^2$	$\begin{bmatrix}3&3&4&6&2\end{bmatrix}$	6
22	$T_1^2T_2+T_3^4+T_4^3$	$\begin{bmatrix}3&6&3&4&2\end{bmatrix}$	6
23	$T_1^2T_2+T_3^9+T_4^2$	$\begin{bmatrix}6&6&2&9&1\end{bmatrix}$	6
24	$T_1^2T_2+T_3^7+T_4^2$	$\begin{bmatrix}6&2&2&7&3\end{bmatrix}$	6
43	$T_1^5T_2+T_3^3+T_4^2$	$\begin{bmatrix}1&1&2&3&1\end{bmatrix}$	8
44	$T_1^3T_2+T_3^5+T_4^2$	$\begin{bmatrix}2&4&2&5&1\end{bmatrix}$	8
45	$T_1^2T_2+T_3^5+T_4^2$	$\begin{bmatrix}4&2&2&5&1\end{bmatrix}$	8
59	$T_1^2T_2+T_3^5+T_4^2$	$\begin{bmatrix}5&10&4&10&1\end{bmatrix}$	10
60	$T_1^2T_2+T_3^3+T_4^2$	$\begin{bmatrix}5&2&4&6&5\end{bmatrix}$	10
61	$T_1T_2+T_3^5+T_4^4$	$\begin{bmatrix}10&10&4&5&1\end{bmatrix}$	10
62	$T_1T_2+T_3^5+T_4^3$	$\begin{bmatrix}5&10&3&5&2\end{bmatrix}$	10
71	$T_1^4T_2+T_3^3+T_4^2$	$\begin{bmatrix}3&12&8&12&1\end{bmatrix}$	12
72	$T_1^2T_2+T_3^9+T_4^2$	$\begin{bmatrix}3&12&2&9&4\end{bmatrix}$	12
73	$T_1T_2+T_3^8+T_4^3$	$\begin{bmatrix}12&12&3&8&1\end{bmatrix}$	12
74	$T_1T_2+T_3^5+T_4^3$	$\begin{bmatrix}3&12&3&5&4\end{bmatrix}$	12

ID	relations	gd-matrix	$-\mathcal{K}^3$
101	$T_1^4T_2+T_3^3+T_4^2$	$\begin{bmatrix}1&2&2&3&2\end{bmatrix}$	16
102	$T_1^2T_2+T_3^3+T_4^2$	$\begin{bmatrix}2&2&2&3&1\end{bmatrix}$	16
103	$T_1T_2+T_3^4+T_4^3$	$\begin{bmatrix}4&8&3&4&1\end{bmatrix}$	16
118	$T_1^2T_2+T_3^5+T_4^2$	$\begin{bmatrix}2&6&2&5&1\end{bmatrix}$	18
119	$T_1^2T_2+T_3^3+T_4^2$	$\begin{bmatrix}2&2&2&3&3\end{bmatrix}$	18
120	$T_1T_2+T_3^5+T_4^4$	$\begin{bmatrix}2&18&4&5&9\end{bmatrix}$	18
152	$T_1T_2+T_3^8+T_4^3$	$\begin{bmatrix}2&22&3&8&11\end{bmatrix}$	22
153	$T_1^3T_2+T_3^3+T_4^2$	$\begin{bmatrix}1&3&2&3&3\end{bmatrix}$	24
154	$T_1^3T_2+T_3^3+T_4^2$	$\begin{bmatrix}2&12&6&9&1\end{bmatrix}$	24
155	$T_1T_2+T_3^9+T_4^2$	$\begin{bmatrix}6&12&2&9&1\end{bmatrix}$	24
156	$T_1T_2+T_3^7+T_4^2$	$\begin{bmatrix}2&12&2&7&3\end{bmatrix}$	24
157	$T_1T_2+T_3^3+T_4^2$	$\begin{bmatrix}3&3&2&3&1\end{bmatrix}$	24
197	$T_1^4T_2+T_3^3+T_4^2$	$\begin{bmatrix}1&8&4&6&1\end{bmatrix}$	32
198	$T_1^2T_2+T_3^5+T_4^2$	$\begin{bmatrix}1&8&2&5&2\end{bmatrix}$	32
199	$T_1^2T_2+T_3^3+T_4^2$	$\begin{bmatrix}1&4&2&3&4\end{bmatrix}$	32
200	$T_1T_2+T_3^5+T_4^2$	$\begin{bmatrix}2&8&2&5&1\end{bmatrix}$	32
216	$T_1^2T_2+T_3^7+T_4^2$	$\begin{bmatrix}1&12&2&7&4\end{bmatrix}$	36
228	$T_1T_2+T_3^7+T_4^3$	$\begin{bmatrix}1&20&3&7&10\end{bmatrix}$	40
229	$T_1T_2+T_3^3+T_4^2$	$\begin{bmatrix}1&5&2&3&5\end{bmatrix}$	40
239	$T_1^2T_2+T_3^3+T_4^2$	$\begin{bmatrix}1&10&4&6&1\end{bmatrix}$	50

Classification list 2.12.5. *Non-toric, $\mathbb{Q}$-factorial, Gorenstein, log terminal Fano threefolds of Picard number one with an effective two-torus action: Specifying data for divisor class group $\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$ and format $(2,2,1,0)$.*

ID	relations	gd-matrix	$-\mathcal{K}^3$
252	$T_1^4T_2^2+T_3^3T_4^2+T_5^2$	$\begin{bmatrix}1&2&2&1&4\\\bar{0}&\bar{0}&\bar{0}&\bar{1}&\bar{1}\end{bmatrix}$	2
253	$T_1^3T_2^2+T_3^3T_4^2+T_5^2$	$\begin{bmatrix}2&1&2&1&4\\\bar{0}&\bar{1}&\bar{0}&\bar{0}&\bar{1}\end{bmatrix}$	2
254	$T_1^4T_2^4+T_3^3T_4+T_5^2$	$\begin{bmatrix}1&1&2&2&4\\\bar{1}&\bar{0}&\bar{0}&\bar{0}&\bar{1}\end{bmatrix}$	2
271	$T_1^4T_2+T_3^3T_4^3+T_5^2$	$\begin{bmatrix}1&2&1&1&3\\\bar{0}&\bar{0}&\bar{1}&\bar{1}&\bar{0}\end{bmatrix}$	4
272	$T_1^4T_2^2+T_3^3T_4+T_5^2$	$\begin{bmatrix}2&1&2&4&5\\\bar{1}&\bar{0}&\bar{0}&\bar{0}&\bar{1}\end{bmatrix}$	4
273	$T_1^4T_2+T_3^2T_4^2+T_5^2$	$\begin{bmatrix}2&2&1&4&5\\\bar{1}&\bar{0}&\bar{0}&\bar{0}&\bar{1}\end{bmatrix}$	4
274	$T_1^3T_2^2+T_3^2T_4^2+T_5^2$	$\begin{bmatrix}2&2&1&4&5\\\bar{0}&\bar{1}&\bar{1}&\bar{0}&\bar{0}\end{bmatrix}$	4
275	$T_1^8T_2^4+T_3^2T_4+T_5^2$	$\begin{bmatrix}1&1&4&4&6\\\bar{1}&\bar{0}&\bar{0}&\bar{0}&\bar{1}\end{bmatrix}$	4

ID	relations	gd-matrix	$-\mathcal{K}^3$
276	$T_1^4T_2^2+T_3^2T_4+T_5^2$	$\begin{bmatrix}2&1&4&2&5\\\bar{1}&\bar{0}&\bar{0}&\bar{0}&\bar{1}\end{bmatrix}$	4
291	$T_1^{10}T_2+T_3^2T_4+T_5^2$	$\begin{bmatrix}1&2&3&6&6\\\bar{0}&\bar{0}&\bar{1}&\bar{0}&\bar{1}\end{bmatrix}$	6
292	$T_1^8T_2^2+T_3^2T_4+T_5^2$	$\begin{bmatrix}1&2&3&6&6\\\bar{0}&\bar{0}&\bar{1}&\bar{0}&\bar{1}\end{bmatrix}$	6
293	$T_1^6T_2^3+T_3^2T_4+T_5^2$	$\begin{bmatrix}1&2&3&6&6\\\bar{0}&\bar{0}&\bar{1}&\bar{0}&\bar{1}\end{bmatrix}$	6
294	$T_1^5T_2^2+T_3^2T_4+T_5^2$	$\begin{bmatrix}2&1&3&6&6\\\bar{0}&\bar{1}&\bar{0}&\bar{0}&\bar{1}\end{bmatrix}$	6
295	$T_1^4T_2^4+T_3^2T_4+T_5^2$	$\begin{bmatrix}1&2&3&6&6\\\bar{0}&\bar{0}&\bar{1}&\bar{0}&\bar{1}\end{bmatrix}$	6
296	$T_1^2T_2+T_3^2T_4+T_5^2$	$\begin{bmatrix}3&2&3&2&4\\\bar{0}&\bar{0}&\bar{1}&\bar{0}&\bar{1}\end{bmatrix}$	6
320	$T_1^4T_2+T_3^3T_4^2+T_5^2$	$\begin{bmatrix}1&4&2&1&4\\\bar{0}&\bar{0}&\bar{0}&\bar{1}&\bar{1}\end{bmatrix}$	8

ID	relations	gd-matrix	$-\mathcal{K}^3$
321	$T_1^3T_2+T_3^3T_4+T_5^2$	$\begin{bmatrix}1&1&1&1&2\\ \bar{0}&\bar{0}&\bar{1}&\bar{1}&\bar{0}\end{bmatrix}$	8
322	$T_1^3T_2+T_3^2T_4+T_5^4$	$\begin{bmatrix}1&1&1&2&1\\ \bar{1}&\bar{1}&\bar{0}&\bar{0}&\bar{0}\end{bmatrix}$	8
323	$T_1^7T_2+T_3^2T_4+T_5^2$	$\begin{bmatrix}1&1&2&4&4\\ \bar{0}&\bar{0}&\bar{1}&\bar{0}&\bar{1}\end{bmatrix}$	8
324	$T_1^6T_2^2+T_3^2T_4+T_5^2$	$\begin{bmatrix}1&1&2&4&4\\ \bar{0}&\bar{0}&\bar{1}&\bar{0}&\bar{1}\end{bmatrix}$	8
325	$T_1^6T_2^2+T_3^2T_4+T_5^2$	$\begin{bmatrix}1&1&2&4&4\\ \bar{0}&\bar{1}&\bar{1}&\bar{0}&\bar{0}\end{bmatrix}$	8
326	$T_1^5T_2^3+T_3^2T_4+T_5^2$	$\begin{bmatrix}1&1&2&4&4\\ \bar{0}&\bar{0}&\bar{1}&\bar{0}&\bar{1}\end{bmatrix}$	8
327	$T_1^4T_2^4+T_3^2T_4+T_5^2$	$\begin{bmatrix}1&1&2&4&4\\ \bar{0}&\bar{0}&\bar{1}&\bar{0}&\bar{1}\end{bmatrix}$	8
328	$T_1^4T_2^4+T_3^2T_4+T_5^2$	$\begin{bmatrix}1&1&2&4&4\\ \bar{1}&\bar{0}&\bar{0}&\bar{0}&\bar{1}\end{bmatrix}$	8
329	$T_1^4T_2+T_3^2T_4+T_5^2$	$\begin{bmatrix}1&2&2&2&3\\ \bar{0}&\bar{0}&\bar{1}&\bar{0}&\bar{1}\end{bmatrix}$	8
330	$T_1^2T_2^2+T_3^2T_4+T_5^4$	$\begin{bmatrix}1&1&1&2&1\\ \bar{0}&\bar{1}&\bar{1}&\bar{0}&\bar{0}\end{bmatrix}$	8
331	$T_1^2T_2+T_3^2T_4^2+T_5^2$	$\begin{bmatrix}2&2&2&1&3\\ \bar{1}&\bar{0}&\bar{0}&\bar{0}&\bar{1}\end{bmatrix}$	8
332	$T_1^2T_2+T_3^2T_4^2+T_5^2$	$\begin{bmatrix}2&2&2&1&3\\ \bar{1}&\bar{0}&\bar{1}&\bar{0}&\bar{0}\end{bmatrix}$	8
333	$T_1^6T_2^2+T_3T_4+T_5^4$	$\begin{bmatrix}1&1&4&4&2\\ \bar{0}&\bar{1}&\bar{0}&\bar{0}&\bar{1}\end{bmatrix}$	8
334	$T_1^4T_2^2+T_3T_4+T_5^8$	$\begin{bmatrix}1&2&4&4&1\\ \bar{1}&\bar{1}&\bar{0}&\bar{0}&\bar{0}\end{bmatrix}$	8
335	$T_1^4T_2^2+T_3T_4+T_5^4$	$\begin{bmatrix}1&4&4&8&3\\ \bar{0}&\bar{1}&\bar{0}&\bar{0}&\bar{1}\end{bmatrix}$	8
350	$T_1^4T_2+T_3^3T_4+T_5^2$	$\begin{bmatrix}1&6&3&1&5\\ \bar{0}&\bar{0}&\bar{1}&\bar{1}&\bar{0}\end{bmatrix}$	12
351	$T_1^2T_2^2+T_3^2T_4+T_5^3$	$\begin{bmatrix}1&2&1&4&2\\ \bar{1}&\bar{1}&\bar{0}&\bar{0}&\bar{0}\end{bmatrix}$	12
352	$T_1^2T_2+T_3^2T_4+T_5^6$	$\begin{bmatrix}2&2&1&4&1\\ \bar{1}&\bar{0}&\bar{1}&\bar{0}&\bar{0}\end{bmatrix}$	12
353	$T_1^2T_2+T_3^2T_4+T_5^4$	$\begin{bmatrix}1&6&3&2&2\\ \bar{0}&\bar{0}&\bar{1}&\bar{0}&\bar{1}\end{bmatrix}$	12
354	$T_1^2T_2+T_3^2T_4+T_5^3$	$\begin{bmatrix}1&1&1&1&1\\ \bar{1}&\bar{0}&\bar{1}&\bar{0}&\bar{0}\end{bmatrix}$	12
355	$T_1^{10}T_2^4+T_3T_4+T_5^2$	$\begin{bmatrix}1&2&6&12&9\\ \bar{0}&\bar{1}&\bar{0}&\bar{0}&\bar{1}\end{bmatrix}$	12
356	$T_1^9T_2+T_3T_4+T_5^3$	$\begin{bmatrix}1&6&3&12&5\\ \bar{0}&\bar{1}&\bar{1}&\bar{0}&\bar{1}\end{bmatrix}$	12
357	$T_1^8T_2^2+T_3T_4+T_5^2$	$\begin{bmatrix}2&1&6&12&9\\ \bar{1}&\bar{1}&\bar{0}&\bar{0}&\bar{0}\end{bmatrix}$	12
358	$T_1^5T_2+T_3T_4+T_5^3$	$\begin{bmatrix}1&1&2&4&2\\ \bar{0}&\bar{1}&\bar{1}&\bar{0}&\bar{1}\end{bmatrix}$	12
359	$T_1^4T_2^4+T_3T_4+T_5^2$	$\begin{bmatrix}1&3&4&12&8\\ \bar{0}&\bar{1}&\bar{0}&\bar{0}&\bar{1}\end{bmatrix}$	12
360	$T_1^4T_2^2+T_3T_4+T_5^2$	$\begin{bmatrix}2&3&2&12&7\\ \bar{1}&\bar{0}&\bar{0}&\bar{0}&\bar{1}\end{bmatrix}$	12
361	$T_1^4T_2+T_3T_4+T_5^2$	$\begin{bmatrix}1&2&3&3&3\\ \bar{0}&\bar{0}&\bar{1}&\bar{1}&\bar{0}\end{bmatrix}$	12
362	$T_1^3T_2+T_3T_4+T_5^3$	$\begin{bmatrix}1&3&3&3&2\\ \bar{0}&\bar{0}&\bar{1}&\bar{1}&\bar{0}\end{bmatrix}$	12
363	$T_1^3T_2+T_3T_4+T_5^3$	$\begin{bmatrix}1&3&3&3&2\\ \bar{0}&\bar{1}&\bar{0}&\bar{1}&\bar{1}\end{bmatrix}$	12
364	$T_1^2T_2^2+T_3T_4+T_5^4$	$\begin{bmatrix}1&3&2&6&2\\ \bar{0}&\bar{1}&\bar{0}&\bar{0}&\bar{1}\end{bmatrix}$	12
381	$T_1^5T_2+T_3^2T_4+T_5^2$	$\begin{bmatrix}1&1&1&4&3\\ \bar{0}&\bar{0}&\bar{1}&\bar{0}&\bar{1}\end{bmatrix}$	16
382	$T_1^4T_2^2+T_3^2T_4+T_5^2$	$\begin{bmatrix}1&1&1&4&3\\ \bar{0}&\bar{0}&\bar{1}&\bar{0}&\bar{1}\end{bmatrix}$	16
383	$T_1^4T_2^2+T_3^2T_4+T_5^2$	$\begin{bmatrix}1&1&1&4&3\\ \bar{0}&\bar{1}&\bar{1}&\bar{0}&\bar{0}\end{bmatrix}$	16
384	$T_1^4T_2+T_3^2T_4+T_5^2$	$\begin{bmatrix}2&2&1&8&5\\ \bar{1}&\bar{0}&\bar{0}&\bar{0}&\bar{1}\end{bmatrix}$	16
385	$T_1^3T_2^3+T_3^2T_4+T_5^2$	$\begin{bmatrix}1&1&1&4&3\\ \bar{0}&\bar{0}&\bar{1}&\bar{0}&\bar{1}\end{bmatrix}$	16
386	$T_1^3T_2^2+T_3^2T_4+T_5^2$	$\begin{bmatrix}2&2&1&8&5\\ \bar{0}&\bar{1}&\bar{0}&\bar{0}&\bar{1}\end{bmatrix}$	16
387	$T_1^2T_2+T_3^2T_4+T_5^2$	$\begin{bmatrix}1&2&1&2&2\\ \bar{0}&\bar{0}&\bar{1}&\bar{0}&\bar{1}\end{bmatrix}$	16
388	$T_1^8T_2^4+T_3T_4+T_5^2$	$\begin{bmatrix}1&1&4&8&6\\ \bar{1}&\bar{0}&\bar{0}&\bar{0}&\bar{1}\end{bmatrix}$	16
389	$T_1^4T_2^2+T_3T_4+T_5^2$	$\begin{bmatrix}2&1&2&8&5\\ \bar{1}&\bar{0}&\bar{0}&\bar{0}&\bar{1}\end{bmatrix}$	16
390	$T_1^3T_2+T_3T_4+T_5^2$	$\begin{bmatrix}1&1&2&2&2\\ \bar{0}&\bar{0}&\bar{1}&\bar{1}&\bar{0}\end{bmatrix}$	16
391	$T_1^2T_2+T_3T_4+T_5^4$	$\begin{bmatrix}1&2&2&2&1\\ \bar{0}&\bar{0}&\bar{1}&\bar{1}&\bar{0}\end{bmatrix}$	16
392	$T_1^2T_2+T_3T_4+T_5^2$	$\begin{bmatrix}1&4&2&4&3\\ \bar{0}&\bar{0}&\bar{1}&\bar{1}&\bar{0}\end{bmatrix}$	16
400	$T_1^6T_2+T_3^2T_4+T_5^2$	$\begin{bmatrix}1&2&1&6&4\\ \bar{0}&\bar{0}&\bar{1}&\bar{0}&\bar{1}\end{bmatrix}$	18
401	$T_1^4T_2^2+T_3^2T_4+T_5^2$	$\begin{bmatrix}1&2&1&6&4\\ \bar{0}&\bar{0}&\bar{1}&\bar{0}&\bar{1}\end{bmatrix}$	18
402	$T_1^3T_2^2+T_3^2T_4+T_5^2$	$\begin{bmatrix}2&1&1&6&4\\ \bar{0}&\bar{1}&\bar{0}&\bar{0}&\bar{1}\end{bmatrix}$	18
403	$T_1^4T_2^4+T_3T_4+T_5^2$	$\begin{bmatrix}1&1&2&6&4\\ \bar{1}&\bar{0}&\bar{0}&\bar{0}&\bar{1}\end{bmatrix}$	18
405	$T_1^3T_2+T_3T_4+T_5^5$	$\begin{bmatrix}1&2&1&4&1\\ \bar{1}&\bar{1}&\bar{0}&\bar{0}&\bar{0}\end{bmatrix}$	20
408	$T_1^5T_2+T_3T_4+T_5^3$	$\begin{bmatrix}1&4&1&8&3\\ \bar{0}&\bar{1}&\bar{1}&\bar{0}&\bar{1}\end{bmatrix}$	24
409	$T_1T_2+T_3T_4+T_5^3$	$\begin{bmatrix}1&2&1&2&1\\ \bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{0}\end{bmatrix}$	24
410	$T_1T_2+T_3T_4+T_5^3$	$\begin{bmatrix}2&1&1&2&1\\ \bar{1}&\bar{1}&\bar{0}&\bar{0}&\bar{0}\end{bmatrix}$	24
411	$T_1T_2+T_3T_4+T_5^2$	$\begin{bmatrix}1&3&1&3&2\\ \bar{0}&\bar{0}&\bar{1}&\bar{1}&\bar{0}\end{bmatrix}$	24

Classification list 2.12.6. *Non-toric, $\mathbb{Q}$-factorial, Gorenstein, log terminal Fano threefolds of Picard number one with an effective two-torus action: Specifying data for divisor class group $\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$ and format $(2,2,1,1,0)$.*

ID	relations	gd-matrix	$-\mathcal{K}^3$
269	$T_1^2T_2+T_3T_4+T_5^4$, $\lambda T_3T_4+T_5^4+T_6^2$	$\begin{bmatrix} 1&2&2&2&1&2 \\ \bar{0}&\bar{0}&\bar{0}&\bar{0}&\bar{1}&\bar{1} \end{bmatrix}$	4
270	$T_1^2T_2+T_3T_4+T_5^2$, $\lambda T_3T_4+T_5^2+T_6^2$	$\begin{bmatrix} 2&2&2&4&3&3 \\ \bar{1}&\bar{0}&\bar{0}&\bar{0}&\bar{0}&\bar{1} \end{bmatrix}$	4
349	$T_1T_2+T_3T_4+T_5^4$, $\lambda T_3T_4+T_5^4+T_6^2$	$\begin{bmatrix} 1&3&1&3&1&2 \\ \bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{0}&\bar{1} \end{bmatrix}$	12

Classification list 2.12.7. *Non-toric, $\mathbb{Q}$-factorial, Gorenstein, log terminal Fano threefolds of Picard number one with an effective two-torus action: Specifying data for divisor class group $\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$ and format $(3,1,1,0)$.*

ID	relations	gd-matrix	$-\mathcal{K}^3$
248	$T_1^3T_2^3T_3+T_4^4+T_5^2$	$\begin{bmatrix} 1&1&2&2&4 \\ \bar{0}&\bar{0}&\bar{0}&\bar{1}&\bar{1} \end{bmatrix}$	2
249	$T_1^3T_2^2T_3+T_4^4+T_5^2$	$\begin{bmatrix} 1&2&1&2&4 \\ \bar{0}&\bar{0}&\bar{0}&\bar{1}&\bar{1} \end{bmatrix}$	2
250	$T_1^3T_2T_3+T_4^4+T_5^2$	$\begin{bmatrix} 2&1&1&2&4 \\ \bar{0}&\bar{0}&\bar{0}&\bar{1}&\bar{1} \end{bmatrix}$	2
251	$T_1^2T_2^2T_3+T_4^8+T_5^2$	$\begin{bmatrix} 2&1&2&1&4 \\ \bar{0}&\bar{0}&\bar{0}&\bar{1}&\bar{1} \end{bmatrix}$	2
263	$T_1^4T_2^4T_3+T_4^3+T_5^2$	$\begin{bmatrix} 1&1&4&4&6 \\ \bar{1}&\bar{0}&\bar{0}&\bar{0}&\bar{1} \end{bmatrix}$	4
264	$T_1^4T_2T_3+T_4^3+T_5^2$	$\begin{bmatrix} 1&1&1&2&3 \\ \bar{0}&\bar{1}&\bar{1}&\bar{0}&\bar{0} \end{bmatrix}$	4
265	$T_1^3T_2^2T_3+T_4^3+T_5^2$	$\begin{bmatrix} 1&1&1&2&3 \\ \bar{1}&\bar{0}&\bar{1}&\bar{0}&\bar{0} \end{bmatrix}$	4
266	$T_1^2T_2^2T_3+T_4^6+T_5^2$	$\begin{bmatrix} 1&1&2&1&3 \\ \bar{1}&\bar{0}&\bar{0}&\bar{0}&\bar{1} \end{bmatrix}$	4
267	$T_1^2T_2^2T_3+T_4^5+T_5^2$	$\begin{bmatrix} 2&1&4&2&5 \\ \bar{1}&\bar{0}&\bar{0}&\bar{0}&\bar{1} \end{bmatrix}$	4
268	$T_1^2T_2T_3+T_4^{10}+T_5^2$	$\begin{bmatrix} 2&2&4&1&5 \\ \bar{1}&\bar{0}&\bar{0}&\bar{0}&\bar{1} \end{bmatrix}$	4
288	$T_1^2T_2^2T_3+T_4^4+T_5^2$	$\begin{bmatrix} 2&1&6&3&6 \\ \bar{0}&\bar{0}&\bar{0}&\bar{1}&\bar{1} \end{bmatrix}$	6
289	$T_1^4T_2^2T_3+T_4^2+T_5^2$	$\begin{bmatrix} 1&3&2&6&6 \\ \bar{0}&\bar{1}&\bar{0}&\bar{1}&\bar{0} \end{bmatrix}$	6
290	$T_1T_2T_3+T_4^4+T_5^2$	$\begin{bmatrix} 2&3&3&2&4 \\ \bar{0}&\bar{0}&\bar{0}&\bar{1}&\bar{1} \end{bmatrix}$	6
311	$T_1^3T_2T_3+T_4^4+T_5^2$	$\begin{bmatrix} 1&4&1&2&4 \\ \bar{0}&\bar{0}&\bar{0}&\bar{1}&\bar{1} \end{bmatrix}$	8
312	$T_1^2T_2^2T_3+T_4^4+T_5^2$	$\begin{bmatrix} 1&1&4&2&4 \\ \bar{0}&\bar{0}&\bar{0}&\bar{1}&\bar{1} \end{bmatrix}$	8
313	$T_1^2T_2^2T_3+T_4^4+T_5^2$	$\begin{bmatrix} 1&1&4&2&4 \\ \bar{1}&\bar{0}&\bar{0}&\bar{0}&\bar{1} \end{bmatrix}$	8
314	$T_1^2T_2T_3+T_4^8+T_5^2$	$\begin{bmatrix} 1&2&4&1&4 \\ \bar{0}&\bar{0}&\bar{0}&\bar{1}&\bar{1} \end{bmatrix}$	8
315	$T_1^2T_2T_3+T_4^4+T_5^2$	$\begin{bmatrix} 1&1&1&1&2 \\ \bar{0}&\bar{0}&\bar{0}&\bar{1}&\bar{1} \end{bmatrix}$	8
316	$T_1^2T_2T_3+T_4^4+T_5^2$	$\begin{bmatrix} 1&1&1&1&2 \\ \bar{0}&\bar{1}&\bar{1}&\bar{1}&\bar{1} \end{bmatrix}$	8
317	$T_1^2T_2T_3+T_4^3+T_5^2$	$\begin{bmatrix} 1&2&2&2&3 \\ \bar{0}&\bar{1}&\bar{1}&\bar{0}&\bar{0} \end{bmatrix}$	8
318	$T_1^4T_2^2T_3+T_4^2+T_5^2$	$\begin{bmatrix} 1&1&2&4&4 \\ \bar{0}&\bar{1}&\bar{0}&\bar{1}&\bar{0} \end{bmatrix}$	8
319	$T_1^3T_2^2T_3+T_4^2+T_5^2$	$\begin{bmatrix} 1&2&1&4&4 \\ \bar{0}&\bar{1}&\bar{0}&\bar{1}&\bar{0} \end{bmatrix}$	8
337	$T_1T_2T_3+T_4^8+T_5^2$	$\begin{bmatrix} 1&5&10&2&8 \\ \bar{0}&\bar{0}&\bar{0}&\bar{1}&\bar{1} \end{bmatrix}$	10
347	$T_1^2T_2^2T_3+T_4^3+T_5^2$	$\begin{bmatrix} 2&1&12&6&9 \\ \bar{1}&\bar{0}&\bar{0}&\bar{0}&\bar{1} \end{bmatrix}$	12
348	$T_1T_2T_3+T_4^4+T_5^2$	$\begin{bmatrix} 1&3&12&4&8 \\ \bar{0}&\bar{0}&\bar{0}&\bar{1}&\bar{1} \end{bmatrix}$	12
380	$T_1^2T_2^2T_3+T_4^3+T_5^2$	$\begin{bmatrix} 1&1&8&4&6 \\ \bar{1}&\bar{0}&\bar{0}&\bar{0}&\bar{1} \end{bmatrix}$	16
399	$T_1T_2T_3+T_4^4+T_5^2$	$\begin{bmatrix} 1&1&6&2&4 \\ \bar{0}&\bar{0}&\bar{0}&\bar{1}&\bar{1} \end{bmatrix}$	18

Classification list 2.12.8. *Non-toric, $\mathbb{Q}$-factorial, Gorenstein, log terminal Fano threefolds of Picard number one with an effective two-torus action: Specifying data for divisor class group $\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$ and format $(2,1,1,1)$.*

ID	relations	gd-matrix	$-\mathcal{K}^3$
246	$T_1^3T_2^2+T_3^4+T_4^2$	$\begin{bmatrix} 2&1&2&4&1\\ \bar{0}&\bar{1}&\bar{0}&\bar{1}&\bar{0} \end{bmatrix}$	2
247	$T_1^3T_2^2+T_3^4+T_4^2$	$\begin{bmatrix} 2&1&2&4&1\\ \bar{0}&\bar{0}&\bar{1}&\bar{1}&\bar{0} \end{bmatrix}$	2
256	$T_1^5T_2+T_3^3+T_4^2$	$\begin{bmatrix} 1&1&2&3&1\\ \bar{0}&\bar{0}&\bar{0}&\bar{1}&\bar{1} \end{bmatrix}$	4
257	$T_1^4T_2^2+T_3^3+T_4^2$	$\begin{bmatrix} 1&4&4&6&1\\ \bar{1}&\bar{0}&\bar{0}&\bar{1}&\bar{0} \end{bmatrix}$	4
258	$T_1^4T_2^2+T_3^3+T_4^2$	$\begin{bmatrix} 1&1&2&3&1\\ \bar{1}&\bar{1}&\bar{0}&\bar{0}&\bar{0} \end{bmatrix}$	4
259	$T_1^4T_2^2+T_3^3+T_4^2$	$\begin{bmatrix} 1&1&2&3&1\\ \bar{1}&\bar{0}&\bar{0}&\bar{1}&\bar{0} \end{bmatrix}$	4
260	$T_1^2T_2^2+T_3^5+T_4^2$	$\begin{bmatrix} 4&1&2&5&2\\ \bar{0}&\bar{0}&\bar{0}&\bar{1}&\bar{1} \end{bmatrix}$	4
261	$T_1^2T_2+T_3^{12}+T_4^2$	$\begin{bmatrix} 4&4&1&6&1\\ \bar{0}&\bar{0}&\bar{1}&\bar{1}&\bar{0} \end{bmatrix}$	4
262	$T_1^2T_2+T_3^{10}+T_4^2$	$\begin{bmatrix} 4&2&1&5&2\\ \bar{0}&\bar{0}&\bar{0}&\bar{1}&\bar{1} \end{bmatrix}$	4
277	$T_1^3T_2+T_3^4+T_4^2$	$\begin{bmatrix} 2&6&3&6&1\\ \bar{0}&\bar{0}&\bar{1}&\bar{1}&\bar{0} \end{bmatrix}$	6
278	$T_1^2T_2+T_3^{12}+T_4^2$	$\begin{bmatrix} 3&6&1&6&2\\ \bar{0}&\bar{0}&\bar{1}&\bar{1}&\bar{0} \end{bmatrix}$	6
279	$T_1^2T_2+T_3^6+T_4^2$	$\begin{bmatrix} 3&6&2&6&1\\ \bar{1}&\bar{0}&\bar{0}&\bar{1}&\bar{0} \end{bmatrix}$	6
280	$T_1^2T_2+T_3^6+T_4^2$	$\begin{bmatrix} 3&6&2&6&1\\ \bar{1}&\bar{0}&\bar{1}&\bar{0}&\bar{0} \end{bmatrix}$	6
281	$T_1^2T_2+T_3^4+T_4^2$	$\begin{bmatrix} 3&2&2&4&3\\ \bar{0}&\bar{0}&\bar{0}&\bar{1}&\bar{1} \end{bmatrix}$	6
282	$T_1^2T_2+T_3^4+T_4^2$	$\begin{bmatrix} 3&2&2&4&3\\ \bar{1}&\bar{0}&\bar{1}&\bar{1}&\bar{1} \end{bmatrix}$	6
283	$T_1^{10}T_2+T_3^2+T_4^2$	$\begin{bmatrix} 1&2&6&6&3\\ \bar{0}&\bar{0}&\bar{1}&\bar{0}&\bar{1} \end{bmatrix}$	6
284	$T_1^6T_2^3+T_3^2+T_4^2$	$\begin{bmatrix} 1&2&6&6&3\\ \bar{0}&\bar{0}&\bar{1}&\bar{0}&\bar{1} \end{bmatrix}$	6
285	$T_1^5T_2^2+T_3^2+T_4^2$	$\begin{bmatrix} 2&1&6&6&3\\ \bar{0}&\bar{1}&\bar{1}&\bar{0}&\bar{0} \end{bmatrix}$	6
286	$T_1^3T_2^2+T_3^2+T_4^2$	$\begin{bmatrix} 2&3&6&6&1\\ \bar{0}&\bar{1}&\bar{1}&\bar{0}&\bar{0} \end{bmatrix}$	6
287	$T_1T_2+T_3^6+T_4^4$	$\begin{bmatrix} 6&6&2&3&1\\ \bar{0}&\bar{0}&\bar{1}&\bar{1}&\bar{0} \end{bmatrix}$	6
297	$T_1^4T_2+T_3^3+T_4^2$	$\begin{bmatrix} 1&2&2&3&2\\ \bar{0}&\bar{0}&\bar{0}&\bar{1}&\bar{1} \end{bmatrix}$	8
298	$T_1^3T_2+T_3^4+T_4^2$	$\begin{bmatrix} 1&1&1&2&1\\ \bar{0}&\bar{0}&\bar{1}&\bar{1}&\bar{0} \end{bmatrix}$	8
299	$T_1^3T_2+T_3^4+T_4^2$	$\begin{bmatrix} 1&1&1&2&1\\ \bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{0} \end{bmatrix}$	8
300	$T_1^2T_2^2+T_3^3+T_4^2$	$\begin{bmatrix} 2&1&2&3&2\\ \bar{0}&\bar{0}&\bar{0}&\bar{1}&\bar{1} \end{bmatrix}$	8
301	$T_1^2T_2^2+T_3^3+T_4^2$	$\begin{bmatrix} 2&1&2&3&2\\ \bar{1}&\bar{0}&\bar{0}&\bar{0}&\bar{1} \end{bmatrix}$	8
302	$T_1^2T_2+T_3^8+T_4^2$	$\begin{bmatrix} 2&4&1&4&1\\ \bar{1}&\bar{0}&\bar{0}&\bar{1}&\bar{0} \end{bmatrix}$	8
303	$T_1^2T_2+T_3^8+T_4^2$	$\begin{bmatrix} 2&4&1&4&1\\ \bar{0}&\bar{0}&\bar{1}&\bar{1}&\bar{0} \end{bmatrix}$	8

ID	relations	gd-matrix	$-\mathcal{K}^3$
304	$T_1^2T_2+T_3^6+T_4^2$	$\begin{bmatrix} 2&2&1&3&2\\ \bar{1}&\bar{0}&\bar{0}&\bar{1}&\bar{0} \end{bmatrix}$	8
305	$T_1^2T_2+T_3^6+T_4^2$	$\begin{bmatrix} 2&2&1&3&2\\ \bar{0}&\bar{0}&\bar{0}&\bar{1}&\bar{1} \end{bmatrix}$	8
306	$T_1^2T_2+T_3^3+T_4^2$	$\begin{bmatrix} 2&2&2&3&1\\ \bar{1}&\bar{0}&\bar{0}&\bar{1}&\bar{0} \end{bmatrix}$	8
307	$T_1^7T_2+T_3^2+T_4^2$	$\begin{bmatrix} 1&1&4&4&2\\ \bar{0}&\bar{0}&\bar{1}&\bar{0}&\bar{1} \end{bmatrix}$	8
308	$T_1^6T_2+T_3^2+T_4^2$	$\begin{bmatrix} 1&2&4&4&1\\ \bar{1}&\bar{0}&\bar{1}&\bar{0}&\bar{0} \end{bmatrix}$	8
309	$T_1^5T_2^3+T_3^2+T_4^2$	$\begin{bmatrix} 1&1&4&4&2\\ \bar{0}&\bar{0}&\bar{1}&\bar{0}&\bar{1} \end{bmatrix}$	8
310	$T_1^3T_2^2+T_3^2+T_4^2$	$\begin{bmatrix} 2&1&4&4&1\\ \bar{0}&\bar{1}&\bar{1}&\bar{0}&\bar{0} \end{bmatrix}$	8
336	$T_1T_2+T_3^6+T_4^4$	$\begin{bmatrix} 2&10&2&3&5\\ \bar{0}&\bar{0}&\bar{1}&\bar{0}&\bar{1} \end{bmatrix}$	10
340	$T_1^3T_2+T_3^3+T_4^2$	$\begin{bmatrix} 1&3&2&3&3\\ \bar{0}&\bar{0}&\bar{0}&\bar{1}&\bar{1} \end{bmatrix}$	12
341	$T_1^4T_2+T_3^2+T_4^2$	$\begin{bmatrix} 1&2&3&3&3\\ \bar{0}&\bar{0}&\bar{0}&\bar{1}&\bar{1} \end{bmatrix}$	12
342	$T_1T_2+T_3^{18}+T_4^2$	$\begin{bmatrix} 6&12&1&9&2\\ \bar{0}&\bar{0}&\bar{0}&\bar{1}&\bar{1} \end{bmatrix}$	12
343	$T_1T_2+T_3^{16}+T_4^2$	$\begin{bmatrix} 4&12&1&8&3\\ \bar{0}&\bar{0}&\bar{0}&\bar{1}&\bar{1} \end{bmatrix}$	12
344	$T_1T_2+T_3^{10}+T_4^2$	$\begin{bmatrix} 4&6&1&5&6\\ \bar{0}&\bar{0}&\bar{0}&\bar{1}&\bar{1} \end{bmatrix}$	12
345	$T_1T_2+T_3^3+T_4^2$	$\begin{bmatrix} 3&3&2&3&1\\ \bar{1}&\bar{1}&\bar{0}&\bar{0}&\bar{0} \end{bmatrix}$	12
346	$T_1T_2+T_3^2+T_4^2$	$\begin{bmatrix} 2&4&3&3&6\\ \bar{0}&\bar{0}&\bar{0}&\bar{1}&\bar{1} \end{bmatrix}$	12
365	$T_1^4T_2+T_3^3+T_4^2$	$\begin{bmatrix} 1&8&4&6&1\\ \bar{1}&\bar{0}&\bar{0}&\bar{1}&\bar{0} \end{bmatrix}$	16
366	$T_1^2T_2+T_3^6+T_4^2$	$\begin{bmatrix} 1&4&1&3&1\\ \bar{1}&\bar{0}&\bar{0}&\bar{1}&\bar{0} \end{bmatrix}$	16
367	$T_1^2T_2+T_3^6+T_4^2$	$\begin{bmatrix} 1&4&1&3&1\\ \bar{1}&\bar{0}&\bar{1}&\bar{0}&\bar{0} \end{bmatrix}$	16
368	$T_1^2T_2+T_3^5+T_4^2$	$\begin{bmatrix} 1&8&2&5&2\\ \bar{0}&\bar{0}&\bar{0}&\bar{1}&\bar{1} \end{bmatrix}$	16
369	$T_1^2T_2+T_3^4+T_4^2$	$\begin{bmatrix} 1&2&1&2&2\\ \bar{0}&\bar{0}&\bar{1}&\bar{1}&\bar{0} \end{bmatrix}$	16
370	$T_1^2T_2+T_3^4+T_4^2$	$\begin{bmatrix} 1&2&1&2&2\\ \bar{0}&\bar{0}&\bar{0}&\bar{1}&\bar{1} \end{bmatrix}$	16
371	$T_1^2T_2+T_3^3+T_4^2$	$\begin{bmatrix} 1&4&2&3&4\\ \bar{1}&\bar{0}&\bar{0}&\bar{0}&\bar{1} \end{bmatrix}$	16
372	$T_1^3T_2+T_3^2+T_4^2$	$\begin{bmatrix} 1&1&2&2&2\\ \bar{0}&\bar{0}&\bar{0}&\bar{1}&\bar{1} \end{bmatrix}$	16
373	$T_1^2T_2+T_3^2+T_4^2$	$\begin{bmatrix} 1&2&2&2&1\\ \bar{1}&\bar{0}&\bar{0}&\bar{1}&\bar{0} \end{bmatrix}$	16
374	$T_1T_2+T_3^{12}+T_4^2$	$\begin{bmatrix} 4&8&1&6&1\\ \bar{0}&\bar{0}&\bar{1}&\bar{1}&\bar{0} \end{bmatrix}$	16
375	$T_1T_2+T_3^{10}+T_4^2$	$\begin{bmatrix} 2&8&1&5&2\\ \bar{0}&\bar{0}&\bar{0}&\bar{1}&\bar{1} \end{bmatrix}$	16
376	$T_1T_2+T_3^6+T_4^2$	$\begin{bmatrix} 2&4&1&3&4\\ \bar{1}&\bar{1}&\bar{0}&\bar{1}&\bar{1} \end{bmatrix}$	16

ID	relations	gd-matrix	$-\mathcal{K}^3$
377	$T_1T_2+T_3^6+T_4^2$	$\begin{bmatrix} 2 & 4 & 1 & 3 & 4 \\ \bar{0} & \bar{0} & \bar{0} & \bar{1} & \bar{1} \end{bmatrix}$	16
378	$T_1T_2+T_3^4+T_4^2$	$\begin{bmatrix} 2 & 2 & 1 & 2 & 1 \\ \bar{1} & \bar{1} & \bar{1} & \bar{1} & \bar{0} \end{bmatrix}$	16
379	$T_1T_2+T_3^4+T_4^2$	$\begin{bmatrix} 2 & 2 & 1 & 2 & 1 \\ \bar{0} & \bar{0} & \bar{1} & \bar{1} & \bar{0} \end{bmatrix}$	16
393	$T_1^2T_2+T_3^8+T_4^2$	$\begin{bmatrix} 1 & 6 & 1 & 4 & 2 \\ \bar{0} & \bar{0} & \bar{1} & \bar{1} & \bar{0} \end{bmatrix}$	18
394	$T_1^2T_2+T_3^4+T_4^2$	$\begin{bmatrix} 1 & 6 & 2 & 4 & 1 \\ \bar{1} & \bar{0} & \bar{0} & \bar{1} & \bar{0} \end{bmatrix}$	18
395	$T_1^2T_2+T_3^4+T_4^2$	$\begin{bmatrix} 1 & 6 & 2 & 4 & 1 \\ \bar{0} & \bar{0} & \bar{1} & \bar{1} & \bar{0} \end{bmatrix}$	18
396	$T_1^2T_2+T_3^2+T_4^2$	$\begin{bmatrix} 1 & 2 & 2 & 2 & 3 \\ \bar{1} & \bar{0} & \bar{1} & \bar{0} & \bar{0} \end{bmatrix}$	18

ID	relations	gd-matrix	$-\mathcal{K}^3$
397	$T_1T_2+T_3^8+T_4^2$	$\begin{bmatrix} 2 & 6 & 1 & 4 & 1 \\ \bar{0} & \bar{0} & \bar{1} & \bar{1} & \bar{0} \end{bmatrix}$	18
398	$T_1T_2+T_3^4+T_4^2$	$\begin{bmatrix} 2 & 2 & 1 & 2 & 3 \\ \bar{0} & \bar{0} & \bar{0} & \bar{1} & \bar{1} \end{bmatrix}$	18
404	$T_1T_2+T_3^3+T_4^2$	$\begin{bmatrix} 1 & 5 & 2 & 3 & 5 \\ \bar{1} & \bar{1} & \bar{0} & \bar{0} & \bar{0} \end{bmatrix}$	20
406	$T_1T_2+T_3^4+T_4^2$	$\begin{bmatrix} 1 & 3 & 1 & 2 & 3 \\ \bar{1} & \bar{1} & \bar{0} & \bar{1} & \bar{1} \end{bmatrix}$	24
407	$T_1T_2+T_3^4+T_4^2$	$\begin{bmatrix} 1 & 3 & 1 & 2 & 3 \\ \bar{0} & \bar{0} & \bar{0} & \bar{1} & \bar{1} \end{bmatrix}$	24
412	$T_1T_2+T_3^2+T_4^2$	$\begin{bmatrix} 1 & 1 & 1 & 1 & 2 \\ \bar{0} & \bar{0} & \bar{0} & \bar{1} & \bar{1} \end{bmatrix}$	32

Classification list 2.12.9. *Non-toric, $\mathbb{Q}$-factorial, Gorenstein, log terminal Fano threefolds of Picard number one with an effective two-torus action: Specifying data for divisor class group $\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$ and format $(2,1,1,1,1)$.*

ID	relations	gd-matrix	$-\mathcal{K}^3$
255	$T_1T_2+T_3^3+T_4^2,$ $\lambda T_3^3+T_4^2+T_5^2$	$\begin{bmatrix} 4 & 2 & 2 & 3 & 3 & 2 \\ \bar{0} & \bar{0} & \bar{0} & \bar{1} & \bar{0} & \bar{1} \end{bmatrix}$	4

Classification list 2.12.10. *Non-toric, $\mathbb{Q}$-factorial, Gorenstein, log terminal Fano threefolds of Picard number one with an effective two-torus action: Specifying data for divisor class group $\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$ and format $(1,1,1,2)$.*

ID	relations	gd-matrix	$-\mathcal{K}^3$
338	$T_1^3+T_2^2+T_3^2$	$\begin{bmatrix} 2 & 3 & 3 & 6 & 4 \\ \bar{0} & \bar{0} & \bar{1} & \bar{1} & \bar{0} \end{bmatrix}$	12

ID	relations	gd-matrix	$-\mathcal{K}^3$
339	$T_1^3+T_2^2+T_3^2$	$\begin{bmatrix} 2 & 3 & 3 & 3 & 1 \\ \bar{0} & \bar{1} & \bar{0} & \bar{1} & \bar{0} \end{bmatrix}$	12

Classification list 2.12.11. *Non-toric, $\mathbb{Q}$-factorial, Gorenstein, log terminal Fano threefolds of Picard number one with an effective two-torus action: Specifying data for divisor class group $\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$ and format $(2,2,1,0)$.*

ID	relations	gd-matrix	$-\mathcal{K}^3$
480	$T_1^4T_2^2+T_3^2T_4^2+T_5^2$	$\begin{bmatrix} 1 & 1 & 2 & 1 & 3 \\ \bar{0} & \bar{1} & \bar{0} & \bar{1} & \bar{0} \\ \bar{0} & \bar{0} & \bar{0} & \bar{1} & \bar{1} \end{bmatrix}$	2
491	$T_1^6T_2^2+T_3^2T_4+T_5^2$	$\begin{bmatrix} 1 & 1 & 2 & 4 & 4 \\ \bar{0} & \bar{1} & \bar{0} & \bar{0} & \bar{1} \\ \bar{0} & \bar{1} & \bar{1} & \bar{0} & \bar{0} \end{bmatrix}$	4
492	$T_1^4T_2^4+T_3^2T_4+T_5^2$	$\begin{bmatrix} 1 & 1 & 2 & 4 & 4 \\ \bar{1} & \bar{0} & \bar{1} & \bar{0} & \bar{0} \\ \bar{1} & \bar{0} & \bar{0} & \bar{0} & \bar{1} \end{bmatrix}$	4
493	$T_1^2T_2^2+T_3^2T_4^2+T_5^2$	$\begin{bmatrix} 1 & 1 & 1 & 1 & 2 \\ \bar{1} & \bar{0} & \bar{1} & \bar{0} & \bar{0} \\ \bar{0} & \bar{0} & \bar{0} & \bar{1} & \bar{1} \end{bmatrix}$	4
494	$T_1^2T_2^2+T_3^2T_4+T_5^4$	$\begin{bmatrix} 1 & 1 & 1 & 2 & 1 \\ \bar{1} & \bar{1} & \bar{0} & \bar{0} & \bar{0} \\ \bar{1} & \bar{0} & \bar{1} & \bar{0} & \bar{0} \end{bmatrix}$	4

ID	relations	gd-matrix	$-\mathcal{K}^3$
495	$T_1^2T_2+T_3^2T_4^2+T_5^2$	$\begin{bmatrix} 2 & 2 & 2 & 1 & 3 \\ \bar{0} & \bar{0} & \bar{1} & \bar{0} & \bar{1} \\ \bar{1} & \bar{0} & \bar{0} & \bar{0} & \bar{1} \end{bmatrix}$	4
501	$T_1^3T_2+T_3T_4+T_5^3$	$\begin{bmatrix} 1 & 3 & 3 & 3 & 2 \\ \bar{0} & \bar{1} & \bar{0} & \bar{1} & \bar{1} \\ \bar{0} & \bar{1} & \bar{1} & \bar{0} & \bar{1} \end{bmatrix}$	6
518	$T_1^4T_2^2+T_3^2T_4+T_5^2$	$\begin{bmatrix} 1 & 1 & 1 & 4 & 3 \\ \bar{0} & \bar{1} & \bar{0} & \bar{0} & \bar{1} \\ \bar{0} & \bar{1} & \bar{1} & \bar{0} & \bar{0} \end{bmatrix}$	8
520	$T_1T_2+T_3T_4+T_5^3$	$\begin{bmatrix} 1 & 2 & 1 & 2 & 1 \\ \bar{1} & \bar{1} & \bar{1} & \bar{1} & \bar{0} \\ \bar{1} & \bar{1} & \bar{0} & \bar{0} & \bar{0} \end{bmatrix}$	12

Classification list 2.12.12. *Non-toric, $\mathbb{Q}$-factorial, Gorenstein, log terminal Fano threefolds of Picard number one with an effective two-torus action: Specifying data for divisor class group $\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$ and format* $(2,2,1,1,0)$.

ID	relations	gd-matrix	$-\mathcal{K}^3$
479	$T_1^2T_2^2+T_3T_4+T_5^2$, $\lambda T_3T_4+T_5^2+T_6^2$	$\begin{bmatrix} 1 & 1 & 2 & 2 & 2 & 2 \\ \bar{0} & \bar{1} & \bar{0} & \bar{0} & \bar{1} & \bar{0} \\ \bar{0} & \bar{0} & \bar{0} & \bar{0} & \bar{1} & \bar{1} \end{bmatrix}$	2

Classification list 2.12.13. *Non-toric, $\mathbb{Q}$-factorial, Gorenstein, log terminal Fano threefolds of Picard number one with an effective two-torus action: Specifying data for divisor class group $\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$ and format* $(2,2,1,1,1,0)$.

ID	relations	gd-matrix	$-\mathcal{K}^3$
478	$T_1T_2+T_3T_4+T_5^2$, $\lambda_1T_3T_4+T_5^2+T_6^2$, $\lambda_2T_5^2+T_6^2+T_7^2$	$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ \bar{0} & \bar{0} & \bar{0} & \bar{0} & \bar{1} & \bar{1} & \bar{0} \\ \bar{1} & \bar{1} & \bar{1} & \bar{1} & \bar{1} & \bar{0} & \bar{0} \end{bmatrix}$	2

Classification list 2.12.14. *Non-toric, $\mathbb{Q}$-factorial, Gorenstein, log terminal Fano threefolds of Picard number one with an effective two-torus action: Specifying data for divisor class group $\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$ and format* $(3,1,1,0)$.

ID	relations	gd-matrix	$-\mathcal{K}^3$
477	$T_1^2T_2^2T_3+T_4^6+T_5^2$	$\begin{bmatrix} 1 & 1 & 2 & 1 & 3 \\ \bar{0} & \bar{0} & \bar{0} & \bar{1} & \bar{1} \\ \bar{1} & \bar{0} & \bar{0} & \bar{0} & \bar{1} \end{bmatrix}$	2
489	$T_1^2T_2^2T_3+T_4^4+T_5^2$	$\begin{bmatrix} 1 & 1 & 4 & 2 & 4 \\ \bar{1} & \bar{0} & \bar{0} & \bar{1} & \bar{0} \\ \bar{0} & \bar{0} & \bar{0} & \bar{1} & \bar{1} \end{bmatrix}$	4
490	$T_1^2T_2T_3+T_4^4+T_5^2$	$\begin{bmatrix} 1 & 1 & 1 & 1 & 2 \\ \bar{0} & \bar{1} & \bar{1} & \bar{0} & \bar{0} \\ \bar{0} & \bar{1} & \bar{1} & \bar{1} & \bar{1} \end{bmatrix}$	4

Classification list 2.12.15. *Non-toric, $\mathbb{Q}$-factorial, Gorenstein, log terminal Fano threefolds of Picard number one with an effective two-torus action: Specifying data for divisor class group $\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$ and format* $(3,1,1,1,0)$.

ID	relations	gd-matrix	$-\mathcal{K}^3$
476	$T_1T_2T_3+T_4^2+T_5^2$, $\lambda T_4^2+T_5^2+T_6^2$	$\begin{bmatrix} 1 & 1 & 2 & 2 & 2 & 2 \\ \bar{0} & \bar{0} & \bar{0} & \bar{1} & \bar{1} & \bar{0} \\ \bar{0} & \bar{0} & \bar{0} & \bar{0} & \bar{1} & \bar{1} \end{bmatrix}$	2

Classification list 2.12.16. *Non-toric, $\mathbb{Q}$-factorial, Gorenstein, log terminal Fano threefolds of Picard number one with an effective two-torus action: Specifying data for divisor class group $\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$ and format* $(2,1,1,1)$.

ID	relations	gd-matrix	$-\mathcal{K}^3$
474	$T_1^4T_2^2+T_3^3+T_4^2$	$\begin{bmatrix} 1&1&2&3&1 \\ \bar{0}&\bar{1}&\bar{0}&\bar{1}&\bar{0} \\ \bar{1}&\bar{0}&\bar{0}&\bar{1}&\bar{0} \end{bmatrix}$	2
475	$T_1^2T_2^2+T_3^6+T_4^2$	$\begin{bmatrix} 2&1&1&3&1 \\ \bar{0}&\bar{1}&\bar{1}&\bar{0}&\bar{0} \\ \bar{0}&\bar{0}&\bar{1}&\bar{1}&\bar{0} \end{bmatrix}$	2
481	$T_1^3T_2+T_3^4+T_4^2$	$\begin{bmatrix} 1&1&1&2&1 \\ \bar{1}&\bar{1}&\bar{0}&\bar{0}&\bar{0} \\ \bar{0}&\bar{0}&\bar{1}&\bar{1}&\bar{0} \end{bmatrix}$	4
482	$T_1^2T_2^2+T_3^4+T_4^2$	$\begin{bmatrix} 1&1&1&2&1 \\ \bar{0}&\bar{0}&\bar{1}&\bar{1}&\bar{0} \\ \bar{0}&\bar{1}&\bar{1}&\bar{0}&\bar{0} \end{bmatrix}$	4
483	$T_1^2T_2^2+T_3^4+T_4^2$	$\begin{bmatrix} 1&1&1&2&1 \\ \bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{0} \\ \bar{0}&\bar{1}&\bar{1}&\bar{0}&\bar{0} \end{bmatrix}$	4
484	$T_1^2T_2^2+T_3^3+T_4^2$	$\begin{bmatrix} 2&1&2&3&2 \\ \bar{1}&\bar{0}&\bar{0}&\bar{1}&\bar{0} \\ \bar{1}&\bar{0}&\bar{0}&\bar{0}&\bar{1} \end{bmatrix}$	4
485	$T_1^2T_2+T_3^8+T_4^2$	$\begin{bmatrix} 2&4&1&4&1 \\ \bar{1}&\bar{0}&\bar{1}&\bar{0}&\bar{0} \\ \bar{1}&\bar{0}&\bar{0}&\bar{1}&\bar{0} \end{bmatrix}$	4
486	$T_1^2T_2+T_3^6+T_4^2$	$\begin{bmatrix} 2&2&1&3&2 \\ \bar{1}&\bar{0}&\bar{0}&\bar{0}&\bar{1} \\ \bar{1}&\bar{0}&\bar{0}&\bar{1}&\bar{0} \end{bmatrix}$	4
487	$T_1^6T_2^2+T_3^2+T_4^2$	$\begin{bmatrix} 1&1&4&4&2 \\ \bar{0}&\bar{1}&\bar{1}&\bar{0}&\bar{0} \\ \bar{0}&\bar{1}&\bar{0}&\bar{0}&\bar{1} \end{bmatrix}$	4
488	$T_1^4T_2^2+T_3^2+T_4^2$	$\begin{bmatrix} 1&2&4&4&1 \\ \bar{0}&\bar{1}&\bar{1}&\bar{0}&\bar{0} \\ \bar{1}&\bar{1}&\bar{0}&\bar{0}&\bar{0} \end{bmatrix}$	4
499	$T_1^4T_2+T_3^2+T_4^2$	$\begin{bmatrix} 1&2&3&3&3 \\ \bar{0}&\bar{0}&\bar{1}&\bar{1}&\bar{0} \\ \bar{0}&\bar{0}&\bar{1}&\bar{0}&\bar{1} \end{bmatrix}$	6
500	$T_1^2T_2^2+T_3^2+T_4^2$	$\begin{bmatrix} 2&1&3&3&3 \\ \bar{0}&\bar{0}&\bar{0}&\bar{1}&\bar{1} \\ \bar{0}&\bar{0}&\bar{1}&\bar{1}&\bar{0} \end{bmatrix}$	6

ID	relations	gd-matrix	$-\mathcal{K}^3$
509	$T_1^2T_2+T_3^6+T_4^2$	$\begin{bmatrix} 1&4&1&3&1 \\ \bar{0}&\bar{0}&\bar{1}&\bar{1}&\bar{0} \\ \bar{1}&\bar{0}&\bar{0}&\bar{1}&\bar{0} \end{bmatrix}$	8
510	$T_1^2T_2+T_3^4+T_4^2$	$\begin{bmatrix} 1&2&1&2&2 \\ \bar{0}&\bar{0}&\bar{1}&\bar{0}&\bar{1} \\ \bar{0}&\bar{0}&\bar{1}&\bar{1}&\bar{0} \end{bmatrix}$	8
511	$T_1^3T_2+T_3^2+T_4^2$	$\begin{bmatrix} 1&1&2&2&2 \\ \bar{0}&\bar{0}&\bar{1}&\bar{1}&\bar{0} \\ \bar{0}&\bar{0}&\bar{1}&\bar{0}&\bar{1} \end{bmatrix}$	8
512	$T_1^2T_2^2+T_3^2+T_4^2$	$\begin{bmatrix} 1&1&2&2&2 \\ \bar{1}&\bar{0}&\bar{1}&\bar{0}&\bar{0} \\ \bar{0}&\bar{0}&\bar{1}&\bar{1}&\bar{0} \end{bmatrix}$	8
513	$T_1^2T_2^2+T_3^2+T_4^2$	$\begin{bmatrix} 1&1&2&2&2 \\ \bar{0}&\bar{0}&\bar{0}&\bar{1}&\bar{1} \\ \bar{0}&\bar{0}&\bar{1}&\bar{1}&\bar{0} \end{bmatrix}$	8
514	$T_1^2T_2^2+T_3^2+T_4^2$	$\begin{bmatrix} 1&1&2&2&2 \\ \bar{0}&\bar{0}&\bar{0}&\bar{1}&\bar{1} \\ \bar{1}&\bar{0}&\bar{0}&\bar{0}&\bar{1} \end{bmatrix}$	8
515	$T_1^2T_2+T_3^2+T_4^2$	$\begin{bmatrix} 1&2&2&2&1 \\ \bar{0}&\bar{0}&\bar{1}&\bar{1}&\bar{0} \\ \bar{1}&\bar{0}&\bar{1}&\bar{0}&\bar{0} \end{bmatrix}$	8
516	$T_1T_2+T_3^6+T_4^2$	$\begin{bmatrix} 2&4&1&3&4 \\ \bar{1}&\bar{1}&\bar{0}&\bar{0}&\bar{0} \\ \bar{1}&\bar{1}&\bar{0}&\bar{1}&\bar{1} \end{bmatrix}$	8
517	$T_1T_2+T_3^4+T_4^2$	$\begin{bmatrix} 2&2&1&2&1 \\ \bar{1}&\bar{1}&\bar{0}&\bar{0}&\bar{0} \\ \bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{0} \end{bmatrix}$	8
519	$T_1T_2+T_3^4+T_4^2$	$\begin{bmatrix} 1&3&1&2&3 \\ \bar{1}&\bar{1}&\bar{0}&\bar{0}&\bar{0} \\ \bar{1}&\bar{1}&\bar{0}&\bar{1}&\bar{1} \end{bmatrix}$	12
523	$T_1T_2+T_3^2+T_4^2$	$\begin{bmatrix} 1&1&1&1&2 \\ \bar{0}&\bar{0}&\bar{1}&\bar{1}&\bar{0} \\ \bar{0}&\bar{0}&\bar{1}&\bar{0}&\bar{1} \end{bmatrix}$	16

Classification list 2.12.17. *Non-toric, $\mathbb{Q}$-factorial, Gorenstein, log terminal Fano threefolds of Picard number one with an effective two-torus action: Specifying data for divisor class group $\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$ and format $(2,1,1,1,1)$.*

ID	relations	gd-matrix	$-\mathcal{K}^3$
473	$T_1T_2+T_3^4+T_4^2$, $\lambda T_3^4+T_4^2+T_5^2$	$\begin{bmatrix} 2&2&1&2&2&1 \\ \bar{0}&\bar{0}&\bar{1}&\bar{0}&\bar{1}&\bar{0} \\ \bar{0}&\bar{0}&\bar{0}&\bar{1}&\bar{1}&\bar{0} \end{bmatrix}$	2
507	$T_1T_2+T_3^2+T_4^2$, $\lambda T_3^2+T_4^2+T_5^2$	$\begin{bmatrix} 1&1&1&1&1&1 \\ \bar{0}&\bar{0}&\bar{1}&\bar{1}&\bar{0}&\bar{0} \\ \bar{0}&\bar{0}&\bar{0}&\bar{1}&\bar{1}&\bar{0} \end{bmatrix}$	8

ID	relations	gd-matrix	$-\mathcal{K}^3$
508	$T_1T_2+T_3^2+T_4^2$, $\lambda T_3^2+T_4^2+T_5^2$	$\begin{bmatrix} 1&1&1&1&1&1 \\ \bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{0}&\bar{0} \\ \bar{0}&\bar{0}&\bar{0}&\bar{1}&\bar{1}&\bar{0} \end{bmatrix}$	8

Classification list 2.12.18. *Non-toric, $\mathbb{Q}$-factorial, Gorenstein, log terminal Fano threefolds of Picard number one with an effective two-torus action: Specifying data for divisor class group $\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$ and format $(1,1,1,2)$.*

ID	relations	gd-matrix	$-\mathcal{K}^3$
496	$T_1^6+T_2^2+T_3^2$	$\begin{bmatrix} 1&3&3&3&2 \\ \bar{0}&\bar{0}&\bar{1}&\bar{1}&\bar{0} \\ \bar{0}&\bar{1}&\bar{1}&\bar{0}&\bar{0} \end{bmatrix}$	6

ID	relations	gd-matrix	$-\mathcal{K}^3$
497	$T_1^3+T_2^2+T_3^2$	$\begin{bmatrix} 2&3&3&3&1 \\ \bar{0}&\bar{1}&\bar{1}&\bar{0}&\bar{0} \\ \bar{0}&\bar{1}&\bar{0}&\bar{1}&\bar{0} \end{bmatrix}$	6

ID	relations	gd-matrix	$-\mathcal{K}^3$
498	$T_1^2+T_2^2+T_3^2$	$\begin{bmatrix} 3 & 3 & 3 & 2 & 1 \\ \bar{1} & \bar{1} & \bar{0} & \bar{0} & \bar{0} \\ \bar{0} & \bar{1} & \bar{1} & \bar{0} & \bar{0} \end{bmatrix}$	6
502	$T_1^4+T_2^2+T_3^2$	$\begin{bmatrix} 1 & 2 & 2 & 1 & 2 \\ \bar{0} & \bar{0} & \bar{1} & \bar{1} & \bar{0} \\ \bar{0} & \bar{1} & \bar{1} & \bar{0} & \bar{0} \end{bmatrix}$	8
503	$T_1^4+T_2^2+T_3^2$	$\begin{bmatrix} 1 & 2 & 2 & 2 & 1 \\ \bar{1} & \bar{0} & \bar{1} & \bar{0} & \bar{0} \\ \bar{1} & \bar{1} & \bar{1} & \bar{1} & \bar{0} \end{bmatrix}$	8
504	$T_1^4+T_2^2+T_3^2$	$\begin{bmatrix} 1 & 2 & 2 & 2 & 1 \\ \bar{0} & \bar{0} & \bar{1} & \bar{1} & \bar{0} \\ \bar{0} & \bar{1} & \bar{1} & \bar{0} & \bar{0} \end{bmatrix}$	8
505	$T_1^2+T_2^2+T_3^2$	$\begin{bmatrix} 2 & 2 & 2 & 1 & 1 \\ \bar{1} & \bar{1} & \bar{0} & \bar{0} & \bar{0} \\ \bar{0} & \bar{1} & \bar{1} & \bar{0} & \bar{0} \end{bmatrix}$	8
506	$T_1^2+T_2^2+T_3^2$	$\begin{bmatrix} 2 & 2 & 2 & 1 & 1 \\ \bar{0} & \bar{0} & \bar{1} & \bar{1} & \bar{0} \\ \bar{0} & \bar{1} & \bar{1} & \bar{0} & \bar{0} \end{bmatrix}$	8
521	$T_1^2+T_2^2+T_3^2$	$\begin{bmatrix} 1 & 1 & 1 & 1 & 2 \\ \bar{1} & \bar{1} & \bar{0} & \bar{0} & \bar{0} \\ \bar{1} & \bar{0} & \bar{0} & \bar{1} & \bar{0} \end{bmatrix}$	16
522	$T_1^2+T_2^2+T_3^2$	$\begin{bmatrix} 1 & 1 & 1 & 2 & 1 \\ \bar{0} & \bar{0} & \bar{1} & \bar{1} & \bar{0} \\ \bar{0} & \bar{1} & \bar{1} & \bar{0} & \bar{0} \end{bmatrix}$	16
524	$T_1^2+T_2^2+T_3^2$	$\begin{bmatrix} 1 & 1 & 1 & 3 & 2 \\ \bar{1} & \bar{1} & \bar{0} & \bar{0} & \bar{0} \\ \bar{1} & \bar{0} & \bar{0} & \bar{1} & \bar{0} \end{bmatrix}$	18

Classification list 2.12.19. *Non-toric, $\mathbb{Q}$-factorial, Gorenstein, log terminal Fano threefolds of Picard number one with an effective two-torus action: Specifying data for divisor class group $\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$ and format $(2,2,1,0)$.*

ID	relations	gd-matrix	$-\mathcal{K}^3$
533	$T_1^2T_2^2+T_3^2T_4^2+T_5^2$	$\begin{bmatrix} 1 & 1 & 1 & 1 & 2 \\ \bar{0} & \bar{0} & \bar{1} & \bar{1} & \bar{0} \\ \bar{1} & \bar{0} & \bar{0} & \bar{1} & \bar{0} \\ \bar{0} & \bar{0} & \bar{1} & \bar{0} & \bar{1} \end{bmatrix}$	2

Classification list 2.12.20. *Non-toric, $\mathbb{Q}$-factorial, Gorenstein, log terminal Fano threefolds of Picard number one with an effective two-torus action: Specifying data for divisor class group $\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$ and format $(2,1,1,1)$.*

ID	relations	gd-matrix	$-\mathcal{K}^3$
532	$T_1^2T_2^2+T_3^4+T_4^2$	$\begin{bmatrix} 1 & 1 & 1 & 2 & 1 \\ \bar{1} & \bar{1} & \bar{0} & \bar{0} & \bar{0} \\ \bar{0} & \bar{0} & \bar{1} & \bar{1} & \bar{0} \\ \bar{1} & \bar{0} & \bar{1} & \bar{0} & \bar{0} \end{bmatrix}$	2
537	$T_1^2T_2^2+T_3^2+T_4^2$	$\begin{bmatrix} 1 & 1 & 2 & 2 & 2 \\ \bar{1} & \bar{0} & \bar{1} & \bar{1} & \bar{1} \\ \bar{1} & \bar{0} & \bar{1} & \bar{0} & \bar{0} \\ \bar{0} & \bar{0} & \bar{1} & \bar{1} & \bar{0} \end{bmatrix}$	4

Classification list 2.12.21. *Non-toric, $\mathbb{Q}$-factorial, Gorenstein, log terminal Fano threefolds of Picard number one with an effective two-torus action: Specifying data for divisor class group $\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$ and format $(2,1,1,1,1)$.*

ID	relations	gd-matrix	$-\mathcal{K}^3$
536	$T_1T_2+T_3^2+T_4^2$, $\lambda T_3^2+T_4^2+T_5^2$	$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ \bar{1} & \bar{1} & \bar{0} & \bar{0} & \bar{0} & \bar{0} \\ \bar{0} & \bar{0} & \bar{1} & \bar{1} & \bar{0} & \bar{0} \\ \bar{1} & \bar{1} & \bar{0} & \bar{1} & \bar{1} & \bar{0} \end{bmatrix}$	4

Classification list 2.12.22. *Non-toric, $\mathbb{Q}$-factorial, Gorenstein, log terminal Fano threefolds of Picard number one with an effective two-torus action: Specifying data for divisor class group $\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$ and format $(1,1,1,2)$.*

ID	relations	gd-matrix	$-\mathcal{K}^3$
534	$T_1^4+T_2^2+T_3^2$	$\begin{bmatrix} 1 & 2 & 2 & 2 & 1 \\ \bar{1} & \bar{0} & \bar{0} & \bar{1} & \bar{0} \\ \bar{1} & \bar{0} & \bar{1} & \bar{0} & \bar{0} \\ \bar{1} & \bar{1} & \bar{1} & \bar{1} & \bar{0} \end{bmatrix}$	4
535	$T_1^2+T_2^2+T_3^2$	$\begin{bmatrix} 2 & 2 & 2 & 1 & 1 \\ \bar{1} & \bar{1} & \bar{1} & \bar{1} & \bar{0} \\ \bar{1} & \bar{1} & \bar{0} & \bar{0} & \bar{0} \\ \bar{1} & \bar{0} & \bar{0} & \bar{1} & \bar{0} \end{bmatrix}$	4
538	$T_1^2+T_2^2+T_3^2$	$\begin{bmatrix} 1 & 1 & 1 & 2 & 1 \\ \bar{1} & \bar{1} & \bar{1} & \bar{1} & \bar{0} \\ \bar{1} & \bar{1} & \bar{0} & \bar{0} & \bar{0} \\ \bar{1} & \bar{0} & \bar{0} & \bar{1} & \bar{0} \end{bmatrix}$	8

Classification list 2.12.23. *Non-toric, $\mathbb{Q}$-factorial, Gorenstein, log terminal Fano threefolds of Picard number one with an effective two-torus action: Specifying data for divisor class group $\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/4\mathbb{Z}$ and format $(2,2,1,0)$.*

ID	relations	gd-matrix	$-\mathcal{K}^3$
529	$T_1^2T_2^2+T_3T_4+T_5^2$	$\begin{bmatrix} 1 & 1 & 2 & 2 & 2 \\ \bar{0} & \bar{0} & \bar{1} & \bar{1} & \bar{0} \\ \bar{0} & \bar{3} & \bar{2} & \bar{0} & \bar{1} \end{bmatrix}$	4

Classification list 2.12.24. *Non-toric, $\mathbb{Q}$-factorial, Gorenstein, log terminal Fano threefolds of Picard number one with an effective two-torus action: Specifying data for divisor class group $\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/4\mathbb{Z}$ and format $(2,2,1,1,0)$.*

ID	relations	gd-matrix	$-\mathcal{K}^3$
528	$T_1T_2+T_3T_4+T_5^2$, $\lambda T_3T_4+T_5^2+T_6^2$	$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ \bar{1} & \bar{1} & \bar{0} & \bar{0} & \bar{0} & \bar{0} \\ \bar{3} & \bar{1} & \bar{3} & \bar{1} & \bar{2} & \bar{0} \end{bmatrix}$	4

Classification list 2.12.25. *Non-toric, $\mathbb{Q}$-factorial, Gorenstein, log terminal Fano threefolds of Picard number one with an effective two-torus action: Specifying data for divisor class group $\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/4\mathbb{Z}$ and format $(3,1,1,0)$.*

ID	relations	gd-matrix	$-\mathcal{K}^3$
527	$T_1T_2T_3+T_4^2+T_5^2$	$\begin{bmatrix} 1 & 1 & 2 & 2 & 2 \\ \bar{1} & \bar{0} & \bar{1} & \bar{1} & \bar{1} \\ \bar{2} & \bar{0} & \bar{0} & \bar{3} & \bar{1} \end{bmatrix}$	4

Classification list 2.12.26. *Non-toric, $\mathbb{Q}$-factorial, Gorenstein, log terminal Fano threefolds of Picard number one with an effective two-torus action: Specifying data for divisor class group $\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/4\mathbb{Z}$ and format $(2,1,1,1)$.*

ID	relations	gd-matrix	$-\mathcal{K}^3$
526	$T_1T_2+T_3^4+T_4^4$	$\begin{bmatrix} 2 & 2 & 1 & 1 & 2 \\ \bar{1} & \bar{1} & \bar{0} & \bar{0} & \bar{0} \\ \bar{3} & \bar{1} & \bar{3} & \bar{0} & \bar{3} \end{bmatrix}$	4
530	$T_1T_2+T_3^2+T_4^2$	$\begin{bmatrix} 1 & 1 & 1 & 1 & 2 \\ \bar{0} & \bar{0} & \bar{0} & \bar{1} & \bar{1} \\ \bar{2} & \bar{0} & \bar{1} & \bar{3} & \bar{0} \end{bmatrix}$	8

Classification list 2.12.27. *Non-toric, $\mathbb{Q}$-factorial, Gorenstein, log terminal Fano threefolds of Picard number one with an effective two-torus action: Specifying data for divisor class group $\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/6\mathbb{Z}$ and format* $(3,1,1,0)$.

ID	relations	gd-matrix	$-\mathcal{K}^3$
531	$T_1T_2T_3+T_4^3+T_5^3$	$\begin{bmatrix} 1&1&1&1&1 \\ \bar{1}&\bar{0}&\bar{1}&\bar{0}&\bar{0} \\ \bar{5}&\bar{5}&\bar{2}&\bar{4}&\bar{0} \end{bmatrix}$	2

Classification list 2.12.28. *Non-toric, $\mathbb{Q}$-factorial, Gorenstein, log terminal Fano threefolds of Picard number one with an effective two-torus action: Specifying data for divisor class group $\mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z}$ and format* $(2,2,1,0)$.

ID	relations	gd-matrix	$-\mathcal{K}^3$
424	$T_1^3T_2+T_3^3T_4^3+T_5^2$	$\begin{bmatrix} 1&3&1&1&3 \\ \bar{0}&\bar{0}&\bar{2}&\bar{1}&\bar{0} \end{bmatrix}$	6
425	$T_1^9T_2^3+T_3T_4+T_5^3$	$\begin{bmatrix} 1&1&6&6&4 \\ \bar{0}&\bar{1}&\bar{0}&\bar{0}&\bar{2} \end{bmatrix}$	6
426	$T_1^3T_2^3+T_3T_4+T_5^3$	$\begin{bmatrix} 1&2&3&6&3 \\ \bar{0}&\bar{1}&\bar{0}&\bar{0}&\bar{2} \end{bmatrix}$	6
432	$T_1^{10}T_2+T_3T_4+T_5^2$	$\begin{bmatrix} 1&4&2&12&7 \\ \bar{0}&\bar{1}&\bar{1}&\bar{0}&\bar{2} \end{bmatrix}$	12
433	$T_1^7T_2+T_3T_4+T_5^2$	$\begin{bmatrix} 1&1&2&6&4 \\ \bar{0}&\bar{1}&\bar{1}&\bar{0}&\bar{2} \end{bmatrix}$	12
434	$T_1T_2+T_3T_4+T_5^2$	$\begin{bmatrix} 1&3&2&2&2 \\ \bar{2}&\bar{0}&\bar{2}&\bar{0}&\bar{1} \end{bmatrix}$	12
437	$T_1^7T_2+T_3T_4+T_5^2$	$\begin{bmatrix} 1&3&1&9&5 \\ \bar{0}&\bar{1}&\bar{1}&\bar{0}&\bar{2} \end{bmatrix}$	18
438	$T_1T_2+T_3T_4+T_5^2$	$\begin{bmatrix} 1&1&1&1&1 \\ \bar{0}&\bar{0}&\bar{1}&\bar{2}&\bar{0} \end{bmatrix}$	18
439	$T_1T_2+T_3T_4+T_5^2$	$\begin{bmatrix} 1&1&1&1&1 \\ \bar{2}&\bar{1}&\bar{2}&\bar{1}&\bar{0} \end{bmatrix}$	18

Classification list 2.12.29. *Non-toric, $\mathbb{Q}$-factorial, Gorenstein, log terminal Fano threefolds of Picard number one with an effective two-torus action: Specifying data for divisor class group $\mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z}$ and format* $(2,2,1,1,0)$.

ID	relations	gd-matrix	$-\mathcal{K}^3$
423	$T_1T_2+T_3T_4+T_5^3$, $\lambda T_3T_4+T_5^3+T_6^3$	$\begin{bmatrix} 2&1&1&2&1&1 \\ \bar{2}&\bar{1}&\bar{1}&\bar{2}&\bar{2}&\bar{0} \end{bmatrix}$	6

Classification list 2.12.30. *Non-toric, $\mathbb{Q}$-factorial, Gorenstein, log terminal Fano threefolds of Picard number one with an effective two-torus action: Specifying data for divisor class group $\mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z}$ and format* $(3,1,1,0)$.

ID	relations	gd-matrix	$-\mathcal{K}^3$
414	$T_1^2T_2^2T_3+T_4^3+T_5^3$	$\begin{bmatrix} 1&1&2&2&2 \\ \bar{0}&\bar{0}&\bar{0}&\bar{1}&\bar{2} \end{bmatrix}$	2
415	$T_1^2T_2T_3+T_4^3+T_5^3$	$\begin{bmatrix} 2&1&1&2&2 \\ \bar{1}&\bar{2}&\bar{2}&\bar{2}&\bar{0} \end{bmatrix}$	2
416	$T_1T_2T_3+T_4^9+T_5^3$	$\begin{bmatrix} 1&4&4&1&3 \\ \bar{1}&\bar{1}&\bar{1}&\bar{0}&\bar{1} \end{bmatrix}$	4
421	$T_1^3T_2^3T_3+T_4^3+T_5^2$	$\begin{bmatrix} 1&1&6&4&6 \\ \bar{1}&\bar{0}&\bar{0}&\bar{2}&\bar{0} \end{bmatrix}$	6
422	$T_1T_2T_3+T_4^3+T_5^3$	$\begin{bmatrix} 1&2&6&3&3 \\ \bar{0}&\bar{0}&\bar{0}&\bar{1}&\bar{2} \end{bmatrix}$	6
429	$T_1T_2T_3+T_4^3+T_5^3$	$\begin{bmatrix} 1&1&4&2&2 \\ \bar{0}&\bar{0}&\bar{0}&\bar{1}&\bar{2} \end{bmatrix}$	8
430	$T_1T_2T_3+T_4^3+T_5^3$	$\begin{bmatrix} 1&1&1&1&1 \\ \bar{1}&\bar{1}&\bar{1}&\bar{2}&\bar{0} \end{bmatrix}$	8

Classification list 2.12.31. *Non-toric, $\mathbb{Q}$-factorial, Gorenstein, log terminal Fano threefolds of Picard number one with an effective two-torus action: Specifying data for divisor class group $\mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z}$ and format $(2,1,1,1)$.*

ID	relations	gd-matrix	$-\mathcal{K}^3$
413	$T_1^2T_2^2+T_3^3+T_4^3$	$\begin{bmatrix} 2 & 1 & 2 & 2 & 1 \\ \bar{0} & \bar{0} & \bar{1} & \bar{2} & \bar{0} \end{bmatrix}$	2
418	$T_1T_2+T_3^{12}+T_4^3$	$\begin{bmatrix} 6 & 6 & 1 & 4 & 1 \\ \bar{0} & \bar{0} & \bar{1} & \bar{2} & \bar{0} \end{bmatrix}$	6
419	$T_1T_2+T_3^9+T_4^3$	$\begin{bmatrix} 3 & 6 & 1 & 3 & 2 \\ \bar{0} & \bar{0} & \bar{0} & \bar{1} & \bar{2} \end{bmatrix}$	6
420	$T_1T_2+T_3^3+T_4^3$	$\begin{bmatrix} 3 & 3 & 2 & 2 & 2 \\ \bar{0} & \bar{0} & \bar{2} & \bar{0} & \bar{1} \end{bmatrix}$	6
427	$T_1^2T_2+T_3^3+T_4^3$	$\begin{bmatrix} 1 & 4 & 2 & 2 & 1 \\ \bar{0} & \bar{0} & \bar{1} & \bar{2} & \bar{0} \end{bmatrix}$	8

ID	relations	gd-matrix	$-\mathcal{K}^3$
428	$T_1^2T_2+T_3^3+T_4^3$	$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ \bar{0} & \bar{0} & \bar{1} & \bar{2} & \bar{0} \end{bmatrix}$	8
431	$T_1T_2+T_3^{12}+T_4^3$	$\begin{bmatrix} 2 & 10 & 1 & 4 & 5 \\ \bar{1} & \bar{2} & \bar{0} & \bar{1} & \bar{1} \end{bmatrix}$	10
435	$T_1T_2+T_3^9+T_4^3$	$\begin{bmatrix} 1 & 8 & 1 & 3 & 4 \\ \bar{1} & \bar{2} & \bar{0} & \bar{1} & \bar{1} \end{bmatrix}$	16
436	$T_1T_2+T_3^3+T_4^3$	$\begin{bmatrix} 1 & 2 & 1 & 1 & 2 \\ \bar{1} & \bar{2} & \bar{2} & \bar{0} & \bar{2} \end{bmatrix}$	16

Classification list 2.12.32. *Non-toric, $\mathbb{Q}$-factorial, Gorenstein, log terminal Fano threefolds of Picard number one with an effective two-torus action: Specifying data for divisor class group $\mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z}$ and format $(1,1,1,2)$.*

ID	relations	gd-matrix	$-\mathcal{K}^3$
417	$T_1^3+T_2^3+T_3^2$	$\begin{bmatrix} 2 & 2 & 3 & 3 & 2 \\ \bar{1} & \bar{0} & \bar{0} & \bar{0} & \bar{2} \end{bmatrix}$	6

Classification list 2.12.33. *Non-toric, $\mathbb{Q}$-factorial, Gorenstein, log terminal Fano threefolds of Picard number one with an effective two-torus action: Specifying data for divisor class group $\mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z}$ and format $(2,2,1,0)$.*

ID	relations	gd-matrix	$-\mathcal{K}^3$
525	$T_1T_2+T_3T_4+T_5^2$	$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ \bar{2} & \bar{1} & \bar{2} & \bar{1} & \bar{0} \\ \bar{2} & \bar{1} & \bar{0} & \bar{0} & \bar{0} \end{bmatrix}$	6

Classification list 2.12.34. *Non-toric, $\mathbb{Q}$-factorial, Gorenstein, log terminal Fano threefolds of Picard number one with an effective two-torus action: Specifying data for divisor class group $\mathbb{Z} \oplus \mathbb{Z}/4\mathbb{Z}$ and format $(2,2,1,0)$.*

ID	relations	gd-matrix	$-\mathcal{K}^3$
452	$T_1^5T_2+T_3^2T_4+T_5^2$	$\begin{bmatrix} 1 & 1 & 1 & 4 & 3 \\ \bar{0} & \bar{2} & \bar{3} & \bar{0} & \bar{1} \end{bmatrix}$	8
453	$T_1^3T_2^3+T_3^2T_4+T_5^2$	$\begin{bmatrix} 1 & 1 & 1 & 4 & 3 \\ \bar{2} & \bar{0} & \bar{3} & \bar{0} & \bar{1} \end{bmatrix}$	8
454	$T_1^2T_2+T_3^2T_4+T_5^2$	$\begin{bmatrix} 1 & 2 & 1 & 2 & 2 \\ \bar{0} & \bar{2} & \bar{3} & \bar{0} & \bar{1} \end{bmatrix}$	8

ID	relations	gd-matrix	$-\mathcal{K}^3$
455	$T_1^{10}T_2^2+T_3T_4+T_5^2$	$\begin{bmatrix} 1 & 1 & 4 & 8 & 6 \\ \bar{0} & \bar{1} & \bar{2} & \bar{0} & \bar{3} \end{bmatrix}$	8
456	$T_1^6T_2^2+T_3T_4+T_5^2$	$\begin{bmatrix} 1 & 2 & 2 & 8 & 5 \\ \bar{0} & \bar{1} & \bar{2} & \bar{0} & \bar{3} \end{bmatrix}$	8
457	$T_1^2T_2^2+T_3T_4+T_5^2$	$\begin{bmatrix} 1 & 1 & 2 & 2 & 2 \\ \bar{0} & \bar{3} & \bar{0} & \bar{2} & \bar{1} \end{bmatrix}$	8

Classification list 2.12.35. *Non-toric, $\mathbb{Q}$-factorial, Gorenstein, log terminal Fano threefolds of Picard number one with an effective two-torus action: Specifying data for divisor class group $\mathbb{Z} \oplus \mathbb{Z}/4\mathbb{Z}$ and format $(2, 2, 1, 1, 0)$.*

ID	relations	gd-matrix	$-\mathcal{K}^3$
451	$T_1T_2+T_3T_4+T_5^2$, $\lambda T_3T_4+T_5^2+T_6^2$	$\begin{bmatrix} 1&1&1&1&1&1 \\ \bar{1}&\bar{3}&\bar{3}&\bar{1}&\bar{2}&\bar{0} \end{bmatrix}$	8

Classification list 2.12.36. *Non-toric, $\mathbb{Q}$-factorial, Gorenstein, log terminal Fano threefolds of Picard number one with an effective two-torus action: Specifying data for divisor class group $\mathbb{Z} \oplus \mathbb{Z}/4\mathbb{Z}$ and format $(3, 1, 1, 0)$.*

ID	relations	gd-matrix	$-\mathcal{K}^3$
443	$T_1T_2T_3+T_4^{10}+T_5^2$	$\begin{bmatrix} 1&3&6&1&5 \\ \bar{3}&\bar{3}&\bar{2}&\bar{0}&\bar{2} \end{bmatrix}$	6
444	$T_1T_2T_3+T_4^2+T_5^2$	$\begin{bmatrix} 1&2&3&3&3 \\ \bar{3}&\bar{2}&\bar{3}&\bar{2}&\bar{0} \end{bmatrix}$	6
448	$T_1^2T_2^2T_3+T_4^3+T_5^2$	$\begin{bmatrix} 1&1&8&4&6 \\ \bar{1}&\bar{0}&\bar{0}&\bar{2}&\bar{3} \end{bmatrix}$	8

ID	relations	gd-matrix	$-\mathcal{K}^3$
449	$T_1T_2T_3+T_4^2+T_5^2$	$\begin{bmatrix} 1&1&2&2&2 \\ \bar{3}&\bar{3}&\bar{0}&\bar{3}&\bar{1} \end{bmatrix}$	8
450	$T_1T_2T_3+T_4^2+T_5^2$	$\begin{bmatrix} 1&1&2&2&2 \\ \bar{3}&\bar{1}&\bar{2}&\bar{3}&\bar{1} \end{bmatrix}$	8

Classification list 2.12.37. *Non-toric, $\mathbb{Q}$-factorial, Gorenstein, log terminal Fano threefolds of Picard number one with an effective two-torus action: Specifying data for divisor class group $\mathbb{Z} \oplus \mathbb{Z}/4\mathbb{Z}$ and format $(2, 1, 1, 1)$.*

ID	relations	gd-matrix	$-\mathcal{K}^3$
440	$T_1T_2+T_3^8+T_4^4$	$\begin{bmatrix} 4&4&1&2&1 \\ \bar{0}&\bar{0}&\bar{1}&\bar{3}&\bar{0} \end{bmatrix}$	4
441	$T_1^3T_2+T_3^2+T_4^2$	$\begin{bmatrix} 1&3&3&3&2 \\ \bar{0}&\bar{2}&\bar{1}&\bar{3}&\bar{0} \end{bmatrix}$	6
442	$T_1T_2+T_3^8+T_4^4$	$\begin{bmatrix} 2&6&1&2&3 \\ \bar{2}&\bar{2}&\bar{0}&\bar{1}&\bar{1} \end{bmatrix}$	6
445	$T_1^3T_2+T_3^2+T_4^2$	$\begin{bmatrix} 1&1&2&2&2 \\ \bar{0}&\bar{2}&\bar{1}&\bar{3}&\bar{0} \end{bmatrix}$	8

ID	relations	gd-matrix	$-\mathcal{K}^3$
446	$T_1^2T_2+T_3^2+T_4^2$	$\begin{bmatrix} 1&2&2&2&1 \\ \bar{0}&\bar{2}&\bar{1}&\bar{3}&\bar{0} \end{bmatrix}$	8
447	$T_1T_2+T_3^4+T_4^4$	$\begin{bmatrix} 2&2&1&1&2 \\ \bar{1}&\bar{3}&\bar{3}&\bar{0}&\bar{3} \end{bmatrix}$	8
458	$T_1T_2+T_3^2+T_4^2$	$\begin{bmatrix} 1&1&1&1&2 \\ \bar{2}&\bar{0}&\bar{3}&\bar{1}&\bar{0} \end{bmatrix}$	16

Classification list 2.12.38. *Non-toric, $\mathbb{Q}$-factorial, Gorenstein, log terminal Fano threefolds of Picard number one with an effective two-torus action: Specifying data for divisor class group $\mathbb{Z} \oplus \mathbb{Z}/5\mathbb{Z}$ and format $(2, 2, 1, 0)$.*

ID	relations	gd-matrix	$-\mathcal{K}^3$
462	$T_1^3T_2^3+T_3T_4+T_5^2$	$\begin{bmatrix} 1&1&1&5&3 \\ \bar{3}&\bar{0}&\bar{4}&\bar{0}&\bar{2} \end{bmatrix}$	10

Classification list 2.12.39. *Non-toric, $\mathbb{Q}$-factorial, Gorenstein, log terminal Fano threefolds of Picard number one with an effective two-torus action: Specifying data for divisor class group $\mathbb{Z} \oplus \mathbb{Z}/5\mathbb{Z}$ and format $(3, 1, 1, 0)$.*

ID	relations	gd-matrix	$-\mathcal{K}^3$
459	$T_1T_2T_3+T_4^5+T_5^5$	$\begin{bmatrix} 1 & 2 & 2 & 1 & 1 \\ \bar{2} & \bar{4} & \bar{4} & \bar{4} & \bar{0} \end{bmatrix}$	2

ID	relations	gd-matrix	$-\mathcal{K}^3$
461	$T_1T_2T_3+T_4^3+T_5^2$	$\begin{bmatrix} 1 & 1 & 10 & 4 & 6 \\ \bar{1} & \bar{0} & \bar{0} & \bar{2} & \bar{3} \end{bmatrix}$	10

Classification list 2.12.40. *Non-toric, $\mathbb{Q}$-factorial, Gorenstein, log terminal Fano threefolds of Picard number one with an effective two-torus action: Specifying data for divisor class group $\mathbb{Z} \oplus \mathbb{Z}/5\mathbb{Z}$ and format $(2,1,1,1)$.*

ID	relations	gd-matrix	$-\mathcal{K}^3$
460	$T_1T_2+T_3^5+T_4^5$	$\begin{bmatrix} 1 & 4 & 1 & 1 & 2 \\ \bar{2} & \bar{3} & \bar{4} & \bar{0} & \bar{4} \end{bmatrix}$	8

Classification list 2.12.41. *Non-toric, $\mathbb{Q}$-factorial, Gorenstein, log terminal Fano threefolds of Picard number one with an effective two-torus action: Specifying data for divisor class group $\mathbb{Z} \oplus \mathbb{Z}/6\mathbb{Z}$ and format $(2,2,1,0)$.*

ID	relations	gd-matrix	$-\mathcal{K}^3$
469	$T_1^4T_2^4+T_3T_4+T_5^2$	$\begin{bmatrix} 1 & 1 & 2 & 6 & 4 \\ \bar{1} & \bar{0} & \bar{4} & \bar{0} & \bar{5} \end{bmatrix}$	6

Classification list 2.12.42. *Non-toric, $\mathbb{Q}$-factorial, Gorenstein, log terminal Fano threefolds of Picard number one with an effective two-torus action: Specifying data for divisor class group $\mathbb{Z} \oplus \mathbb{Z}/6\mathbb{Z}$ and format $(3,1,1,0)$.*

ID	relations	gd-matrix	$-\mathcal{K}^3$
465	$T_1T_2T_3+T_4^3+T_5^3$	$\begin{bmatrix} 1 & 1 & 4 & 2 & 2 \\ \bar{3} & \bar{0} & \bar{0} & \bar{5} & \bar{1} \end{bmatrix}$	4
466	$T_1T_2T_3+T_4^3+T_5^3$	$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ \bar{2} & \bar{5} & \bar{5} & \bar{4} & \bar{0} \end{bmatrix}$	4

ID	relations	gd-matrix	$-\mathcal{K}^3$
468	$T_1T_2T_3+T_4^4+T_5^2$	$\begin{bmatrix} 1 & 1 & 6 & 2 & 4 \\ \bar{2} & \bar{0} & \bar{0} & \bar{5} & \bar{1} \end{bmatrix}$	6

Classification list 2.12.43. *Non-toric, $\mathbb{Q}$-factorial, Gorenstein, log terminal Fano threefolds of Picard number one with an effective two-torus action: Specifying data for divisor class group $\mathbb{Z} \oplus \mathbb{Z}/6\mathbb{Z}$ and format $(2,1,1,1)$.*

ID	relations	gd-matrix	$-\mathcal{K}^3$
463	$T_1^2T_2+T_3^3+T_4^3$	$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ \bar{0} & \bar{3} & \bar{5} & \bar{1} & \bar{0} \end{bmatrix}$	4
464	$T_1T_2+T_3^6+T_4^6$	$\begin{bmatrix} 2 & 4 & 1 & 1 & 2 \\ \bar{2} & \bar{4} & \bar{5} & \bar{0} & \bar{5} \end{bmatrix}$	4

ID	relations	gd-matrix	$-\mathcal{K}^3$
467	$T_1T_2+T_3^4+T_4^2$	$\begin{bmatrix} 2 & 2 & 1 & 2 & 3 \\ \bar{2} & \bar{4} & \bar{0} & \bar{3} & \bar{3} \end{bmatrix}$	6
470	$T_1T_2+T_3^3+T_4^3$	$\begin{bmatrix} 1 & 2 & 1 & 1 & 2 \\ \bar{5} & \bar{1} & \bar{4} & \bar{0} & \bar{4} \end{bmatrix}$	8

Classification list 2.12.44. *Non-toric, $\mathbb{Q}$-factorial, Gorenstein, log terminal Fano threefolds of Picard number one with an effective two-torus action: Specifying data for divisor class group $\mathbb{Z} \oplus \mathbb{Z}/8\mathbb{Z}$ and format $(3,1,1,0)$.*

ID	relations	gd-matrix	$-\mathcal{K}^3$
471	$T_1T_2T_3+T_4^4+T_5^4$	$\begin{bmatrix} 1 & 1 & 2 & 1 & 1 \\ \bar{3} & \bar{7} & \bar{6} & \bar{6} & \bar{0} \end{bmatrix}$	2

ID	relations	gd-matrix	$-\mathcal{K}^3$
472	$T_1T_2T_3+T_4^6+T_5^2$	$\begin{bmatrix} 1 & 1 & 4 & 1 & 3 \\ \bar{2} & \bar{0} & \bar{0} & \bar{3} & \bar{5} \end{bmatrix}$	4

2.13 Hilbert-Poincaré series

Here we present the Hilbert-Poincaré series of our Fano varieties. Recall that the Hilbert-Poincaré series of a finitely generated $\mathbb{Z}_{\geq 0}$-graded $\mathbb{K}$-algebra $A = \oplus_k A_k$ is the formal power series

$$\mathrm{HP}_A(t) \ := \ \sum_{k \geq 0} \dim_{\mathbb{K}}(A_k) t^k.$$

Assume that $f_1, \ldots, f_r \in A$ are homogeneous of degrees $w_1, \ldots, w_r$ respectively and generate A as an algebra. Then there is a polynomial $q_A \in \mathbb{Z}[t]$ such that

$$\mathrm{HP}_A(t) \ = \ \frac{q_A(t)}{\prod_{i=1}^r (1 - t^{w_i})}.$$

Given a Fano variety X, we associate with it the Hilbert-Poincaré series $\mathrm{HP}_X(t)$ of its anticanonical ring A_X and we define the corresponding polynomial $q_X(t)$ with respect to a minimal system of homogeneous generators of the anticanonical ring A_X.

Proposition 2.13.1. *The following table lists for each possible pair (g, c) of genus and codimension the classification IDs from Section 2.12 of the varieties X attaining (g, c) and the cancelled presentation of the associated Hilbert-Poincaré series together with its first eight terms.*

(g,c)	$\mathrm{HP}_X(t)$	IDs
$(2,1)$	$\frac{1+t^3}{1-t^4}$ $1+4t+10t^2+21t^3+39t^4+66t^5+104t^6+155t^7+\ldots$	2, 9, 10, 11, 246, 251, 252, 253, 254, 459, 471, 473, 474, 475, 477, 478, 479, 480, 531, 532, 533
$(2,2)$	$\frac{1+t^3}{1-t^4}$ $1+4t+10t^2+21t^3+39t^4+66t^5+104t^6+155t^7+\ldots$	1, 3, 4, 5, 6, 7, 8, 247, 248, 249, 250, 413, 414, 415, 476
$(3,1)$	$\frac{1+t+t^2+t^3}{1-t^4}$ $1+5t+15t^2+35t^3+69t^4+121t^5+195t^6+295t^7+\ldots$	12, 416, 440, 472, 529
$(3,2)$	$\frac{1+t+t^2+t^3}{1-t^4}$ $1+5t+15t^2+35t^3+69t^4+121t^5+195t^6+295t^7+\ldots$	13, 14, 15, 16, 17, 18, 255, 256, 257, 258, 259, 260, 261, 262, 263, 264, 265, 266, 267, 268, 269, 270, 271, 272, 273, 274, 275, 276, 463, 464, 465, 466, 481, 482, 483, 484, 485, 486, 487, 488, 489, 490, 491, 492, 493, 494, 495, 526, 527, 528, 534, 535, 536, 537
$(4,2)$	$\frac{1+2t+2t^2+t^3}{1-t^4}$ $1+6t+20t^2+49t^3+99t^4+176t^5+286t^6+435t^7+\ldots$	19, 21, 22, 28, 30, 32, 33, 34, 35, 36, 37, 280, 282, 287, 290, 417, 418, 419, 420, 421, 423, 424, 425, 426, 442, 443, 467, 468, 469, 501, 525
$(4,4)$	$\frac{1+2t+2t^2+t^3}{1-t^4}$ $1+6t+20t^2+49t^3+99t^4+176t^5+286t^6+435t^7+\ldots$	20, 23, 24, 25, 26, 27, 29, 31, 38, 39, 40, 41, 42, 277, 278, 279, 281, 283, 284, 285, 286, 288, 289, 291, 292, 293, 294, 295, 296, 422, 441, 444, 496, 497, 498, 499, 500

(g,c)	$\mathrm{HP}_X(t)$	IDs
$(5,3)$	$\frac{1+3t+3t^2+t^3}{1-t^4}$ $1+7t+25t^2+63t^3+129t^4+231t^5+377t^6+575t^7+\ldots$	301, 317, 321, 322, 325, 330, 332, 333, 334, 335, 448, 451, 452, 453, 454, 455, 456, 457, 503, 516, 517, 530
$(5,6)$	$\frac{1+3t+3t^2+t^3}{1-t^4}$ $1+7t+25t^2+63t^3+129t^4+231t^5+377t^6+575t^7+\ldots$	43, 44, 45, 46, 47, 48, 49, 50, 51, 52, 53, 54, 55, 56, 57, 58, 297, 298, 299, 300, 302, 303, 304, 305, 306, 307, 308, 309, 310, 311, 312, 313, 314, 315, 316, 318, 319, 320, 323, 324, 326, 327, 328, 329, 331, 427, 428, 429, 430, 445, 446, 447, 449, 450, 460, 470, 502, 504, 505, 506, 507, 508, 509, 510, 511, 512, 513, 514, 515, 518, 538
$(6,4)$	$\frac{1+4t+4t^2+t^3}{1-t^4}$ $1+8t+30t^2+77t^3+159t^4+286t^5+468t^6+715t^7+\ldots$	59, 60, 61, 62, 63, 64, 65, 66, 67, 68, 69, 70, 336, 337, 431, 461, 462
$(7,5)$	$\frac{1+5t+5t^2+t^3}{1-t^4}$ $1+9t+35t^2+91t^3+189t^4+341t^5+559t^6+855t^7+\ldots$	71, 72, 73, 74, 75, 76, 77, 78, 79, 80, 81, 82, 83, 84, 85, 86, 87, 88, 89, 90, 91, 92, 93, 94, 95, 96, 97, 98, 99, 100, 338, 339, 340, 341, 342, 343, 344, 345, 346, 347, 348, 349, 350, 351, 352, 353, 354, 355, 356, 357, 358, 359, 360, 361, 362, 363, 364, 432, 433, 434, 519, 520
$(9,7)$	$\frac{1+7t+7t^2+t^3}{1-t^4}$ $1+11t+45t^2+119t^3+249t^4+451t^5+741t^6+1135t^7+\ldots$	101, 102, 103, 104, 105, 106, 107, 108, 109, 110, 111, 112, 113, 114, 115, 116, 117, 365, 366, 367, 368, 369, 370, 371, 372, 373, 374, 375, 376, 377, 378, 379, 380, 381, 382, 383, 384, 385, 386, 387, 388, 389, 390, 391, 392, 435, 436, 458, 521, 522, 523
$(10,8)$	$\frac{1+8t+8t^2+t^3}{1-t^4}$ $1+12t+50t^2+133t^3+279t^4+506t^5+832t^6+1275t^7+\ldots$	118, 119, 120, 121, 122, 123, 124, 125, 126, 127, 128, 129, 130, 131, 132, 133, 134, 135, 136, 137, 138, 139, 140, 141, 142, 143, 393, 394, 395, 396, 397, 398, 399, 400, 401, 402, 403, 437, 438, 439, 524
$(11,9)$	$\frac{1+9t+9t^2+t^3}{1-t^4}$ $1+13t+55t^2+147t^3+309t^4+561t^5+923t^6+1415t^7+\ldots$	144, 145, 146, 147, 148, 149, 150, 151, 404, 405
$(12,10)$	$\frac{1+10t+10t^2+t^3}{1-t^4}$ $1+14t+60t^2+161t^3+339t^4+616t^5+1014t^6+1555t^7+\ldots$	152
$(13,11)$	$\frac{1+11t+11t^2+t^3}{1-t^4}$ $1+15t+65t^2+175t^3+369t^4+671t^5+1105t^6+1695t^7+\ldots$	153, 154, 155, 156, 157, 158, 159, 160, 161, 162, 163, 164, 165, 166, 167, 168, 169, 170, 171, 172, 173, 174, 175, 176, 177, 178, 179, 180, 181, 182, 183, 184, 185, 186, 187, 406, 407, 408, 409, 410, 411
$(16,14)$	$\frac{1+14t+14t^2+t^3}{1-t^4}$ $1+18t+80t^2+217t^3+459t^4+836t^5+1378t^6+2115t^7+\ldots$	188, 189, 190, 191, 192, 193, 194, 195, 196
$(17,15)$	$\frac{1+15t+15t^2+t^3}{1-t^4}$ $1+19t+85t^2+231t^3+489t^4+891t^5+1469t^6+2255t^7+\ldots$	197, 198, 199, 200, 201, 202, 203, 204, 205, 206, 207, 208, 209, 210, 211, 212, 213, 214, 215, 412
$(19,17)$	$\frac{1+17t+17t^2+t^3}{1-t^4}$ $1+21t+95t^2+259t^3+549t^4+1001t^5+1651t^6+2535t^7+\ldots$	216, 217, 218, 219, 220, 221, 222, 223, 224, 225, 226, 227

(g,c)	$\mathrm{HP}_X(t)$	IDs
$(21,19)$	$\frac{1+19t+19t^2+t^3}{1-t^4}$ $1+23t+105t^2+287t^3+609t^4+1111t^5+1833t^6+2815t^7+\ldots$	228, 229, 230, 231
$(22,20)$	$\frac{1+20t+20t^2+t^3}{1-t^4}$ $1+24t+110t^2+301t^3+639t^4+1166t^5+1924t^6+2955t^7+\ldots$	232, 233, 234
$(25,23)$	$\frac{1+23t+23t^2+t^3}{1-t^4}$ $1+27t+125t^2+343t^3+729t^4+1331t^5+2197t^6+3375t^7+\ldots$	235, 236, 237, 238
$(26,24)$	$\frac{1+24t+24t^2+t^3}{1-t^4}$ $1+28t+130t^2+357t^3+759t^4+1386t^5+2288t^6+3515t^7+\ldots$	239, 240, 241, 242
$(28,26)$	$\frac{1+26t+26t^2+t^3}{1-t^4}$ $1+30t+140t^2+385t^3+819t^4+1496t^5+2470t^6+3795t^7+\ldots$	243, 244, 245

Proof. Observe that the anticanonical ring is the Veronese subalgebra of the Cox ring associated to the subgroup generated by the anticanonical class. Thus, we can use the Cox ring data from the classification lists in Section 2.12 to compute a minimal system of generators and the associated relations. This provides us in particular with genus and codimension. Moreover, it allows us to compute the Hilbert-Poincaré series; we used the computer algebra system Singular. □

Corollary 2.13.2. *The Hilbert-Poincaré series of a non-toric, $\mathbb{Q}$-factorial, log-terminal, Gorenstein, Fano threefold X of Picard number one with an effective action of a two-dimensional torus only depends on the genus g of X and can be explicitly written down as*

$$\mathrm{HP}_X(t) \;=\; \frac{1+(g-2)(t+t^2)+t^3}{(1-t)^4}.$$

CHAPTER

THREE

LOCALLY FACTORIAL FANO FOURFOLDS OF PICARD NUMBER TWO

We classify the locally factorial Fano fourfolds of Picard number two with a hypersurface Cox ring that admit an effective action of a three-dimensional torus. The chapter is organized as follows. In Section 3.1 we present our classification results. Section 3.2 serves to provide the necessary background on Cox rings. In Section 3.3 we establish two general facts, essentially supporting our classification: First, Proposition 3.3.1 shows that in our setting we always have a torsion-free Picard group and second, Proposition 3.3.2 supplies us with an explicit smoothing procedure. The complete proof of the classification spans the Sections 3.4 – 3.9. The classification tables are presented in Section 3.10. The results of this chapter have been achieved in cooperation with equal contributions by Christian Mauz and the author and are published in the joint work [17].

3.1 Main result

We study locally factorial Fano fourfolds of Picard number two that admit an effective action of a three-dimensional torus. Locally factorial means that every Weil divisor is locally principal. Whereas in the toric case smoothness and local factoriality coincide, the latter setting turns out to be much more general for torus actions of complexity one; for instance, the varieties need not be log terminal any more and we find infinite series of non-isomorphic Fanos in fixed dimensions. We settle the case of a Cox ring defined by a single relation. Our main result considerably extends the corresponding one in the smooth case [35, Thm. 1.2].

Theorem 3.1.1. *There are* 447 *sporadic cases and* 106 *infinite series of locally factorial Fano fourfolds of Picard number two coming with an effective action of a three-dimensional torus and a Cox ring defined by a single relation.*

Our varieties in question are uniquely determined by the generator degrees and the relation in their Cox ring. Classification lists 3.10.1 to 3.10.11 provide the complete and redundancy free presentation of the specifying data for Theorem 3.1.1. A data file containing the complete classification data is also available at [18].

For the proof of Theorem 3.1.1 we distinguish two main cases. The first one treats an ample relation degree. There, we provide a smoothing procedure via Bertini's theorem which allows us to infer first constraints on the relevant invariants from the classification of smooth Fano fourfoulds of Picard number two with a hypersurface Cox ring [45, Thm. 1.1]. The situation becomes more involved when the relation degree is not ample. In this situation, we have to classify case by case according to the possible constellations of the Cox ring generator degrees in the effective cone, making heavy use of the combinatorial description of varieties with a torus action of complexity one from [41, 46], see also [6, Sec. 3.4].

3.2 Background on Cox rings

By a *Mori dream space* we mean an irreducible, normal, projective complex variety X with finitely generated divisor class group $\mathrm{Cl}(X)$ and finitely generated Cox ring $\mathcal{R}(X)$. We give a brief summary on the combinatorial approach [6, 20, 40] to Mori dream spaces, adapted to our needs. By a *K-graded affine algebra*, where K is a finitely generated abelian group, we mean an affine $\mathbb{C}$-algebra R coming with a direct sum decomposition into $\mathbb{C}$-vector subspaces

$$R = \bigoplus_{w \in K} R_w$$

such that $R_w R_{w'} \subseteq R_{w+w'}$ holds for all $w, w' \in K$. An element $f \in R$ is called *homogeneous* if $f \in R_w$ holds for some $w \in K$. In that case w is the *degree* of f and we write $w = \deg(f)$. Geometrically, we have the affine variety $\bar{X}$ with R as its algebra of global functions and the quasitorus H with K as its character group:

$$\bar{X} \ = \ \mathrm{Spec}\, R, \qquad\qquad H \ = \ \mathrm{Spec}\, \mathbb{C}[K].$$

The K-grading of R defines an algebraic action of H on $\bar{X}$. By $K_{\mathbb{Q}} := K \otimes_{\mathbb{Z}} \mathbb{Q}$ we denote the $\mathbb{Q}$-vector space associated with K.

(i) The *effective cone* of R is

$$\mathrm{Eff}(R) \ := \ \mathrm{cone}(\, w \in K;\ R_w \neq 0 \,) \ \subseteq \ K_{\mathbb{Q}}.$$

(ii) For $x \in \bar{X}$ we have the *orbit cone*

$$\omega_x \ := \ \mathrm{cone}(\, w \in K;\ f(x) \neq 0 \text{ for some } f \in R_w \,) \ \subseteq \ K_{\mathbb{Q}}.$$

(iii) For $w \in \mathrm{Eff}(R)$ we have the *GIT-cone*

$$\lambda_w \ := \ \bigcap_{x \in \bar{X},\, w \in \omega_x} \omega_x \ \subseteq \ K_{\mathbb{Q}}.$$

The K-grading of R is called *pointed* if $R_0 = \mathbb{C}$ holds and $\mathrm{Eff}(R)$ contains no line. The effective cone, as well as orbit cones and GIT-cones are convex polyhedral cones and there are only finitely many of them. The GIT-cones form a (quasi-) fan $\Lambda(R)$ in $K_{\mathbb{Q}}$ called the *GIT-fan* of R, having the effective cone $\mathrm{Eff}(R)$ as its support. We recall Cox's quotient presentation [29] for projective toric varieties.

Construction 3.2.1. Let $S = \mathbb{C}[T_1, \ldots, T_r]$ and consider a pointed K-grading on S, such that the variables $T_1, \ldots, T_r$ are homogeneous. Write $w_i := \deg(T_i) \in K$ for the generator degrees, also when considered in $K_{\mathbb{Q}}$. We denote the *grading map* by

$$Q \colon \mathbb{Z}^r \ \to \ K, \qquad e_i \ \mapsto \ w_i.$$

We have the action of the quasitorus H on the affine toric variety $\bar{Z}$, where

$$H \ := \ \mathrm{Spec}\, \mathbb{C}[K], \qquad\qquad \bar{Z} \ := \ \mathrm{Spec}\, S \ = \ \mathbb{C}^r.$$

We assume that any $r-1$ of the degrees $w_1, \ldots, w_r$ generate K as a group, ie. the K-grading is *almost free*. Moreover we assume that the *moving cone*

$$\mathrm{Mov}(S) \ := \ \bigcap_{i=1}^{r} \mathrm{cone}(\, w_j;\ j \neq i\,) \ \subseteq \ K_{\mathbb{Q}}$$

is of full dimension. Fix a GIT-cone $\tau \in \Lambda(S)$ with $\tau^\circ \subseteq \mathrm{Mov}(S)^\circ$. There is the H-invariant open set of *semistable points* $\widehat{Z}$ and the corresponding *good quotient* Z:

$$\widehat{Z} \ := \ \bar{Z}^{ss}(\tau) \ = \ \{\, x \in \bar{Z};\ \lambda \subseteq \omega_x \,\}, \qquad\qquad Z \ := \ \widehat{Z} /\!\!/ H.$$

The quotient variety Z is a projective toric variety of dimension $r - \dim(K_{\mathbb{Q}})$ with divisor class group $\mathrm{Cl}(Z) = K$ and Cox ring $\mathcal{R}(Z) = S$.

The following construction produces Mori dream spaces as hypersurfaces in projective toric varieties. A K-graded algebra R is called *K-factorial*, or the K-grading of R is called *factorial*, if R is integral and every homogeneous non-zero non-unit is a product of K-primes. A *K-prime* is a homogeneous non-zero non-unit $f \in R$ with the property that $f \mid gh$ for homogeneous $g, h \in R$ implies that $f \mid g$ or $f \mid h$ holds.

Construction 3.2.2. See [6, Sec. 3.2, 3.3] and [45, Constr. 4.1, Rem. 4.2]. In the setting of Construction 3.2.1 fix $0 \neq \mu \in K$ and $g \in S_\mu$ and set

$$R_g \ := \ S/\langle g \rangle, \qquad \bar{X}_g \ := \ V(g) \ \subseteq \ \bar{Z}, \qquad \widehat{X}_g \ := \ \bar{X}_g \cap \widehat{Z}, \qquad X_g \ := \ \widehat{X}_g /\!\!/ H \subseteq Z.$$

Then the factor algebra R_g inherits a K-grading from S and the quotient $X_g \subseteq Z$ is a closed subvariety. Moreover, there is a GIT-cone $\lambda \in \Lambda(R_g)$ with

$$\widehat{X}_g \ = \ \bar{X}_g^{ss}(\lambda) \ = \ \{\, x \in \bar{X}_g;\ \lambda \subseteq \omega_x \,\}.$$

We assume that R_g is integral and normal with $R_g^* = \mathbb{C}^*$, the induced K-grading is factorial and $T_1, \ldots, T_r$ define a minimal system of pairwise non-associated K-primes in R_g. Then X_g is a normal, projective variety with dimension, divisor class group and Cox ring given by

$$\dim(X_g) \;=\; \dim(Z) - 1, \qquad \mathrm{Cl}(X_g) \;=\; K, \qquad \mathcal{R}(X_g) \;=\; R_g.$$

Moreover, the cones of effective, movable, semiample and ample divisor classes of X are given in the rational divisor class group $\mathrm{Cl}(X_g)_{\mathbb{Q}} = K_{\mathbb{Q}}$ by

$$\mathrm{Eff}(X_g) \;=\; \mathrm{Eff}(R_g), \qquad \mathrm{Mov}(X_g) \;=\; \mathrm{Mov}(R_g) \;=\; \mathrm{Mov}(S),$$

$$\mathrm{SAmple}(X_g) \;=\; \lambda, \qquad \mathrm{Ample}(X_g) \;=\; \lambda^\circ.$$

Remark 3.2.3. Let $X = X_g$ as in Construction 3.2.2. The *minimal ambient toric variety* of X is the unique minimal open toric subvariety $Z_g \subseteq Z$ containing X. For the ample cones of X, Z_g and Z we have

$$\tau^\circ \;=\; \mathrm{Ample}(Z) \;\subseteq\; \mathrm{Ample}(Z_g) \;=\; \mathrm{Ample}(X) \;=\; \lambda^\circ.$$

Remark 3.2.4. A Mori dream space X with divisor class group $\mathrm{Cl}(X) = K$ has a *hypersurface Cox ring* if there is an *irredundant* K-graded presentation

$$\mathcal{R}(X) \;=\; \mathbb{C}[T_1, \ldots, T_r]/\langle g \rangle,$$

meaning that the ideal $\langle g \rangle$ contains no element $T_i - h_i$ with $h_i \in \mathbb{K}[T_1, \ldots, T_r]$ not depending on T_i. If such a presentation exists, then we have $X = X_g$ as in Construction 3.2.2.

Proposition 3.2.5. *See* [6, Prop. 3.3.3.2]. *Let $X = X_g$ as in Construction 3.2.2. Then the anticanonical class of X is given in $K = \mathrm{Cl}(X)$ by*

$$-\mathcal{K} \;=\; w_1 + \cdots + w_r - \mu.$$

There is a decomposition of $X = X_g$ into locally closed subsets as follows. Denote by γ the positive orthant in $\mathbb{Q}^r$. A face $\gamma_0 \preceq \gamma$ is called an $\bar{X}_g$*-face* if there is $x \in \bar{X}_g$ with

$$x_i \neq 0 \iff e_i \in \gamma_0.$$

The orbit cones of $\bar{X}_g$ are precisely the cones $Q(\gamma_0)$, where γ_0 is an $\bar{X}_g$-face. An X*-face* is an $\bar{X}_g$-face γ_0 with $\lambda^\circ \subseteq Q(\gamma_0)^\circ$. Write $\mathrm{rlv}(X)$ for the set of X-faces. We have

$$X \;=\; \bigcup_{\gamma_0 \in \mathrm{rlv}(X)} X(\gamma_0), \qquad X(\gamma_0) \;:=\; \{x \in \bar{X}_g;\; x_i \neq 0 \iff e_i \in \gamma_0\} /\!\!/ H.$$

Proposition 3.2.6. *See* [6, Cor. 3.3.1.8]. *For $X = X_g$ as in Construction 3.2.2 the following hold.*

(i) *X is $\mathbb{Q}$-factorial if and only if the cone $\lambda = \mathrm{SAmple}(X) \subseteq K_{\mathbb{Q}}$ is full-dimensional.*
(ii) *X is locally factorial if and only if for every X-face $\gamma_0 \preceq \gamma$, the group K is generated by $Q(\gamma_0 \cap \mathbb{Z}^r)$.*

For locally factorial X of Picard number two, Proposition 3.2.6 (ii) in particular yields the following two statements.

Lemma 3.2.7. *See* [45, Lemma 5.6]. *Let $X = X_g$ as in Construction 3.2.2. Assume X is locally factorial and of Picard number two. Let $1 \le i, j \le r$ with $\lambda \subseteq \operatorname{cone}(w_i, w_j)$. Then either w_i, w_j generate K as a group, or g has precisely one monomial of the form $T_i^{l_i} T_j^{l_j}$, where $l_i + l_j > 0$.*

Let $X = X_g$ from Construction 3.2.2 be of Picard number two. Then we decompose the effective cone into the convex sets

$$\operatorname{Eff}(R_g) \ = \ \lambda^- \cup \lambda^\circ \cup \lambda^+,$$

where λ^- and λ^+ are the convex polyhedral cones not intersecting λ° and the intersection $\lambda^+ \cap \lambda^-$ consists only of the origin.

Lemma 3.2.8. *See* [45, Lemma 5.7]. *Let $X = X_g$ as in Construction 3.2.2. Assume X is locally factorial and of Picard number two. Let $1 \le i < j < k \le r$. Then w_i, w_j, w_k generate K as a group, provided that one of the following holds:*

(i) *$w_i, w_j \in \lambda^-$, $w_k \in \lambda^+$ and g has no monomial of the form $T_k^{l_k}$,*
(ii) *$w_i \in \lambda^-$, $w_j, w_k \in \lambda^+$ and g has no monomial of the form $T_i^{l_i}$,*
(iii) *$w_i \in \lambda^-$, $w_j \in \lambda^\circ$, $w_k \in \lambda^+$.*

Moreover, if (iii) holds, then g has a monomial of the form $T_j^{l_j}$ where l_j is divisible by the order of the factor group $K/\langle w_i, w_k\rangle$.

We turn to rational varieties with a complexity one torus action. For the general theory see [41, 42, 46], also [6, Sec. 3.4]. Here we focus on the case of hypersurface Cox rings.

Proposition 3.2.9. *For a Mori dream space X with a hypersurface Cox ring the following are equivalent:*

(i) *X admits a torus action of complexity one.*
(ii) *The Cox ring of X has an irredundant $\operatorname{Cl}(X)$-graded presentation*

$$\mathcal{R}(X) \ = \ \mathbb{C}[T_1, \ldots, T_r]/\langle g \rangle,$$

where g is a trinomial consisting of pairwise coprime monomials.

Proof. We write $K = \operatorname{Cl}(X)$ for the divisor class group and $R = \mathcal{R}(X)$ for the Cox ring of X. Assume that (i) holds. Then by [6, Thm. 4.4.1.6] there is an irredundant K-graded presentation

$$R \ = \ \mathbb{C}[T_1, \ldots, T_r]/\langle g_1, \ldots, g_t \rangle,$$

such that the variables define pairwise non-associated K-prime generators and the polynomials $g_1, \ldots, g_t \in \mathbb{C}[T_1, \ldots, T_r]$ are homogeneous trinomials, each one consisting of pairwise coprime monomials. Moreover, since X has a hypersurface Cox ring, there is an irredundant K-graded presentation

$$R \;=\; \mathbb{C}[T_1, \ldots, T_{r'}]/\langle g \rangle.$$

The graded isomorphism between these two presentations of R lifts to a graded isomorphism $\mathbb{C}[T_1, \ldots, T_r] \to \mathbb{C}[T_1, \ldots, T_{r'}]$, see [44, Lemma 2.4]. This yields $r = r'$ and $t = 1$. Now assume that (ii) holds. Using [6, Constr. 3.2.4.2] we construct the *gradiator* of g. This is the maximal effective grading of $\mathbb{K}[T_1, \ldots, T_r]$ for which the variables $T_1, \ldots, T_r$ and the polynomial g are homogeneous. Geometrically, this grading yields an effective action of a quasitorus H_g on X. The coprimeness of the monomials of g guarantees that the quasitorus H_g contains a torus T of dimension $\dim(T) = \dim(X) - 1$. Thus X admits an effective torus action of complexity one. □

We conclude this section by quoting two results used in the proof of Theorem 3.1.1. For a torsion-free grading group K the notions of K-factoriality and factoriality coincide, see [6, Thm. 3.4.1.11].

Remark 3.2.10. See [41, Thm. 1.1]. For $l_1, l_2, l_3 \in \mathbb{Z}^r_{\geq 0}$ assume that the monomials $T^{l_1}, T^{l_2}, T^{l_3} \in \mathbb{C}[T_1, \ldots, T_r]$ are pairwise coprime. Then the ring

$$R \;=\; \mathbb{C}[T_1, \ldots, T_r]/\langle T^{l_1} + T^{l_2} + T^{l_3} \rangle$$

is a unique factorization domain if and only if the integers $\gcd(l_1)$, $\gcd(l_2)$ and $\gcd(l_3)$ are pairwise coprime.

Remark 3.2.11. See [45, Prop. 2.4]. Let $X = X_g$ as in Construction 3.2.2. Then we have

$$\mu \;\in\; \bigcap_{1 \leq i < j \leq r} \operatorname{cone}(\, w_k;\ k \neq i, k \neq j\,) \;\subseteq\; K_{\mathbb{Q}}.$$

3.3 Picard group and smoothability

In this section we establish two general facts, being essential for our proof of Theorem 3.1.1. The first is Proposition 3.3.1, which shows that in our setting the Picard group is always torsion-free. The second is Proposition 3.3.2, which in particular gives rise to an explicit smoothing procedure in the case of an ample relation degree $\mu = \deg(g)$ in the Cox ring; see also Remark 3.10.13.

Proposition 3.3.1. *Let $X = X_g$ as in Construction 3.2.2. Assume that X is $\mathbb{Q}$-factorial, Fano, of Picard number two and admits a torus action of complexity one. Then the Picard group* $\operatorname{Pic}(X)$ *is torsion-free.*

Proof. By [6, Cor. 3.3.1.6] we have the identity

$$\operatorname{Pic}(X) \;=\; \bigcap_{\gamma_0\ X\text{-face}} Q(\operatorname{lin}(\gamma_0) \cap \mathbb{Z}^r).$$

It therefore suffices to show that there is a two-dimensional X-face. By Proposition 3.2.9 we may assume that $g \in S$ is a trinomial consisting of pairwise coprime monomials. We write $\rho_1, \ldots, \rho_s$ for the rays generated by the generator degrees $w_1, \ldots, w_7$. The effective cone of $R := R_g$ is given by $\operatorname{Eff}(R) = \rho_1 + \rho_s$. We distinguish two cases. First, assume $\mu = \deg(g)$ is contained in $\operatorname{Eff}(R)^\circ$. In this case we can find generator degrees w_i, w_j that satisfy the following conditions:

(i) $\lambda^\circ \subseteq \operatorname{cone}(w_i, w_j)$.

(ii) $\mu \in \operatorname{cone}(w_i, w_j)^\circ$.

(iii) g does not contain a monomial of the form $T_i^{l_i} T_j^{l_j}$.

Explicitly, we do the following: Taking $w_i \in \rho_1$ and $w_j \in \rho_s$ satisfies (i) and (ii). If g contains a monomial of the form $T_i^{l_i} T_j^{l_j}$, then, since μ is contained in the interior of $\operatorname{Eff}(R)$, the exponents l_i and l_j are positive. The definition of $\operatorname{Mov}(R)$ and Remark 3.2.11 ensure that we can replace either w_i or w_j with a different generator degree, such that the new pair $(w_{i'}, w_{j'})$ still satisfies (i) and (ii). Since the monomials of g are pairwise coprime, this pair also satisfies (iii). The face γ_0 of the positive orthant γ spanned by $e_{i'}$ and $e_{j'}$ is thus a two-dimensional X-face.

Now assume that μ lies on one of the bounding rays of $\operatorname{Eff}(R_g)$. We may assume that $\mu \in \rho_1$ holds. Since g is a trinomial and its monomials are pairwise coprime, the ray ρ_1 contains at least three generator degrees. If ρ_1 contains four or more generator degrees, then there is $w_i \in \rho_1$ such that g does not contain a monomial of the form $T_i^{l_i}$. Choose any $w_j \in \rho_s$. Then the face $\gamma_0 := \operatorname{cone}(e_i, e_j)$ is again a two-dimensional X-face. Now assume that ρ_1 contains exactly three generator degrees, say w_1, w_2 and w_3. The ample cone λ of X is of the form $\lambda = \rho_k + \rho_{k+1}$ for some $k = 1, \ldots, s-1$. If $\lambda \neq \rho_1 + \rho_2$, then we take w_i and w_j from each of the bounding rays of λ. The face $\gamma_0 := \operatorname{cone}(e_i, e_j)$ is again a two-dimensional X-face. It remains to consider the case $\lambda = \rho_1 + \rho_2$. Applying a unimodular transformation, we achieve that ρ_1 is the ray generated by e_1. We write $w_1 = (a_1, 0)$, $w_2 = (a_2, 0)$ and $w_3 = (a_3, 0)$. Switching the roles of w_1, w_2 and w_3 if necessary, we may assume that $a_1 \geq a_2 \geq a_3$ holds. Homogeneity of g yields $l_1 a_1 = l_2 a_2 = l_3 a_3$. As X is Fano, it's anticanonical class is ample. By Proposition 3.2.5, this means

$$(1 - l_1) w_1 + w_2 + w_3 + w_4 + \cdots + w_r \ \in \lambda^\circ \ = \ (\rho_1 + \rho_2)^\circ.$$

The point $w := w_4 + \cdots + w_r$ is contained in the cone $\rho_2 + (-\rho_1)$. Thus, for the sum to lie in the interior of λ, we must have $(1 - l_1) w_1 + w_2 + w_3 \in \rho_1$. This is equivalent to $(l_1 - 1) a_1 < a_2 + a_3$. Since a_1 is at least as big as a_2 and a_3, this yields $l_1 = 2$. Homogeneity of g thus yields $a_2 = 2a_1/l_2$ and $a_3 = 2a_1/l_3$. With this, the exponents l_2 and l_3 satisfy the following inequality:

$$1 \ < \ \frac{2}{l_2} + \frac{2}{l_3}.$$

Since $a_2 \geq a_3$, we have $l_3 \geq l_2$. Moreover, both exponents are at least two by irredundancy of the presentation of R. The triple (l_1, l_2, l_3) is therefore one of the following:

$$(l_1, l_2, l_3) \ = \ (2, 2, y), \qquad (l_1, l_2, l_3) \ = \ (2, 3, 3),$$

$$(l_1, l_2, l_3) \;=\; (2, 3, 4), \qquad (l_1, l_2, l_3) \;=\; (2, 3, 5),$$

where $y \geq 2$. Thus, by [5, Thm. 3.13] the variety X has at most log terminal singularities. With this, we are in the situation of [70, Prop. 2.1.2], which tells us that the Picard group $\mathrm{Pic}(X)$ is torsion-free. $\square$

Proposition 3.3.2. *Let $X = X_g$ as in Construction 3.2.2 be locally factorial and of Picard number two. Assume that*

$$\mu \;\in\; \mathrm{SAmple}(X) \cap \mathrm{Mov}(X)^\circ.$$

Then there is a non-empty open subset $U \subseteq S_\mu$ such that for all $h \in U$ the variety $X_h \subseteq Z$ is smooth with divisor class group $\mathrm{Cl}(X_h) = K$ and Cox ring $\mathcal{R}(X_h) = R_h$.

The remainder of this section is devoted to the proof of Proposition 3.3.2. We adopt the notation of Construction 3.2.1 and Construction 3.2.2. A homogeneous polynomial $h \in S_\mu$ is called *spread*, if every monomial $T^\nu \in S$ of degree $\mu = \deg(h)$ is a convex combination of monomials of h. We say that R_h is spread, if h is spread, see [45, Def. 4.3]. Here we identify a monomial $T^\nu = T_1^{\nu_1} \cdots T_r^{\nu_r}$ with its exponent vector $\nu \in \mathbb{Q}^r$. If $h, h' \in S_\mu$ are spread, then the minimal ambient toric varieties Z_h of X_h and $Z_{h'}$ of $X_{h'}$ coincide. Thus the toric variety $Z_\mu := Z_h$ is well-defined. It is called the *μ-minimal ambient toric variety*, see [45, Def. 4.18]. The following two Propositions, originally [45, Prop. 4.11] and [45, Cor. 4.19], are essential to the proof of Proposition 3.3.2.

Proposition 3.3.3. *See* [45, Cor. 4.19]. *In the setting of Construction 3.2.2, assume* $\mathrm{rank}(K) = 2$ *and that $Z_\mu \subseteq Z$ is smooth. If $\mu \in \tau$ holds, then μ is basepoint free. Moreover, then there is a non-empty open subset of polynomials $g \in S_\mu$ such that X_g is smooth.*

Proposition 3.3.4. *See* [45, Prop. 4.11]. *In the setting of Construction 3.2.2 assume that K is of rank at most $r - 4$ and torsion-free, there is $g \in S_\mu$ such that $T_1, \ldots, T_r$ define primes in R_g, we have $\mu \in \tau^\circ$ and μ is basepoint free on Z. Then there is a non-empty open subset of polynomials $g \in S_\mu$ such that R_g is a UFD.*

For the rest of this section it is assumed that we have $X = X_g$ as in Construction 3.2.2 and that X is locally factorial and of Picard number two. For any homogeneous $h \in S_\mu$ we denote by $\lambda_h \in \Lambda(R_h)$ the smallest GIT-cone that contains τ. Note that local factoriality of X in particular implies $\mathbb{Q}$-factoriality. Thus by Proposition 3.2.6 (i) the cone λ is full-dimensional.

Lemma 3.3.5. *Let $h \in S_\mu$ such that each monomial of g is also a monomial of h. Then $\lambda \subseteq \lambda_h$ holds.*

Proof. The cone $\lambda_h \in \Lambda(R_h)$ is the smallest GIT-cone that contains τ. Thus in the case $\tau = \lambda$ there is nothing to show. So we may assume that $\tau \subsetneq \lambda$ holds. We write $\lambda = \mathrm{cone}(w_i, w_j)$ and $\tau = \mathrm{cone}(w_k, w_l)$. Since τ is a proper subset of λ, one of its ray generators is contained in the interior of λ, say $w_k \in \lambda^\circ$. By [45, Prop. 2.8] the degree w_k is the only generator degree that is contained in the interior of λ. Moreover, μ lies on the ray of w_k and the relation g contains a monomial of the form $T_k^{l_k}$. Since h contains each monomial of g, it also contains the monomial $T_k^{l_k}$. Therefore, each projected $\bar{X}_h$-face $Q(\gamma_0)$, that contains τ, necessarily also contains generator degrees on both sides of w_k. Since w_k is the only generator degree in the interior of λ, the cone $Q(\gamma_0)$ also contains λ. This implies the assertion. □

Proposition 3.3.6. *The μ-minimal ambient toric variety $Z_\mu \subseteq Z$ is smooth.*

Proof. Let $h \in S_\mu$ spread such that each monomial of g is also a monomial of h and let $\gamma_0 \preceq \gamma$ with $\lambda_h^\circ \subseteq Q(\gamma_0)^\circ$. Write $\gamma_0 = \operatorname{cone}(e_{i_1}, \dots, e_{i_m})$. By Proposition 3.2.6 (ii) we have to show that either $w_{i_1}, \dots, w_{i_m}$ generate K as a group, or γ_0 is not an $\bar{X}_h$-face. Assume that $w_{i_1}, \dots, w_{i_m}$ do not generate K. We show that γ_0 is not an $\bar{X}_h$-face. By Lemma 3.3.5, we have $\lambda^\circ \subseteq Q(\gamma_0)^\circ$. Since λ is of full dimension, γ_0 is at least two-dimensional. In particular we have $m \geq 2$. None of the degrees $w_{i_1}, \dots, w_{i_m}$ lies in λ°: If one of them did, say $w_{i_1} \in \lambda^\circ$, then by [45, Prop. 2.8] it is the only generator degree in the interior of λ and g contains a monomial of the form $T_{i_1}^{l_{i_1}}$. Moreover, in this case we have $m \geq 3$. We may assume that w_{i_2} and w_{i_3} each lie in one of the bounding rays of $Q(\gamma_0)$. The degrees w_{i_2}, w_{i_3} do not generate K as a group. Thus, since X is locally factorial, by Proposition 3.2.6 (ii) the cone spanned by e_{i_2} and e_{i_3} is not a $\bar{X}$-face. This means that g contains a monomial of the form $T_{i_2}^{l_{i_2}} T_{i_3}^{l_{i_3}}$. But then the cone spanned by e_{i_1}, e_{i_2} and e_{i_3} is an $\bar{X}$-face and thus $w_{i_1}, w_{i_2}, w_{i_3}$ generate K. A contradiction. Thus none of the degrees $w_{i_1}, \dots, w_{i_m}$ lies in λ°. We may assume that the generator degrees are sorted in such a way that for all $v \in \lambda^\circ$ we have $\det(v, w_{i_j}) < 0$ if $j \leq k$ and $\det(v, w_{i_j}) > 0$ if $j > k$ for some fixed $1 \leq k \leq m$. We show that either $k = 1$ or $k + 1 = m$ holds. If $1 < k$ and $k + 1 < m$, then we have $m \geq 4$ and neither $w_{i_1}, w_{i_{m-1}}$, nor w_{i_2}, w_{i_m} generate K. By local factoriality of X and Proposition 3.2.6 (ii), the relation g then contains monomials of the form $T_{i_1}^{l_{i_1}} T_{i_{m-1}}^{i_{m-1}}$ and $T_{i_2}^{l_{i_2}} T_{i_m}^{i_m}$. Thus $\gamma_0' = \operatorname{cone}(w_{i_1}, w_{i_2}, w_{i_{m-1}}, w_{i_m})$ is an $\bar{X}$-face and $w_{i_1}, w_{i_2}, w_{i_{m-1}}, w_{i_m}$ generate K. A contradiction. We may thus assume that $k = 1$ holds. If $m = 2$, then g contains a monomial of the form $T_{i_1}^{l_{i_1}} T_{i_2}^{l_{i_2}}$ and this is the only monomial in S_μ only depending on these two variables. Therefore γ_0 is not an $\bar{X}_h$-face. If $m > 2$, then by Proposition 3.2.6 (ii), the relation g contains a monomial of the form $T_{i_1}^{l_{i_1}}$ and this is the only monomial in S_μ only depending on the variables $T_{i_1}, \dots, T_{i_m}$. Thus also in this case γ_0 is not an $\bar{X}_h$-face. □

Lemma 3.3.7. *If $\mu \in \lambda$ holds, then we also have $\mu \in \tau$.*

Proof. In case τ and λ coincide, there is nothing to show. So we assume that $\tau \subsetneq \lambda$ holds. Write $\lambda = \operatorname{cone}(w_i, w_j)$ and $\tau = \operatorname{cone}(w_k, w_l)$. Since τ is a proper subset of λ, one of the generator degrees in its bounding rays lies in the interior of λ, say $w_k \in \lambda^\circ$. By [45, Prop. 2.8], it is the only generator degree that lies in λ° and g contains a monomial of the form $T_k^{l_k}$. This shows that $\mu \in \tau$ holds. □

Proposition 3.3.8. *Assume that $\mu \in \lambda \cap \operatorname{Mov}(R_g)^\circ$ holds. Then there is a non-empty open subset $U \subseteq S_\mu$ such that R_h is a UFD for each $h \in U$.*

Proof. By Lemma 3.3.7 the relation degree μ is contained in the cone τ. We distinguish two cases. First assume that $\mu \in \tau^\circ$ holds. As Z_μ is smooth by Proposition 3.3.6, the class μ is basepoint free by Proposition 3.3.3. We can thus apply Proposition 3.3.4, which yields the assertion. Now assume that $\mu \in \partial\tau$ holds. Let $\tau_\mu \in \Lambda(S)$ the unique GIT-cone that contains μ in its interior, ie. τ_μ is the bounding ray of τ containing μ. We write $Z(\tau_\mu) := \bar{Z}^{ss}(\tau_\mu)/\!\!/H$ for the projective toric variety associated with τ_μ as in Construction 3.2.1. We show that μ is basepoint free on $Z(\tau_\mu)$. Note that μ is semiample. Thus by [30, Thm. 6.3.12] it suffices to show that μ is Cartier on $Z(\tau_\mu)$. By [6, Cor. 3.3.1.6] this is the case if and only if

$$\mu \in \bigcap_{\gamma_0 \in \operatorname{rlv}(Z(\tau_\mu))} Q(\gamma_0 \cap \mathbb{Z}^r)$$

holds. Let $\gamma_0 \in \operatorname{rlv}(Z(\tau_\mu))$. If $Q(\gamma_0)$ is two-dimensional, then $\lambda^\circ \subseteq Q(\gamma_0)^\circ$ holds and by Proposition 3.3.6 we have $Q(\gamma_0 \cap \mathbb{Z}^r) = K$. So assume that $Q(\gamma_0)$ is one-dimensional, ie. $Q(\gamma_0) = \tau_\mu$. We distinguish two cases. First assume $\mu \in \lambda^\circ$. Then by [45, Prop. 2.8], τ_μ contains a single generator degree w_k and μ is a multiple of w_k. Thus in this case $\mu \in Q(\gamma_0 \cap \mathbb{Z}^r)$ holds. Now we assume that $\mu \in \partial\lambda$ holds. Then τ_μ is one of the bounding rays of λ. Write $\gamma_0 = \operatorname{cone}(e_{i_1}, \dots, e_{i_m})$ and let w_k a generator degree in the other bounding ray of μ. If $w_{i_1}, \dots, w_{i_m}, w_k$ generate K, then μ is a linear combination of $w_{i_1}, \dots, w_{i_m}$ and thus $\mu \in Q(\gamma_0 \cap \mathbb{Z}^r)$ holds. If they do not generate K, then by Proposition 3.2.6 (ii) the relation g contains a single monomial only depending on $T_{i_1}, \dots, T_{i_m}, T_k$. Since μ is contained in τ_μ, this monomial can not depend on T_k. Thus μ is again a linear combination of the degrees $w_{i_1}, \dots, w_{i_m}$ and thus $\mu \in Q(\gamma_0 \cap \mathbb{Z}^r)$ holds. This shows that μ is basepoint free on $Z(\tau_\mu)$. We again apply Proposition 3.3.4, which yields the assertion. □

Proof of Proposition 3.3.2. The set $U_1 \subseteq S_\mu$ of polynomials h such that R_h is a UFD is open and non-empty by Proposition 3.3.8. The set $U_2 \subseteq S_\mu$ of polynomials h such that $T_1, \dots, T_r$ form a minimal system of non-associated K-prime generators in R_h is open by [45, Prop. 4.10] and since R_g has that property, the set U_2 is non-empty. Finally, the set $U_3 \subseteq S_\mu$ of polynomials h such that X_h is smooth is open and non-empty by Proposition 3.3.3 and Proposition 3.3.6. Now for any h in the intersection

$$U = U_1 \cap U_2 \cap U_3$$

the affine K-algebra R_h is a UFD and the variables $T_1, \dots, T_r$ define pairwise non-associated primes in R_h. Being a UFD implies that R_h is normal and that the K-grading is factorial. The grading is also pointed, as this is inherited from S. In particular this implies that $R_h^* = \mathbb{C}^*$ holds. We are thus in the situation of Construction 3.2.2. So R_h is the Cox ring of the smooth projective variety X_h. □

3.4 Proof of Theorem 3.1.1: Preparation

We are ready to enter the proof of Theorem 3.1.1. We start by fixing the setting.

Setting 3.4.1. Let X be a locally factorial Fano fourfold of Picard number $\rho = 2$ with a hypersurface Cox ring $R = \mathcal{R}(X)$. Write $K = \mathrm{Cl}(X)$ and let

$$R \;=\; \mathbb{C}[T_1, \ldots, T_7]/\langle g \rangle$$

an irredundant K-graded presentation of R such that the variables $T_1, \ldots, T_7$ define pairwise non-associated K-prime generators of R. We have $X = X_g$ as in Construction 3.2.2. We assume that X is of complexity $c = 1$. By Proposition 3.3.1 the group K is torsion-free and we identify $K = \mathbb{Z}^2$. By [6, Thm. 3.4.1.11] the ring R is a UFD. By Proposition 3.2.9 and Remark 3.2.10 we may thus assume that g satisfies the following two conditions.

(C1) The relation g is of the form

$$g \;=\; T^{l_1} + T^{l_2} + T^{l_3}$$

with $l_1, l_2, l_3 \in \mathbb{Z}^7_{\geq 0}$ such that each variable $T_1, \ldots, T_7$ divides at most one monomial of g.

(C2) The integers $\gcd(l_1)$, $\gcd(l_2)$ and $\gcd(l_3)$ are pairwise coprime.

We turn to the grading map Q of R. Write $w_i := Q(e_i) = \deg(T_i)$ and $\mu := \deg(g)$ for the degrees in K, also when regarded in $K_{\mathbb{Q}}$. Suitably ordering $w_1, \ldots, w_7$ we ensure

$$\det(w_i, w_j) \;\geq\; 0$$

whenever $i \leq j$. Some of the degrees w_i may share a common ray. We denote by s the number of distinct rays $\rho_1, \ldots, \rho_s$ generated by the degrees $w_1, \ldots, w_7$,

$$s \;:=\; \#\{\,\mathrm{cone}(w_i);\; i = 1, \ldots, 7\,\}.$$

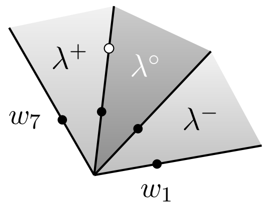
Moreover, we denote the number of generator degrees w_j contained in the ray ρ_i by n_i. We have $s \leq 7$ and

$$n_1 + \cdots + n_s \;=\; 7.$$

Each ray ρ_j in the GIT-fan $\Lambda(R)$ is of the form $\rho_j = \mathrm{cone}(w_i)$ for some w_i, but the converse may not hold. As X is locally factorial, it is in particular $\mathbb{Q}$-factorial. By Proposition 3.2.6 (i) this means that the cone $\lambda = \mathrm{SAmple}(X)$ is full-dimensional. As a GIT-cone in $K_{\mathbb{Q}} = \mathbb{Q}^2$, the cone λ is the intersection of two projected $\bar{X}_g$-faces and thus each bounding ray of λ contains at least one of the degrees w_i. We decompose the effective cone $\mathrm{Eff}(R)$ into the three convex sets

$$\mathrm{Eff}(X) \;=\; \lambda^- \cup \lambda^\circ \cup \lambda^+,$$

where λ^- and λ^+ are the convex polyhedral cones not intersecting $\lambda^\circ = \mathrm{Ample}(X)$ and the intersection $\lambda^+ \cap \lambda^-$ consists only of the origin. Each of the cones λ^+ and λ^- contains at least two of the generator degrees $w_1, \ldots, w_7$. However, λ^+ as well as λ^- may be one-dimensional. The following picture illustrates the situation for the case $s = 4$.



The black dots represent the generator degrees $w_1, \ldots, w_7$. The white dot represents the relation degree μ. In this example the cones λ^+ and λ^- are full-dimensional.

Remark 3.4.2. Let X as in Setting 3.4.1.

(i) The variety X is uniquely determined by it's *specifying data* (Q, g): The variety $X(Q, g) := X_g$ as in Construction 3.2.2 satisfies $X \cong X(Q, g)$.

(ii) Up to reversing order, the tuple $(n_1, \ldots, n_s)$ is invariant under automorphisms of K. We call it the *degree constellation* of X.

Remark 3.4.3. Given specifying data (Q, g) and (Q', g'), we need criteria to decide computationally whether or not the varieties $X(Q, g)$ and $X(Q', g')$ are isomorphic. Here we can make use of Proposition 2.3.1; see also [16, Prop. 3.4]: If $X(Q, g)$ and $X(Q', g')$ are isomorphic, then g and g' coincide up to permutation of variables.

Setting 3.4.1 divides the proof of Theorem 3.1.1 into six cases, according to the number s of rays spanned by the degrees $w_1, \ldots, w_7$. The case $s = 7$ does not occur, see Proposition 3.4.8. The other five cases $s = 2, \ldots, 6$ are treated in the coming Sections 3.5 to 3.9. Before we jump into the specific cases, we first gather some general observations that will be used throughout the proof.

Lemma 3.4.4. *Let $b > a > 1$ coprime integers. If $ab \leq 2 + a + b$ holds, then we have $a = 2$ and $b = 3$.*

Proof. Dividing both sides of the inequality in the assertion by ab, we obtain

$$1 \leq \frac{2}{ab} + \frac{1}{b} + \frac{1}{a}.$$

We have $ab \geq 6$ as well as $1/b < 1/a$. With this we obtain $a = 2$. The original inequality turns into $2b \leq 4 + b$. As $b > a$ holds and a and b are coprime, this is only fulfilled for $b = 3$. □

Lemma 3.4.5. *In the situation of Setting 3.4.1, for each ray ρ_j, at most two of the generator degrees w_i contained in ρ_j are non-primitive lattice points.*

Proof. Assume that ρ_j contains three non-primitive generator degrees w_{i_1}, w_{i_2} and w_{i_3}. Applying a unimodular transformation if necessary, we may assume that ρ_j is the ray generated by the first standard basis vector. Write

$$w_{i_1} = (a_{i_1}, 0), \qquad w_{i_2} = (a_{i_2}, 0), \qquad w_{i_3} = (a_{i_3}, 0).$$

As the degrees $w_{i_1}, w_{i_2}, w_{i_3}$ are non-primitive, we have $a_{i_1}, a_{i_2}, a_{i_3} > 1$. By [45, Prop. 2.8] the ray ρ_j does not lie in the interior of λ. Thus there is a generator degree w_k such that λ is contained in the two-dimensional cone $\tau_m = \mathrm{cone}(w_{i_m}, w_k)$ for $m = 1, 2, 3$. Since w_{i_m} is not primitive, the degrees w_{i_m} and w_k do not generate K as a group. Lemma 3.2.7 thus tells us that the relation g contains monomials of the form $T_{i_m}^{l_{i_m}} T_k^{l_k(m)}$. As the monomials of g a pairwise coprime, at least two of the exponent $l_k(m)$ must be zero. By homogeneity of g we obtain $l_k(1) = l_k(2) = l_k(3) = 0$ and μ lies in ρ_j. In particular, we have

$$\mu \;=\; l_{i_1} w_{i_1} \;=\; l_{i_2} w_{i_2} \;=\; l_{i_3} w_{i_3}.$$

By the condition (C2) from Setting 3.4.1, the integers $l_{i_1}, l_{i_2}, l_{i_3}$ are pairwise coprime. Moreover, they are all bigger than one by irredundancy of the presentation of R. Let p a prime divisor of l_{i_1}. Then p must divide both a_{i_2} and a_{i_3}. In particular, a_{i_2} and a_{i_3} are not coprime. Thus the three degrees w_{i_2}, w_{i_3} and w_k do not generate K as a group. By Lemma 3.2.8 the relation g must therefore contain a monomial of the form $w_k^{l_k}$. This is a contradiction to the position of μ. Thus at least one of $w_{i_1}, w_{i_2}, w_{i_3}$ is primitive. $\square$

Lemma 3.4.6. *In the situation of Setting 3.4.1, assume that $\mu \in \lambda^+ \backslash \lambda$ holds. Then the following hold.*

(i) *The cone λ is regular and every generator degree lying on its boundary is primitive.*

(ii) *All generator degrees contained in λ^- coincide. In particular, $n_1 \geq 2$ holds and $\lambda^- = \rho_1$ is a bounding ray of λ.*

Proof. We prove (i). Let $1 \leq i < j \leq 7$ such that $\lambda = \mathrm{cone}(w_i, w_j)$ holds. Since the relation degree μ is not contained in λ, the relation g does not contain a monomial of the form $T_i^{l_i} T_j^{l_j}$. By Lemma 3.2.7, the generator degrees w_i and w_j generate K as a group.
We prove (ii). Let $1 \leq i < j \leq 7$ such that $\lambda = \mathrm{cone}(w_i, w_j)$ holds. By (i) we may assume that $w_i = (1, 0)$ and $w_j = (0, 1)$ holds. We write $w_1 = (a_1, -b_1)$ for some $a_1, b_1 \in \mathbb{Z}_{\geq 0}$. Applying Lemma 3.2.7 to the generator degrees w_1 and w_j shows that $a_1 = 1$ holds. So in order to verify item (ii) it suffices to show that $b_1 = 0$ holds. For this we first show that $\mathrm{Eff}(R)$ contains a lattice point $v \in \mathbb{Z}^2$ of the form $v = (-a, 1)$ for some $a \in \mathbb{Z}_{\geq 1}$. Note that w_5, w_6, w_7 do not lie in λ. Write $w_k = (-a_k, b_k)$ for $k = 5, 6, 7$. If $b_k = 1$ holds for one of those, then we have found such a point v. Otherwise we have $b_5, b_6, b_7 > 1$. Thus $\det(w_i, w_k) = b_k > 1$ holds and by Lemma 3.2.7, the relation g contains monomials of the form $T_i^{k_i} T_5^{k_5}$, $T_i^{l_i} T_6^{l_6}$ and $T_i^{m_i} T_7^{m_7}$. Since only at most one monomial of g is divisible by T_i, we conclude that two of the exponents k_i, l_i and m_i must be zero. Homogeneity of g thus implies that two of the generator degrees w_5, w_6, w_7 lie on the ray through μ. Let $v = (v_1, v_2) \in \mathbb{Z}^2$ denote the primitive lattice vector on this ray. We apply Lemma 3.2.8 to w_i and the two generator degrees on the ray through μ to infer $v_2 = \det(w_i, v) = 1$. Thus v is a lattice point of the desired form. From $w_1 \in \lambda^-$ and $v \in \lambda^+$ we infer

$$0 \;<\; \det(w_1, v) \;=\; 1 - ab_1.$$

As a is positive, this inequality can only be fulfilled by $b_1 = 0$. Hence $w_1 = (1, 0)$ holds, which proves the assertion. $\square$

Lemma 3.4.7. *In the situation of Setting 3.4.1 assume that $\mu \in \rho_1$ holds. Then we have $s \leq 3$ and $n_1 \geq 4$.*

Proof. Since the relation g is a trinomial consisting of pairwise coprime monomials, the ray ρ_1 contains at least the three generator degrees w_1, w_2, w_3. Applying a unimodular transformation we may assume that ρ_1 is the ray generated by the first standard basis vector. We write $w_i = (a_i, 0)$ with positive integers a_1, a_2, a_3. By Lemma 3.4.5 at least one of a_1, a_2, a_3 is equal to one. Renaming variables if necessary, we may assume $a_1 = 1$. If no other generator degree is contained in ρ_1, then g must be of the form

$$g \;=\; T_1^{l_1} + T_2^{l_2} + T_3^{l_3}.$$

Homogeneity of g thus yields $l_1 = a_2 l_2$, which contradicts condition (C2) from Setting 3.4.1. Thus the ray ρ_1 contains at least the four generator degrees w_1, w_2, w_3, w_4. In particular we have $s \leq 4$. Assume that $s = 4$ holds. Applying Lemma 3.2.8 to the triples (w_1, w_2, w_5), (w_1, w_2, w_6) and (w_1, w_2, w_7) shows that w_5, w_6, w_7 are primitive. We apply a unimodular transformation to achieve

$$Q \;=\; \begin{bmatrix} 1 & 1 & a_3 & a_4 & a & b & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 \end{bmatrix}, \qquad \mu \;=\; (\mu_1, 0),$$

where $a > b > 0$ and $a_4 \geq a_3$. Lemma 3.2.8 applied to the triple (w_3, w_4, w_7) shows that a_3 and a_4 are coprime. The moving cone is given by $\mathrm{Mov}(R) = \rho_1 + \rho_3$ and it is subdivided by ρ_2 into two two-dimensional chambers. One of them is λ. If $\lambda = \rho_2 + \rho_3$ holds, then by Lemma 3.4.6 the degrees w_6 and w_7 must coincide, which contradicts the assumption $s = 4$. Thus $\lambda = \rho_1 + \rho_2$ holds. To satisfy the conditions (C1) and (C2) from Setting 3.4.1, the relation g must be of the form

$$g \;=\; T_1^{l_1} T_2^{l_2} + T_3^{l_3} + T_4^{l_4},$$

where l_3 and l_4 are coprime. By homogeneity of g we obtain $l_3 a_3 = l_4 a_4$. In particular $a_3 = l_4$ and $a_4 = l_3$ and $\mu_1 = a_3 a_4$ holds. By Proposition 3.2.5, the anticanonical class of X is given by

$$-\mathcal{K} \;=\; \begin{bmatrix} 2 + a_3 + a_4 - \mu_1 + a + b \\ 3 \end{bmatrix}.$$

From X being Fano, ie. $-\mathcal{K} \in \lambda$, we infer the inequality

$$0 \;\leq\; 1 + a_3 + a_4 - \mu_1 + b - 2a.$$

Since $b - 2a$ is negative, we must have $\mu_1 < 1 + a_3 + a_4$. We are thus in the setting of Lemma 3.4.4, which yields $a_3 = 2$ and $a_4 = 3$ and $\mu_1 = 6$. However, this yields $2a \leq b$, which contradicts the fact that $a > b$ holds. Thus we must have $s \leq 3$. $\square$

Proposition 3.4.8. *In the situation of Setting 3.4.1 we have $s \leq 6$.*

Proof. Let X as in 3.4.1 and assume that $s = 7$ holds. Then each of the seven rays $\rho_1, \ldots, \rho_7$ contains a single generator degree. In particular, Lemma 3.4.6 (ii) tells us that we must have $\mu \in \lambda$. The Moving cone is given by $\mathrm{Mov}(R) = \rho_2 + \rho_6$ and by Remark 3.2.11 the relation degree μ is contained in the cone $\rho_3 + \rho_5$. In particular, μ lies in the interior of $\mathrm{Mov}(R)$. We are thus in the situation of Proposition 3.3.2 which tells us that for a general polynomial $h \in \mathbb{C}[T_1, \ldots, T_7]$ of degree $\deg(h) = \mu$, the variety X_h is smooth with divisor class group $\mathrm{Cl}(X_h) = K$ and Cox ring $\mathcal{R}(X_h) = R_h$. Moreover, by Proposition 3.2.5 X_h is Fano. Thus, X_h is a smooth Fano fourfold of Picard number two with a spread hypersurface Cox ring. In particular, up to unimodular equivalence, the grading matrix $Q = (w_1, \ldots, w_7)$ together with the relation degree $\mu = \deg(g)$ appear in the classification list presented in [45, Thm. 1.1]. However, there is no entry in that list with generator degrees $w_1, \ldots, w_7$ distributed among seven different rays. Thus we have $s \leq 6$. $\square$

3.5 Proof of Theorem 3.1.1: Case $s = 2$

Setting 3.4.1 and Proposition 3.4.8 divide the proof of Theorem 3.1.1 into the five cases $s = 2, \ldots, 6$, according to the number of rays spanned by the degrees $w_1, \ldots, w_7$. In this section we treat the case $s = 2$.

Theorem 3.5.1. *The table from 3.10.1 provides specifying data (Q, g) for 18 locally factorial Fano fourfolds of Picard number $\rho = 2$ and complexity $c = 1$ with a hypersurface Cox ring and $s = 2$. Moreover, any locally factorial Fano fourfold with a hypersurface Cox ring and invariants $(\rho, c, s) = (2, 1, 2)$ is isomorphic to precisely one $X(Q, g)$ with (Q, g) from that table.*

Proof. With the tools provided in Section 3.2 we verify that each specifying data (Q, g) from the table in 3.10.1 defines a locally factorial Fano fourfold $X(Q, g)$ with a hypersurface Cox ring and invariants $(\rho, c, s) = (2, 1, 2)$. Moreover, with the help of Remark 3.4.3 we verify that distinct specifying data from the table in 3.10.1 define pairwise non-isomorphic varieties. This proves the first assertion in Theorem 3.5.1. For the second assertion let X as in Setting 3.4.1 with invariants $(\rho, c, s) = (2, 1, 2)$. We show that X is isomorphic to $X(Q, g)$ with (Q, g) from the table in 3.10.1. By assumption the generator degrees $w_1, \ldots, w_7$ lie on two distinct rays. In particular, the GIT-fan of X contains a single full-dimensional cone. Since λ is full-dimensional, we have

$$\mathrm{Eff}(R) \;=\; \mathrm{Mov}(R) \;=\; \lambda.$$

We distinguish two cases, depending on the position of the relation degree μ relative to λ.

Case 3.5.1.1: $\mu \in \lambda^\circ$. We can apply Proposition 3.3.2, which tells us that for a general polynomial $h \in \mathbb{C}[T_1, \ldots, T_7]$ of degree $\deg(h) = \mu$, the variety X_h is a smooth Fano fourfold of Picard number two with a spread hypersurface Cox ring. Thus the grading matrix $Q = (w_1, \ldots, w_7)$ together with the relation degree $\mu = \deg(g)$ appear in the classification list presented in [45, Thm. 1.1]. For each such entry (Q, μ) with $s = 2$ we determine all trinomials g of degree $\deg(g) = \mu$ that satisfy the conditions of Setting 3.4.1 and filter the resulting list by isomorphy. This yields the specifying data with ID's 1 to 7 in Classification list 3.10.1. The variety X is isomorphic to precisely one variety with specifying data (Q, g) from that list.

Case 3.5.1.2: $\mu \in \partial\lambda$. The relation degree μ is contained in one of the bounding rays of the effective cone. Reversing orientation if necessary, we may assume that $\mu \in \rho_1$ holds. Let $m = n_1$. By Remark 3.2.11 we have $m \geq 3$. Moreover, by the definition of $\mathrm{Mov}(R)$, we also have $m \leq 5$. Applying Lemma 3.2.8 to the generator degrees w_1, w_2, w_i, where $i \geq m+1$ shows that the effective cone is regular and that every generator degree contained ρ_2 is primitive. For the grading matrix Q and relation degree μ we can thus write

$$Q \;=\; \begin{bmatrix} a_1 & \dots & a_m & 0 & \dots & 0 \\ 0 & \dots & 0 & 1 & \dots & 1 \end{bmatrix}, \qquad \mu \;=\; (\mu_1, 0),$$

with $a_1, \dots, a_m \in \mathbb{Z}_{\geq 1}$ and we may assume $a_1 \leq \dots \leq a_m$. Again applying Lemma 3.2.8 to the generator degrees w_i, w_j, w_7 with $1 \leq i < j \leq m$ shows that the integers $a_1, \dots, a_m$ are pairwise coprime. Moreover, the relation g only depends on the variables $T_1, \dots, T_m$. Assume $m = 3$ holds. Since the monomials of g are pairwise coprime, we have

$$g \;=\; T_1^{l_1} + T_2^{l_2} + T_3^{l_3}$$

with $l_1 a_1 = l_2 a_2 = l_3 a_3$. Since R is a UFD, the exponents l_1, l_2, l_3 are pairwise coprime; see Remark 3.2.10. This is not possibly due to the coprimeness of a_1, a_2, a_3. Thus $m \geq 4$ holds and the degree constellation (n_1, n_2) of X is one of the following:

$$(n_1, n_2) \;=\; (5, 2), \qquad\qquad (n_1, n_2) \;=\; (4, 3).$$

Case 3.5.1.2.1: $(n_1, n_2) = (5, 2)$. Applying Lemma 3.2.8 to the generator degrees w_i, w_6, w_7 for $i = 1, \dots, 5$ shows that at most three of the degrees $w_1, \dots, w_5$ are non-primitive. Thus $a_1 = a_2 = 1$ holds. If $a_3 > 1$ holds, then g is of the form $g = T_3^{l_3} + T_4^{l_4} + T_5^{l_5}$ with pairwise coprime exponents l_3, l_4, l_5. This is not possible due to the coprimeness of a_3, a_4, a_5. Therefore $a_3 = 1$ holds and we have

$$Q \;=\; \begin{bmatrix} 1 & 1 & 1 & a_4 & a_5 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix}, \qquad \mu \;=\; (\mu_1, 0).$$

The anticanonical class is given by $-\mathcal{K} = (3 + a_4 + a_5 - \mu_1, 2)$. The Fano condition on X yields

$$\mu_1 \;\leq\; 2 + a_4 + a_5. \tag{3.5.1.1}$$

We distinguish three cases, depending on the values of a_4 and a_5.

Case 3.5.1.2.1.1: $a_4, a_5 > 1$. We can apply Lemma 3.4.4 to obtain $a_4 = 2$, $a_5 = 3$ and $\mu_1 = 6$. Grading matrix and relation degree are thus given by

$$Q = \begin{bmatrix} 1 & 1 & 1 & 2 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix}, \qquad \mu = (6, 0).$$

Applying Lemma Lemma 3.2.7 to the pairs (w_4, w_7) and (w_5, w_7) shows that g is of the form

$$g = T_1^{l_1} T_2^{l_2} T_3^{l_3} + T_4^3 + T_5^2.$$

By homogeneity of g we have $l_1 + l_2 + l_3 = 6$. Since R is a UFD, l_1, l_2, l_3 must be coprime; see Remark 3.2.10. Filtering by isomorphy, this leads to the specifying data no. 11, 12 and 13.

Case 3.5.1.2.1.2: $a_4 = 1, a_5 > 1$. Applying Lemma 3.2.7 to the pair (w_5, w_6) shows that g contains a monomial of the form $T_5^{l_5}$ with $l_5 > 1$. In particular we have $\mu_1 = l_5 a_5$ and Equation 3.5.1.1 yields

$$(l_5 - 1) a_5 \leq 3.$$

There are thus two cases for a_5 and l_5, namely $(a_5, l_5) = (2, 2)$ and $(a_5, l_5) = (3, 2)$. In the first case the grading matrix and relation degree are given by

$$Q = \begin{bmatrix} 1 & 1 & 1 & 1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix}, \qquad \mu = (4, 0).$$

We check all trinomials g of degree $(4, 0)$ that contain the monomial T_5^2 for the conditions from Setting 3.4.1 and filter the resulting list by isomorphy. This leads to the single specifying data no. 14. In the second case the grading matrix and relation degree are given by

$$Q = \begin{bmatrix} 1 & 1 & 1 & 1 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix}, \qquad \mu = (6, 0).$$

Again we check all trinomials g of degree $(6, 0)$ that contain the monomial T_5^2 for the conditions from Setting 3.4.1 and filter the resulting list by isomorphy. This leads to the specifying data no. 9 and no. 10.

Case 3.5.1.2.1.3: $a_4 = a_5 = 1$. Equation 3.5.1.1 yields $\mu_1 \leq 4$. By assumption the presentation of R is irredundant. Thus $\mu_1 \geq 2$ holds. We have

$$Q = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix}, \qquad \mu = (\mu_1, 0), \qquad 2 \leq \mu_1 \leq 4.$$

For each value of μ_1 we determine all trinomials g of degree μ satisfying the conditions from Setting 3.4.1 and filter the resulting list by isomorphy. This leads to the specifying data no. 15, 16 and 17.

Case 3.5.1.2.2: $(n_1, n_2) = (4, 3)$. With the same arguments as in the case 2.1 we obtain $a_1 = a_2 = 1$ and that a_3 and a_4 are coprime. The relation g only depends on the variables $T_1, \dots, T_4$. Since its monomials are pairwise coprime, the relation g contains at least two monomials that only depend on a single variable. Moreover, as R is a UFD, their exponents must be coprime, see Remark 3.2.10. This is only possible if g is of the form

$$g \ = \ T_1^{l_1} T_2^{l_2} + T_3^{l_3} + T_4^{l_4}.$$

By homogeneity of g we have $\mu_1 = l_3 a_3 = l_4 a_4$. Coprimeness of l_3 and l_4 yields $a_3, a_4 > 1$ and that they are coprime. We can thus apply Lemma 3.4.4 to obtain $a_3 = 2$, $a_4 = 3$ and $\mu_1 = 6$. Grading matrix and relation degree are thus given by

$$Q \ = \ \begin{bmatrix} 1 & 1 & 2 & 3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 \end{bmatrix}, \qquad \mu \ = \ (6, 0).$$

The exponents of the first monomial of g are coprime and satisfy $l_1 + l_2 = 6$. Filtering by isomorphy, this leads to specifying data no. 18. $\square$

3.6 Proof of Theorem 3.1.1: Case $s = 3$

Setting 3.4.1 and Proposition 3.4.8 divide the proof of Theorem 3.1.1 into the five cases $s = 2, \dots, 6$, according to the number of rays spanned by the degrees $w_1, \dots, w_7$. In this section we treat the case $s = 3$.

Theorem 3.6.1. *The tables from 3.10.2, 3.10.3 and 3.10.4 provide specifying data* (Q, g) *for* 223 *sporadic cases and* 4 *infinite series of locally factorial Fano fourfolds of Picard number* $\rho = 2$ *and complexity* $c = 1$ *with a hypersurface Cox ring and* $s = 3$. *Moreover, any locally factorial Fano fourfold with a hypersurface Cox ring and invariants* $(\rho, c, s) = (2, 1, 3)$ *is isomorphic to precisely one* $X(Q, g)$ *with* (Q, g) *from these tables.*

The proof of Theorem 3.6.1 splits into two parts. First, with the tools provided in Section 3.2 we verify that each specifying data (Q, g) from the tables in 3.10.2, 3.10.3 and 3.10.4 defines a locally factorial Fano fourfold $X(Q, g)$ with a hypersurface Cox ring and invariants $(\rho, c, s) = (2, 1, 3)$. Moreover, with the help of Remark 3.4.3 we verify that distinct specifying data from the tables in 3.10.2, 3.10.3 and 3.10.4 define pairwise non-isomorphic varieties. The second part is to show that any locally factorial Fano fourfold with a hypersurface Cox ring and invariants $(\rho, c, s) = (2, 1, 3)$ is isomorphic to $X(Q, g)$ with (Q, g) from these tables. We divide the proof of this into the two general cases

$$\mu \ \in \ \mathrm{SAmple}(X), \qquad\qquad \mu \ \notin \ \mathrm{SAmple}(X).$$

The case $\mu \in \mathrm{SAmple}(X)$ will be treated in Proposition 3.6.2. In Proposition 3.6.3 we treat the case $\mu \notin \mathrm{SAmple}(X)$.

Proposition 3.6.2. *Let X as in Setting 3.4.1 with $s = 3$. Assume that $\mu \in \lambda$ holds. Then X is isomorphic to an $X(Q, g)$ with specifying data (Q, g) appearing in Classification list 3.10.2.*

Proof. We divide the proof into the two cases $\mu \in \mathrm{Mov}(R)^\circ$ and $\mu \in \partial\,\mathrm{Mov}(R)$.

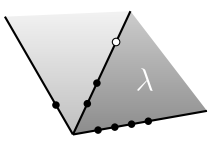
Case 3.6.2.1: $\mu \in \mathrm{Mov}(R)^\circ$. We are in the situation of Proposition 3.3.2. Thus, for a general polynomial $h \in \mathbb{C}[T_1, \ldots, T_7]$ of degree $\deg(h) = \mu$, the projective variety X_h is smooth with divisor class group $\mathrm{Cl}(X_h) = K$ and Cox ring $\mathcal{R}(X_h) = R_h$. Moreover, by Proposition 3.2.5 X_h is Fano. Thus, X_h is a smooth Fano fourfold of Picard number two with a spread hypersurface Cox ring. In particular, up to unimodular equivalence, the grading matrix $Q = (w_1, \ldots, w_7)$ together with the relation degree $\mu = \deg(g)$ appear in the classification list presented in [45, Thm. 1.1]. For each such entry (Q, μ) with $s = 3$ we determine all trinomials g of degree $\deg(g) = \mu$ that satisfy the conditions (C1) and (C2) from Setting 3.4.1 and filter the resulting list by isomorphy. This yields the specifying data no. 19 to 80 in Classification list 3.10.2.

Case 3.6.2.2: $\mu \in \partial\,\mathrm{Mov}(R)$. The relation degree μ is contained in one of the rays ρ_1, ρ_2, ρ_3. Reversing the order of the variables if necessary, we may assume that $\lambda = \rho_1 + \rho_2$ holds and that μ is contained in either ρ_1 or ρ_2.

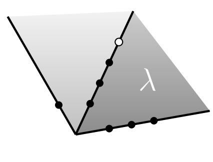
Case 3.6.2.2.1: $\mu \in \rho_2$. The ray ρ_2 is a bounding ray of $\mathrm{Mov}(R)$. Thus in this configuration the cones λ and $\mathrm{Mov}(R)$ coincide. By the definition of $\mathrm{Mov}(R)$ we must have $n_3 = 1$ and $n_1 \geq 2$. Moreover, Remark 3.2.11 yields $n_2 \geq 2$. Applying Lemma 3.2.8 to the generator degrees w_1, w_2, w_7 shows that the cone $\mathrm{Eff}(R)$ is regular and that w_7 is primitive. We may thus assume that $\mathrm{Eff}(R)$ is the positive quadrant and that $w_7 = (0, 1)$ holds. Applying Lemma 3.2.8 to the triples (w_1, w_5, w_6) and (w_2, w_5, w_6) shows that the primitive generator v of ρ_2 is of the form $v = (c, 1)$ for some $c \geq 1$. We obtain

$$w_1 \;=\; w_2 \;=\; (1, 0), \qquad w_5 \;=\; (a_5 c, a_5), \qquad w_6 \;=\; (a_6 c, a_6),$$

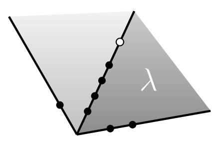
where $a_5, a_6, c \in \mathbb{Z}_{\geq 1}$ and a_5, a_6 are coprime. We may assume that $a_6 \geq a_5$ holds. There are three possible degree constellations (n_1, n_2, n_3) for X, displayed in the following pictures.



$(n_1, n_2, n_3) = (4, 2, 1)$



$(n_1, n_2, n_3) = (3, 3, 1)$



$(n_1, n_2, n_3) = (2, 4, 1)$

The black dots represent the generator degrees $w_1, \ldots, w_7$, the white circle represents the relation degree μ. We distinguish three cases, according to the degree constellation.

Case 3.6.2.2.1.1: $(n_1, n_2, n_3) = (4, 2, 1)$. We apply Lemma 3.2.8 to the triples

(w_3, w_5, w_6) and (w_4, w_5, w_6) to obtain $w_3 = w_4 = (1, 0)$. Grading matrix and relation degree are given by

$$Q = \begin{bmatrix} 1 & 1 & 1 & 1 & a_5c & a_6c & 0 \\ 0 & 0 & 0 & 0 & a_5 & a_6 & 1 \end{bmatrix}, \qquad \mu = (kc, k)$$

for some $k \in \mathbb{Z}_{\geq 1}$. As g is a trinomial with pairwise coprime monomials of degree μ, each monomial of g is divisible by precisely one of T_5, T_6, T_7. Due to the constellation of the generator degrees $w_1, \ldots, w_7$, the relation g thus contains a monomial of the form $T_5^{l_5}$ and a monomial of the form $T_6^{l_6}$, where $l_5, l_6 > 1$. The relation degree therefore satisfies $k = l_5a_5 = l_6a_6$. Note that by Remark 3.2.10, the exponents l_5 and l_6 are coprime. The coprimeness of a_5 and a_6 thus yields

$$l_5 = a_6, \qquad l_6 = a_6, \qquad k = a_5a_6.$$

By Proposition 3.2.5 the anticanonical class of X is given by

$$-\mathcal{K} = \begin{bmatrix} 4 + (a_5 + a_6 - k)c \\ 1 + a_5 + a_6 - k \end{bmatrix}.$$

From X being Fano, ie. $-\mathcal{K} \in \lambda^\circ$, we infer the inequality $k \leq a_5 + a_6$. We have $a_5 \leq a_6$. Note that a_5 and a_6 are both bigger than one and coprime. Thus $a_5 < a_6$ holds. Since k is divisible by both a_5 and a_6 we have $k > a_5 + a_6$. A contradiction. Thus this case does not occur.

Case 3.6.2.2.1.2: $(n_1, n_2, n_3) = (3, 3, 1)$. We apply Lemma 3.2.8 to the triple (w_3, w_5, w_6) to obtain $w_3 = (1, 0)$. By Lemma 3.4.5 at least one of w_4, w_5, w_6 is primitive. We may assume that w_4 is primitive. Grading matrix and relation degree are given by

$$Q = \begin{bmatrix} 1 & 1 & 1 & c & a_5c & a_6c & 0 \\ 0 & 0 & 0 & 1 & a_5 & a_6 & 1 \end{bmatrix}, \qquad \mu = (kc, k)$$

for some $k \in \mathbb{Z}_{\geq 1}$. Note that by irredundancy of the presentation of R we have $k \geq 2$. Let $5 \leq i \leq 6$. Assume that $a_i > 1$ holds. Then by Lemma 3.2.8, applied to the tuple (w_1, w_i), the relation g has a monomial of the form $T_i^{l_i}$ with $l_i \geq 2$. By homogeneity of g we have $k = l_ia_i$. If $a_i = 1$, then clearly k is a multiple of a_i. The relation degree thus satisfies

$$k = l_5a_5 = l_6a_6$$

with $l_5 \geq l_6 \geq 2$. By Proposition 3.2.5 the anticanonical class $-\mathcal{K}$ of X is given by

$$-\mathcal{K} = \begin{bmatrix} 3 + (1 + a_5 + a_6 - k)c \\ 2 + a_5 + a_6 - k \end{bmatrix}.$$

From X being Fano, ie. $-\mathcal{K} \in \lambda$, we infer the inequalities

$$k \leq 1 + a_5 + a_6, \tag{3.6.2.1}$$

$$c \leq 2. \tag{3.6.2.2}$$

We distinguish three cases, depending on the values of a_5 and a_6.

Case 3.6.2.2.1.2.1: $a_5 = a_6 = 1$. Equation 3.6.2.1 yields the bound $k \leq 3$. Grading matrix and relation degree are given by

$$Q = \begin{bmatrix} 1 & 1 & 1 & c & c & c & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 \end{bmatrix}, \qquad \mu = (kc, k)$$

with $2 \leq k \leq 3$ and $1 \leq c \leq 2$. For the possible values of k and c we check each homogeneous trinomial g of degree $\deg(g) = \mu$ for the conditions (C1) and (C2) from Setting 3.4.1 and filter by isomorphy. Depending on the values of k and c this leads to the following specifying data from Classification list 3.10.2:

(k,c)	$(2,1)$	$(3,1)$	$(2,2)$	$(3,2)$
ID	149	150, 151	152, 153	154, 155, 156, 157, 158

Case 3.6.2.2.1.2.2: $a_5 = 1$, $a_6 > 1$. Equation 3.6.2.1 yields $a_6 = 2$ and $l_6 = 2$. Grading matrix and relation degree are given by

$$Q = \begin{bmatrix} 1 & 1 & 1 & c & c & 2c & 0 \\ 0 & 0 & 0 & 1 & 1 & 2 & 1 \end{bmatrix}, \qquad \mu = (4c, 4).$$

For the two values of c we check each homogeneous trinomial g of degree $\deg(g) = \mu$ for the conditions (C1) and (C2) from Setting 3.4.1 and filter by isomorphy. For $c = 1$ this leads to specifying data no. 159 and 160. For $c = 2$ we get the specifying data no. 161 to no. 166.

Case 3.6.2.2.1.2.3: $a_5, a_6 > 1$. We have $a_5 = l_6$, $a_6 = l_5$ and $k = a_5 a_6$. Thus we can apply Lemma 3.4.4 to obtain $a_5 = 2$, $a_6 = 3$ and $k = 6$. Grading matrix and relation degree are given by

$$Q = \begin{bmatrix} 1 & 1 & 1 & c & 2c & 3c & 0 \\ 0 & 0 & 0 & 1 & 2 & 3 & 1 \end{bmatrix}, \qquad \mu = (6c, 6).$$

For the two values of c we check each homogeneous trinomial g of degree $\deg(g) = \mu$ for the conditions (C1) and (C2) from Setting 3.4.1 and filter by isomorphy. For $c = 1$ this leads to specifying data no. 167 to 180. For $c = 2$ we get the specifying data no. 181 to no. 219.

Case 3.6.2.2.1.3: $(n_1, n_2, n_3) = (2, 4, 1)$. By Lemma 3.4.5 at least two of w_3, w_4, w_5, w_6 are primitive. We may assume that w_3 and w_4 are primitive. Grading matrix and relation degree are given by

$$Q = \begin{bmatrix} 1 & 1 & c & c & a_5 c & a_6 c & 0 \\ 0 & 0 & 1 & 1 & a_5 & a_6 & 1 \end{bmatrix}, \qquad \mu = (kc, k)$$

for some $k \in \mathbb{Z}_{\geq 1}$. By irredundancy of the presentation of R we have $k \geq 2$. By Proposition 3.2.5 the anticanonical class $-\mathcal{K}$ of X is given by

$$-\mathcal{K} \;=\; \begin{bmatrix} 2+(2+a_5+a_6-k)c \\ 3+a_5+a_6-k \end{bmatrix}.$$

From X being Fano, ie. $-\mathcal{K} \in \lambda$, we infer the inequalities

$$k \;\leq\; 2+a_5+a_6, \tag{3.6.2.3}$$
$$c \;\leq\; 1. \tag{3.6.2.4}$$

Thus $c = 1$ holds. We distinguish three cases, depending on the values of a_5 and a_6.

Case 3.6.2.2.1.3.1: $a_5 = a_6 = 1$. Equation 3.6.2.3 yields the bound $k \leq 4$. Grading matrix and relation degree are given by

$$Q \;=\; \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}, \qquad \mu \;=\; (k,k),$$

with $2 \leq k \leq 4$. For the three values of k we check each homogeneous trinomial g of degree $\deg(g) = \mu$ for the conditions (C1) and (C2) from Setting 3.4.1 and filter by isomorphy. For $k = 2$ we get specifying data no. 81 to 84, for $k = 3$ we get specifying data no. 85 to 91 and for $k = 4$ we get specifying data no. 92 to 101.

Case 3.6.2.2.1.3.2: $a_5 = 1$, $a_6 > 1$. Equation 3.6.2.3 yields $l_6 = 2$ and $a_6 \leq 3$. Grading matrix and relation degree are given by

$$Q \;=\; \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & a_6 & 0 \\ 0 & 0 & 1 & 1 & 1 & a_6 & 1 \end{bmatrix}, \qquad \mu \;=\; (2a_6, 2a_6).$$

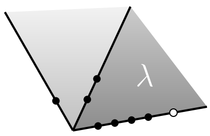
For the two values of a_6 we check each homogeneous trinomial g of degree $\deg(g) = \mu$ for the conditions (C1) and (C2) from Setting 3.4.1 and filter by isomorphy. For $a_6 = 2$ this leads to specifying data no. 102 to 107. For $a_6 = 3$ we get the specifying data no. 108 to no. 129.

Case 3.6.2.2.1.3.3: $a_5, a_6 > 1$. By Lemma 3.2.8 applied to (w_1, w_5, w_6), the integers a_5, a_6 are coprime. With Remark 3.2.10 we obtain $a_5 = l_6$, $a_6 = l_5$ and $k = a_5a_6$. Thus we can apply Lemma 3.4.4 to obtain $a_5 = 2$, $a_6 = 3$ and $k = 6$. Grading matrix and relation degree are given by

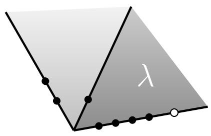
$$Q \;=\; \begin{bmatrix} 1 & 1 & 1 & 1 & 2 & 3 & 0 \\ 0 & 0 & 1 & 1 & 2 & 3 & 1 \end{bmatrix}, \qquad \mu \;=\; (6,6).$$

We check each homogeneous trinomial g of degree $\deg(g) = \mu$ for the conditions (C1) and (C2) from Setting 3.4.1 and filter by isomorphy. This leads to specifying data no. 130 to 148.

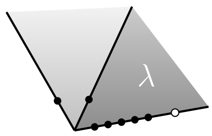
Case 3.6.2.2.2: $\mu \in \rho_1$. By Remark 3.2.11 we have $n_1 \geq 3$. Assume $n_1 = 3$. Then g is of the form $g = T_1^{l_1} + T_2^{l_2} + T_3^{l_3}$. By irredundancy of the presentation of R, the exponents l_1, l_2, l_3 are all at least two. By Lemma 3.4.5, at least one of w_1, w_2, w_3 is primitive, say w_1. Then l_1 is a multiple of l_2. In particular, l_1 and l_2 are not coprime. By Remark 3.2.10 this is a contradiction to factoriality of R. Thus $n_1 \geq 4$ holds. Applying Lemma 3.2.8 to the triple (w_1, w_2, w_7) shows that the cone $\mathrm{Eff}(R)$ is regular and that w_7 is primitive. We may thus assume that $\mathrm{Eff}(R)$ is the positive quadrant and that $w_7 = (0, 1)$ holds. By Lemma 3.4.5 at least two of the generator degrees contained in ρ_1 are primitive. We may thus assume that $w_1 = w_2 = (1, 0)$ holds. There are three possible degree constellations (n_1, n_2, n_3) for X, displayed in the following pictures.



$(4, 2, 1)$



$(4, 1, 2)$



$(5, 1, 1)$

The black dots represent the generator degrees $w_1, \ldots, w_7$, the white circle represents the relation degree μ. We distinguish three cases, according to the degree constellation.

Case 3.6.2.2.2.1: $(n_1, n_2, n_3) = (4, 2, 1)$. Applying Lemma 3.2.8 to the triples (w_1, w_2, w_5) and (w_1, w_2, w_6) shows that $w_5 = w_6 = (c, 1)$ holds for some $c \geq 1$. Grading matrix and relation degree are thus given by

$$Q \;=\; \begin{bmatrix} 1 & 1 & a_3 & a_4 & c & c & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 \end{bmatrix}, \qquad \mu \;=\; (\mu_1, 0).$$

By Proposition 3.2.5 the anticanonical class $-\mathcal{K}$ of X is given by

$$-\mathcal{K} \;=\; \begin{bmatrix} 2 + 2c + a_3 + a_4 - \mu_1 \\ 3 \end{bmatrix}.$$

From X being Fano, ie. $-\mathcal{K} \in \lambda$, we infer the inequality

$$\mu_1 \;\leq\; a_3 + a_4. \tag{3.6.2.5}$$

Note that the relation g only depends on the variables $T_1, \ldots, T_4$. As g is a trinomial of coprime monomials, it contains at least two monomials which only depend on a single variable. Having in mind the restrictions imposed on g by Remark 3.2.10, the only possible form for g is

$$g \;=\; T_1^{l_1} T_2^{l_2} + T_3^{l_3} + T_4^{l_4},$$

where l_3 and l_4 are larger than one and coprime. By homogeneity of g, the integers a_3, a_4 are also larger than one and due to Lemma 3.2.8 applied to (w_3, w_4, w_7), they also must be coprime. Thus we have $a_3 = l_4$, $a_4 = l_3$ and $\mu_1 = a_3a_4$. In particular $\mu_1 > a_3 + a_4$ holds. This is a contradiction to Equation 3.6.2.5. Thus this case does not occur.

Case 3.6.2.2.2.2: $(n_1, n_2, n_3) = (4, 1, 2)$. Applying Lemma 3.2.8 to the triples (w_1, w_2, w_5) and (w_1, w_2, w_6) shows that $w_5 = (c, 1)$ holds for some $c \geq 1$ and $w_6 = (0, 1)$. Grading matrix and relation degree are thus given by

$$Q = \begin{bmatrix} 1 & 1 & a_3 & a_4 & c & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 \end{bmatrix}, \qquad \mu = (\mu_1, 0).$$

By Proposition 3.2.5 the anticanonical class $-\mathcal{K}$ of X is given by

$$-\mathcal{K} = \begin{bmatrix} 2 + c + a_3 + a_4 - \mu_1 \\ 3 \end{bmatrix}.$$

From X being Fano, ie. $-\mathcal{K} \in \lambda$, we infer the inequality

$$\mu_1 \leq a_3 + a_4 - 1.$$

As in the previous case, the relation g only depends on the variables $T_1, \ldots, T_4$. Since g is a trinomial consisting of coprime monomials, it contains at least two monomials which only depend on a single variable. Having in mind the restrictions imposed on g by Remark 3.2.10, the only possible form for g is

$$g = T_1^{l_1} T_2^{l_2} + T_3^{l_3} + T_4^{l_4},$$

where l_3 and l_4 are larger than one and coprime. By homogeneity of g, the integers a_3, a_4 are also larger than one and due to Lemma 3.2.8 applied to (w_3, w_4, w_7), they also must be coprime. Thus we have $a_3 = l_4$, $a_4 = l_3$ and $\mu_1 = a_3a_4$. In particular $\mu_1 > a_3 + a_4$ holds. This is a contradiction to Equation 3.6.2.5. Thus this case does not occur.

Case 3.6.2.2.2.3: $(n_1, n_2, n_3) = (5, 1, 1)$. By Lemma 3.4.5, at least one of w_3, w_4, w_5 is primitive, We may thus assume $w_3 = (1, 0)$. Applying Lemma 3.2.8 to the triple (w_1, w_2, w_6) shows that $w_6 = (c, 1)$ holds for some $c \geq 1$. Grading matrix and relation degree are thus given by

$$Q = \begin{bmatrix} 1 & 1 & 1 & a_4 & a_5 & c & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix}, \qquad \mu = (\mu_1, 0).$$

By Proposition 3.2.5 the anticanonical class $-\mathcal{K}$ of X is given by

$$-\mathcal{K} = \begin{bmatrix} 3 + c + a_4 + a_5 - \mu_1 \\ 2 \end{bmatrix}.$$

From X being Fano, ie. $-\mathcal{K} \in \lambda$, we infer the inequality

$$\mu_1 \leq 2 - c + a_4 + a_5. \tag{3.6.2.6}$$

Note that μ_1 is a multiple of both a_4 and a_5. This is only possible if $c \geq 2$ holds. We distinguish three cases, depending on the values of a_4 and a_5.

Case 3.6.2.2.2.3.1: $a_4 = a_5 = 1$. Equation 3.6.2.6 yields the bound $\mu_1 \leq 3$. Grading matrix and relation degree are given by

$$Q \;=\; \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & c & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix}, \qquad \mu \;=\; (k, 0),$$

with $2 \leq k \leq 3$. For the possible values of k and c we check each homogeneous trinomial g of degree $\deg(g) = \mu$ for the conditions (C1) and (C2) from Setting 3.4.1 and filter by isomorphy. For $(k, c) = (2, 1)$ we get specifying data no. 225, for $(k, c) = (3, 1)$ we get specifying data no. 224 and for $(k, c) = (2, 2)$ we get specifying data no. 226.

Case 3.6.2.2.2.3.2: $a_4 = 1$, $a_5 > 1$. Equation 3.6.2.6 yields $c = 1$ and $a_5 = 2$ and $l_5 = 2$. Grading matrix and relation degree are given by

$$Q \;=\; \begin{bmatrix} 1 & 1 & 1 & 1 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix}, \qquad \mu \;=\; (4, 0).$$

We check each homogeneous trinomial g of degree $\deg(g) = \mu$ for the conditions (C1) and (C2) from Setting 3.4.1 and filter by isomorphy. This leads to specifying data no. 223.

Case 3.6.2.2.2.3.3: $a_4, a_5 > 1$. By Lemma 3.2.8 applied to (w_4, w_5, w_7), the integers a_4, a_5 are coprime. With Remark 3.2.10 we obtain $a_4 = l_5$, $a_5 = l_4$ and $\mu_1 = a_4 a_5$. Thus we can apply Lemma 3.4.4 to obtain $a_4 = 2$, $a_5 = 3$ and $k = 6$. Plugging these values into Equation 3.6.2.6, we obtain $c = 1$. Grading matrix and relation degree are given by

$$Q \;=\; \begin{bmatrix} 1 & 1 & 1 & 2 & 3 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix}, \qquad \mu \;=\; (6, 0).$$

We check each homogeneous trinomial g of degree $\deg(g) = \mu$ for the conditions (C1) and (C2) from Setting 3.4.1 and filter by isomorphy. This yields the specifying data no. 220, 221 and 222. $\square$

Proposition 3.6.3. *Let X as in Setting 3.4.1 with $s = 3$. Assume that $\mu \notin \lambda$ holds. Then X is isomorphic to an $X(Q, g)$ with specifying data (Q, g) appearing in Classification list 3.10.3 or in Classification list 3.10.4.*

Proof. Reversing the order of the variables if necessary, we may assume that $\lambda = \rho_1 + \rho_2$ holds. By assumption μ is not contained in λ, so we have $\mu \in (\rho_2 + \rho_3) \backslash \rho_2$. By Lemma 3.4.6 we have $n_1 \geq 2$ and $w_1 = w_2$ is the primitive point in ρ_1. Moreover, by Remark 3.2.11 we have $n_3 \geq 3$. Applying Lemma 3.2.8 to the triple (w_1, w_6, w_7) shows that the cone $\mathrm{Eff}(R)$ is regular. We may thus assume that $\mathrm{Eff}(R)$ is the positive quadrant in $\mathbb{Q}^2$ and that $w_1 = w_2 = (1,0)$ holds. We distinguish the two cases $\mu \in \rho_3$ and $\mu \in (\rho_2 + \rho_3)^\circ$.

Case 3.6.3.1: $\mu \in \rho_3$. By Lemma 3.4.7 we have $n_3 \geq 4$. Thus X has degree constellation $(n_1, n_2, n_3) = (2, 1, 4)$. By Lemma 3.4.5 at least two of the generator degrees $w_4, \ldots, w_7$ are primitive. We may therefore assume that $w_4 = w_5 = (0, 1)$ holds. Applying Lemma 3.2.8 to the triple (w_1, w_2, w_3), we obtain $w_3 = (c, 1)$ for some $c \geq 1$. Grading matrix and relation degree are given by

$$Q = \begin{bmatrix} 1 & 1 & c & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & b_6 & b_7 \end{bmatrix}, \qquad \mu = (0, k)$$

for some $b_6, b_7, k \geq 1$. We may assume that $b_6 \leq b_7$ holds. The relation g only depends on the variables $T_4, \ldots, T_7$. As g is a trinomial with pairwise coprime monomials, it contains two monomials that each only depend on a single variable. To fulfill the conditions (C1) and (C2) on g from Setting 3.4.1, the only possible form for g is

$$g = T_4^{l_4} T_5^{l_5} + T_6^{l_6} + T_7^{l_7},$$

where l_6, l_7 and $\gcd(l_4, l_5)$ are pairwise coprime. Moreover, due do irredundancy of the presentation of R, the exponents l_6 and l_7 are at least two and homogeneity of g yields $l_6 b_6 = l_7 b_7$. Furthermore, applying Lemma 3.2.8 to the triple (w_1, w_6, w_7) shows that b_6 and b_7 are coprime. Thus we have $b_5 = l_6$ and $b_6 = l_5$ and $k = b_5 b_6$. By Proposition 3.2.5, the anticanonical class of X is given by

$$-\mathcal{K} = \begin{bmatrix} 2 + c \\ 3 + b_6 + b_7 - k \end{bmatrix}.$$

Form X being Fano, ie. $-\mathcal{K} \in \lambda^\circ$, we infer the inequalities

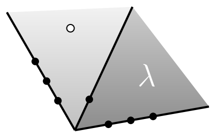
$$k \leq 2 + b_6 + b_7, \tag{3.6.3.1}$$
$$0 \leq 1 - 2c - (b_6 + b_7 - k)c. \tag{3.6.3.2}$$

Equation 3.6.3.1 together with Lemma 3.4.4 yields $b_6 = 2$, $b_7 = 3$ and $k = 6$. Plugging these values into Equation 3.6.3.2, we obtain $c = 1$. Grading matrix and relation degree are thus given by

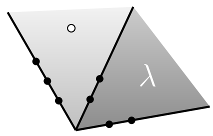
$$Q = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 2 & 3 \end{bmatrix}, \qquad \mu = (0, 6).$$

We check each homogeneous trinomial g of degree $\deg(g) = \mu$ for the conditions (C1) and (C2) from Setting 3.4.1 and filter by isomorphy. This leads to specifying data no. 241.

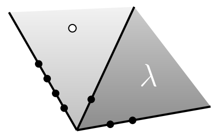
Case 3.6.3.2: $\mu \in (\rho_2 + \rho_3)^\circ$. By Remark 3.2.11 we have $n_3 \geq 3$. Applying Lemma 3.2.8 to the triple (w_1, w_2, w_i) with $w_i \in \rho_2 \cup \rho_3$ or the triple (w_j, w_6, w_7), where $w_j \in \rho_1$, shows that every generator degree is primitive. In particular we have $w_5 = w_6 = w_7 = (0, 1)$. The primitive point $v \in \rho_2$ is of the form $v = (a, 1)$ for some $a \geq 1$. There are three possible degree constellations (n_1, n_2, n_3) for X, displayed in the following pictures.



$(3, 1, 3)$



$(2, 2, 3)$



$(2, 1, 4)$

The black dots represent the generator degrees $w_1, \ldots, w_7$, the white circle represents the relation degree μ. We distinguish three cases, according to the degree constellation.

Case 3.6.3.2.1: $(n_1, n_2, n_3) = (3, 1, 3)$. Grading matrix and anticanonical class of X are given by

$$Q = \begin{bmatrix} 1 & 1 & 1 & a & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 \end{bmatrix}, \qquad -\mathcal{K} = \begin{bmatrix} 3 + a - \mu_1 \\ 4 - \mu_2 \end{bmatrix}.$$

From X being Fano, ie. $-\mathcal{K} \in \lambda^\circ$, we infer the inequalities

$$\mu_2 \leq 3, \tag{3.6.3.3}$$

$$\mu_1 \leq 2 - (3 - \mu_2)a. \tag{3.6.3.4}$$

In particular, we have $2 \leq \mu_2 \leq 3$. We first consider the case $\mu_2 = 2$. As μ_1 is positive, Equation 3.6.3.4 yields $a = 1$ and $\mu_1 = 1$. Grading matrix and relation degree are thus given by

$$Q = \begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 \end{bmatrix}, \qquad \mu = (1, 2).$$

Up to isomorphy this leads to specifying data no. 227 and 228. Now consider the case $\mu_2 = 3$. Then Equation 3.6.3.4 yields $1 \leq \mu_1 \leq 2$. We first consider the case $\mu = (1, 3)$. The relation g is a trinomial with pairwise coprime monomials. Due to the position of λ, each monomial of g is divisible by one of $T_1, \ldots, T_4$. If T_4 divides a monomial of g, then by homogeneity we have $a = 1$. Up to isomorphy this yields specifying data no. 229. If T_4 does not appear in g, then each monomial of the relation is divisible by precisely one of $T_1, \ldots, T_3$. Moreover, by the same argument, each monomial is divisible by precisely one of $T_5, \ldots, T_7$. Thus up to permutation of variables, the relation g is of the form

$$g = T_1 T_5^3 + T_2 T_6^3 + T_3 T_7^3.$$

Any choice for $a \geq 1$ yields valid specifying data. This is series S1. Finally we consider the case $\mu = (2, 3)$. Again, if T_4 divides a monomial of g, then homogeneity yields $a \leq 2$. For the two possible values of a we check each homogeneous trinomial g of degree $\deg(g) = \mu$ for the conditions (C1) and (C2) from Setting 3.4.1 and filter by isomorphy. For $a = 1$ we get specifying data no. 230 to 232. For $a = 2$ we get specifying data no. 233 and 234. If T_4 does not appear in g, then up to permutation of variables, the relation g is of the form

$$g \;=\; T_1^2T_5^3 + T_2^2T_6^3 + T_3^2T_7^3$$

Any choice for $a \geq 1$ yields valid specifying data. This is series S2.

Case 3.6.3.2.2: $(n_1, n_2, n_3) = (2, 2, 3)$. Grading matrix and anticanonical class of X are given by

$$Q \;=\; \begin{bmatrix} 1 & 1 & a & a & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}, \qquad -\mathcal{K} \;=\; \begin{bmatrix} 2 + 2a - \mu_1 \\ 5 - \mu_2 \end{bmatrix}.$$

From X being Fano, ie. $-\mathcal{K} \in \lambda^\circ$, we infer the inequalities

$$\mu_2 \;\leq\; 4, \tag{3.6.3.5}$$

$$\mu_1 \;\leq\; 1 - (3 - \mu_2)a. \tag{3.6.3.6}$$

In particular we have $3 \leq \mu_2 \leq 4$. We first consider the case $\mu_2 = 3$. Then by Equation 3.6.3.6 we have $\mu_1 = 1$. The relation g is a trinomial with pairwise coprime monomials. Due to the position of λ, each monomial of g is divisible by one of $T_1, \ldots, T_4$. In particular, at least one monomial of g is divisible by T_3 or T_4. With homogeneity of g we obtain $a = 1$. Grading matrix and relation degree are thus given by

$$Q \;=\; \begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}, \qquad \mu \;=\; (1, 3).$$

This leads to specifying data no. 235 and 236. We discuss the case $\mu_2 = 4$. In that case Equation 3.6.3.6 yields $\mu_1 \leq a + 1$. Each monomial of g is divisible by exactly one of T_5, T_6, T_7. Moreover, g contains a monomial that is not divisible by T_1 or T_2. Thus we may assume that g contains a monomial of the form $T_3^{l_3}T_4^{l_4}T_7^{l_7}$, where $l_3 + l_4 > 0$ and $l_3 + l_4 + l_7 = 4$. The relation degree thus satisfies $\mu = ((l_3 + l_4)a, 4)$. With the bound on μ_1, we obtain

$$(l_3 + l_4)a \;\leq\; a + 1.$$

Assume $l_3 + l_4 > 1$ holds. This is only possible for $l_3 + l_4 = 2$ and $a = 1$. In this case the relation has degree $\mu = (2, 4)$. Up to permutation of variables of the same degree, the homogeneous trinomials g of degree $\deg(g) = (2, 4)$ with coprime monomials are:

$$g \;=\; T_1^2T_5^4 + T_2^2T_6^4 + T_3T_4T_7^2, \qquad g \;=\; T_1^2T_5^4 + T_2^2T_6^4 + T_4^2T_7^2,$$

$$g \;=\; T_1^2T_5^4 + T_3T_2T_6^3 + T_4^2T_7^2, \qquad g \;=\; T_1^2T_5^4 + T_3^2T_6^2 + T_4^2T_7^2.$$

None of these satisfy the condition (C1) in 3.4.1. Thus $l_3 + l_4 = 1$ holds. Switching the roles of T_3 and T_4 if necessary, we may assume that g contains the monomial $T_4T_7^3$.

The other two monomials of g are divisible by precisely one of T_5 and T_6. Checking all trinomials of degree $\mu = (a, 4)$ with these properties, that satisfy the conditions (C1) and (C2) in 3.4.1 and filtering by isomorphy leads to series S3 and S4.

Case 3.6.3.2.3: $(n_1, n_2, n_3) = (2, 1, 4)$. Grading matrix and anticanonical class of X are given by

$$Q = \begin{bmatrix} 1 & 1 & a & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}, \qquad -\mathcal{K} = \begin{bmatrix} 2 + a - \mu_1 \\ 5 - \mu_2 \end{bmatrix}.$$

From X being Fano, ie. $-\mathcal{K} \in \lambda^\circ$, we infer the inequalities

$$\mu_2 \leq 4, \tag{3.6.3.7}$$

$$\mu_1 \leq 1 - (4 - \mu_2)a. \tag{3.6.3.8}$$

These inequalities are only simultaneously fulfilled for $\mu_1 = 1$ and $\mu_2 = 4$. The relation g is a trinomial with pairwise coprime monomials. Due to the position of λ, each monomial of g is divisible by precisely one of T_1, T_2, T_3. Homogeneity of g thus yields $a = 1$. This leads to specifying data no. 237 to no. 240. $\square$

3.7 Proof of Theorem 3.1.1: Case $s = 4$

Setting 3.4.1 and Proposition 3.4.8 divide the proof of Theorem 3.1.1 into the five cases $s = 2, \ldots, 6$, according to the number of rays spanned by the degrees $w_1, \ldots, w_7$. In this section we treat the case $s = 4$.

Theorem 3.7.1. *The tables from 3.10.5, 3.10.6 and 3.10.7 provide specifying data (Q, g) for 169 sporadic cases and 32 infinite series of locally factorial Fano fourfolds of Picard number $\rho = 2$ and complexity $c = 1$ with a hypersurface Cox ring and $s = 4$. Moreover, any locally factorial Fano fourfold with a hypersurface Cox ring and invariants $(\rho, c, s) = (2, 1, 4)$ is isomorphic to precisely one $X(Q, g)$ with (Q, g) from these tables.*

The proof of Theorem 3.7.1 splits into two parts. First, with the tools provided in Section 3.2 we verify that each specifying data (Q, g) from the tables in 3.10.5, 3.10.6 and 3.10.7 defines a locally factorial Fano fourfold $X(Q, g)$ with a hypersurface Cox ring and invariants $(\rho, c, s) = (2, 1, 4)$. Moreover, with the help of Remark 3.4.3 we verify that distinct specifying data from the tables in 3.10.5, 3.10.6 and 3.10.7 define pairwise non-isomorphic varieties. The second part is to show that any locally factorial Fano fourfold with a hypersurface Cox ring and invariants $(\rho, c, s) = (2, 1, 4)$ is isomorphic to $X(Q, g)$ with (Q, g) from these tables. We divide the proof of this into the two general cases

$$\mu \in \mathrm{SAmple}(X), \qquad \mu \notin \mathrm{SAmple}(X).$$

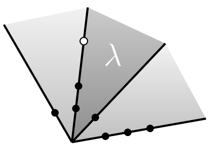
The case $\mu \in \mathrm{SAmple}(X)$ will be treated in Proposition 3.7.2. In Proposition 3.7.3 we treat the case $\mu \notin \mathrm{SAmple}(X)$.

Proposition 3.7.2. *Let X as in Setting 3.4.1 with $s = 4$. Assume that $\mu \in \lambda$ holds. Then X is isomorphic to an $X(Q, g)$ with specifying data (Q, g) appearing in Classification list 3.10.5.*

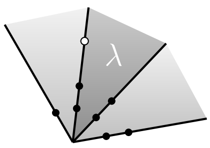
Proof. We divide the proof into the two cases $\mu \in \operatorname{Mov}(R)^\circ$ and $\mu \in \partial \operatorname{Mov}(R)$.

Case 3.7.2.1: $\mu \in \operatorname{Mov}(R)^\circ$. We are in the situation of Proposition 3.3.2. Thus, for a general polynomial $h \in \mathbb{C}[T_1, \ldots, T_7]$ of degree $\deg(h) = \mu$, the projective variety X_h is smooth with divisor class group $\operatorname{Cl}(X_h) = K$ and Cox ring $\mathcal{R}(X_h) = R_h$. Moreover, by Proposition 3.2.5 X_h is Fano. Thus, X_h is a smooth Fano fourfold of Picard number two with a spread hypersurface Cox ring. In particular, up to unimodular equivalence, the grading matrix $Q = (w_1, \ldots, w_7)$ together with the relation degree $\mu = \deg(g)$ appear in the classification list presented in [45, Thm. 1.1]. For each such entry (Q, μ) with $s = 4$ we determine all trinomials g of degree $\deg(g) = \mu$ that satisfy the conditions (C1) and (C2) from Setting 3.4.1 and filter the resulting list by isomorphy. This yields the specifying data no. 242, 243 and 246-251.

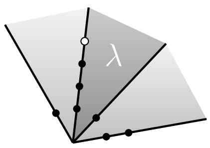
Case 3.7.2.2: $\mu \in \partial \operatorname{Mov}(R)$. The relation degree μ is contained in one of the rays $\rho_1, \ldots, \rho_4$. By Lemma 3.4.7, μ is neither contained in ρ_1 nor in ρ_4. Reversing the order of the variables if necessary, we may assume that $\mu \in \rho_3$ holds. By assumption μ lies in the boundary of $\operatorname{Mov}(R)$. Thus we have $n_4 = 1$. Moreover, Remark 3.2.11 yields $n_3 \geq 2$. The relation degree μ lies in the boundary of λ, which is contained in $\operatorname{Mov}(R)$. So $\lambda = \rho_2 + \rho_3$ holds. There are six possible degree constellations (n_1, n_2, n_3, n_4) for X, displayed in the following pictures.



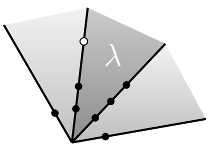
$(3, 1, 2, 1)$



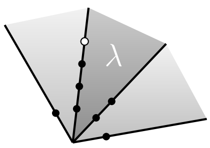
$(2, 2, 2, 1)$



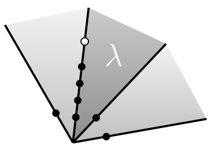
$(2, 1, 3, 1)$



$(1, 3, 2, 1)$



$(1, 2, 3, 1)$



$(1, 1, 4, 1)$

The black dots represent the generator degrees $w_1, \ldots, w_7$, the white circle represents the relation degree μ. We distinguish six cases, according to the degree constellation.

Case 3.7.2.2.1: $(n_1, n_2, n_3, n_4) = (3, 1, 2, 1)$. Applying Lemma 3.2.8 to the triple (w_1, w_2, w_7) shows that the cone $\operatorname{Eff}(R)$ is regular and that w_7 is primitive. We may thus assume that $\operatorname{Eff}(R)$ is the positive quadrant and that $w_7 = (0, 1)$ holds. Moreover, Lemma 3.2.8 applied to (w_i, w_5, w_6) with $i = 1, 2, 3$ shows that $w_1 = w_2 = w_3 = (1, 0)$ holds, that the primitive point $v \in \rho_3$ is of the form $v = (c, 1)$ for some $c \geq 1$ and that w_5, w_6 are coprime multiples of v. The grading matrix is thus given by

$$Q \;=\; \begin{bmatrix} 1 & 1 & 1 & a_4 & a_5c & a_6c & 0 \\ 0 & 0 & 0 & b_4 & a_5 & a_6 & 1 \end{bmatrix}, \qquad a_4, a_5, a_6, b_4, c \in \mathbb{Z}_{\geq 1}$$

with $\gcd(a_5, a_6) = 1$. Lemma 3.2.8 applied to (w_4, w_5, w_6) shows that w_4 is primitive. Thus $\gcd(a_4, b_4) = 1$ holds. For the relation degree we have $\mu = (kc, k)$. THe relation g is a trinomial with pairwise coprime monomials. Due to the position of μ, each monomial of g is divisible by precisely one of T_5, T_6, T_7. Thus there are $l_5, l_6 \geq 2$ with $k = l_5a_5 = l_6a_6$. In particular we have $k \geq a_5 + a_6$. The anticanonical class of X is given by

$$-\mathcal{K} \;=\; \begin{bmatrix} 3 + a_4 + (a_5 + a_6 - k)c \\ b_4 + a_5 + a_6 - k \end{bmatrix}.$$

From X being Fano, ie. $-\mathcal{K} \in \lambda^\circ$, we infer the inequality

$$0 \;<\; \det(w_4, -\mathcal{K}) \;=\; -3b_4 + (a_5 + a_6 - k)(a_4 - b_4c).$$

By the ordering of the generator degrees, we have $a_4 - b_4c > 0$. As $k \geq a_5 + a_6$, the scond summand on the right hand side is negative. This is a contradiction. Thus the degree constellation $(3, 1, 2, 1)$ does not occur.

Case 3.7.2.2.2: $(n_1, n_2, n_3, n_4) = (2, 2, 2, 1)$. Applying Lemma 3.2.8 to the triple (w_1, w_2, w_7) shows that the cone $\mathrm{Eff}(R)$ is regular and that w_7 is primitive. We may thus assume that $\mathrm{Eff}(R)$ is the positive quadrant and that $w_7 = (0, 1)$ holds. Moreover, Lemma 3.2.8 applied to (w_i, w_5, w_6) with $i = 1, 2$ shows that $w_1 = w_2 = (1, 0)$ holds, that the primitive point $v \in \rho_3$ is of the form $v = (c, 1)$ for some $c \geq 1$ and that w_5, w_6 are coprime multiples of v. Finally, applying Lemma 3.2.8 applied to (w_3, w_4, w_7) shows that the primitive point $u \in \rho_2$ is of the form $v = (1, b)$ for some $b \geq 1$ and that w_3, w_4 are coprime multiples of v. The grading matrix is thus given by

$$Q \;=\; \begin{bmatrix} 1 & 1 & a_3 & a_4 & a_5c & a_6c & 0 \\ 0 & 0 & a_3b & a_4b & a_5 & a_6 & 1 \end{bmatrix}, \qquad a_3, a_4, a_5, a_6, b, c \in \mathbb{Z}_{\geq 1}$$

with $\gcd(a_3, a_4) = 1$ and $\gcd(a_5, a_6) = 1$. By the ordering of the generator degrees, we have

$$0 \;<\; \det(u, v) \;=\; 1 - bc.$$

This is a contradiction, as $bc \geq 1$ holds. Thus the degree constellation $(2, 2, 2, 1)$ does not occur.

Case 3.7.2.2.3: $(n_1, n_2, n_3, n_4) = (2, 1, 3, 1)$. Applying Lemma 3.2.8 to the triple (w_1, w_2, w_7) shows that the cone $\mathrm{Eff}(R)$ is regular and that w_7 is primitive. We may thus assume that $\mathrm{Eff}(R)$ is the positive quadrant and that $w_7 = (0, 1)$ holds. Moreover, Lemma 3.2.8 applied to (w_i, w_5, w_6) with $i = 1, 2$ shows that $w_1 = w_2 = (1, 0)$ holds, that the primitive point $v \in \rho_3$ is of the form $v = (c, 1)$ for some $c \geq 1$. By Lemma 3.4.5, at least one of w_4, w_5, w_6 is primitive. We may assume that $w_4 = v$ holds. Write $w_5 = (a_5c, a_5)$ and $w_5 = (a_6c, a_6)$. Lemma 3.2.8 applied to (w_1, w_5, w_6) shows that a_5, a_6 are coprime. The grading matrix is thus given by

$$Q \;=\; \begin{bmatrix} 1 & 1 & a_3 & c & a_5c & a_6c & 0 \\ 0 & 0 & b_3 & 1 & a_5 & a_6 & 1 \end{bmatrix}, \qquad a_3, a_5, a_6, b_3, c \in \mathbb{Z}_{\geq 1}$$

with $\gcd(a_5, a_6) = 1$. We may assume $a_5 \leq a_6$. For the relation degree we have $\mu = (kc, k)$ for some $k \geq 2$. If w_5 is not primitive, then by Lemma 3.2.7 applied to w_1 and w_5, the relation g contains a monomial of the form $T_5^{l_5}$. The same holds for w_6. Thus in any case there are $l_5, l_6 \geq 2$ with $k = l_5a_5 = l_6a_6$. The anticanonical class of X is given by

$$-\mathcal{K} \;=\; \begin{bmatrix} 2 + a_3 + (1 + a_5 + a_6 - k)c \\ 2 + b_3 + a_5 + a_6 - k \end{bmatrix}.$$

From X being Fano, ie. $-\mathcal{K} \in \lambda^\circ$ we infer the inequalities

$$0 \;<\; a_3 - 2b_3 + (1 + a_5 + a_6 - k)(a_3 - b_3c), \tag{3.7.2.1}$$

$$c \;\leq\; 1 + a_3 - b_3c. \tag{3.7.2.2}$$

By Lemma 3.2.8 applied to (w_3, w_4, w_5), the degrees w_3 and w_4 generate K as a group. Thus we have

$$1 \;=\; \det(w_3, w_4) \;=\; a_3 - b_3c. \tag{3.7.2.3}$$

Plugging this into Equations 3.7.2.1 and 3.7.2.2, we obtain

$$k \;\leq\; 1 + a_5 + a_6, \tag{3.7.2.4}$$

$$c \;\leq\; 2. \tag{3.7.2.5}$$

Note that by Equation 3.7.2.3 we have $a_3 = b_3c + 1$. In particular, a_3 is at least two. This means that w_3 and w_7 do not generate K as a group. Thus, by Lemma 3.2.7, the relation g contains a monomial of the form $T_3^{l_3}T_7^{l_7}$. By homogeneity of g we obtain $kc = l_3a_3$ and $k = l_3b_3 + l_7$. Combining these two equations yields

$$l_3 \;=\; l_7c, \qquad k \;=\; l_7(b_3c + 1). \tag{3.7.2.6}$$

The relation g is a trinomial with pairwise coprime monomials. Due to the position of μ, the remaining two monomials only depend on the variables T_4, T_5, T_6. We may thus assume that g contains the monomial $T_6^{l_6}$. We distinguish three cases, depending on the values of a_5 and a_6.

Case 3.7.2.2.3.1: $a_5, a_6 > 1$. By Lemma 3.2.7 applied to the tuples (w_1, w_5) and (w_1, w_6), the relation g contains the monomials $T_5^{l_5}$ and $T_6^{l_6}$ and by condition (C2) from Setting 3.4.1 the exponents l_5 and l_6 are coprime. With the coprimeness of a_5 and a_6 we obtain $k = a_5 a_6$. We are thus in the situation of 3.4.4, which yields $a_5 = 2$, $a_6 = 3$ and $k = 6$. Plugging these values into Equation 3.7.2.1 yields $0 < a_3 - 2b_3$. Combining this with Equations 3.7.2.3 and 3.7.2.5 yields $c = 2$. Grading matrix and relation degree are thus given by

$$Q = \begin{bmatrix} 1 & 1 & a_3 & 2 & 4 & 6 & 0 \\ 0 & 0 & b_3 & 1 & 2 & 3 & 1 \end{bmatrix}, \qquad \mu = (12, 6).$$

For the relation we have $g = T_3^{l_3} T_7^{l_7} + T_5^3 + T_6^2$. Equation 3.7.2.6 yields

$$l_3 = 2l_7, \qquad 6 = l_7(2b_3 + 1).$$

In particular, l_7 is a proper divisor of 6. Moreover, by the second identity, l_7 is even. Since b_3 is at least one ,we have $l_7 = 2$. This yields $l_3 = 4$. But then g does not satisfy the condition (C2) from Setting 3.4.1. A contradiction. Thus, this case does not occur.

Case 3.7.2.2.3.2: $a_5 = 1$, $a_6 > 1$. Equation 3.7.2.4 in this case reads $(l_6 - 1)a_6 \leq 2$, which yields $a_6 = 2$ and $l_6 = 2$. Plugging these values into Equation 3.7.2.1 yields $0 < a_3 - 2b_3$. Combining this with Equations 3.7.2.3 and 3.7.2.5 yields $c = 2$. Grading matrix and relation degree are thus given by

$$Q = \begin{bmatrix} 1 & 1 & a_3 & 2 & 2 & 4 & 0 \\ 0 & 0 & b_3 & 1 & 1 & 2 & 1 \end{bmatrix}, \qquad \mu = (8, 4).$$

Equation 3.7.2.6 yields

$$l_3 = 2l_7, \qquad 4 = l_7(2b_3 + 1).$$

Note that the second identity cannot be fulfilled for $b_3 > 0$. A contradiction. Thus, this case does not occur.

Case 3.7.2.2.3.3: $a_5 = a_6 = 1$. Equation 3.7.2.4 in this case yields $k \leq 3$. First assume $k = 2$. Then Equation 3.7.2.6 yields $l_7 = 1$, $b_3 = 1$ and $c = 1$. Thus grading matrix and relation degree are given by

$$Q = \begin{bmatrix} 1 & 1 & 2 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}, \qquad \mu = (2, 2).$$

The relation g is of the form $g = T_3 T_7 + T_4^{l_4} T_5^{l_5} + T_6^2$. Homogeneity of g together with condition (C2) form Setting 3.4.1 yield $l_4 = l_5 = 1$. This is specifying data no. 244. Now assume $k = 3$. In this case Equation 3.7.2.1 yields $0 < a_3 - 2b_3$. Combining this with Equations 3.7.2.3 and 3.7.2.5 yields $c = 2$. With the help of Equation 3.7.2.6 we obtain $l_7 = 1$ and $b_3 = 1$. Grading matrix and relation degree are thus given by

$$Q = \begin{bmatrix} 1 & 1 & 3 & 2 & 2 & 2 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}, \qquad \mu = (6, 3),$$

The relation g is of the form $g = T_3^2 T_7 + T_4^{l_4} T_5^{l_5} + T_6^3$ and we may assume that $l_4 \geq l_5$ holds. Homogeneity of g together with condition (C2) form Setting 3.4.1 yield $l_4 = 2$ and $l_5 = 1$. This is specifying data no. 245.

Case 3.7.2.2.4: $(n_1, n_2, n_3, n_4) = (1, 3, 2, 1)$. Applying Lemma 3.2.8 to the triple (w_1, w_5, w_6) shows that the cone $\rho_1 + \rho_3$ is regular and that w_1 is primitive. We may thus assume that $\rho_1 + \rho_3$ is the positive quadrant and that $w_1 = (1, 0)$ holds. Moreover, Lemma 3.2.8 applied to (w_i, w_5, w_6) with $i = 2, 3, 4$ shows that $w_i = (1, b)$ holds for some $b \geq 1$. The grading matrix is thus given by

$$Q = \begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 & -a_7 \\ 0 & b & b & b & b_5 & b_6 & b_7 \end{bmatrix}, \qquad a_7, b, b_5, b_6, b_7 \in \mathbb{Z}_{\geq 1}.$$

Applying Lemma 3.2.8 to the triple w_2, w_3, w_7 shows that w_7 is primitive and that w_2 and w_7 generate K as a group. We thus have

$$1 = \det(w_2, w_7) = b_7 + a_7 b.$$

However, since a_7, b, b_7 are positive, the right hand side is at least two. A contradiction. Thus the degree constellation $(1, 3, 2, 1)$ does not occur.

Case 3.7.2.2.5: $(n_1, n_2, n_3, n_4) = (1, 2, 3, 1)$. Applying Lemma 3.2.8 to the triple (w_1, w_5, w_6) shows that the cone $\rho_1 + \rho_3$ is regular and that w_1 is primitive. We may thus assume that $\rho_1 + \rho_3$ is the positive quadrant and that $w_1 = (1, 0)$ holds. By Lemma 3.4.5, at least one of w_4, w_5, w_6 is primitive. We may thus assume that $w_4 = (0, 1)$ holds. Moreover, Lemma 3.2.8 applied to (w_i, w_5, w_6) with $i = 2, 3$ shows that $w_i = (1, b)$ holds for some $b \geq 1$. The grading matrix is thus given by

$$Q = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 & -a_7 \\ 0 & b & b & 1 & b_5 & b_6 & b_7 \end{bmatrix}, \qquad a_7, b, b_5, b_6, b_7 \in \mathbb{Z}_{\geq 1}.$$

Applying Lemma 3.2.8 to the triple w_2, w_3, w_7 shows that w_7 is primitive and that w_2 and w_7 generate K as a group. We thus have

$$1 = \det(w_2, w_7) = b_7 + a_7 b.$$

However, since a_7, b, b_7 are positive, the right hand side is at least two. A contradiction. Thus the degree constellation $(1, 2, 3, 1)$ does not occur.

Case 3.7.2.2.6: $(n_1, n_2, n_3, n_4) = (1, 1, 4, 1)$. Applying Lemma 3.2.8 to the triple (w_1, w_3, w_4) shows that the cone $\rho_1 + \rho_3$ is regular and that w_1 is primitive. We may thus assume that $\rho_1 + \rho_3$ is the positive quadrant and that $w_1 = (1, 0)$ holds. By Lemma 3.4.5, at least two of w_3, w_4, w_5, w_6 are primitive. We may thus assume that $w_3 = w_4 = (0, 1)$ holds. Moreover, Lemma 3.2.8 applied to (w_2, w_3, w_4) shows that $w_2 = (1, b)$ holds for some $b \geq 1$. The grading matrix is thus given by

$$Q = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 & -a_7 \\ 0 & b & 1 & 1 & b_5 & b_6 & b_7 \end{bmatrix}, \qquad a_7, b, b_5, b_6, b_7 \in \mathbb{Z}_{\geq 1}.$$

By Lemma 3.2.8 for (w_1, w_5, w_6) the integers b_5 and b_6 are coprime. We may assume $b_5 \le b_6$. We have

$$\det(w_2, w_7) \;=\; b_7 + a_7 b \;>\; 1.$$

In particular, w_2 and w_7 do not generate K as a group. Thus by Lemma 3.2.7 the relation g contains a monomial of the form $T_2^{l_2} T_7^{l_7}$. Since the monomials of g are coprime, the relation does not contain a monomial of the form $T_1^{l_1} T_7^{l_7}$. Lemma 3.2.7 thus yields

$$1 \;=\; \det(w_1, w_7) \;=\; b_7.$$

For the relation degree we have $\mu = (0, k)$ for some $k \ge 2$. We have already determined one monomial of the trinomial g. Due to the position of μ, the other two monomials of g only depend on the variables $T_4, \ldots, T_6$. If w_5 is not primitive, then by Lemma 3.2.7 applied to (w_1, w_5), g contains a monomial of the form $T_5^{l_5}$. The same holds for w_6. Thus, in any case there are $l_5, l_6 \ge 2$ with $k = l_5 b_5 = l_6 b_6$. In particular we have $k \ge b_5 + b_6$. By Proposition 3.2.5 the anticanonical class of X is given by

$$-\mathcal{K} \;=\; \begin{bmatrix} 2 - a_7 \\ 3 + b + b_5 + b_6 - k \end{bmatrix}.$$

From X being Fano, ie. $-\mathcal{K} \in \lambda^\circ$, we infer the inequalities

$$k \;\le\; 2 + (a_7 - 1)b + b_5 + b_6, \tag{3.7.2.7}$$

$$a_7 \;\le\; 1. \tag{3.7.2.8}$$

In particular, Equation 3.7.2.8 yields $a_7 = 1$. Grading matrix and relation degree are given by

$$Q \;=\; \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 & -1 \\ 0 & b & 1 & 1 & b_5 & b_6 & 1 \end{bmatrix}, \qquad \mu \;=\; (0, k).$$

The relation g contains a monomial of the form $T_2^{l_2} T_7^{l_7}$. Homogeneity of g yields

$$l_2 \;=\; l_7, \qquad k \;=\; l_2(b + 1). \tag{3.7.2.9}$$

Plugging the value for a_7 into Equation 3.7.2.7, we obtain the inequality

$$k \;\le\; 2 + b_5 + b_6. \tag{3.7.2.10}$$

We distinguish three cases, depending on the values of b_5 and b_6.

Case 3.7.2.2.6.1: $b_5, b_6 > 1$. Applying Lemma 3.2.7 to the pairs (w_1, w_5) and (w_1, w_6) shows that g contains the monomials $T_5^{l_5}$ and $T_6^{l_6}$. By condition (C2) from Setting 3.4.1 the exponents l_5 and l_6 are coprime. This yields $b_5 = l_6$ and $b_6 = l_5$ and $k = b_5 b_6$. We are thus in the situation of 3.4.4, which yields $b_5 = 2$, $b_6 = 3$ and $k = 6$. By 3.7.2.9 the relation g satisfies

$$g \;=\; T_2^{l_2} T_7^{l_2} + T_5^3 + T_6^2, \qquad 6 = l_2(b + 1).$$

Thus l_2 is a proper divisor of 6 and by condition (C2) from *Setting* 3.4.1, l_2 is neither two nor three. Thus $l_2 = 1$ and $b = 5$ holds. This means that w_2, w_6, w_7 do not generate K as a group. By Lemma 3.2.8 g contains a single monomial that only depends on T_2, T_6 and T_7. But this is not the case. A contradiction. Thus, this case does not occur.

Case 3.7.2.2.6.2: $b_5 = 1$, $b_6 > 1$. Equation 3.7.2.10 reduces to $(l_6 - 1)b_6 \leq 3$. Thus we have $l_6 = 2$ and $2 \leq b_6 \leq 3$. By Lemma 3.2.7 applied to (w_1, w_6), the relation g contains the monomial T_6^2. It is thus of the form

$$g \ = \ T_2^{l_2}T_7^{l_2} + T_3^{l_3}T_4^{l_4}T_5^{l_5} + T_6^2.$$

Assume $b_6 = 2$. Then $k = 4$ holds and 3.7.2.9 yields $4 = l_2(b+1)$. By condition (C2) from Setting 3.4.1, l_2 is not divisible by two. This means that w_2, w_6, w_7 do not generate K as a group. By Lemma 3.2.8 g contains a single monomial that only depends on T_2, T_6 and T_7. But this is not the case. A contradiction. Thus we must have $b_6 = 3$. Then $k = 6$ holds and 3.7.2.9 yields $6 = l_2(b+1)$. So l_2 is a proper divisor of six, different from two. We have seen that the case $b = 5$ leads to a contradiction. Thus we have $l_2 = 3$ and $b = 1$. Grading matrix and relation are thus given by

$$Q \ = \ \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 & -1 \\ 0 & 1 & 1 & 1 & 1 & 3 & 1 \end{bmatrix}, \qquad g \ = \ T_2^3T_7^3 + T_3^{l_3}T_4^{l_4}T_5^{l_5} + T_6^2,$$

where $l_3 + l_4 + l_5 = 6$. Filtering by isomorphy, this leads to specifying data no. 262 to 264.

Case 3.7.2.2.6.3: $b_5 = b_6 = 1$. Grading matrix and relation degree are given by

$$Q \ = \ \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 & -1 \\ 0 & b & 1 & 1 & 1 & 1 & 1 \end{bmatrix}, \qquad \mu \ = \ (0, k).$$

The relation g contains the monomial $T_2^{l_2}T_7^{l_2}$, where l_2 is a proper divisor of k. Equation 3.7.2.10 reduces to $k \leq 4$. We distinguish three cases.

Case 3.7.2.2.6.3.1: $k = 2$. In this case we have $l_2 = 1$ and $b = 1$. Grading matrix and relation degree are given by

$$Q \ = \ \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 & -1 \\ 0 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}, \qquad \mu \ = \ (0, 2).$$

The relation g contains the monomial T_3T_7 and the other two monomials only depend on T_4, T_5, T_6. Up to isomorphy this leads to specifying data no. 252 and 253.

Case 3.7.2.2.6.3.2: $k = 3$. In this case we have $l_2 = 1$ and $b = 2$. Grading matrix and relation degree are given by

$$Q \ = \ \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 & -1 \\ 0 & 2 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}, \qquad \mu \ = \ (0, 3).$$

The relation g contains the monomial T_3T_7 and the other two monomials only depend on T_4, T_5, T_6. Up to isomorphy this leads to specifying data no. 255 to 257.

Case 3.7.2.2.6.3.3: $k = 4$. In this case we either have $(l_2, b) = (2, 1)$ or $(l_2, b) = (1, 3)$. In the first case grading matrix and relation degree are given by

$$Q = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 & -1 \\ 0 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}, \qquad \mu = (0, 4).$$

The relation g contains the monomial $T_3^2T_7^2$ and the other two monomials only depend on T_4, T_5, T_6. Up to isomorphy this leads to specifying data no. 254. In the second case grading matrix and relation degree are given by

$$Q = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 & -1 \\ 0 & 3 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}, \qquad \mu = (0, 4).$$

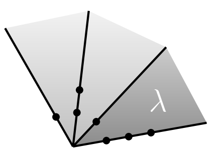
The relation g contains the monomial T_3T_7 and the other two monomials only depend on $T_4, \ldots, T_6$. Up to isomorphy this leads to specifying data no. 258 to 261. $\square$

Proposition 3.7.3. *Let X as in Setting 3.4.1 with $s = 4$. Assume that $\mu \notin \lambda$ holds. Then X is isomorphic to an $X(Q, g)$ with specifying data (Q, g) appearing in Classification list 3.10.6 or in Classification list 3.10.7.*

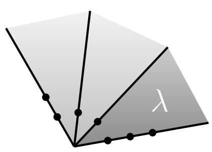
Proof. We have $\mu \notin \lambda$. Reversing the ordering of the variables if necessary, we may assume that $\mu \in \lambda^+ \backslash \lambda$ holds. We are thus in the situation of Lemma 3.4.6. Thus $\lambda = \rho_1 + \rho_2$ holds. Moreover, we have $n_1 \geq 2$ and all generator degrees contained in ρ_1 are primitive. By Lemma 3.4.7, μ is contained in the interior of $\mathrm{Eff}(R)$. Thus, applying Lemma 3.2.8 to the triples (w_1, w_2, w_i), where $w_i \in \rho_4$, shows that the cone $\mathrm{Eff}(R)$ is regular and that w_i is primitive. We may thus assume that $\mathrm{Eff}(R)$ is the positive quadrant and that

$$w_1 = w_2 = (1, 0), \qquad w_7 = (0, 1)$$

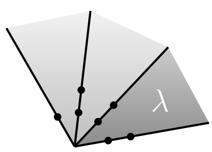
holds. Since μ is contained in the interior of $\mathrm{Eff}(R)$, but lies outside of λ, we have $\mu \in (\rho_2 + \rho_4)^\circ$. Remark 3.2.11 thus yields $n_3 + n_4 \geq 3$. There are seven possible degree constellations (n_1, n_2, n_3, n_4) for X, displayed in the following pictures.



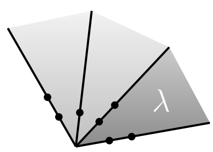
$(3, 1, 2, 1)$



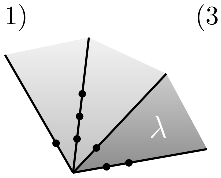
$(3, 1, 1, 2)$



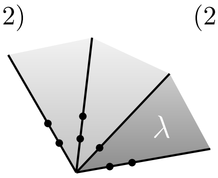
$(2, 2, 2, 1)$



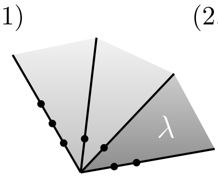
$(2, 2, 1, 2)$



$(2, 1, 3, 1)$



$(2, 1, 2, 2)$



$(2, 1, 1, 3)$

The black dots represent the generator degrees $w_1, \ldots, w_7$. We distinguish seven cases, according to the degree constellation.

Case 3.7.3.1: $(n_1, n_2, n_3, n_4) = (3, 1, 2, 1)$. Applying Lemma 3.2.7 to the pair (w_3, w_4) shows that $w_4 = (a, 1)$ holds for some $a \geq 1$. Moreover, applying Lemma 3.2.8 to the triple (w_1, w_5, w_6) shows that the primitive point $v \in \rho_3$ is of the form $v = (c, 1)$ for some $c \geq 1$. The grading matrix is thus given by

$$Q \;=\; \begin{bmatrix} 1 & 1 & 1 & a & a_5c & a_6c & 0 \\ 0 & 0 & 0 & 1 & a_5 & a_6 & 1 \end{bmatrix}, \qquad a, a_5, a_6, c \in \mathbb{Z}_{\geq 1}.$$

Applying Lemma 3.2.8 to (w_1, w_5, w_6) shows that the integers a_5 and a_6 are coprime. By Remark 3.2.11 we have $\mu \in (\rho_2 + \rho_3) \backslash \rho_2$. We may assume $a_6 \geq a_5$. We distinguish the two cases $\mu \in (\rho_2 + \rho_3)^\circ$ and $\mu \in \rho_3$.

Case 3.7.3.1.1: $\mu \in (\rho_2 + \rho_3)^\circ$. Applying Lemma 3.2.8 to (w_1, w_2, w_5) and (w_1, w_2, w_6) yields $a_5 = a_6 = 1$. Grading matrix and anticanonical class of X are thus given by

$$Q \;=\; \begin{bmatrix} 1 & 1 & 1 & a & c & c & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 \end{bmatrix}, \qquad -\mathcal{K} \;=\; \begin{bmatrix} 3 + a + 2c - \mu_1 \\ 4 - \mu_2 \end{bmatrix}.$$

From X being Fano, ie. $\mu \in \lambda^\circ$ we infer the inequalities

$$\mu_2 \;\leq\; 3, \tag{3.7.3.1}$$
$$\mu_1 \;\leq\; 2 + 2c + (\mu_2 - 3)a. \tag{3.7.3.2}$$

From the position of μ, we obtain the inequality $\mu_2 c + 1 \leq \mu_1$. Moreover, by the ordering of the generator degrees, we have $c \leq a - 1$. Combining this with Equation 3.7.3.2 yields

$$0 \;\leq\; \mu_2 - a - 1.$$

Having in mind 3.7.3.1, this yields $\mu_2 = 3$ and $a = 2$. With this we directly get $c = 1$ and $\mu_1 = 4$. Grading matrix and relation degree are thus given by

$$Q \;=\; \begin{bmatrix} 1 & 1 & 1 & 2 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 \end{bmatrix}, \qquad \mu \;=\; (4, 3).$$

Checking all trinomials g of degree $\deg(g) = \mu$ that satisfy the conditions (C1) and (C2) in 3.4.1 and filtering by isomorphy leads to specifying data no. 265 to 270.

Case 3.7.3.1.2: $\mu \in \rho_3$. The relation degree satisfies $\mu = (kc, k)$ for some $k \geq 2$. Grading matrix and anticanonical class of X are thus given by

$$Q \;=\; \begin{bmatrix} 1 & 1 & 1 & a & a_5c & a_6c & 0 \\ 0 & 0 & 0 & 1 & a_5 & a_6 & 1 \end{bmatrix}, \qquad -\mathcal{K} \;=\; \begin{bmatrix} 3 + a + (a_5 + a_6 - k)c \\ 2 + a_5 + a_6 - k \end{bmatrix}.$$

From X being Fano, ie. $-\mathcal{K} \in \lambda^\circ$ we infer the inequalities

$$\begin{aligned} k &\leq 1 + a_5 + a_6, && (3.7.3.3)\\ 0 &\leq 2 - c + (1 + a_5 + a_6 - k)(c - a). && (3.7.3.4) \end{aligned}$$

The relation g is a trinomial with coprime monomials. Due to the position of μ, the relation g has monomials of the form $T_5^{l_5}$ and $T_6^{l_6}$ with $l_5, l_6 \geq 2$. By Remark 3.2.10, the exponents l_5 and l_6 are coprime. Using homogeneity of g we see that l_5 divides a_6 and l_6 divides a_5. In particular, we have $a_5, a_6 > 1$. As a_5 and a_6 are coprime, we obtain $k = a_5 a_6$. By Equation 3.7.3.3 we are in the situation of Lemma 3.4.4, which tells us that $a_5 = 2$ and $a_6 = 3$ as well as $k = 6$ hold. Plugging these values into Equation 3.7.3.4, we obtain $c \leq 2$. Thus, grading matrix and relation degree are given by

$$Q = \begin{bmatrix} 1 & 1 & 1 & a & 2c & 3c & 0 \\ 0 & 0 & 0 & 1 & 2 & 3 & 1 \end{bmatrix}, \qquad \mu = (6c, 6), \qquad c \leq 2.$$

If T_4 does not appear in g, then it is of the form

$$g = T_1^{l_1} T_2^{l_2} T_3^{l_3} T_7^6 + T_5^3 + T_6^2,$$

where $l_1 + l_2 + l_3 = 6c$. Apart from $a > c$, there is no restriction on the value of a. For each possible combination of exponents l_1, l_2, l_3 we check the trinomial g for the conditions (C1) and (C2) from Setting 3.4.1 and filter by isomorphy. For $c = 1$ we obtain the series S5 to S7. For $c = 2$ we obtain the series S8 to S17. If T_4 does appear in g, then homogeneity of g yields the inequality $a \leq 6c$. There are thus 15 possible combinations for the values of a and c. Checking in each case all trinomials g of degree $\deg(g) = \mu$ for the conditions (C1) and (C2) from Setting 3.4.1 and filtering by isomorphy, we obtain the following specifying data:

(a,c)	$(2,1)$	$(3,1)$	$(4,1)$	$(5,1)$	$(6,1)$	$(3,2)$
ID	271-275	276-278	279-280	281	282	283-300
(a,c)	$(4,2)$	$(5,2)$	$(6,2)$	$(7,2)$	$(8,2)$	$(9,2)$
ID	301-312	313-321	322-328	329-333	334-337	338-340
(a,c)	$(10,2)$	$(11,2)$	$(12,2)$			
ID	341-342	343	344			

Case 3.7.3.2: $(n_1, n_2, n_3, n_4) = (3, 1, 1, 2)$. Applying Lemma 3.2.7 to the pair (w_1, w_4) shows that $w_4 = (a, 1)$ holds for some $a \geq 1$. The grading matrix is thus of the form

$$Q = \begin{bmatrix} 1 & 1 & 1 & a & a_5 & 0 & 0 \\ 0 & 0 & 0 & 1 & b_5 & 1 & 1 \end{bmatrix}, \qquad a, a_5, b_5 \in \mathbb{Z}_{\geq 1}.$$

By Remark 3.2.11 we have $\mu \in (\rho_2 + \rho_3) \backslash \rho_2$. We distinguish the two cases $\mu \in (\rho_2 + \rho_3)^\circ$ and $\mu \in \rho_3$.

Case 3.7.3.2.1: $\mu \in (\rho_2 + \rho_3)^\circ$. By Lemma 3.2.8, applied to the triple (w_1, w_2, w_5), we have $b_5 = 1$. Grading matrix and anticanonical class of X are given by

$$Q = \begin{bmatrix} 1 & 1 & 1 & a & c & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 \end{bmatrix}, \qquad -\mathcal{K} = \begin{bmatrix} 3 + a + c - \mu_1 \\ 4 - \mu_2 \end{bmatrix}.$$

From X being Fano, ie. $-\mathcal{K} \in \lambda^\circ$, we infer the inequalities

$$\mu_2 \leq 3, \tag{3.7.3.5}$$
$$\mu_1 \leq 2 + c + (\mu_2 - 3)a. \tag{3.7.3.6}$$

From the position of μ, we get the inequality $\mu_2 c + 1 \leq \mu_1$. Combining this with Equation 3.7.3.6, we obtain

$$0 \leq 1 + (1 - \mu_2)c + (\mu_2 - 3)a.$$

The right hand side is negative due to 3.7.3.5. A contradiction. Thus this case does not occur.

Case 3.7.3.2.2: $\mu \in \rho_3$. The relation degree satisfies $\mu = (kc, k)$ for some $k \geq 2$. Grading matrix and anticanonical class of X are given by

$$Q = \begin{bmatrix} 1 & 1 & 1 & a & a_5 c & 0 & 0 \\ 0 & 0 & 0 & 1 & a_5 & 1 & 1 \end{bmatrix}, \qquad -\mathcal{K} = \begin{bmatrix} 3 + a + (a_5 - k)c \\ 3 + a_5 - k \end{bmatrix}.$$

From X being Fano we infer the inequalities

$$k \leq 2 + a_5, \tag{3.7.3.7}$$
$$0 \leq 2 - 2c + (k - a_5 - 2)(a - c). \tag{3.7.3.8}$$

By the ordering of the generator degrees we have $c \leq a - 1$. Thus, combining these two inequalities, we obtain $c = 1$ and $k = a_5 + 2$. The relation g is a trinomial with pairwise coprime monomials. Due to the position of μ, each monomial of g is divisible by precisely one of T_5, T_6, T_7. In particular, g has a monomial of the form $T_5^{l_5}$ with $l_5 \geq 2$. Thus $k = a_5 l_5$ holds. This, together with the identity $k = a_5 + 2$ yields $(l_5 - 1)a_5 = 2$. There are two cases, either $(a_5, l_5) = (2, 2)$ or $(a_5, l_5) = (1, 3)$. We note that one monomial of g is divisible by T_4: Assume this is not the case. Switching the roles of T_6 and T_7 and permuting T_1, T_2, T_3 if necessary, we may assume that g contains the monomial $T_1^{kc} T_6^k$. But then g does not satisfy condition (C2) from Setting 3.4.1. A contradiction. Thus T_4 appears in g. In particular, homogeneity of g gives the bound $a \leq l_5 a_5$. In case $(a_5, l_5) = (2, 2)$ grading matrix and relation degree are given by

$$Q = \begin{bmatrix} 1 & 1 & 1 & a & 2 & 0 & 0 \\ 0 & 0 & 0 & 1 & 2 & 1 & 1 \end{bmatrix}, \qquad \mu = (4, 4), \qquad 2 \leq a \leq 4,$$

For each value of a we check all trinomials homogeneous g of degree $\deg(g) = \mu$ for the conditions (C1) and (C2) from Setting 3.4.1 and filter by isomorphy. For $a = 2$ we obtain specifying data no. 345. For $a = 3$ we obtain specifying data no. 346. For $a = 4$ we obtain specifying data no. 347 and 348. In case $(a_5, l_5) = (1, 3)$ grading matrix and relation degree are given by

$$Q = \begin{bmatrix} 1 & 1 & 1 & a & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 \end{bmatrix}, \qquad \mu = (3,3), \qquad 2 \leq a \leq 3.$$

For each value of a we check all trinomials homogeneous g of degree $\deg(g) = \mu$ for the conditions (C1) and (C2) from Setting 3.4.1 and filter by isomorphy. For $a = 2$ we obtain specifying data no. 349. For $a = 3$ we obtain specifying data no. 350 and 351.

Case 3.7.3.3: $(n_1, n_2, n_3, n_4) = (2, 2, 2, 1)$. By Lemma 3.4.6 (i) we have $w_3 = w_4 = (a, 1)$ for some $a \geq 1$. Applying Lemma 3.2.8 to the triple (w_1, w_5, w_6) shows that the primitive point $v \in \rho_3$ is of the form $v = (c, 1)$ for some $c \geq 1$. The grading matrix of X is thus given by

$$Q = \begin{bmatrix} 1 & 1 & a & a & a_5c & a_6c & 0 \\ 0 & 0 & 1 & 1 & a_5 & a_6 & 1 \end{bmatrix}, \qquad a, a_5, a_6, c \in \mathbb{Z}_{\geq 1}.$$

By Lemma 3.2.8 applied to (w_1, w_5, w_6) the integers a_5 and a_6 are coprime. We may assume $a_5 \leq a_6$. By Remark 3.2.11 we have $\mu \in (\rho_2 + \rho_3) \backslash \rho_2$. We distinguish the two cases $\mu \in (\rho_2 + \rho_3)^\circ$ and $\mu \in \rho_3$.

Case 3.7.3.3.1: $\mu \in (\rho_2 + \rho_3)^\circ$. Lemma 3.2.8 applied to the triples (w_1, w_2, w_5) and (w_1, w_2, w_6) yields $a_5 = a_6 = 1$. Grading matrix and anticanonical class of X are given by

$$Q = \begin{bmatrix} 1 & 1 & a & a & c & c & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}, \qquad -\mathcal{K} = \begin{bmatrix} 2 + 2a + 2c - \mu_1 \\ 5 - \mu_2 \end{bmatrix}.$$

From X being Fano, ie. $-\mathcal{K} \in \lambda^\circ$, we infer the inequalities

$$\mu_2 \leq 4, \tag{3.7.3.9}$$
$$\mu_1 \leq 1 + 2c + (\mu_2 - 3)a. \tag{3.7.3.10}$$

The position of μ yields the inequality $\mu_2 c + 1 \leq \mu_1$. Combining this with Equation 3.7.3.10, we obtain

$$0 \leq (2 - \mu_2)c + (\mu_2 - 3)a.$$

Due to Equation 3.7.3.10 and the fact that c is strictly smaller than a, this is only fulfilled for $\mu_2 = 4$. With this, Equation 3.7.3.10 turns into

$$\mu_1 \leq 1 + 2c + a. \tag{3.7.3.11}$$

The relation g is a trinomial with pairwise coprime monomials. Due to the position of μ, each monomial of g is divisible by precisely one of T_5, T_6, T_7 and by at least one of

$T_1, \ldots, T_4$. Thus there is a monomial divisible by either T_3 or T_4. For the monomial m with T_5 we write $m = T_1^{l_1} T_2^{l_2} T_3^{l_3} T_4^{l_4} T_5^{l_5}$. Set $l := l_1 + l_2$ and $l' := l_3 + l_4$. Homogeneity of g then yields $l' = 4 - l_5$ and $\mu_1 = l + l'a + l_5 c$. With Equation 3.7.3.11 we obtain the inequality

$$0 \ \leq \ 1 - l + (2 - l_5)c + (l_5 - 3)a.$$

Since a is strictly bigger than c, this yields $3 \leq l_5 \leq 4$. Similarly we obtain $3 \leq l_6 \leq 4$ and $2 \leq l_7 \leq 4$. We distinguish between the different possible values for l_5, l_6, l_7. As a first step, we consider the cases $l_7 = 2$ and $l_7 \geq 3$.

Case 3.7.3.3.1.1: $l_7 = 2$. For the monomial m of g that contains T_7 we write $m = T_1^{l_1} T_2^{l_2} T_3^{l_3} T_4^{l_4} T_7^2$. By homogeneity of g we have $l_3 + l_4 = 2$. If l_3 and l_4 are both positive, then, by coprimeness of the monomials of the trinomial g, it must be of the form

$$g \ = \ T_1^{l_1'} T_5^4 + T_2^{l_2'} T_6^4 + T_3 T_4 T_7^2.$$

The relation has degree $\mu = (2a, 4)$. Thus $l_1' = l_2' = 2a - 4c$ holds. In particular they are even, which contradicts condition (C2) form Setting 3.4.1. Thus we have $l_3 = 0$ or $l_4 = 0$. We may assume $l_3 = 0$ and $l_4 = 2$. To avoid a contradiction to condition (C2), the variable T_3 must appear among the other two monomials of g. Switching the roles of T_5 and T_6 if necessary, we may assume that g contains the monomials $T_1^{l_1} T_2^{l_2} T_5^4$ and $T_2^{l_2'} T_3 T_6^3$, where at least one of l_2, l_2' is zero. If $l_2' > 0$, then $l_2 = 0$ holds and l_1 is even, leading to the same contradiction as before. Thus $l_2' = 0$ holds. The relation g is thus of the form

$$g \ = \ T_1^{l_1} T_2^{l_2} T_5^4 + T_3 T_6^3 + T_2^{l_2'} T_4^2 T_7^2.$$

Homogeneity of g, together with Equation 3.7.3.11 yield the following conditions on a and c:

$$a + 3c \ \leq \ 1 + 2c + a, \qquad a + 3c \ = \ 2a + l_2'.$$

This yields $c = 1$ and $a + l_2' = 3$. Thus we have $2 \leq a \leq 3$ and $l_2' = 3 - a$. For $a = 2$ grading matrix and relation are given by

$$Q \ = \ \begin{bmatrix} 1 & 1 & 2 & 2 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}, \qquad g \ = \ T_1 T_5^4 + T_3 T_6^3 + T_2 T_4^2 T_7^2.$$

This is specifying data no. 352. For $a = 3$ grading matrix and relation are given by

$$Q \ = \ \begin{bmatrix} 1 & 1 & 3 & 3 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}, \qquad g \ = \ T_1 T_2 T_5^4 + T_3 T_6^3 + T_4^2 T_7^2.$$

This is specifying data no. 353.

Case 3.7.3.3.1.2: $l_7 \geq 3$. We divide this case further depending on the values of l_5, l_6 and l_7. We have seen earlier, that $3 \leq l_5, l_6 \leq 4$ holds. Switching the roles of T_5 and T_6 if necessary, we may assume that $l_5 \geq l_6$ holds. Moreover, due to the position of μ, the cases $(l_5, l_6, l_7) = (3, 3, 3)$ and $(l_5, l_6, l_7) = (4, 4, 4)$ cannot occur. We thus distinguish tthe following four cases:

$$\begin{aligned} (l_5, l_6, l_7) &= (4, 3, 3), & (l_5, l_6, l_7) &= (4, 4, 3), \\ (l_5, l_6, l_7) &= (3, 3, 4), & (l_5, l_6, l_7) &= (4, 3, 4). \end{aligned}$$

Case 3.7.3.3.1.2.1: $(l_5, l_6, l_7) = (4, 3, 3)$. Switching roles of T_1 and T_2 as well as T_3 and T_4 if necessary, we can write g as

$$g = T_1^{l_1} T_2^{l_2} T_5^4 + T_2^{k_2} T_3 T_6^3 + T_2^{m_2} T_4 T_7^3,$$

where $l_1 > 0$ and at most one of l_2, k_2, m_2 is non-zero. By homogeneity of g we have $m_2 = 3c + k_2$. Thus $m_2 = 3c$ and $l_2 = k_2 = 0$ holds and the relation degree is $\mu = (a + 3c, 4)$. Comparing this to the first monomial of g, we obtain $l_1 = a - c$. Plugging the value for μ_1 into Equation 3.7.3.11 yields $c = 1$. Grading matrix and relation are thus given by

$$Q = \begin{bmatrix} 1 & 1 & a & a & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}, \qquad g = T_1^{a-1} T_5^4 + T_3 T_6^3 + T_2^3 T_4 T_7^3.$$

This is series S21.

Case 3.7.3.3.1.2.2: $(l_5, l_6, l_7) = (4, 4, 3)$. Switching roles of T_3 and T_4 if necessary, we can write g as

$$g = T_1^{l_1} T_5^4 + T_2^{l_2} T_6^4 + T_4 T_7^3.$$

Homogeneity of g yields $l_1 = l_2 = a - 4c$. In particular we have $a \geq 4c + 1$. Grading matrix and relation are thus given by

$$Q = \begin{bmatrix} 1 & 1 & a & a & c & c & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}, \qquad g = T_1^{a-4c} T_5^4 + T_2^{a-4c} T_6^4 + T_4 T_7^3.$$

For g to satisfy condition (C2) from Setting 3.4.1, a must be odd. This is series S18.

Case 3.7.3.3.1.2.3: $(l_5, l_6, l_7) = (3, 3, 4)$. Switching roles of T_1 and T_2 as well as T_3 and T_4 if necessary, we can write g as

$$g = T_1^{l_1} T_3 T_5^3 + T_1^{k_1} T_4 T_6^3 + T_1^{m_1} T_2^{m_2} T_7^4,$$

where $m_2 > 0$ and at most one of l_1, k_1, m_1 is non-zero. By homogeneity of g we have $l_1 = k_1$ and thus $l_1 = k_1 = 0$ holds. The relation degree is $\mu = (a + 3c, 4)$. Plugging the value for μ_1 into Equation 3.7.3.11 yields $c = 1$. Comparing μ to the degree of the third monomial, we see $m_1 = a + 3c - m_2$. Grading matrix and relation are thus given by

$$Q = \begin{bmatrix} 1 & 1 & a & a & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}, \qquad g = T_3 T_5^3 + T_4 T_6^3 + T_1^{a+3c-l} T_2^l T_7^4.$$

This is series S19.

Case 3.7.3.3.1.2.4: $(l_5, l_6, l_7) = (4, 3, 4)$. Switching roles of T_1 and T_2 as well as T_3 and T_4 if necessary, we can write g as

$$g = T_1^{l_1} T_5^4 + T_3 T_6^3 + T_2^{l_2} T_7^4,$$

The relation degree is $\mu = (a + 3c, 4)$. Homogeneity of g yields $l_1 = a - c$ and $l_2 = a + 3c$. Plugging the value for μ_1 into Equation 3.7.3.11 yields $c = 1$. Grading matrix and relation are thus given by

$$Q = \begin{bmatrix} 1 & 1 & a & a & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}, \qquad g = T_1^{a-1} T_5^4 + T_3 T_6^3 + T_2^{a+3} T_7^4.$$

For g to satisfy condition (C2) from Setting 3.4.1, a must be even. This is series S20.

Case 3.7.3.3.2: $\mu \in \rho_3$. The relation degree satisfies $\mu = (kc, k)$ for some $k \geq 2$. Grading matrix and relation degree are thus given by

$$Q = \begin{bmatrix} 1 & 1 & a & a & a_5 c & a_6 c & 0 \\ 0 & 0 & 1 & 1 & a_5 & a_6 & 1 \end{bmatrix}, \qquad \mu = (kc, k).$$

As g is a trinomial with pairwise coprime monomials, the position of μ requires that g contains monomials of the form $T_5^{l_5}$ and $T_6^{l_6}$. By irredundancy of the presentation of R we have $l_5, l_6 \geq 2$. Homogeneity of g yields $k = l_5 a_5 = l_6 a_6$. Due to condition (C2) from Setting 3.4.1 the exponents l_5 and l_6 are coprime. Coprimeness of a_5 and a_6 yields $a_5 = l_6$ and $a_6 = l_5$ and $k = a_5 a_6$. We are thus in the situation of Lemma 3.4.4, which yields $a_5 = 2$, $a_6 = 3$ and $k = 6$. Grading matrix and anticanonical class, due to Proposition 3.2.5, are given by

$$Q = \begin{bmatrix} 1 & 1 & a & a & 2c & 3c & 0 \\ 0 & 0 & 1 & 1 & 2 & 3 & 1 \end{bmatrix}, \qquad -\mathcal{K} = \begin{bmatrix} 2 + 2a - c \\ 2 \end{bmatrix}.$$

From X being Fano, ie. $-\mathcal{K} \in \lambda^\circ$, we infer the inequality

$$0 < \det(-\mathcal{K}, w_3) = 2 - c,$$

which yields $c = 1$. The relation is of the form

$$g = T_1^{l_1} T_2^{l_2} T_3^{l_3} T_4^{l_4} T_7^{l_7} + T_5^3 + T_6^2,$$

where $l_1 + l_2 = l_7$ and $l_3 + l_4 + l_7 = 6$. We distinguish the two cases $l_3 + l_4 = 0$ and $l_3 + l_4 > 0$.

Case 3.7.3.3.2.1: $l_3 + l_4 = 0$. We have $l_1 + l_2 = l_7 = 6$. For g to satisfy condition (C2) from Setting 3.4.1, up two switching T_1 and T_2, we have $l_1 = 5$ and $l_2 = 1$. Grading matrix and relation are given by

$$Q = \begin{bmatrix} 1 & 1 & a & a & 2 & 3 & 0 \\ 0 & 0 & 1 & 1 & 2 & 3 & 1 \end{bmatrix}, \qquad g = T_1^5T_2T_7^6 + T_5^3 + T_6^2.$$

This is series S22.

Case 3.7.3.3.2.2: $l_3 + l_4 > 0$. By homogeneity of g we have $l_1 + l_2 + (l_3 + l_4)a = 6$, which yields $a \leq 6$. For each possible value of a we determine all homogeneous trinomials g of degree $\deg(g) = \mu$ and filter for isomorphy. According to the value of a we obtain the following specifying data

a	2	3	4	5	6
ID	354-360	361-363	364-365	366	367

Case 3.7.3.4: $(n_1, n_2, n_3, n_4) = (2, 2, 1, 2)$. By Lemma 3.4.6 (i) we have $w_3 = w_4 = (a, 1)$ for some $a \geq 1$. The grading matrix of X is thus of the form

$$Q = \begin{bmatrix} 1 & 1 & a & a & a_5 & 0 & 0 \\ 0 & 0 & 1 & 1 & b_5 & 1 & 1 \end{bmatrix}, \qquad a, a_5, b_5 \in \mathbb{Z}_{\geq 1}.$$

By Remark 3.2.11 we have $\mu \in (\rho_2 + \rho_3) \backslash \rho_2$. We distinguish the two cases $\mu \in (\rho_2 + \rho_3)^\circ$ and $\mu \in \rho_3$.

Case 3.7.3.4.1: $\mu \in (\rho_2 + \rho_3)^\circ$. Applying Lemma 3.2.8 to the triple (w_1, w_2, w_5) yields $b_5 = 1$. Set $c := a_5$. Grading matrix and the anticanonical class, due to Proposition 3.2.5, are given by

$$Q = \begin{bmatrix} 1 & 1 & a & a & c & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}, \qquad -\mathcal{K} = \begin{bmatrix} 2 + 2a + c - \mu_1 \\ 5 - \mu_2 \end{bmatrix}.$$

From X being Fano, ie. $-\mathcal{K} \in \lambda^\circ$, we infer the inequalities

$$\mu_2 \leq 4, \qquad (3.7.3.12)$$
$$\mu_1 \leq 1 + c + (\mu_2 - 3)a. \qquad (3.7.3.13)$$

The position of μ yields the inequality $\mu_2 c + 1 \leq \mu_1$. Combining this with Equation 3.7.3.13 yields

$$0 \leq (1 - \mu_2)c + (\mu_2 - 3)a.$$

Having in mind 3.7.3.12, this inequality yields $\mu_2 = 4$. Plugging this into Equation 3.7.3.13, we obtain the bound

$$\mu_1 \leq 1 + a + c. \qquad (3.7.3.14)$$

The relation g is a trinomial consisting of pairwise coprime monomials. Due to the position of μ, each monomial of g is divisible by precisely one of T_5, T_6, T_7. We establish bounds for the exponents l_5, l_6, l_7. First consider the monomial m of g containing T_5. It is of the form

$$m \;=\; T_1^{l_1}T_2^{l_2}T_3^{l_3}T_4^{l_4}T_5^{l_5}.$$

Set $l := l_1 + l_2$ and $l' := l_3 + l_4$. By homogeneity of g we obtain $l + l'a + l_5c = \mu_1$ and $l' + l_5 = 4$. In combination with the bound 3.7.3.14 we obtain the inequality

$$0 \;\leq\; 1 + (l_5 - 3)a + (1 - l_5)c.$$

This is only fulfilled by $l_5 = 4$. With the same arguments we obtain the following bound on l_6:

$$0 \;\leq\; 1 + (l_6 - 3)a + c.$$

Having in mind that $a > 4c$ holds due to the position of μ, this inequality yields $l_6 \geq 3$. Since $w_6 = w_7$, we also obtain $l_7 \geq 3$. Moreover, switching the roles of T_6 and T_7 if necessary, we may assume that $l_6 \geq l_7$ holds. Note that the case $(l_5, l_6, l_7) = (4, 4, 4)$ cannot occur. We thus distinguish the two cases $(l_5, l_6, l_7) = (4, 3, 3)$ and $(l_5, l_6, l_7) = (4, 4, 3)$.

Case 3.7.3.4.1.1: $(l_5, l_6, l_7) = (4, 3, 3)$. Switching roles of T_3 and T_4 if necessary, we can write g as

$$g \;=\; T_1^{l_1}T_2^{l_2}T_5^4 + T_3T_6^3 + T_4T_7^3.$$

The relation degree is $\mu = (a, 4)$. Comparing this to the degree of the first monomial, we obtain $l_1 = a - l_2$. Grading matrix and relation are thus given by

$$Q \;=\; \begin{bmatrix} 1 & 1 & a & a & c & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}, \qquad g \;=\; T_1^{a-l}T_2^{l}T_5^4 + T_3T_6^3 + T_4T_7^3.$$

This is series S24.

Case 3.7.3.4.1.2: $(l_5, l_6, l_7) = (4, 4, 3)$. Switching roles of T_1 and T_2 as well as T_3 and T_4 if necessary, we can write g as

$$g \;=\; T_1^{l_1}T_5^4 + T_2^{l_2}T_6^4 + T_4T_7^3.$$

The relation degree is $\mu = (a, 4)$. Homogeneity of g yields $l_1 = a - 4c$ and $l_2 = a$. Grading matrix and relation are thus given by

$$Q \;=\; \begin{bmatrix} 1 & 1 & a & a & c & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}, \qquad g \;=\; T_1^{a-4c}T_5^4 + T_2^{a}T_6^3 + T_4T_7^3.$$

This is series S23.

Case 3.7.3.4.2: $\mu \in \rho_3$. The relation degree μ and the generator degree w_5 lie on the same ray. Since g is a trinomial consisting of pairwise coprime monomials, each monomial of g is divisible by precisely one of T_5, T_6, T_7. In particular, g has a monomial of the form

$T_5^{l_5}$ with $l_5 \geq 2$. Thus $\mu = (l_5a_5, l_5b_5)$ holds. Grading matrix and anticanonical class of X, due to 3.2.5, are given by

$$Q \;=\; \begin{bmatrix} 1 & 1 & a & a & a_5 & 0 & 0 \\ 0 & 0 & 1 & 1 & b_5 & 1 & 1 \end{bmatrix}, \qquad -\mathcal{K} \;=\; \begin{bmatrix} 2+2a+(1-l_5)a_5 \\ 4+(1-l_5)b_5 \end{bmatrix}.$$

From X being Fano, ie. $-\mathcal{K} \in \lambda^\circ$, we infer the inequalities

$$(l_5-1)b_5 \;\leq\; 3, \tag{3.7.3.15}$$
$$0 \;\leq\; 1-2a+(l_5-1)(ab_5-a_5). \tag{3.7.3.16}$$

In particular, by the first inequality, we have $1 \leq b_5 \leq 3$. We distinguish three cases, depending on the value of b_5.

Case 3.7.3.4.2.1: $b_5 = 1$. In this case we have $a > a_5$. Equation 3.7.3.16 reads

$$0 \;\leq\; 1+(l_5-3)a-(l_5-1)a_5.$$

Having in mind 3.7.3.15, this yields $l_5 = 4$. Plugging the value for l_5 back into that inequality, we also obtain $a \geq 3a_5 - 1$. We establish bounds for the values of the exponents l_6 and l_7. For the monomial m of g containing T_6 we write

$$m \;=\; T_1^{l_1}T_2^{l_2}T_3^{l_3}T_4^{l_4}T_6^{l_6}.$$

We set $l := l_1+l_2$ and $l' := l_3+l_4$. Homogeneity of g yields $l' = 4-l_6$ and $4a_5 = l+(4-l_6)a$. Together with the inequality $a \geq 3a_5 - 1$, we see that $l_6 \geq 2$ holds. Assume $l_6 = 2$. Then we have $l_3 + l_4 = 2$. To satisfy condition (C2) from Setting 3.4.1, we must have $l_3 = l_4 = 1$. Moreover, we obtain the inequality

$$4a_5 \;\geq\; 6a_5 + l - 2,$$

which is only fulfilled for $a_5 = 1$, $a = 2$ and $l = 0$. Up to switching the roles of T_1 and T_2, grading matrix and relation are thus given by

$$Q \;=\; \begin{bmatrix} 1 & 1 & 2 & 2 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}, \qquad g \;=\; T_5^4 + T_3T_4T_6^2 + T_1^5T_2T_7^4.$$

This is specifying data no. 368. Switching the roles of T_6 and T_7, this also includes the case $l_7 = 2$. Thus we may now assume that $l_6, l_7 \geq 3$ holds. Moreover, we may assume $l_6 \geq l_7$. Note that the case $(l_5, l_6, l_7) = (4, 4, 4)$ cannot occur. We thus distinguish the two cases $(l_5, l_6, l_7) = (4, 3, 3)$ and $(l_5, l_6, l_7) = (4, 4, 3)$.

Case 3.7.3.4.2.1.1: $(l_5, l_6, l_7) = (4, 3, 3)$. Switching roles of T_1 and T_2 as well as T_3 and T_4 if necessary, we can write g as

$$g \;=\; T_5^4 + T_1^{l_1}T_2^{l_2}T_3T_6^3 + T_2^{m_2}T_4T_7^3,$$

where at most one of l_2, m_2 is non-zero. The relation degree is $\mu = (4c, 4)$. Homogeneity of g yields $l_2 = 0$ and $l_1 = m_2 = a - 4c$. In particular $a > 4c$ holds. Grading matrix and relation are given by

$$Q = \begin{bmatrix} 1 & 1 & a & a & c & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}, \qquad g = T_5^4 + T_1^{a-4c} T_3 T_6^3 + T_2^{a-4c} T_4 T_7^3.$$

This is series S25.

Case 3.7.3.4.2.1.2: $(l_5, l_6, l_7) = (4, 4, 3)$. Switching roles of T_1 and T_2 as well as T_3 and T_4 if necessary, we can write g as

$$g = T_5^4 + T_1^{l_1} T_2^{l_2} T_6^4 + T_2^{m_2} T_4 T_7^3,$$

where at most one of l_2, m_2 is non-zero. To satisfy the condition (C2) from Setting 3.4.1, we must have $m_2 = 0$ and l_1, l_2 must be odd. The relation degree is $\mu = (4c, 4)$. Homogeneity of g yields $a = 4c$ and $l_1 = 4c - l_2$. Grading matrix and relation are given by

$$Q = \begin{bmatrix} 1 & 1 & 4c & 4c & c & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}, \qquad g = T_5^4 + T_1^{4c-l} T_2^l T_6^4 + T_4 T_7^3.$$

This is series S26.

Case 3.7.3.4.2.2: $b_5 = 2$. By Equation 3.7.3.15 we have $l_5 = 2$. Plugging these values into Equation 3.7.3.16, we obtain $a_5 = 1$. Thus the relation g has degree $\mu = (2, 4)$. Note that in order to fulfill the condition (C2) from Setting 3.4.1, at least one of T_3, T_4 must appear in g. This then yields the bound $a \le 2$. Grading matrix and relation degree are thus given by

$$Q = \begin{bmatrix} 1 & 1 & a & a & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 2 & 1 & 1 \end{bmatrix}, \qquad \mu = (2, 4), \qquad a \le 2.$$

For each of the two values of a we determine all homogeneous trinomials g of degree $\deg(g) = \mu$ that satisfy conditions (C1) and (C2) from Setting 3.4.1 and filter by isomorphy. For $a = 1$ this leads to specifying data no. 369. For $a = 2$ this leads to specifying data no. 370.

Case 3.7.3.4.2.3: $b_5 = 3$. By Equation 3.7.3.15 we have $l_5 = 2$. Thus the relation g has degree $\mu = (2a_5, 6)$. We establish bounds for the exponents l_6 and l_7. By homogeneity of g we have $l_6, l_7 \le 6$. Switching the roles of T_6 and T_7 if necessary, we may assume $l_6 \ge l_7$. Consider the monomial m of g that contains T_7. It is of the form

$$m = T_1^{l_1} T_2^{l_2} T_3^{l_3} T_4^{l_4} T_7^{l_7}.$$

We set $l := l_1 + l_2$ and $l' := l_3 + l_4$. Homogeneity of g yields $l' = 6 - l_7$ and $2a_5 = l + (6 - l_7)a$. Together with Equation 3.7.3.15 we obtain

$$(4 - l_7)a \le 2. \qquad (3.7.3.17)$$

Thus we have $l_7 \geq 2$. With the same arguments we obtain $l_7 \geq 2$. We distinguish two cases, depending on the value of l_7.

Case 3.7.3.4.2.3.1: $l_7 \geq 4$. In this case we also have $l_6 \geq 4$. To fulfill the condition (C2) from Setting 3.4.1, at least one of T_3, T_4 must appear in g. Thus $l_7 \leq 5$ holds. If $l_7 = 4$, then both T_3 and T_4 must be paired with T_7 in order for g to satisfy condition (C2) from 3.4.1. Thus in this case $l_6 = 6$ holds. We thus distinguish the following three cases:

$$(l_5, l_6, l_7) \;=\; (2, 6, 4), \qquad (l_5, l_6, l_7) \;=\; (2, 5, 5), \qquad (l_5, l_6, l_7) \;=\; (2, 6, 5).$$

Case 3.7.3.4.2.3.1.1: $(l_5, l_6, l_7) = (2, 6, 4)$. The relation is of the form

$$g \;=\; T_5^2 + T_1^{l_1} T_2^{l_2} T_6^6 + T_3 T_4 T_7^4.$$

The relation degree is $\mu = (2a_5, 2)$. Homogeneity yields $l_1 = 2a_5 - l_2$ and $a = a_5$. Grading matrix and relation are thus given by

$$Q \;=\; \begin{bmatrix} 1 & 1 & a & a & a & 0 & 0 \\ 0 & 0 & 1 & 1 & 3 & 1 & 1 \end{bmatrix}, \qquad g \;=\; T_5^2 + T_1^{2a-l} T_2^l T_6^6 + T_3 T_4 T_7^4.$$

This is series S28.

Case 3.7.3.4.2.3.1.2: $(l_5, l_6, l_7) = (2, 5, 5)$. Switching roles of T_1 and T_2 as well as T_3 and T_4 if necessary, we can write g as

$$g \;=\; T_5^2 + T_1^{l_1} T_3 T_6^5 + T_2^{l_2} T_4 T_7^5.$$

The relation degree is $\mu = (2a_5, 2)$. Homogeneity yields $l_1 = l_2 = 2a_5 - a$. In particular we have $a < 2a_5$. Setting $c := a_5$, grading matrix and relation are given by

$$Q \;=\; \begin{bmatrix} 1 & 1 & a & a & c & 0 & 0 \\ 0 & 0 & 1 & 1 & 3 & 1 & 1 \end{bmatrix}, \qquad g \;=\; T_5^2 + T_1^{2c-a} T_3 T_6^5 + T_2^{2c-a} T_4 T_7^5.$$

This is series S27.

Case 3.7.3.4.2.3.1.3: $(l_5, l_6, l_7) = (2, 6, 5)$. In order to satisfy condition (C2) from Setting 3.4.1, T_1 and T_2 must both be paired with T_6. Switching roles of T_3 and T_4 if necessary, we can write g as

$$g \;=\; T_5^2 + T_1^{l_1} T_2^{l_2} T_6^6 + T_4 T_7^5.$$

The relation degree is $\mu = (2a_5, 2)$. Homogeneity yields $a = 2a_5$ and $l_1 = 2a_5 - l_2$. Setting $c := a_5$, grading matrix and relation are given by

$$Q \;=\; \begin{bmatrix} 1 & 1 & 2c & 2c & c & 0 & 0 \\ 0 & 0 & 1 & 1 & 3 & 1 & 1 \end{bmatrix}, \qquad g \;=\; T_5^2 + T_1^{2c-l} T_2^l T_6^6 + T_4 T_7^5.$$

This is series S29.

Case 3.7.3.4.2.3.2: $l_7 < 4$. In this case $4 - l_7$ is positive. Thus Equation 3.7.3.17 yields $a \leq 2$. Moreover, by Equation 3.7.3.16 we have $a_5 \leq a + 1 \leq 3$. By the ordering of the degrees we have $3a > a_5$ and the restriction $l_7 < 4$ yields $2a_5 \geq 3a$. The possible values for a and a_5 are thus $(a, a_5) = (1, 2)$ and $(a, a_5) = (2, 3)$. In both cases we determine all homogeneous trinomials g of degree $\deg(g) = \mu$ with $l_7 \leq 3$ that satisfy conditions (C1) and (C2) from Setting 3.4.1 and filter by isomorphy. For $(a, a_5) = (1, 2)$ we obtain specifying data no. 371 to 373. For $(a, a_5) = (2, 3)$ we obtain specifying data no. 374 to 377.

Case 3.7.3.5: $(n_1, n_2, n_3, n_4) = (2, 1, 3, 1)$. By Lemma 3.4.6 (i) we have $w_3 = (a, 1)$ for some $a \geq 1$. Applying Lemma 3.2.8 to the triple (w_1, w_4, w_5) shows that the primitive point $v \in \rho_3$ is of the form $v = (c, 1)$ for some $c \geq 1$. By Lemma 3.4.5, at least one of w_4, w_5, w_6 is primitive. We may assume that $w_4 = v$ holds. The grading matrix of X is thus of the form

$$Q = \begin{bmatrix} 1 & 1 & a & c & a_5c & a_6c & 0 \\ 0 & 0 & 1 & 1 & a_5 & a_6 & 1 \end{bmatrix}, \qquad a, a_5, a_6, c \in \mathbb{Z}_{\geq 1}.$$

Applying Lemma 3.2.8 to the triple (w_1, w_5, w_6) shows that a_5 and a_6 are coprime. We may assume $a_5 \leq a_6$. By Remark 3.2.11 we have $\mu \in (\rho_2 + \rho_3) \backslash \rho_2$. We distinguish the two cases $\mu \in (\rho_2 + \rho_3)^\circ$ and $\mu \in \rho_3$.

Case 3.7.3.5.1: $\mu \in (\rho_2 + \rho_3)^\circ$. Applying Lemma 3.2.8 to the triples (w_1, w_2, w_5) and (w_1, w_2, w_6) shows that $a_5 = a_6 = 1$ holds. Grading matrix and anticanonical class of X, due to Proposition 3.2.5, are thus given by

$$Q = \begin{bmatrix} 1 & 1 & a & c & c & c & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}, \qquad -\mathcal{K} = \begin{bmatrix} 2 + a + 3c - \mu_1 \\ 5 - \mu_2 \end{bmatrix}.$$

From X being Fano, ie. $-\mathcal{K} \in \lambda^\circ$ we infer the inequalities

$$\mu_2 \leq 4, \tag{3.7.3.18}$$
$$\mu_1 \leq 1 + 3c + (\mu_2 - 4)a. \tag{3.7.3.19}$$

The position of μ yields the inequality $\mu_2 c + 1 \leq \mu_1$. Combining this with Equation 3.7.3.19, we obtain

$$0 \leq (3 - \mu_2)c + (\mu_2 - 4)a.$$

Having in mind Equation 3.7.3.18 and $a > c$, this inequality is only fulfilled for $\mu_2 = 4$. Plugging this into Equation 3.7.3.19 yields $\mu_1 \leq 3c + 1$. However, by the position of μ, we have $\mu_1 > \mu_2 c = 4c$. A contradiction. Thus the case $\mu \in (\rho_2 + \rho_3)^\circ$ thus not occur.

Case 3.7.3.5.2: $\mu \in \rho_3$. The relation degree satisfies $\mu = (kc, k)$ for some $k \geq 2$.

Grading matrix and anticanonical class of X, due to Proposition 3.2.5, are given by

$$Q = \begin{bmatrix} 1 & 1 & a & c & a_5c & a_6c & 0 \\ 0 & 0 & 1 & 1 & a_5 & a_6 & 1 \end{bmatrix}, \qquad -\mathcal{K} = \begin{bmatrix} 2+a+(1+a_5+a_6-k)c \\ 3+a_5+a_6-k \end{bmatrix}.$$

From X being Fano, ie $-\mathcal{K} \in \lambda^\circ$, we infer the inequalities

$$k \le 2 + a_5 + a_6, \tag{3.7.3.20}$$
$$0 \le 1 - a + (1 + a_5 + a_6 - k)(c - a). \tag{3.7.3.21}$$

We distinguish three cases, depending on the values of a_5 and a_6.

Case 3.7.3.5.2.1: $a_5 = a_6 = 1$. Equations 3.7.3.20 and 3.7.3.21 yield

$$0 \le 1 + (k-4)a + (3-k)c, \qquad k \le 4.$$

Since $a > c$, this is only fulfilled for $k = 4$ and in this case we have $c = 1$. Grading matrix and relation degree are thus given by

$$Q = \begin{bmatrix} 1 & 1 & a & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}, \qquad \mu = (4,4).$$

If T_3 does not appear in g, then permuting T_4, T_5, T_6 if necessary, the relation g is of the form

$$g = T_1^{l_1}T_2^{l_2}T_7^4 + T_4^{l_4}T_5^{l_5} + T_6^4$$

with $l_1 + l_2 = l_4 + l_5 = 4$. To fulfill condition (C2) from Setting 3.4.1, up to switching T_1 and T_2, respectively T_4 and T_5, we have $l_1 = l_4 = 3$ and $l_2 = l_5 = 1$. This is series S30. If T_3 does appear in g, then homogeneity of g yields the bound $a \le 4$. For each possible value of a we determine all homogeneous trinomials g of degree $\deg(g) = \mu$ that contain T_3 and satisfy conditions (C1) and (C2) from Setting 3.4.1 and filter by isomorphy. For $a = 2$ we obtain specifying data no. 378 and 379. For $a = 3$ we obtain specifying data no. 380. For $a = 4$ we obtain specifying data no. 381.

Case 3.7.3.5.2.2: $a_5 = 1$, $a_6 > 1$. Applying Lemma 3.2.7 to the pair (w_1, w_6) shows that g contains a monomial of the form $T_6^{l_6}$ with $l_6 \ge 2$. In particular we have $k = l_6a_6$. Equation 3.7.3.20 turns into $(l_6 - 1)a_6 \le 3$, which yields $l_6 = 2$ and $2 \le a_6 \le 3$. Plugging this into Equation 3.7.3.21, we obtain the inequality

$$0 \le 1 + (2 - a_6)c + (a_6 - 3)a.$$

Since $a > 1$, this yields $a_6 = 3$. Moreover we obtain $c = 1$. Grading matrix and relation degree are thus given by

$$Q = \begin{bmatrix} 1 & 1 & a & 1 & 1 & 3 & 0 \\ 0 & 0 & 1 & 1 & 1 & 3 & 1 \end{bmatrix}, \qquad \mu = (6,6).$$

If T_3 does not appear in g, then permuting T_4, T_5, T_6 if necessary, the relation g is of the form

$$g = T_1^{l_1} T_2^{l_2} T_7^6 + T_4^{l_4} T_5^{l_5} + T_6^2$$

with $l_1 + l_2 = l_4 + l_5 = 6$. Checking each possible combination of exponents (l_1, l_2, l_4, l_5) for condition (C2) from Setting 3.4.1 and filtering by isomorphy, we obtain series S31 to S33. If T_3 does appear in g, then homogeneity of g yields the bound $a \le 6$. For each possible value of a we determine all homogeneous trinomials g of degree $\deg(g) = \mu$ that contain T_3 and satisfy conditions (C1) and (C2) from Setting 3.4.1 and filter by isomorphy. Depending on the value of a, we obtain the following specifying data:

a	2	3	4	5	6
ID	382-390	391-394	395-398	399-400	401-402

Case 3.7.3.5.2.3: $a_5, a_6 > 1$. Applying Lemma 3.2.7 to the pairs (w_1, w_5) and (w_1, w_6) shows that g contains monomials of the form $T_6^{l_5}$ and $T_6^{l_6}$ with $l_5, l_6 \ge 2$. In particular we have $k = l_5 a_5 = l_6 a_6$. By condition (C2) from Setting 3.4.1, the exponents l_5 and l_6 are coprime. With coprimeness of a_5 and a_6 we obtain $a_5 = l_6$ and $a_6 = l_5$ and $k = a_5 a_6$. We are thus in the situation of Lemma 3.4.4, which yields $a_5 = 2$, $a_6 = 3$ and $k = 6$. Plugging these values into Equation 3.7.3.21, we obtain $a \le 1$. This is a contradiction to $a > c \ge 1$. Thus this case does not occur.

Case 3.7.3.6: $(n_1, n_2, n_3, n_4) = (2, 1, 2, 2)$. By Lemma 3.4.6 (i) we have $w_3 = (a, 1)$ for some $a \ge 1$. Applying Lemma 3.2.8 to the triple (w_1, w_4, w_5) shows that the primitive point $v \in \rho_3$ is of the form $v = (c, 1)$ for some $c \ge 1$. The grading matrix of X is thus of the form

$$Q = \begin{bmatrix} 1 & 1 & a & a_4 c & a_5 c & 0 & 0 \\ 0 & 0 & 1 & a_4 & a_5 & 1 & 1 \end{bmatrix}, \qquad a, a_4, a_5, c \in \mathbb{Z}_{\ge 1},$$

Applying Lemma 3.2.8 to the triple (w_1, w_4, w_5) shows that a_4 and a_5 are coprime. We may assume $a_4 \le a_5$. By Remark 3.2.11 we have $\mu \in (\rho_2 + \rho_3) \backslash \rho_2$. We distinguish the two cases $\mu \in (\rho_2 + \rho_3)^\circ$ and $\mu \in \rho_3$.

Case 3.7.3.6.1: $\mu \in (\rho_2 + \rho_3)^\circ$. Lemma 3.2.8 applied to the triples (w_1, w_2, w_4) and (w_1, w_2, w_5) yields $a_4 = a_5 = 1$. Grading matrix and anticanonical class of X, due to Proposition 3.2.5, are given by

$$Q = \begin{bmatrix} 1 & 1 & a & c & c & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}, \qquad -\mathcal{K} = \begin{bmatrix} 2 + a + 2c - \mu_1 \\ 5 - \mu_2 \end{bmatrix}.$$

From X being Fano, ie. $-\mathcal{K} \in \lambda^\circ$, we infer the inequalities

$$\mu_2 \le 4, \tag{3.7.3.22}$$
$$\mu_1 \le 1 + 2c + (\mu_2 - 4)a. \tag{3.7.3.23}$$

The position of μ yields the inequality $\mu_2 c + 1 \leq \mu_1$ Combined with Equation 3.7.3.23, we obtain the inequality

$$0 \ \leq \ (2-\mu_2)c + (\mu_2 - 4)a,$$

the right hand side of which is negative for $\mu_2 \leq 4$. This is a contradiction to Equation 3.7.3.22. Thus the case $\mu \in (\rho_2 + \rho_3)^\circ$ does not occur.

Case 3.7.3.6.2: $\mu \in \rho_3$. The relation degree satisfies $\mu = (kc, k)$ for some $k \geq 2$. Grading matrix and anticanonical class of X, due to Proposition 3.2.5, are thus given by

$$Q = \begin{bmatrix} 1 & 1 & a & a_4 c & a_5 c & 0 & 0 \\ 0 & 0 & 1 & a_4 & a_5 & 1 & 1 \end{bmatrix}, \qquad -\mathcal{K} = \begin{bmatrix} 2 + a + (a_4 + a_5 - k)c \\ 3 + a_4 + a_5 - k \end{bmatrix}.$$

From X being Fano, ie. $-\mathcal{K} \in \lambda^\circ$, we infer the inequalities

$$k \ \leq \ 2 + a_4 + a_5, \tag{3.7.3.24}$$

$$0 \ \leq \ 1 - (2 + a_4 + a_5 - k)a + (a_4 + a_5 - k)c. \tag{3.7.3.25}$$

If $a_4 > 1$, then Lemma 3.2.7 applied to the pair (w_1, w_4) shows that g contains a monomial to the form $T_4^{l_4}$ with $l_4 \geq 2$. Thus in this case μ is a multiple of w_4. If $a_4 = 1$, then w_4 is primitive and μ is a multiple of w_4 as well. The same holds for w_5. There are thus $l_4, l_5 \geq 2$ with $k = l_4 a_4 = l_5 a_5$. In particular we have $k \geq a_4 + a_5$. This, together with Equation 3.7.3.24 shows that the right hand side of Equation 3.7.3.25 is strictly negative. A contradiction. Thus the case $\mu \in \rho_3$ thus not occur.

Case 3.7.3.7: $(n_1, n_2, n_3, n_4) = (2, 1, 1, 3)$. By Lemma 3.4.6 (i) we have $w_3 = (a, 1)$ for some $a \geq 1$. The grading matrix of X is thus of the form

$$Q = \begin{bmatrix} 1 & 1 & a & a_4 & 0 & 0 & 0 \\ 0 & 0 & 1 & b_4 & 1 & 1 & 1 \end{bmatrix}, \qquad a, a_4, b_4 \in \mathbb{Z}_{\geq 1},$$

By Lemma 3.4.7 we have $\mu \in (\rho_2 + \rho_4)^\circ$. We distinguish the three cases

$$\mu \in (\rho_2 + \rho_3)^\circ, \qquad \mu \in \rho_3, \qquad \mu \in (\rho_3 + \rho_4)^\circ.$$

Case 3.7.3.7.1: $\mu \in (\rho_2 + \rho_3)^\circ$. Lemma 3.2.8 applied to the triple (w_1, w_2, w_4) yields $b_4 = 1$. We set $c := a_4$. Grading matrix and anticanonical class of X, due to Proposition 3.2.5, are thus given by

$$Q = \begin{bmatrix} 1 & 1 & a & c & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}, \qquad -\mathcal{K} = \begin{bmatrix} 2 + a + c - \mu_1 \\ 5 - \mu_2 \end{bmatrix}.$$

From X being Fano, ie $-\mathcal{K} \in \lambda^\circ$, we infer the inequalities

$$\mu_2 \ \leq \ 4, \tag{3.7.3.26}$$

$$\mu_1 \ \leq \ 1 + c + (\mu_2 - 4)a. \tag{3.7.3.27}$$

The position of μ yields the inequality $\mu_2 c + 1 \leq \mu_1$. Combined with Equation 3.7.3.27, we obtain the inequality

$$0 \;\leq\; (1 - \mu_2) + (\mu_2 - 4)a,$$

the right hand side of which is negative for $\mu_2 \leq 4$. This is a contradiction to Equation 3.7.3.26. Thus the case $\mu \in (\rho_2 + \rho_3)^\circ$ does not occur.

Case 3.7.3.7.2: $\mu \in \rho_3$. Relation degree μ and generator degree w_4 lie on the same ray. If w_4 is primitive, then μ is a multiple of w_4. If w_4 is not primitive, then Lemma 3.2.7 applied to the pair (w_1, w_4) shows that g contains a monomial of the form $T_4^{l_4}$ with $l_4 \geq 2$. Thus in any case there is $l_4 \geq 2$ such that $\mu = (l_4 a_4, l_4 b_4)$ holds. Grading matrix and anticanonical class of X, due to Setting 3.4.1, are thus given by

$$Q \;=\; \begin{bmatrix} 1 & 1 & a & a_4 & 0 & 0 & 0 \\ 0 & 0 & 1 & b_4 & 1 & 1 & 1 \end{bmatrix}, \qquad -\mathcal{K} \;=\; \begin{bmatrix} 2 + a + (1 - l_4)a_4 \\ 4 + (1 - l_4)b_4 \end{bmatrix}.$$

From X being Fano, ie. $-\mathcal{K} \in \lambda^\circ$, we infer the inequalities

$$(l_4 - 1)b_4 \;\leq\; 3, \tag{3.7.3.28}$$

$$0 \;\leq\; 1 - 3a + (l_4 - 1)(ab_4 - a_4). \tag{3.7.3.29}$$

By Equation 3.7.3.28 we have $b_4 \leq 3$. Assume $b_4 = 1$ holds. Then by Equation 3.7.3.28 we have $l_4 \leq 4$ and Equation 3.7.3.29 yields

$$0 \;\leq\; 1 + (l_4 - 4)a - (1 - l_4)a_4.$$

The right hand side is strictly negative for all possible values of l_4. A contradiction. Thus $b_4 > 1$ holds. This yields $l_4 = 2$. For $b_4 = 2$, Equation 3.7.3.29 yields

$$0 \;\leq\; 1 - a - a_4.$$

The right hand side is strictly negative. A contradiction. Thus $b_4 = 3$ holds. Equation 3.7.3.29 then yields $a_4 = 1$. Grading matrix and relation degree are thus given by

$$Q \;=\; \begin{bmatrix} 1 & 1 & a & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 3 & 1 & 1 & 1 \end{bmatrix}, \qquad \mu \;=\; (2, 6).$$

Assume T_3 does not appear in g. Then, in order for g to satisfy condition (C1) from Setting 3.4.1, up to permuting variables of the same degree g is given by

$$g \;=\; T_1^2 T_5^6 + T_2^2 T_6^{l_6} T_7^{l_7} + T_4^2,$$

where $l_6 + l_7 = 6$. This trinomial does not satisfy condition (C2) from Setting 3.4.1. A contradiction. Thus the variable T_3 appears in g. By homogeneity of g we obtain the bound $a \leq 2$. For each of the two possible values of a we determine all homogeneous trinomials g of degree $\deg(g) = \mu$ that contain T_3 and satisfy condition (C1) and (C2) from Setting 3.4.1 and filter by isomorphy. For $a = 1$ we obtain specifying data no. 403 and 404. For $a = 2$ we obtain specifying data no. 405 to 410.

Case 3.7.3.7.3: $\mu \in (\rho_3 + \rho_4)^\circ$. Lemma 3.2.7 applied to the pair (w_1, w_4) yields $b_4 = 1$. We set $c := a_4$. Grading matrix and anticanonical class of X, due to Proposition 3.2.5, are thus given by

$$Q = \begin{bmatrix} 1 & 1 & a & c & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}, \qquad -\mathcal{K} = \begin{bmatrix} 2 + a + c - \mu_1 \\ 5 - \mu_2 \end{bmatrix}.$$

From X being Fano, ie. $-\mathcal{K} \in \lambda^\circ$, we infer the inequalities

$$\mu_2 \leq 4, \tag{3.7.3.30}$$
$$\mu_1 \leq 1 + c + (\mu_2 - 4)a. \tag{3.7.3.31}$$

The component μ_1 is positive. Since $a > c$ holds, Equation 3.7.3.31 is only fulfilled for $\mu_2 \geq 4$. With Equation 3.7.3.30 we obtain $\mu_2 = 4$. Plugging the value for μ_2 into Equation 3.7.3.31, we obtain the bound $\mu_1 \leq c+1$. The relation g is a trinomial consisting of pairwise coprime monomials. Due to the position of μ, each monomial of g is divisible by precisely one of T_5, T_6, T_7. If T_3 does not appear in g, then up to permuting variables of the same degree, g is of the form

$$g = T_1^{l_1} T_5^4 + T_2^{l_2} T_6^4 + T_4^{l_4} T_7^{4-l_4}.$$

By homogeneity of g we obtain $l_1 = l_2 = l_4 c$. Moreover, the bound on μ_1 yields $(l_4 - 1)c \leq 1$. If $l_4 > 1$, then this yields $l_4 = 2$ and $c = 1$. We obtain $l_1 = l_2 = 2$, which is a contradiction to condition (C2) from Setting 3.4.1. Thus $l_4 = 1$ holds. Grading matrix and relation are thus given by

$$Q = \begin{bmatrix} 1 & 1 & a & c & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}, \qquad g = T_1^c T_5^4 + T_2^c T_6^4 + T_4 T_7^3.$$

For g to satisfy condition (C2) from Setting 3.4.1, c must be odd. This is series S34. If T_3 does appear in g, then the bound on μ_1 yields $a = \mu_1 = c + 1$. Up to permuting variables of the same degree, the relation g is of the form

$$g = T_1^{c+1} T_5^4 + T_2^{l_2} T_4^{l_4} T_6^{l_6} + T_3 T_7^3,$$

where $l_2 + l_4 c = c + 1$ and $l_4 + l_6 = 4$. In particular we have $0 \leq l_4 \leq 1$. For $l_4 = 1$ we obtain series S35. For $l_4 = 0$ we obtain series S36. □

3.8 Proof of Theorem 3.1.1: Case $s = 5$

Setting 3.4.1 and Proposition 3.4.8 divide the proof of Theorem 3.1.1 into the five cases $s = 2, \ldots, 6$, according to the number of rays spanned by the degrees $w_1, \ldots, w_7$. In this section we treat the case $s = 5$.

Theorem 3.8.1. *The tables from 3.10.8, 3.10.9 and 3.10.10 provide specifying data (Q, g) for 37 sporadic cases and 39 infinite series of locally factorial Fano fourfolds of Picard number $\rho = 2$ and complexity $c = 1$ with a hypersurface Cox ring and $s = 5$. Moreover, any locally factorial Fano fourfold with a hypersurface Cox ring and invariants $(\rho, c, s) = (2, 1, 5)$ is isomorphic to precisely one $X(Q, g)$ with (Q, g) from these tables.*

The proof of Theorem 3.8.1 splits into two parts. First, with the tools provided in Section 3.2 we verify that each specifying data (Q, g) from the tables in 3.10.8, 3.10.9 and 3.10.10 defines a locally factorial Fano fourfold $X(Q, g)$ with a hypersurface Cox ring and invariants $(\rho, c, s) = (2, 1, 5)$. Moreover, with the help of Remark 3.4.3 we verify that distinct specifying data from the tables in 3.10.8, 3.10.9 and 3.10.10 define pairwise non-isomorphic varieties. The second part is to show that any locally factorial Fano fourfold with a hypersurface Cox ring and invariants $(\rho, c, s) = (2, 1, 5)$ is isomorphic to $X(Q, g)$ with (Q, g) from these tables. We divide the proof of this into the two general cases

$$\mu \ \in \ \mathrm{SAmple}(X), \qquad\qquad \mu \ \notin \ \mathrm{SAmple}(X).$$

The case $\mu \in \mathrm{SAmple}(X)$ will be treated in Proposition 3.8.2. In Proposition 3.8.3 we treat the case $\mu \notin \mathrm{SAmple}(X)$.

Proposition 3.8.2. *Let X as in Setting 3.4.1 with $s = 5$. Assume that $\mu \in \lambda$ holds. Then X is isomorphic to an $X(Q, g)$ with specifying data (Q, g) appearing in Classification list 3.10.8.*

Proof. We show that $\mu \in \mathrm{Mov}(R)^\circ$ holds. Assume $\mu \in \partial\,\mathrm{Mov}$. The relation degree μ is contained in one of the rays $\rho_1, \ldots, \rho_5$. By Lemma 3.4.7, μ is neither contained in ρ_1 nor in ρ_5. Reversing the order of the variables if necessary, we may assume that $\mu \in \rho_2 \cup \rho_3$ holds. By assumption μ lies in the boundary of $\mathrm{Mov}(R)$. This is not possible if μ lies in ρ_3. Thus we have $\mu \in \rho_2$. Moreover we have $n_1 = 1$ and, by Remark 3.2.11, also $n_2 \geq 2$ holds. The relation degree μ lies in the boundary of λ, which is contained in $\mathrm{Mov}(R)$. So we have $\lambda = \rho_2 + \rho_3$. Denote by v_i the primitive point on the ray ρ_1. Since $n_2 \geq 2$ holds and μ lies on ρ_2, we can apply Lemma 3.2.8 to v_2, v_i for $i = 3, 4, 5$, which tells us that each of the cones $\rho_2 + \rho_i$ is regular. By applying a suitable unimodular transformation we achieve

$$[v_1, v_2, v_3, v_4, v_5] \ = \ \begin{bmatrix} a_1 & 1 & a & b & 0 \\ -b_1 & 0 & 1 & 1 & 1 \end{bmatrix},$$

where a_1, b_1, a, b are positive integers. Let $w_i \in \rho_3$ and $w_j \in \rho_4$. Since $\det(v_1, v_3) > 1$ and $\det(v_1, v_4) > 1$, Lemma 3.2.7 tells us that the relation g contains monomials of the form $T_1^{l_1} T_3^{l_3}$ and $T_1^{l_1'} T_4^{l_4'}$. Due to the position of μ, we have $l_1, l_1', l_3, l_4' > 0$. This is a contradiction to the fact that the monomials of g are pairwise coprime. Thus $\mu \in \mathrm{Mov}(R)^\circ$ holds.

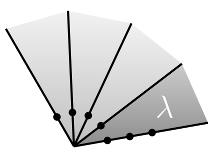
By assumption we have $\mu \in \lambda$ and we have just seen that $\mu \in \mathrm{Mov}(R)^\circ$ holds. We are therefore in the situation of Proposition 3.3.2. Thus, for a general polynomial $h \in \mathbb{C}[T_1, \dots, T_7]$ of degree $\deg(h) = \mu$, the projective variety X_h is smooth with divisor class group $\mathrm{Cl}(X_h) = K$ and Cox ring $\mathcal{R}(X_h) = R_h$. Moreover, by Proposition 3.2.5 X_h is Fano. Thus, X_h is a smooth Fano fourfold of Picard number two with a spread hypersurface Cox ring. In particular, up to unimodular equivalence, the grading matrix $Q = (w_1, \dots, w_7)$ together with the relation degree $\mu = \deg(g)$ appear in the classification list presented in [45, Thm. 1.1]. For each such entry (Q, μ) with $s = 4$ we determine all trinomials g of degree $\deg(g) = \mu$ that satisfy the conditions (C1) and (C2) from Setting 3.4.1 and filter the resulting list by isomorphy. This yields the specifying data no. 411 to 414 in Classification list 3.10.8. □

Proposition 3.8.3. *Let X as in Setting 3.4.1 with $s = 5$. Assume that $\mu \notin \lambda$ holds. Then X is isomorphic to an $X(Q, g)$ with specifying data (Q, g) appearing in Classification list 3.10.9 or in Classification list 3.10.10.*

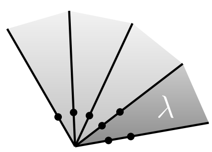
Proof. We have $\mu \notin \lambda$. Reversing the ordering of the variables if necessary, we may assume that $\mu \in \lambda^+ \backslash \lambda$ holds. We are thus in the situation of Lemma 3.4.6. Thus $\lambda = \rho_1 + \rho_2$ holds. Moreover, we have $n_1 \geq 2$ and all generator degrees contained in ρ_1 are primitive. By Lemma 3.4.7, μ is contained in the interior of $\mathrm{Eff}(R)$. Thus applying Lemma 3.2.8 to the triples (w_1, w_2, w_i), where $w_i \in \rho_5$, shows that the cone $\mathrm{Eff}(R)$ is regular and that w_i is primitive. We may thus assume that $\mathrm{Eff}(R)$ is the positive quadrant and that

$$w_1 \ = \ w_2 \ = \ (1,0), \qquad w_7 \ = \ (0,1)$$

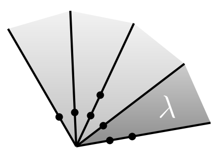
holds. Since μ is contained in the interior of $\mathrm{Eff}(R)$, but lies outside of λ, we have $\mu \in (\rho_2 + \rho_5)^\circ$. As there are only seven generator degrees, $n_5 \leq 2$ holds. Thus Remark 3.2.11 even yields $\mu \in (\rho_2 + \rho_4) \backslash \rho_2$. There are five possible degree constellations $(n_1, n_2, n_3, n_4, n_5)$ for X, displayed in the following pictures.



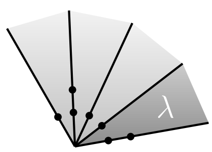
$(3, 1, 1, 1, 1)$



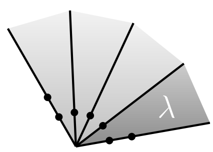
$(2, 2, 1, 1, 1)$



$(2, 1, 2, 1, 1)$



$(2, 1, 1, 2, 1)$



$(2, 1, 1, 1, 2)$

The black dots represent the generator degrees $w_1, \ldots, w_7$. We distinguish five cases, according to the degree constellation.

Case 3.8.3.1: $(n_1, n_2, n_3, n_4, n_5) = (3, 1, 1, 1, 1)$. Applying Lemma 3.2.7 to the pair (w_1, w_4) shows that $w_4 = (a_4, 1)$ holds with $a_4 \geq 1$. By Remark 3.2.11 we have $\lambda \in (\rho_2 + \rho_3) \backslash \rho_2$. Applying Lemma 3.2.8 to the triple (w_1, w_2, w_6) thus shows that $w_6 = (a_6, 1)$ holds with $a_6 \geq 1$. The grading matrix is given by

$$Q = \begin{bmatrix} 1 & 1 & 1 & a_4 & a_5 & a_6 & 0 \\ 0 & 0 & 0 & 1 & b_5 & 1 & 1 \end{bmatrix}, \qquad a_4, a_5, a_6, b_5 \in \mathbb{Z}_{\geq 1},$$

We distinguish the two cases $\mu \in (\rho_2 + \rho_3)^\circ$ and $\mu \in \rho_3$.

Case 3.8.3.1.1: $\mu \in (\rho_2 + \rho_3)^\circ$. Applying Lemma 3.2.8 to the triple (w_1, w_2, w_6) yields $b_5 = 1$. Grading matrix and anticanonical class of X, due to Proposition 3.2.5, are given by

$$Q = \begin{bmatrix} 1 & 1 & 1 & a_4 & a_5 & a_6 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 \end{bmatrix}, \qquad -\mathcal{K} = \begin{bmatrix} 3 + a_4 + a_5 + a_6 - \mu_1 \\ 4 - \mu_2 \end{bmatrix}.$$

From X being Fano, ie. $-\mathcal{K} \in \lambda^\circ$, we infer the inequalities

$$\begin{aligned} \mu_2 &\leq 3, && (3.8.3.1) \\ \mu_1 &\leq 2 + a_5 + a_6 + (\mu_2 - 3)a_4. && (3.8.3.2) \end{aligned}$$

By the ordering of the generator degrees we have $a_4 > a_5 > a_6$. With this, Equation 3.8.3.2 turns into

$$\mu_1 \leq (\mu_2 - 1)a_4 - 1.$$

However, due to the position of μ, we have $\mu_1 > \mu_2 a_4$. This is a contradiction. Thus the case $\mu \in (\rho_2 + \rho_3)^\circ$ does not occur.

Case 3.8.3.1.2: $\mu \in \rho_3$. The relation g is a trinomial consisting of pairwise coprime monomials. Due to the position of μ, the relation g thus contains a monomial of the form $T_5^{l_5}$ with $l_5 \geq 2$. So the relation degree satisfies $\mu = (l_5 a_5, l_5 b_5)$. Grading matrix and anticanonical class of X, due to Proposition 3.2.5, are given by

$$Q = \begin{bmatrix} 1 & 1 & 1 & a_4 & a_5 & a_6 & 0 \\ 0 & 0 & 0 & 1 & b_5 & 1 & 1 \end{bmatrix}, \qquad -\mathcal{K} = \begin{bmatrix} 3 + a_4 + (1 - l_5)a_5 + a_6 \\ 3 + (1 - l_5)b_5 \end{bmatrix}.$$

From X being Fano, ie. $-\mathcal{K} \in \lambda^\circ$, we infer the inequalities

$$\begin{aligned} (l_5 - 1)b_5 &\leq 2 && (3.8.3.3) \\ 0 &\leq 2 + a_6 - 2a_4 + (l_5 - 1)(a_4 b_5 - a_5). && (3.8.3.4) \end{aligned}$$

By Equation 3.8.3.3 we have $b_5 \leq 2$ and $l_5 \leq 3$. Assume $b_5 = 1$. The Equation 3.8.3.4 yields

$$0 \ \leq \ 2 + (l_5 - 3)a_4 - (l_5 - 1)a_5 + a_6.$$

By the ordering of the generator degrees we have $a_4 > a_5 > a_6$. Thus the right hand side is strictly negative. A contradiction. Thus $b_5 = 2$ holds. By Equation 3.8.3.3 we have $l_5 = 2$. Equation 3.8.3.4 turns into

$$a_5 \ \leq \ 2 + a_6.$$

By the ordering of the generator degrees, we have $0 < \det(w_5, w_6) = a_5 - 2a_6$. This yields $a_6 = 1$ and $a_5 = 3$. Grading matrix and relation degree are thus given by

$$Q \ = \ \begin{bmatrix} 1 & 1 & 1 & a_4 & 3 & 1 & 0 \\ 0 & 0 & 0 & 1 & 2 & 1 & 1 \end{bmatrix}, \qquad \mu \ = \ (6,\, 4).$$

The relation g contains the monomial T_5^2. If T_4 does not occur in g, then up to switching the roles of T_1, T_2 and T_3, it either contains the monomial $T_3^6 T_7^4$ or the monomial $T_3^2 T_6^4$. Both of those contradict condition (C2) from Setting 3.4.1. Thus T_4 appears in g. In particular, by homogeneity of g, we obtain the bound $a_4 \leq 6$. For each possible value of a_4 we determine all homogeneous trinomials g of degree $\deg(g) = \mu$ that satisfy conditions (C1) and (C2) from Setting 3.4.1 and filter by isomorphy. Depending on the value of a_4, we obtain the following specifying data:

a_4	2	3	4	5	6
ID	415-418	419-422	423	424	425

Case 3.8.3.2: $(n_1, n_2, n_3, n_4, n_5) = (2, 2, 1, 1, 1)$. Applying Lemma 3.2.7 to the pairs (w_1, w_3) and (w_1, w_4) shows that $w_3 = w_4 = (a, 1)$ holds for some $a \geq 1$. By Remark 3.2.11 we have $\lambda \in (\rho_2 + \rho_3) \backslash \rho_2$. Applying Lemma 3.2.8 to the triple (w_1, w_2, w_6) thus shows that $w_6 = (c, 1)$ holds for some $c \geq 1$. The grading matrix is given by

$$Q \ = \ \begin{bmatrix} 1 & 1 & a & a & a_5 & c & 0 \\ 0 & 0 & 1 & 1 & b_5 & 1 & 1 \end{bmatrix}, \qquad a, a_5, b_5, c \in \mathbb{Z}_{\geq 1},$$

We distinguish the two cases $\mu \in (\rho_2 + \rho_3)^\circ$ and $\mu \in \rho_3$.

Case 3.8.3.2.1: $\mu \in (\rho_2 + \rho_3)^\circ$. Applying Lemma 3.2.8 to the triple (w_1, w_2, w_5) yields $b_5 = 1$. Set $b := a_5$. Grading matrix and anticanonical class of X, due to Proposition 3.2.5, are given by

$$Q \ = \ \begin{bmatrix} 1 & 1 & a & a & b & c & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}, \qquad -\mathcal{K} \ = \ \begin{bmatrix} 2 + 2a + b + c - \mu_1 \\ 5 - \mu_2 \end{bmatrix}.$$

From X being Fano, ie. $-\mathcal{K} \in \lambda^\circ$, we infer the inequalities

$$\mu_2 \ \leq \ 4, \tag{3.8.3.5}$$
$$\mu_1 \ \leq \ 1 + b + c + (\mu_2 - 3)a. \tag{3.8.3.6}$$

By the position of μ, we have the inequality $\mu_1 \geq \mu_2 b + 1$. This, together with the inequality $b > c$, Equation 3.8.3.2 turns into

$$1 \ \leq \ (2 - \mu_2)b + (\mu_2 - 3)a.$$

Having in mind Equation 3.8.3.5, this is only fulfilled for $\mu_2 = 4$. With this, Equation 3.8.3.6 yields the bound

$$\mu \ \leq \ 1 + a + b + c. \tag{3.8.3.7}$$

The relation g is a trinomial consisting of pairwise coprime monomials. Due to the position of μ, each monomial of g is therefore divisible by precisely one of T_5, T_6, T_7. We establish bounds on the exponents l_5, l_6, l_7. Since $\mu_2 = 4$, homogeneity of g yields $l_5, l_6, l_7 \leq 4$. Consider the monomial m of g containing T_5. It is of the form

$$m \ = \ T_1^{l_1} T_2^{l_2} T_3^{l_3} T_4^{l_4} T_5^{l_5}.$$

Set $l := l_1 + l_2$ and $l' := l_3 + l_4$. By homogeneity of g we have $l' = 4 - l_5$ and the bound 3.8.3.7 yields

$$0 \ \leq \ 1 + (l_5 - 3)a + (1 - l_5)b + c.$$

This inequality is only fulfilled for $l_5 = 4$. Similarly we obtain $l_6 \geq 3$ and $l_7 \geq 2$. Assume $l_7 = 2$ holds. Switching the roles of T_1 and T_2 as well as T_3 and T_4 if necessary, we can write g as

$$g \ = \ T_1^{l_1} T_2^{l_2} T_5^4 + T_2^{m_2} T_3^{4-l_6} T_6^{l_6} + T_2^{k_2} T_3^{k_3} T_4^{k_4} T_7^2,$$

where $l_1 > 0$, at most one of l_2, m_2, k_2 is non-zero, $k_3 + k_4 = 2$ holds and either $k_3 = 0$ or $l_6 = 4$. If k_3 and k_4 are positive, then $l_6 = 4$ as well as $m_2 > 0$ holds. By homogeneity of g, both l_1 and m_2 are even. This violates condition (C2) from Setting 3.4.1. Thus either $k_3 = 0$ or $k_4 = 0$ holds. We may assume $k_3 = 0$ and $k_4 = 2$. The case $l_6 = 4$ again leads to the same contradiction. Thus $l_6 = 3$ holds. Moreover, if $l_2 = 0$, then by homogeneity l_1 is even, leading to a violation of condition (C2) from Setting 3.4.1. Thus g is of the form

$$g \ = \ T_1^{l_1} T_2^{l_2} T_5^4 + T_3 T_6^3 + T_4^2 T_7^2.$$

Comparing the degrees of the second and third monomial, we obtain $a = 3c$. Thus $\mu_1 = 6c$ holds. Moreover, the degree of the first monomial yields $4b < 6c$. Combining this with Equation 3.8.3.7, we obtain $c = 1$. Since $b \geq 2$ holds, this yields $8 \leq 4b < 6c = 6$. A contradiction. Thus the case $l_7 = 2$ does not occur. Moreover, the case $(l_5, l_6, l_7) = (4, 4, 4)$ cannot occur. We therefore distinguish the following three cases

$$(l_5, l_6, l_7) \ = \ (4, 3, 3), \qquad (l_5, l_6, l_7) \ = \ (4, 3, 4), \qquad (l_5, l_6, l_7) \ = \ (4, 4, 3).$$

Case 3.8.3.2.1.1: $(l_5, l_6, l_7) = (4, 3, 3)$. Switching roles of T_1 and T_2 as well as T_3 and T_4 if necessary, we can write g as

$$g \ = \ T_1^{l_1} T_2^{l_2} T_5^4 + T_2^{m_2} T_3 T_6^3 + T_2^{k_2} T_4 T_7^3,$$

where $l_1 > 0$ and at most one of l_2, m_2, k_2 is non-zero. By homogeneity of g we obtain $l_2 = m_2 = 0$ and $k_2 = 3c$. This also yields $l_1 = a - 4b + 3c$. Grading matrix and relation are thus given by

$$Q = \begin{bmatrix} 1 & 1 & a & a & b & c & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}, \qquad g = T_1^{a-4b+3c}T_5^4 + T_3T_6^3 + T_2^{3c}T_4T_7^3.$$

This is series S37.

Case 3.8.3.2.1.2: $(l_5, l_6, l_7) = (4, 3, 4)$. Switching roles of T_1 and T_2 as well as T_3 and T_4 if necessary, we can write g as

$$g = T_1^{l_1}T_5^4 + T_3T_6^3 + T_2^{l_2}T_7^4,$$

By homogeneity of g we obtain $l_1 = a - 4b + 3c$ and $l_2 = a + 3c$. Grading matrix and relation are thus given by

$$Q = \begin{bmatrix} 1 & 1 & a & a & b & c & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}, \qquad g = T_1^{a-4b+3c}T_5^4 + T_3T_6^3 + T_2^{a+3c}T_7^4.$$

This is series S38.

Case 3.8.3.2.1.3: $(l_5, l_6, l_7) = (4, 4, 3)$. Switching roles of T_1 and T_2 as well as T_3 and T_4 if necessary, we can write g as

$$g = T_1^{l_1}T_5^4 + T_2^{l_2}T_6^4 + T_4T_7^3.$$

By homogeneity of g we obtain $l_1 = a - 4b$ and $l_2 = a - 4c$. Grading matrix and relation are thus given by

$$Q = \begin{bmatrix} 1 & 1 & a & a & b & c & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}, \qquad g = T_1^{a-4b}T_5^4 + T_2^{a-4c}T_6^4 + T_2^{3c}T_4T_7^3.$$

This is series S39.

Case 3.8.3.2.2: $\mu \in \rho_3$. The relation degree μ and the generator degree l_5 lie on the same ray. If l_5 is not primitive, then applying Lemma 3.2.7 to the pair w_1, w_5 shows that g contains a monomial of the form $T_5^{l_5}$. In particular, μ is a multiple of w_5. If w_5 is primitive, then clearly μ is a multiple of w_5. Thus in any case there is $l_5 \geq 2$ with $\mu = l_5 w_5 = (l_5 a_5, l_5 b_5)$. Grading matrix and anticanonical class of X, due to Proposition 3.2.5, are given by

$$Q = \begin{bmatrix} 1 & 1 & a & a & a_5 & c & 0 \\ 0 & 0 & 1 & 1 & b_5 & 1 & 1 \end{bmatrix}, \qquad -\mathcal{K} = \begin{bmatrix} 2 + 2a + (1 - l_5)a_5 + c \\ 4 + (1 - l_5)b_5 \end{bmatrix}.$$

From X being Fano, ie. $-\mathcal{K} \in \lambda^\circ$, we infer the inequalities

$$(l_5 - 1)b_5 \leq 3, \tag{3.8.3.8}$$

$$0 \leq 1 - 2a + c + (l_5 - 1)(ab_5 - a_5). \tag{3.8.3.9}$$

Thus, by Equation 3.8.3.8 we have $b_5 \leq 3$. We distinguish three cases, depending to the value of b_5.

Case 3.8.3.2.2.1: $b_5 = 1$. By Equation 3.8.3.8 we have $l_5 \leq 4$. Set $b := 5$. Euqation 3.8.3.9 yields the inequality

$$0 \;\leq\; 1 + (l_5 - 3)a + (1 - l_5)b + c,$$

the right hand side of which is only non-negative for $l_5 \geq 4$. Thus $l_5 = 4$ holds. Grading matrix and relation degree are thus given by

$$Q \;=\; \begin{bmatrix} 1 & 1 & a & a & b & c & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}, \qquad \mu \;=\; (4b, 4).$$

Moreover, plugging the value for l_5 into Equation 3.8.3.9, we obtain the bound

$$\mu_1 \;\leq\; 1 + a + b + c. \tag{3.8.3.10}$$

The relation g is a trinomial consisting of pairwise coprime monomials. Due to the position of μ, each monomial of g is divisible by precisely one of T_5, T_6, T_7. We have already seen that g contains the monomial T_5^4. Thus one of the remaining monomials is divisible by T_6, the other one by T_7. We establish bounds for the exponents l_6, l_7. Since $mu_2 = 4$ holds, we have $l_6, l_7 \leq 4$. Consider the monomial m of g containing T_6. It is of the form

$$m \;=\; T_1^{l_1} T_2^{l_2} T_3^{l_3} T_4^{l_4} T_6^{l_6}.$$

Set $l := l_1 + l_2$ and $l' := l_3 + l_4$. By homogeneity of g we have $l' = 4 - l_6$ and Equation 3.8.3.10 yields

$$0 \;\leq\; 1 + (l_6 - 3)a + b + (1 - l_6)c.$$

The right hand side is only non-negative for $l_6 \geq 3$. Similarly we obtain $l_7 \geq 2$. We distinguish the following six cases:

$$(l_5, l_6, l_7) \;=\; (4, 3, 2), \qquad (l_5, l_6, l_7) \;=\; (4, 3, 3), \qquad (l_5, l_6, l_7) \;=\; (4, 3, 4),$$
$$(l_5, l_6, l_7) \;=\; (4, 4, 2), \qquad (l_5, l_6, l_7) \;=\; (4, 4, 3), \qquad (l_5, l_6, l_7) \;=\; (4, 4, 4).$$

Case 3.8.3.2.2.1.1: $(l_5, l_6, l_7) = (4, 3, 2)$. Switching the roles of T_3 and T_4 if necessary, we can write g as

$$g \;=\; T_5^4 + T_1^{l_1} T_2^{l_2} T_3 T_6^3 + T_1^{m_1} T_2^{m_2} T_4^2 T_7^2,$$

where either $l_1 = 0$ or $m_1 = 0$, as well as either $l_2 = 0$ or $m_2 = 0$. To satisfy condition (C2) from Setting 3.4.1, the exponents m_1 and m_2 must be odd. Thus $l_1 = l_2 = 0$ holds. Let $l := l_1' + l_2'$. Homogeneity of g yields the identities

$$4b \;=\; a + 3c, \tag{3.8.3.11}$$
$$4b \;=\; 2a + l. \tag{3.8.3.12}$$

By Equation 3.8.3.11 we have $a = 4b - 3c$. Plugging this into Equation 3.8.3.12, we obtain $l = 6c - 4b$. In particular, we get the bounds

$$3c \ < \ 4b \ < \ 6c.$$

Combining Equation 3.8.3.11 with the bound 3.8.3.10, we obtain $2c \leq 1 + b$. Together with the bound on b, this yields $b < 3$. Thus we have $b = 2$ and $c = 1$. But then $4b > 6c$ holds. A contradiction. Thus the case $(l_5, l_6, l_7) = (4, 3, 2)$ does not occur.

Case 3.8.3.2.2.1.2: $(l_5, l_6, l_7) = (4, 3, 3)$. Switching the roles of T_3 and T_4 if necessary, we can write g as

$$g \ = \ T_5^4 + T_1^{l_1} T_2^{l_2} T_3 T_6^3 + T_1^{m_1} T_2^{m_2} T_4 T_7^3,$$

where either $l_1 = 0$ or $m_1 = 0$, as well as either $l_2 = 0$ or $m_2 = 0$. By homogeneity of g we have $m_1 + m_2 > 0$ and we may assume that m_2 is positive. Thus we have $l_2 = 0$. If $l_1 = 0$ holds, then homogeneity of g yields $a = 4b - 3c$ and $m_1 = 3c - m_2$. Grading matrix and relation are thus given by

$$Q \ = \ \begin{bmatrix} 1 & 1 & 4b-3c & 4b-3c & b & c & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}, \qquad g \ = \ T_5^4 + T_3 T_6^3 + T_1^{3c-l} T_2^l T_4 T_7^3.$$

This is series S40. If $l_1 > 0$ holds, then we have $m_1 = 0$ and homogeneity of g yields $l_1 = 4b - a - 3c$ and $l_2 = 4b - a$. Grading matrix and relation are thus given by

$$Q \ = \ \begin{bmatrix} 1 & 1 & a & a & b & c & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}, \qquad g \ = \ T_5^4 + T_1^{(4b-a-3c)} T_3 T_6^3 + T_2^{(4b-a)} T_4 T_7^3.$$

This is series S42.

Case 3.8.3.2.2.1.3: $(l_5, l_6, l_7) = (4, 3, 4)$. Switching the roles of T_3 and T_4 if necessary, we can write g as

$$g \ = \ T_5^4 + T_1^{l_1} T_2^{l_2} T_3 T_6^3 + T_1^{m_1} T_2^{m_2} T_7^4,$$

where either $l_1 = 0$ or $m_1 = 0$, as well as either $l_2 = 0$ or $m_2 = 0$. To satisfy the condition (C2) from Setting 3.4.1, the exponents m_1, m_2 must be positive and odd. Thus $l_1 = l_2 = 0$ holds. Homogeneity of g yields $a = 4b - 3c$ and $m_1 = 4b - m_2$. Grading matrix and relation are thus given by

$$Q \ = \ \begin{bmatrix} 1 & 1 & 4b-3c & 4b-3c & b & c & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}, \qquad g \ = \ T_5^4 + T_3 T_6^3 + T_1^{4b-l} T_2^l T_7^4.$$

This is series S41.

Case 3.8.3.2.2.1.4: $(l_5, l_6, l_7) = (4, 4, 2)$. The relation g is of the form

$$g \ = \ T_5^4 + T_1^{l_1} T_2^{l_2} T_6^4 + T_1^{m_1} T_2^{m_2} T_3^{m_3} T_4^{m_4} T_7^2,$$

where $m_3 + m_4 = 2$ and $l_1 m_1 = 0$ and $l_2 m_2 = 0$. To satisfy condition (C2) from Setting 3.4.1, the exponents l_1 and l_2 must be positive and odd. Thus $m_1 = m_2 = 0$ holds. This in turn yields $m_3 = m_4 = 1$ with the same argument. By homogeneity of g we thus obtain $a = 2b$. Plugging this into the bound 3.8.3.10, we obtain $b = c + 1$. Using homogeneity of g again, we obtain $l_1 + l_2 = 4$. Switching the roles of T_1 and T_2 if necessary, we may assume $l_1 = 3$ and $l_2 = 1$. Grading matrix and relation are thus given by

$$Q = \begin{bmatrix} 1 & 1 & 2c+2 & 2c+2 & c+1 & c & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}, \qquad g = T_5^4 + T_1^3 T_2 T_6^4 + T_3 T_4 T_7^2.$$

This is series S43.

Case 3.8.3.2.2.1.5: $(l_5, l_6, l_7) = (4, 4, 3)$. Switching roles of T_3 and T_4 if necessary, we can write g as

$$g = T_5^4 + T_1^{l_1} T_2^{l_2} T_6^4 + T_2^{m_2} T_4 T_7^3,$$

where $l_1 > 0$ and either $l_2 = 0$ or $m_2 = 0$. To satisfy condition (C2) from Setting 3.4.1, the exponents l_1 and l_2 must be positive and odd. Thus $m_2 = 0$ holds. Homogeneity of g yields $a = 4b$ and $l_1 = 4b - 4c - l_2$. Grading matrix and relation are thus given by

$$Q = \begin{bmatrix} 1 & 1 & 4b & 4b & b & c & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}, \qquad g = T_5^4 + T_1^{4b-4c-l} T_2^l T_6^4 + T_4 T_7^3.$$

This is series S44.

Case 3.8.3.2.2.1.6: $(l_5, l_6, l_7) = (4, 4, 4)$. In this case the variables T_3 and T_4 do not appear in g. Thus, up to switching T_1 and T_2, the relation g is of the form

$$g = T_5^4 + T_1^{l_1} T_6^4 + T_2^{l_2} T_7^4.$$

By homogeneity of g, the exponents l_1 and l_2 are even. This violates condition (C2) from Setting 3.4.1. Thus the case $(l_5, l_6, l_7) = (4, 4, 4)$ does not occur.

Case 3.8.3.2.2.2: $b_5 = 2$. Equation 3.8.3.8 yields $l_5 = 2$. Plugging the values for b_5 and l_5 into Equation 3.8.3.9, we obtain $a_5 = c + 1$. This yields $\det(w_5, w_6) \leq 0$, contradicting the ordering of the generator degrees. Thus the case $b_5 = 2$ does not occur.

Case 3.8.3.2.2.3: $b_5 = 3$. Equation 3.8.3.8 yields $l_5 = 2$. Set $b := a_5$. Grading matrix and relation degree are given by

$$Q = \begin{bmatrix} 1 & 1 & a & a & b & c & 0 \\ 0 & 0 & 1 & 1 & 3 & 1 & 1 \end{bmatrix}, \qquad \mu = (2b, 6).$$

Plugging the values for b_5 and l_5 into Equation 3.8.3.9 and combining it with homogeneity of g, we obtain the bound

$$\mu_1 = 2b \leq 2 + 2a + 2c. \tag{3.8.3.13}$$

The relation g is a trinomial consisting of pairwise coprime monomials. Due to the position of μ, each monomial is divisible by precisely one of T_5, T_6, T_7. The relation g contains the monomial T_5^2. The other two monomials are thus each divisible by precisely one of T_6 and T_7. We establish bounds on the exponents l_6 and l_7. Since $\mu_2 = 6$ holds, we have $l_6, l_7 \leq 6$. Consider the monomial m of g containing T_6. It is of the form

$$m \ = \ T_1^{l_1} T_2^{l_2} T_3^{l_3} T_4^{l_4} T_6^{l_6}.$$

Set $l := l_1 + l_2$ and $l' := l_3 + l_4$. By homogeneity of g we have $l' = 6 - l_6$. Equation 3.8.3.13 yields

$$0 \ \leq \ 2 + (l_6 - 4)a + (2 - l_6)c.$$

The right hand side is only non-negative for $l_6 \geq 4$. Similarly we obtain $l_7 \geq 2$. We distinguish three cases:

$$\text{(i)} \quad l_6 \ = \ 4, \qquad \text{(ii)} \quad l_7 \ = \ 2, \qquad \text{(iii)} \quad l_6 \ \geq \ 5, \ l_7 \ \geq \ 3.$$

Case 3.8.3.2.2.3.1: $l_6 = 4$. With the notation from above, we have

$$2b \ = \ \mu_1 \ = \ l + (6 - l_6)a + l_6 c \ = \ l + 2a + 4c.$$

The bound on μ_1 3.8.3.13 yields $c = 1$. With this, we obtain $l = 0$ and $b = a + 2$. Grading matrix and relation degree are thus given by

$$Q \ = \ \begin{bmatrix} 1 & 1 & a & a & a+2 & 1 & 0 \\ 0 & 0 & 1 & 1 & 3 & 1 & 1 \end{bmatrix}, \qquad \mu \ = \ (2a + 4, 6).$$

The monomial of g containing T_6 is of the form $m = T_3^{l_3} T_4^{l_4} T_6^4$ with $l_3 + l_4 = 2$. By condition (C2) from Setting 3.4.1 we have $l_3 = l_4 = 1$. The relation g is threfore of the form

$$g \ = \ T_5^2 + T_3 T_4 T_6^4 + T_1^{l_1} T_2^{l_2} T_7^6.$$

By homogeneity of g we have $l_1 = 2a + 4 - l_2$ and by condition (C2) from Setting 3.4.1, l_1 is odd. This is series S45.

Case 3.8.3.2.2.3.2: $l_7 = 2$. Similar to above, we consider the monomial m of g containig T_7 and write

$$m \ = \ T_1^{l_1} T_2^{l_2} T_3^{l_3} T_4^{l_4} T_7^2.$$

We set $l := l_1 + l_2$ and $l' := l_3 + l_4$. By homogeneity of g we obtain $l' = 4$ and

$$2b \ = \ \mu_1 \ = \ l + 4a.$$

The bound on μ_1 3.8.3.13 yields $a = c + 1$ and $l = 0$. Homogeneity then yields $b = 2a = 2c + 2$. By the ordering of the generator degrees we have

$$0 \ < \ \det(w_5, w_6) \ = \ 2 - c.$$

This yields $c = 1$. Grading matrix and relation degree are thus given by

$$Q = \begin{bmatrix} 1 & 1 & 2 & 2 & 4 & 1 & 0 \\ 0 & 0 & 1 & 1 & 3 & 1 & 1 \end{bmatrix}, \qquad \mu = (8,6).$$

By condition (C2) from Setting 3.4.1, both l_3 and l_4 are positive and odd. Up to switching the roles of T_3 and T_4, the relation g is of the form

$$g = T_5^2 + T_1^{l_1} T_2^{l_2} T_6^6 + T_3^3 T_4 T_7^2$$

with $l_1 + l_2 = 2$. Again using condition (C2) yields $l_1 = l_2 = 1$. This is specifying data no. 426.

Case 3.8.3.2.2.3.3: $l_6 \geq 5, l_7 \geq 3$. We further distinguish the following eight exponent constellations for (l_5, l_6, l_7):

$$(2,5,3), \qquad (2,5,4), \qquad (2,5,5), \qquad (2,5,6),$$

$$(2,6,3), \qquad (2,6,4), \qquad (2,6,5), \qquad (2,6,6).$$

Case 3.8.3.2.2.3.3.1: $(l_5, l_6, l_7) = (2,5,3)$. Switching roles of T_3 and T_4 if necessary, we can write g as

$$g = T_5^2 + T_1^{l_1} T_2^{l_2} T_3 T_6^5 + T_1^{m_1} T_2^{m_2} T_4^3 T_7^3, \tag{3.8.3.14}$$

where $l_1 m_1 = 0$ and $l_2 m_2 = 0$. Homogeneity of g together with the bound on μ_1 3.8.3.13 yield the inequalities

$$\begin{aligned} l_1 + l_2 + a + 5c &\leq 2 + 2a + 2c \\ m_1 + m_2 + 3a &\leq 2 + 2a + 2c. \end{aligned}$$

From the first one, we obtain $a \geq 3c - 2$, from the second one $a \leq 2c + 2$. Combining these two, we get the bound $c \leq 4$. Plugging this back into the second inequality, we obtain $a \leq 10$. The bound on μ_1 then yields $b \leq 30$. For all possible combinations of a, b and c within these bounds we determine all homogeneous trinomials g of degree $\deg(g) = \mu$ of the form 3.8.3.14 that satisfy conditions (C1) and (C2) from Setting 3.4.1 and filter by isomorphy. Not all combinations of values for a, b, c do actually produce valid specifying data. Depending on the values of a, b, c we obtain the following specifying data

(a,b,c)	$(2,4,1)$	$(3,5,1)$	$(4,6,1)$	$(4,7,2)$	$(5,8,2)$	$(6,9,2)$
ID	427	428	429-430	431-432	433	434-435
(a,b,c)	$(7,11,3)$	$(8,12,3)$	$(10,15,4)$			
ID	436	437	438			

Case 3.8.3.2.2.3.3.2: $(l_5, l_6, l_7) = (2, 5, 4)$. Switching roles of T_3 and T_4 if necessary, we can write g as

$$g \;=\; T_5^2 + T_1^{l_1}T_2^{l_2}T_3T_6^5 + T_1^{m_1}T_2^{m_2}T_4^2T_7^4,$$

where $l_1m_1 = 0$ and $l_2m_2 = 0$. To satisfy condition (C2) from Setting 3.4.1, the exponents m_1 and m_2 must both be positive and odd. Thus $l_1 = l_2 = 0$ holds. Homogeneity of g yields $a = 2b - 5c$ and $m_1 = 10c - 2b - m_2$. Grading matrix and relation are thus given by

$$Q \;=\; \begin{bmatrix} 1 & 1 & 2b-5c & 2b-5c & b & c & 0 \\ 0 & 0 & 1 & 1 & 3 & 1 & 1 \end{bmatrix}, \qquad g \;=\; T_5^2 + T_3T_6^5 + T_1^{10c-2b-l}T_2^lT_4^2T_7^4.$$

The Fano condition on X, ie $-\mathcal{K} \in \lambda^\circ$ yields $b \geq 4c - 1$. This is series S46.

Case 3.8.3.2.2.3.3.3: $(l_5, l_6, l_7) = (2, 5, 5)$. Switching roles of T_3 and T_4 if necessary, we can write g as

$$g \;=\; T_5^2 + T_1^{l_1}T_2^{l_2}T_3T_6^5 + T_1^{m_1}T_2^{m_2}T_4T_7^5,$$

where $l_1m_1 = 0$ and $l_2m_2 = 0$. Comparing degrees of the second and third monomial shows that $m_1 + m_2 > 0$ holds. We may assume $m_2 > 0$. Then we have $l_2 = 0$. If $l_1 = 0$, then g is of the form

$$g \;=\; T_5^2 + T_3T_6^5 + T_1^{m_1}T_2^{m_2}T_4T_7^5.$$

Homogeneity of g yields $a = 2b - 5c$ and $m_1 = 5c - m_2$. Grading matrix and relation are thus given by

$$Q \;=\; \begin{bmatrix} 1 & 1 & 2b-5c & 2b-5c & b & c & 0 \\ 0 & 0 & 1 & 1 & 3 & 1 & 1 \end{bmatrix}, \qquad g \;=\; T_5^2 + T_3T_6^5 + T_1^{5c-l}T_2^lT_4T_7^5.$$

This is series S47. If $l_1 > 0$, then g is of the form

$$g \;=\; T_5^2 + T_1^{l_1}T_3T_6^5 + T_2^{m_2}T_4T_7^5.$$

Homogeneity of g yields $l_1 = 2b - 5c - a$ and $m_2 = 2b - a$. This in particular yields $a < 2b - 5c$. Note that the right hand side is positive by the ordering of the generator degrees. Grading matrix and relation are given by

$$Q \;=\; \begin{bmatrix} 1 & 1 & a & a & b & c & 0 \\ 0 & 0 & 1 & 1 & 3 & 1 & 1 \end{bmatrix}, \qquad g \;=\; T_5^2 + T_1^{2b-5c-a}T_3T_6^5 + T_2^{2b-a}T_4T_7^5.$$

The Fano condition on X, ie. $-\mathcal{K} \in \lambda^\circ$, yields $b \geq 4c$. This is series S49.

Case 3.8.3.2.2.3.3.4: $(l_5, l_6, l_7) = (2, 5, 6)$. Switching roles of T_1 and T_2 as well as T_3 and T_4 if necessary, we can write g as

$$g \;=\; T_5^2 + T_1^{l_1}T_3T_6^5 + T_1^{m_1}T_2^{m_2}T_7^6,$$

where $m_2 > 0$ and $l_1 m_1 = 0$. To satisfy condition (C2) from Setting 3.4.1, m_1 and m_2 must both be positive and odd. Thus $l_1 = 0$ holds. Homogeneity of g yields $a = 2b - 5c$ and $m_1 = 2b - m_2$. Grading matrix and relation are thus given by

$$Q \;=\; \begin{bmatrix} 1 & 1 & 2b-5c & 2b-5c & b & c & 0 \\ 0 & 0 & 1 & 1 & 3 & 1 & 1 \end{bmatrix}, \qquad g \;=\; T_5^2 + T_3 T_6^5 + T_1^{2b-l} T_2^l T_7^6.$$

This is series S48.

Case 3.8.3.2.2.3.3.5: $(l_5, l_6, l_7) = (2, 6, 3)$. Switching roles of T_1 and T_2 if necessary, we can write g as

$$g \;=\; T_5^2 + T_1^{l_1} T_2^{l_2} T_6^6 + T_2^{m_2} T_3^{m_3} T_4^{m_4} T_7^3,$$

where $l_1 > 0$, at most one of l_2, m_2 is non-zero and $m_3 + m_4 = 3$ holds. By condition (C2) from Setting 3.4.1, the exponents l_1 and l_2 are both positive and odd. Thus $m_2 = 0$ holds. Comparing the degrees of the first and third monomial of g, we see that there is an integer d such that $b = 3d$ and $a = 2d$ holds. The bound on μ_1 thus yields $6d \leq 2 + 4d + 2c$, or equivalently $d \leq c + 1$. Comparing the first and second monomial of g yields $d > c$. Thus $d = c + 1$ holds. This also yields $l_1 + l_2 = 6$. Thus grading matrix and relation are given by

$$Q \;=\; \begin{bmatrix} 1 & 1 & 2c+2 & 2c+2 & 3c+3 & c & 0 \\ 0 & 0 & 1 & 1 & 3 & 1 & 1 \end{bmatrix}, \quad g \;=\; T_5^2 + T_1^{l_1} T_2^{l_2} T_6^6 + T_3^{m_3} T_4^{m_4} T_7^3,$$

where $l_1 + l_2 = 6$ and $m_3 + m_4 = 3$. Having in mind condition (C2) from Setting 3.4.1, up to isomorphy this leads to series S50 to S52.

Case 3.8.3.2.2.3.3.6: $(l_5, l_6, l_7) = (2, 6, 4)$. Switching roles of T_1 and T_2 if necessary, we can write g as

$$g \;=\; T_5^2 + T_1^{l_1} T_2^{l_2} T_6^6 + T_2^{m_2} T_3^{m_3} T_4^{m_4} T_7^4,$$

where $l_1 > 0$, at most one of l_2, m_2 is non-zero and $m_3 + m_4 = 2$ holds. To satisfy condition (C2) from Setting 3.4.1, the exponents l_1 and l_2 must both be positive and odd. Thus $m_2 = 0$ holds. By the same argument we also obtain $m_3 = m_4 = 1$. Homogeneity of g yields $a = b$ and $l_1 = 2a - 6c - l_2$. Grading matrix and relation are thus given by

$$Q \;=\; \begin{bmatrix} 1 & 1 & a & a & a & c & 0 \\ 0 & 0 & 1 & 1 & 3 & 1 & 1 \end{bmatrix}, \qquad g \;=\; T_5^2 + T_1^{2a-6c-l} T_2^l T_6^6 + T_3 T_4 T_7^4.$$

This leads to series S53.

Case 3.8.3.2.2.3.3.7: $(l_5, l_6, l_7) = (2, 6, 5)$. Switching roles of T_1 and T_2 as well as T_3 and T_4 if necessary, we can write g as

$$g \;=\; T_5^2 + T_1^{l_1} T_2^{l_2} T_6^6 + T_2^{m_2} T_4 T_7^5,$$

where $l_1 > 0$ and at most one of l_2, m_2 is non-zero. To satisfy condition (C2) from Setting 3.4.1, the exponents l_1 and l_2 must both be positive and odd. Thus $m_2 = 0$ holds. Homogeneity of g yields $a = 2b$ and $l_1 = a - l_2$. Grading matrix and relation are thus given by

$$Q \;=\; \begin{bmatrix} 1 & 1 & 2b & 2b & b & c & 0 \\ 0 & 0 & 1 & 1 & 3 & 1 & 1 \end{bmatrix}, \qquad g \;=\; T_5^2 + T_1^{2b-l} T_2^l T_6^6 + T_4 T_7^5.$$

This leads to series S54.

Case 3.8.3.2.2.3.3.8: $(l_5, l_6, l_7) = (2, 6, 6)$. Switching roles of T_1 and T_2 if necessary, we can write g as

$$g \;=\; T_5^2 + T_1^{l_1} T_6^6 + T_2^{l_2} T_7^6.$$

By homogeneity of g, the exponents l_1 and l_2 are both odd. This violates condition (C2) from Setting 3.4.1. Thus the case $(l_6, l_7) = (6, 6)$ does not occur.

Case 3.8.3.3: $(n_1, n_2, n_3, n_4, n_5) = (2, 1, 2, 1, 1)$. Applying Lemma 3.2.7 to the pair (w_1, w_3) shows that $w_3 = (a, 1)$ holds for some $a \geq 1$. Moreover, applying Lemma 3.2.8 to the triple (w_1, w_4, w_5) shows that the primitive point $v \in \rho_3$ is of the form $v = (b, 1)$ for some $b \geq 1$. By Remark 3.2.11 we have $\lambda \in (\rho_2 + \rho_3) \backslash \rho_2$. Thus applying Lemma 3.2.8 to the triple (w_1, w_2, w_6) shows that $w_6 = (c, 1)$ holds for some $c \geq 1$. The grading matrix is given by

$$Q \;=\; \begin{bmatrix} 1 & 1 & a & a_4 b & a_5 b & c & 0 \\ 0 & 0 & 1 & a_4 & a_5 & 1 & 1 \end{bmatrix}, \qquad a, a_4, a_5, b, c \in \mathbb{Z}_{\geq 1}.$$

By Lemma 3.2.8 applied to the triple (w_1, w_4, w_5), the integers a_4 and a_5 are coprime. We distinguish the two cases $\mu \in (\rho_2 + \rho_3)^\circ$ and $\mu \in \rho_3$.

Case 3.8.3.3.1: $\mu \in (\rho_2 + \rho_3)^\circ$. Applying Lemma 3.2.8 to the triples (w_1, w_2, w_4) and (w_1, w_2, w_5) shows that $a_4 = a_5 = 1$ holds. Grading matrix and anticanonical class of X, due to Proposition 3.2.5, are given by

$$Q \;=\; \begin{bmatrix} 1 & 1 & a & b & b & c & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}, \qquad -\mathcal{K} \;=\; \begin{bmatrix} 2 + a + 2b + c - \mu_1 \\ 5 - \mu_2 \end{bmatrix}.$$

From X being Fano, ie. $-\mathcal{K} \in \lambda^\circ$, we infer the inequalities

$$\begin{aligned} \mu_2 &\leq 4, && (3.8.3.15) \\ \mu_1 &\leq 1 + (\mu_2 - 4)a + 2b + c. && (3.8.3.16) \end{aligned}$$

The position of μ yields the inequality $\mu_2 b + 1 \leq \mu_1$. Combining this with Equation 3.8.3.16, we obtain

$$0 \;\leq\; (\mu_2 - 4)a + (2 - \mu_2)b + c.$$

Having in mind the ordering of the generator degrees, the right hand side is negative for $\mu_2 \leq 4$. This is a contradiction to Equation 3.8.3.15. Thus the case $\mu \in (\rho_2 + \rho_3)^\circ$ does not occur.

Case 3.8.3.3.2: $\mu \in \rho_3$. The relation degree satisfies $\mu = (kb, k)$ for some $k \geq 2$. Grading matrix and anticanonical class of X, due to Proposition 3.2.5, are given by

$$Q = \begin{bmatrix} 1 & 1 & a & a_4 b & a_5 b & c & 0 \\ 0 & 0 & 1 & a_4 & a_5 & 1 & 1 \end{bmatrix}, \qquad -\mathcal{K} = \begin{bmatrix} 2 + a + (a_4 + a_5 - k)b + c \\ 3 + a_4 + a_5 - k \end{bmatrix}.$$

From X being Fano, ie. $\mu \in \lambda^\circ$, we infer the inequalities

$$\begin{aligned} k &\leq 2 + a_4 + a_5, & (3.8.3.17) \\ 0 &\leq 1 + c - 2a + (a_4 + a_5 - k)(b - a). & (3.8.3.18) \end{aligned}$$

The second inequality can also be written as

$$0 \leq c + 2 - 2b + (2 + a_4 + a_5 - k)(b - a). \qquad (3.8.3.19)$$

By the ordering of the generator degrees we have $c + 2 - 2b < 0$ as well as $b - a < 0$. Thus, by Equation 3.8.3.17, the right hand side of Equation 3.8.3.19 is negative. A contradiction. Thus the case $\mu \in \rho_3$ does not occur. This Case 3.

Case 3.8.3.4: $(n_1, n_2, n_3, n_4, n_5) = (2, 1, 1, 2, 1)$. Applying Lemma 3.2.7 to the pair (w_1, w_3) shows that $w_3 = (a, 1)$ holds for some $a \geq 1$. Moreover, applying Lemma 3.2.8 to the triple (w_1, w_5, w_6) shows that the primitive point $v \in \rho_4$ is of the form $v = (c, 1)$ for some $c \geq 1$. The grading matrix is given by

$$Q = \begin{bmatrix} 1 & 1 & a & a_4 & a_5 c & a_6 c & 0 \\ 0 & 0 & 1 & b_4 & a_5 & a_6 & 1 \end{bmatrix}, \qquad a, a_4, a_5, a_6, b_4, c \in \mathbb{Z}_{\geq 1}.$$

Lemma 3.2.8 for the triple (w_1, w_5, w_6) shows that a_5 and a_6 are coprime. We may assume that $a_5 \leq a_6$ holds. By Remark 3.2.11 we have $\mu \in (\rho_2 + \rho_4) \backslash \rho_2$. We distinguish the following four cases:

$$\mu \in (\rho_2 + \rho_3)^\circ, \qquad \mu \in \rho_3, \qquad \mu \in (\rho_3 + \rho_4)^\circ, \qquad \mu \in \rho_4.$$

Case 3.8.3.4.1: $\mu \in (\rho_2 + \rho_3)^\circ$. Applying Lemma 3.2.8 to the generator degree triples (w_1, w_2, w_4), (w_1, w_2, w_5) and (w_1, w_2, w_6) shows that $b_4 = a_5 = a_6 = 1$ holds. Grading matrix and anticanonical class of X, due to Proposition 3.2.5, are thus given by

$$Q = \begin{bmatrix} 1 & 1 & a & b & c & c & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}, \qquad -\mathcal{K} = \begin{bmatrix} 2 + a + b + 2c - \mu_1 \\ 5 - \mu_2 \end{bmatrix}.$$

From X being Fano, ie. $\mu \in \lambda^\circ$, we infer the inequalities

$$\begin{aligned} \mu_2 &\le 4, & (3.8.3.20)\\ \mu_1 &\le 1 + (\mu_2 - 4)a + 2b + c. & (3.8.3.21)\end{aligned}$$

The position of μ yields $\mu_2 b + 1 \le \mu_1$. Combining this with Equation 3.8.3.21, we obtain

$$0 \le (\mu_2 - 4)a + (2 - \mu_2)b + c.$$

Due to the ordering of the generator degrees, the reight hand side of this inequality is negative for $\mu_2 \le 4$. This is a contradiction to Equation 3.8.3.20. Thus the case $\mu \in (\rho_2 + \rho_3)^\circ$ does not occur.

Case 3.8.3.4.2: $\mu \in \rho_3$. The relation degree μ and the generator degree w_4 lie on the same ray. If w_4 is not primitive, then Lemma 3.2.7 applied to the pair (w_1, w_4) shows that g contains a monomial of the form $T_4^{l_4}$. Thus μ is a multiple of w_4. If w_4 is primitive, then clearly μ is a multiple of w_4. So in any case there is $k \ge 2$ with $\mu = kw_4$. Moreover, applying Lemma 3.2.8 to the generator degree triples (w_1, w_2, w_5) and (w_1, w_2, w_6) shows that $a_5 = a_6 = 1$ holds. Grading matrix and anticanonical class of X, due to Proposition 3.2.5, are thus given by

$$Q = \begin{bmatrix} 1 & 1 & a & a_4 & c & c & 0 \\ 0 & 0 & 1 & b_4 & 1 & 1 & 1 \end{bmatrix}, \qquad -\mathcal{K} = \begin{bmatrix} 2 + a + 2c + (1-k)a_4 \\ 4 + (1-k)b_4 \end{bmatrix}.$$

From X being Fano, ie. $-\mathcal{K} \in \lambda^\circ$, we infer the inequalities

$$\begin{aligned} (k-1)b_4 &\le 3, & (3.8.3.22)\\ 0 &\le 1 + 2c - 3a + (k-1)(ab_4 - a_4). & (3.8.3.23)\end{aligned}$$

Since $\det(w_4, w_5) > 0$ holds, we can rewrite Equation 3.8.3.23 to obtain the inequality

$$0 \le 2 - k - c + (3 - (k-1)b_4)(c - a).$$

By the ordering of the generator degrees and Equation 3.8.3.22, the right hand side of this inequality is strictly negative. A contradiction. Thus the case $\mu \in \rho_3$ does not occur.

Case 3.8.3.4.3: $\mu \in (\rho_3 + \rho_4)^\circ$. Applying Lemma 3.2.8 to the generator degree triples (w_1, w_2, w_4), (w_1, w_2, w_5) and (w_1, w_2, w_6) shows that $b_4 = a_5 = a_6 = 1$ holds. Grading matrix and anticanonical class, due to Proposition 3.2.5, are thus given by

$$Q = \begin{bmatrix} 1 & 1 & a & b & c & c & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}, \qquad -\mathcal{K} = \begin{bmatrix} 2 + a + b + 2c - \mu_1 \\ 5 - \mu_2 \end{bmatrix}.$$

From X being Fano, ie. $\mu \in \lambda^\circ$, we infer the inequalities

$$\begin{aligned} \mu_2 &\le 4, & (3.8.3.24)\\ \mu_1 &\le 1 + b + 2c + (\mu_2 - 4)a. & (3.8.3.25)\end{aligned}$$

The position of μ yields $\mu_2 c + 1 \leq \mu_1$. Combining this with Equation 3.8.3.25, we obtain

$$0 \ \leq \ 1 + (\mu_2 - 4)a + b + (2 - \mu_2)c.$$

Having in mind the ordering of the generator degrees, the right hand side of this inequality is negative for $\mu_2 < 4$. Together with Equation 3.8.3.24 we obtain $\mu_2 = 4$. Plugging this into Equation 3.8.3.25, we obtain the bound

$$\mu_1 \ \leq \ 1 + b + 2c. \tag{3.8.3.26}$$

The relation g is a trinomial consisting of pairwise coprime monomials. Due to the position of μ, each monomial of g is thus divisible by precisely one of T_5, T_6, T_7. We establish bounds for l_5, l_6, l_7. Since $\mu_2 = 4$ holds, we have $l_5, l_6, l_7 \leq 4$. Consider the monomial m of g containing T_5. We write

$$m \ = \ T_1^{l_1} T_2^{l_2} T_3^{l_3} T_4^{l_4} T_5^{l_5}.$$

Set $l := l_1 + l_2$. By homogeneity of g we have $l_3 + l_4 + l_5 = 4$ and $l + l_3 a + l_4 b + l_5 c = \mu_1$. Combining this with the bound 3.8.3.26, we obtain

$$l_5 c + (4 - l_5) b \ \leq \ l_5 c + l_3 a + l_4 b \ \leq \ \mu_1 \ \leq \ 1 + b + 2c. \tag{3.8.3.27}$$

This inequality is only fulfilled for $l_5 \geq 3$. Similarly we obtain $l_6 \geq 3$ and $l_7 \geq 2$. Interchanging T_5 and T_6 if necessary, we may assume that $l_5 \geq l_6$ holds. Assume $l_5 = 3$. Then we have $l_6 = 3$. Switching roles of T_3 and T_4 as well as T_5 and T_6 if necessary, we can write

$$g \ = \ T_3 T_5^3 + T_1^{a-b} T_4 T_6^3 + T_2^{a+3c} T_7^4.$$

With the bound on μ_1 we obtain

$$a + 3c \ = \ \mu_1 \ \leq \ 1 + b + 2c \ \leq \ a + 2c.$$

A contradiction. Thus $l_5 = 4$ holds. In particular, the monomial of g containing T_5 is of the form $T_1^{l_1} T_2^{l_2} T_5^4$ and we may assume that $l_1 > 0$ holds. We distinguish the cases $l_6 = 3$ and $l_6 = 4$.

Case 3.8.3.4.3.1: $l_6 = 3$. Consider the monomial m of g containing T_6. It is of the form

$$m \ = \ T_2^{l_2} T_3^{l_3} T_4^{l_4} T_6^3$$

with $l_3 + l_4 = 1$. The bound 3.8.3.26 yields $l_3 a + l_4 b + c \leq b + 1$. This inequality is only fulfilled for $l_3 = 0$, $l_4 = 1$ and $c = 1$. Thus m is of the form

$$m \ = \ T_2^{l_2} T_4 T_6^3$$

We distinguish three cases, depending on the value of l_7.

Case 3.8.3.4.3.1.1: $(l_5, l_6, l_7) = (4, 3, 2)$. The relation g is of the form

$$g \;=\; T_1^{l_1}T_2^{l_2}T_5^4 + T_2^{k_2}T_4T_6^3 + T_2^{m_2}T_3^2T_7^2,$$

where at most one of l_2, k_2, m_2 is non-zero. Applying the bound 3.8.3.26 to the degree of the third monomial, we obtain

$$2a \;\le\; 2a + m_2 \;=\; \mu_1 \;\le\; 1 + b + 2c \;=\; b + 3.$$

This yields $a \le 2$. However, by the ordering of the generator degrees, $a \ge 3$ holds. A contradiction. Thus the case $(l_5, l_6, l_7) = (4, 3, 2)$ does not occur.

Case 3.8.3.4.3.1.2: $(l_5, l_6, l_7) = (4, 3, 3)$. The relation g is of the form

$$g \;=\; T_1^{l_1}T_2^{l_2}T_5^4 + T_4T_6^3 + T_2^{m_2}T_3T_7^3,$$

where $l_2 = 0$ or $m_2 = 0$. Homogeneity of g yields $a = b+3-m_2 \le b+3$. By the ordering of the generator degrees, we have $a \ge b+1$. We thus distinguish the three cases $a = b+1$, $a = b+2$ and $a = b+3$.

Case 3.8.3.4.3.1.2.1: $a = b + 1$. We have $m_2 = 2$. Grading matrix and relation are given by

$$Q \;=\; \begin{bmatrix} 1 & 1 & b+1 & b & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}, \qquad g \;=\; T_1^{b-1}T_5^4 + T_4T_6^3 + T_2^2T_3T_7^3.$$

This is series S55.

Case 3.8.3.4.3.1.2.2: $a = b + 2$. We have $m_2 = 1$. Grading matrix and relation are given by

$$Q \;=\; \begin{bmatrix} 1 & 1 & b+2 & b & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}, \qquad g \;=\; T_1^{b-1}T_5^4 + T_4T_6^3 + T_2T_3T_7^3.$$

This is series S56.

Case 3.8.3.4.3.1.2.3: $a = b + 3$. We have $m_2 = 0$. Grading matrix and relation are given by

$$Q \;=\; \begin{bmatrix} 1 & 1 & b+3 & b & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}, \qquad g \;=\; T_1^{b-1-l}T_2^lT_5^4 + T_4T_6^3 + T_3T_7^3.$$

This is series S57.

Case 3.8.3.4.3.1.3: $(l_5, l_6, l_7) = (4, 3, 4)$. The relation g is of the form

$$g \;=\; T_1^{l_1}T_5^4 + T_4T_6^3 + T_2^{l_2}T_7^4.$$

Homogeneity of g yields $l_1 = b-1$ and $l_2 = b+3$. Grading matrix and relation are thus given by

$$Q = \begin{bmatrix} 1 & 1 & a & b & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}, \qquad g = T_1^{b-1}T_5^4 + T_4T_6^3 + T_2^{b+3}T_7^4.$$

To satisfy condition (C2) from Setting 3.4.1, b must be even. This is series S58.

Case 3.8.3.4.3.2: $l_6 = 4$. Since $\mu_2 = 4$ holds and T_1 appears in the monomial of T_5, the monomial m of g containing T_6 is now of the form

$$m = T_2^{l_2}T_6^4$$

We distinguish three cases, depending on the value of l_7.

Case 3.8.3.4.3.2.1: $(l_5, l_6, l_7) = (4, 4, 2)$. The relation g is of the form

$$g = T_1^{l_1}T_5^4 + T_2^{l_2}T_6^4 + T_3^{l_3}T_4^{l_4}T_7^2,$$

where $l_3 + l_4 = 2$. If $l_3 = 0$, then $l_4 = 2$ holds and the relation has degree $\mu = (2b, 4)$. Homogeneity of g yields $l_1 = l_2 = 2b$. This violates condition (C2) from Setting 3.4.1. Similarly for the case $l_4 = 0$. Thus $l_3 = l_4 = 1$ holds. Applying the bound 3.8.3.26 to the degrees of the second and third monomial of g, we obtain

$$2c \leq b < a \leq 2c+1.$$

This is only fulfilled for $b = 2c$ and $a = 2c+1$. Moreover we obtain $l_1 = l_2 = 1$. Grading matrix and relation are thus given by

$$Q = \begin{bmatrix} 1 & 1 & 2c+1 & 2c & c & c & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}, \qquad g = T_1T_5^4 + T_2T_6^4 + T_3T_4T_7^2.$$

This is series S59.

Case 3.8.3.4.3.2.2: $(l_5, l_6, l_7) = (4, 4, 3)$. The relation g is of the form

$$g = T_1^{l_1}T_5^4 + T_2^{l_2}T_6^4 + T_3^{l_3}T_4^{l_4}T_7^3,$$

where $l_3 + l_4 = 1$. Homogeneity of g yields $l_1 = l_2 =: l$ and $l = l_3a + l_4b - 4c$. If $l_3 = 1$, then $l_4 = 0$ holds. This yields $l = a - 4c$. Grading matrix and relation are given by

$$Q = \begin{bmatrix} 1 & 1 & a & b & c & c & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}, \qquad g = T_1^{a-4c}T_5^4 + T_2^{a-4c}T_6^4 + T_3T_7^3.$$

To satisfy condition (C2) from Setting 3.4.1, a must be odd. This is series S60. Now assume $l_4 = 1$. Then $l_3 = 0$ holds. This yields $l = b - 4c$. Grading matrix and relation are given by

$$Q = \begin{bmatrix} 1 & 1 & a & b & c & c & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}, \qquad g = T_1^{b-4c}T_5^4 + T_2^{b-4c}T_6^4 + T_4T_7^3.$$

To satisfy condition (C2) from Setting 3.4.1, b must be odd. This is series S61.

Case 3.8.3.4.3.2.3: $(l_5, l_6, l_7) = (4, 4, 4)$. Since $\mu_2 = 4$, the variables T_3, T_4 do not appear in g. The relation g consists of pairwise coprime monomials. It is not possible to form g with only the variables T_1, T_2, T_5, T_6, T_7. Thus the case $(l_5, l_6, l_7) = (4, 4, 4)$ does not occur.

Case 3.8.3.4.4: $\mu \in \rho_4$. The relation degree is of the form $\mu = (kc, k)$ for some $k \geq 2$. The relation g is a trinomial consisting of pairwise coprime monomials. Due to the position of μ, the relation g contains monomials of the form $T_5^{l_5}$ and $T_6^{l_6}$ with $l_5, l_6 > 1$. By homogeneity of g we obtain $k = l_5 a_5 = l_6 a_6$. To satisfy condition (C2) from Setting 3.4.1, the exponents l_5 and l_6 must be coprime. The coprimeness of a_5 and a_6 yields $a_5 = l_6$, $a_6 = l_5$ and $k = a_5 a_6$. We are thus in the situation of Lemma 3.4.4, which yields $a_5 = 2$, $a_6 = 3$ and $k = 6$. Grading matrix and anticanonical class of X, due to Proposition 3.2.5, are thus given by

$$Q = \begin{bmatrix} 1 & 1 & a & b & 2c & 3c & 0 \\ 0 & 0 & 1 & 1 & 2 & 3 & 1 \end{bmatrix}, \qquad -\mathcal{K} = \begin{bmatrix} 2 + a + b - c \\ 2 \end{bmatrix}.$$

From X being Fano we infer the inequality

$$0 \leq 1 + b - a - c.$$

As a is strictly larger than b, the right hand side of this inequality is strictly negative. Thus the case $\mu \in \rho_4$ does not occur.

Case 3.8.3.5: $(n_1, n_2, n_3, n_4, n_5) = (2, 1, 1, 1, 2)$. Applying Lemma 3.2.7 to the pair (w_1, w_3) shows that $w_3 = (a, 1)$ holds for some $a \geq 1$. The grading matrix is given by

$$Q = \begin{bmatrix} 1 & 1 & a & a_4 & a_5 & 0 & 0 \\ 0 & 0 & 1 & b_4 & b_5 & 1 & 1 \end{bmatrix}, \qquad a, a_4, a_5, b_4, b_5 \in \mathbb{Z}_{\geq 1},$$

By Remark 3.2.11 we have $\mu \in (\rho_2 + \rho_4) \backslash \rho_2$. We distinguish the following four cases:

$$\mu \in (\rho_2 + \rho_3)^\circ, \qquad \mu \in \rho_3, \qquad \mu \in (\rho_3 + \rho_4)^\circ, \qquad \mu \in \rho_4.$$

Case 3.8.3.5.1: $\mu \in (\rho_2 + \rho_3)^\circ$. Applying Lemma 3.2.8 to the generator degree triples (w_1, w_2, w_4) and (w_1, w_2, w_5) shows that $b_4 = b_5 = 1$ holds. Grading matrix and anticanonical class of X, due to Proposition 3.2.5, are thus given by

$$Q = \begin{bmatrix} 1 & 1 & a & b & c & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}, \qquad -\mathcal{K} = \begin{bmatrix} 2 + a + b + c - \mu_1 \\ 5 - \mu_2 \end{bmatrix}.$$

From X being Fano, ie. $\mu \in \lambda^\circ$, we infer the inequalities

$$\mu_2 \leq 4, \tag{3.8.3.28}$$
$$\mu_1 \leq 1 + (\mu_2 - 4)a + b + c. \tag{3.8.3.29}$$

The position of μ yields $\mu_2 b + 1 \leq \mu_1$. Combining this with Equation 3.8.3.29, we obtain

$$0 \ \leq \ (\mu_2 - 4)a + (1 - \mu_2)b + c.$$

Having in mind the ordering of the generator degrees, the right hand side of this inequality is negative for $\mu_2 \leq 4$. This is a contradiction to Equation 3.8.3.28. Thus the case $\mu \in (\rho_2 + \rho_3)^\circ$ does not occur.

Case 3.8.3.5.2: $\mu \in \rho_3$. Applying Lemma 3.2.8 to the triple (w_1, w_2, w_5) shows that $b_5 = 1$ holds. The relation degree μ and the generator degree w_4 lie on the same ray. If w_4 is not primitive, then Lemma 3.2.7 applied to the pair (w_1, w_4) shows that g contains a monomial of the form $T_4^{l_4}$. In particular, μ is a multiple of w_4. If w_4 is primitive, then clearly μ is a multiple of w_4. So in any case there is $k \geq 2$ with $\mu = k w_4$. Grading matrix and anticanonical class of X, due to Proposition 3.2.5, are thus given by

$$Q \ = \ \begin{bmatrix} 1 & 1 & a & a_4 & c & 0 & 0 \\ 0 & 0 & 1 & b_4 & 1 & 1 & 1 \end{bmatrix}, \qquad -\mathcal{K} \ = \ \begin{bmatrix} 2 + a + c + (1-k)a_4 \\ 4 + (1-k)b_4 \end{bmatrix}.$$

From X being Fano, ie. $\mu \in \lambda^\circ$, we infer the inequalities

$$\begin{aligned} (k-1)b_4 \ &\leq \ 3, && (3.8.3.30) \\ 0 \ &\leq \ 1 + c - 3a + (k-1)(ab_4 - a_4). && (3.8.3.31) \end{aligned}$$

By the ordering of the generator degrees we have $\det(w_4, w_5) > 0$. With this, we can rewrite Equation 3.8.3.31 to obtain

$$0 \leq (3 - (k-1)b_4)(a - c) + 1 - 2c.$$

Note that a is strictly larger than c. Thus by Equation 3.8.3.30, the right hand side of this inequality is strictly negative. A contradiction. Therfore the case $\mu \in \rho_3$ does not occur.

Case 3.8.3.5.3: $\mu \in (\rho_3 + \rho_4)^\circ$. Applying Lemma 3.2.8 to the triples (w_1, w_2, w_4) and (w_1, w_2, w_5) shows that $b_4 = b_5 = 1$ holds. Grading matrix and anticanonical class of X, due to Proposition 3.2.5, are thus given by

$$Q \ = \ \begin{bmatrix} 1 & 1 & a & b & c & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}, \qquad -\mathcal{K} \ = \ \begin{bmatrix} 2 + a + b + c - \mu_1 \\ 5 - \mu_2 \end{bmatrix}.$$

From X being Fano, ie. $\mu \in \lambda^\circ$, we infer the inequalities

$$\begin{aligned} \mu_2 \ &\leq \ 4, && (3.8.3.32) \\ \mu_1 \ &\leq \ 1 + (\mu_2 - 4)a + b + c. && (3.8.3.33) \end{aligned}$$

The position of μ yields $\mu_2 c + 1 \leq \mu_1$. Combining this with Equation 3.8.3.33, we obtain

$$0 \ \leq \ (\mu_2 - 4)a + b + (1 - \mu_2)c.$$

Having in mind the ordering of the generator degrees, the right hand side of this inequality is negative for $\mu_2 < 4$. Together with 3.8.3.32 this yields $\mu_2 = 4$. Plugging this into Equation 3.8.3.33, we obtain the bound

$$\mu_1 \ \leq \ 1 + b + c. \tag{3.8.3.34}$$

The relation g is a trinomial consisting of pairwise coprime monomials. Due to the position of μ, each monomial of g is therefore divisible by precisely one of T_5, T_6, T_7. We establish bounds for the exponents l_5, l_6, l_7. Since $\mu_2 = 4$ holds, we immediately obtain $l_5, l_6, l_7 \leq 4$. Consider the monomial m of g divisible by T_5. We write

$$m \ = \ T_1^{l_1} T_2^{l_2} T_3^{l_3} T_4^{l_4} T_5^{l_5}.$$

By homogeneity of g we have $l_3 + l_4 + l_5 = 4$. Combining this with the bound 3.8.3.34, we obtain

$$l_5 c + (4 - l_5) b \ \leq \ l_5 c + l_3 a + l_4 b \ \leq \ \mu_1 \ \leq \ 1 + b + c. \tag{3.8.3.35}$$

This inequality is only fulfilled for $l_5 = 4$. In particular, the monomial of g containing T_5 is of the form $T_1^{l_1} T_2^{l_2} T_5^4$ and we may assume that $l_1 > 0$ holds. Similarly we obtain $l_6, l_7 \geq 2$. Switching the roles of T_6 and T_7 if necessary, we may assume that $l_6 \geq l_7$ holds. Note that due to the bound 3.8.3.34, the variable T_3 appears in g with exponent at most one. In particular this yields $l_6 \geq 3$. We show that $l_7 \geq 3$ holds. Assume $l_7 = 2$. The monomial m of g containing T_7 is of the form

$$m \ = \ T_1^{l_1} T_2^{l_2} T_3^{l_3} T_4^{l_4} T_7^2,$$

where $l_3 + l_4 = 2$. With 3.8.3.34 we have

$$l_3 a + l_4 b \ \leq \ 1 + b + c.$$

This yields $l_3 = 0$ and $l_4 = 2$ as well as $b = c + 1$. Thus we can write

$$g \ = \ T_1^{l_1} T_2^{l_2} T_5^4 + T_2^{k_2} T_3^{4 - l_6} T_6^{l_6} + T_2^{m_2} T_4^2 T_7^2,$$

where at most one of l_2, k_2, m_2 is non-zero. Since $b = c + 1$ holds, the bound 3.8.3.34 now reads $\mu_1 \leq 2c + 2$. Applying this to the first monomial of g, we obtain

$$2c + l_1 \ \leq \ 2.$$

A contradiction, since l_1 and c are both positive. Thus $l_7 \geq 3$ holds. Note that l_5, l_6, l_7 cannot all be equal to four. We therefore distinguish the two cases $(l_5, l_6, l_7) = (4, 3, 3)$ and $(l_5, l_6, l_7) = (4, 4, 3)$.

Case 3.8.3.5.3.1: $(l_5, l_6, l_7) = (4, 3, 3)$. Switching roles of T_6 and T_7 if necessary, we can write

$$g \ = \ T_1^{l_1} T_5^4 + T_2^{l_2} T_3 T_6^3 + T_2^{m_2} T_4 T_7^3,$$

where $l_2 = 0$ or $m_2 = 0$. Comparing degrees of the second and third monomial shows that $l_2 = 0$ and $m_2 = a - b$ holds. Moreover we have $l_1 = a - 4c$ by homogeneity of g. Grading matrix and relation are thus given by

$$Q = \begin{bmatrix} 1 & 1 & a & b & c & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}, \qquad g = T_1^{a-4c}T_5^4 + T_3T_6^3 + T_2^{a-b}T_4T_7^3.$$

This is series S62.

Case 3.8.3.5.3.2: $(l_5, l_6, l_7) = (4, 4, 3)$. The relation g is of the form

$$g = T_1^{l_1}T_5^4 + T_2^{l_2}T_6^4 + T_3^{l_3}T_4^{l_4}T_7^3,$$

where $l_3 + l_4 = 1$. If $l_3 = 1$, then $l_4 = 0$ holds and homogeneity of g yields $l_1 = a - 4c$ and $l_2 = a$. Grading matrix and relation are thus given by

$$Q = \begin{bmatrix} 1 & 1 & a & b & c & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}, \qquad g = T_1^{a-4c}T_5^4 + T_2^aT_6^4 + T_3T_7^3.$$

To satisfy condition (C2) from Setting 3.4.1, a must be odd. This is series S63. If $l_4 = 1$, then $l_3 = 0$ holds and homogeneity of g yields $l_1 = b - 4c$ and $l_2 = b$. Grading matrix and relation are thus given by

$$Q = \begin{bmatrix} 1 & 1 & a & b & c & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}, \qquad g = T_1^{b-4c}T_5^4 + T_2^bT_6^4 + T_4T_7^3.$$

To satisfy condition (C2) from Setting 3.4.1, b must be odd. This is series S64.

Case 3.8.3.5.4: $\mu \in \rho_4$. Applying Lemma 3.2.8 to the generator degree triple (w_1, w_2, w_4) shows that $b_4 = 1$ holds. The relation degree μ and the generator degree w_5 lie on the same ray. If w_5 is not primitive, then Lemma 3.2.7 applied to the pair (w_1, w_5) shows that g contains a monomial of the form $T_5^{l_5}$. In particular, μ is a multiple of w_5. If w_5 is primitive, then clearly μ is a multiple of w_5. So in any case there is $k \geq 2$ with $\mu = kw_5$. Grading matrix and anticanonical class of X, due to Proposition 3.2.5, are thus given by

$$Q = \begin{bmatrix} 1 & 1 & a & b & a_5 & 0 & 0 \\ 0 & 0 & 1 & 1 & b_5 & 1 & 1 \end{bmatrix}, \qquad -\mathcal{K} = \begin{bmatrix} 2 + a + b + (1-k)a_5 \\ 4 + (1-k)b_5 \end{bmatrix}.$$

From X being Fano, ie. $\mu \in \lambda^\circ$, we infer the inequalities

$$(k-1)b_5 \leq 3, \tag{3.8.3.36}$$
$$0 \leq 1 - 3a + b + (k-1)(ab_5 - a_5). \tag{3.8.3.37}$$

Equation 3.8.3.36 yields $b_5 \leq 3$. We distinguish three cases, depending on the value of b_5.

Case 3.8.3.5.4.1: $b_5 = 1$. Equation 3.8.3.36 yields $k \leq 4$. Set $c := a_5$. Plugging the value for b_5 into Equation 3.8.3.37, we obtain

$$0 \ \leq \ 1 + (k-4)a + b + (1-k)c.$$

By the ordering of the generator degrees, the right hand side of this inequality is negative for $k < 4$. Thus $k = 4$ holds. Grading matrix and relation degree are given by

$$Q \ = \ \begin{bmatrix} 1 & 1 & a & b & c & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}, \qquad \mu \ = \ (4c, 4).$$

Moreover, Equation 3.8.3.37 yields the bound

$$\mu_1 \ \leq \ 1 + b + c. \tag{3.8.3.38}$$

The relation g is a trinomial consisting of pairwise coprime monomials. Due to the position of μ, each monomial of g is therefore divisible by precisely one of T_5, T_6, T_7. We have already seen that g contains the monomial T_5^4. We establish bounds for the exponents l_6 and l_7. Since $\mu_2 = 4$ holds, we immediately obtain $l_6, l_7 \leq 4$. Consider the monomial m of g divisible by T_6. We write

$$m \ = \ T_1^{l_1} T_2^{l_2} T_3^{l_3} T_4^{l_4} T_6^{l_6}.$$

By homogeneity of g we have $l_3 + l_4 + l_6 = 4$. Combining this with the bound 3.8.3.38, we obtain

$$(4 - l_6)b \ \leq \ l_3 a + l_4 b \ \leq \ \mu_1 \ \leq \ 1 + b + c.$$

This inequality is only fulfilled for $l_6 \geq 2$. Similarly we obtain $l_7 \geq 2$. Switching the roles of T_6 and T_7 if necessary, we may assume that $l_6 \geq l_7$ holds. Note that due to the bound 3.8.3.38, the variable T_3 appears in g with exponent at most one. In particular this yields $l_6 \geq 3$. We show that $l_7 \geq 3$ holds. Assume $l_7 = 2$. The monomial m of g containing T_7 is of the form

$$m \ = \ T_1^{l_1} T_2^{l_2} T_3^{l_3} T_4^{l_4} T_7^{2},$$

where $l_3 + l_4 = 2$. With 3.8.3.38 we have

$$l_3 a + l_4 b \ \leq \ 1 + b + c.$$

This yields $l_3 = 0$ and $l_4 = 2$ as well as $b = c + 1$. Thus we can write

$$g \ = \ T_5^4 + T_1^{l_1} T_2^{l_2} T_3^{4-l_6} T_6^{l_6} + T_1^{m_1} T_2^{m_2} T_4^2 T_7^2,$$

where $l_1 m_1 = 0$ and $l_2 m_2 = 0$. Since $b = c + 1$ holds, the bound 3.8.3.38 now reads $\mu_1 \leq 2c + 2$. As $\mu_1 = 4c$ holds, this yields $c = 1$. Thus the relation has degree $\mu = (4, 4)$. Comparing this to the degree of the third monomial shows $m_1 = m_2 = 0$. But then

g violates condition (C2) from Setting 3.4.1. A contradiction. Thus $l_7 \geq 3$ holds. We distinguish the following three cases:

$$(l_5, l_6, l_7) \ = \ (4,3,3), \qquad (l_5, l_6, l_7) \ = \ (4,4,3), \qquad (l_5, l_6, l_7) \ = \ (4,4,4).$$

Case 3.8.3.5.4.1.1: $(l_5, l_6, l_7) = (4,3,3)$. Switching roles of T_6 and T_7 if necessary, we write g as

$$g \ = \ T_5^4 + T_1^{l_1} T_2^{l_2} T_3 T_6^3 + T_1^{m_1} T_2^{m_2} T_4 T_7^3,$$

where $l_1 m_1 = 0$ and $l_2 m_2 = 0$. Comparing degrees of the second and third monomial shows that $m_1 + m_2 > 0$ holds. We may assume $m_2 > 0$. Then $l_2 = 0$ holds. Assume $l_1 = 0$. Then homogeneity of g yields $a = 4c$ and $m_1 = 4c - b - m_2$. Grading matrix and relation are given by

$$Q \ = \ \begin{bmatrix} 1 & 1 & 4c & b & c & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}, \qquad g \ = \ T_5^4 + T_3 T_6^3 + T_1^{4c-b-l} T_2^l T_4 T_7^3.$$

This is series S65. If $l_1 > 0$, then we have $m_1 = 0$ and homogeneity of g yields $l_1 = 4c - a$ and $l_2 = 4c - b$. Grading matrix and relation are given by

$$Q \ = \ \begin{bmatrix} 1 & 1 & a & b & c & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}, \qquad g \ = \ T_5^4 + T_1^{4c-a} T_3 T_6^3 + T_2^{4c-b} T_4 T_7^3.$$

This is series S67.

Case 3.8.3.5.4.1.2: $(l_5, l_6, l_7) = (4,4,3)$. Switching roles of T_1 and T_2 if necessary, we can write g as

$$g \ = \ T_5^4 + T_1^{l_1} T_2^{l_2} T_6^4 + T_2^{m_2} T_3^{m_3} T_4^{m_4} T_7^3,$$

where $l_1 > 0$, either $l_2 = 0$ or $m_2 = 0$ and $m_3 + m_4 = 1$ holds. To satisfy condition (C2) from Setting 3.4.1, the exponents l_1 and l_2 must both be positive and odd. Thus $m_2 = 0$ holds. In case $m_3 = 1$, we have $m_4 = 0$. Homogeneity of g yields $a = 4c$ and $l_1 = 4c - l_2$. Grading matrix and relation are given by

$$Q \ = \ \begin{bmatrix} 1 & 1 & 4c & b & c & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}, \qquad g \ = \ T_5^4 + T_1^{4c-l} T_2^l T_6^4 + T_3 T_7^3.$$

To satisfy condition (C2) from Setting 3.4.1, l must be odd. This is series S66. If $m_4 = 1$, then $m_3 = 0$ holds and homogeneity of g yields $b = 4c$ and $l_1 = 4c - l_2$.

$$Q \ = \ \begin{bmatrix} 1 & 1 & a & 4c & c & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}, \qquad g \ = \ T_5^4 + T_1^{4c-l} T_2^l T_6^4 + T_4 T_7^3.$$

To satisfy condition (C2) from Setting 3.4.1, l must be odd. This is series S68.

Case 3.8.3.5.4.1.3: $(l_5, l_6, l_7) = (4, 4, 4)$. Since $\mu_2 = 4$, the variables T_3 and T_4 do not appear in g. Up to switching the roles of T_1 and T_2, the relation g is given by

$$g \ = \ T_5^4 + T_1^{4c} T_6^4 + T_2^{4c} T_7^4.$$

This violates condition (C2) from Setting 3.4.1. Thus the case $(l_5, l_6, l_7) = (4, 4, 4)$ thus not occur.

Case 3.8.3.5.4.2: $b_5 = 2$. Equation 3.8.3.36 yields $k = 2$. Plugging the values for b_5 and k into Equation 3.8.3.37, we obtain

$$0 \ \leq \ 1 + b - a - a_5 \ < \ 0.$$

A contradiction. Thus the case $b_5 = 2$ does not occur.

Case 3.8.3.5.4.3: $b_5 = 3$. Equation 3.8.3.36 yields $k = 2$. Set $c := a_5$. Grading matrix and the relation degree are given by

$$Q \ = \ \begin{bmatrix} 1 & 1 & a & b & c & 0 & 0 \\ 0 & 0 & 1 & 1 & 3 & 1 & 1 \end{bmatrix}, \qquad \mu \ = \ (2c, 6).$$

Plugging the values for b_5 and k into Equation 3.8.3.37, we obtain the bound

$$\mu_1 \ \leq \ 1 + b + c. \tag{3.8.3.39}$$

In particular, we obtain $c \leq b + 1$. The relation g is a trinomial consisting of pairwise coprime monomials. Due to the position of μ, each monomial of g is therefore divisible by precisely one of T_5, T_6, T_7. We have already seen that g contains the monomial T_5^2. We establish bounds for the exponents l_6 and l_7. Since $\mu_2 = 6$ holds, we immediately obtain $l_6, l_7 \leq 6$. Consider the monomial m of g containing T_6. It is of the form

$$m \ = \ T_1^{l_1} T_2^{l_2} T_3^{l_3} T_4^{l_4} T_6^{l_6},$$

where $l_3 + l_4 + l_6 = 6$. Homogeneity of g together with Equation 3.8.3.39 yield

$$(6 - l_6)b \ \leq \ \mu_1 \ \leq \ 1 + b + c \ \leq \ 2b + 2.$$

This yields $l_6 \geq 2$. Similarly we obtain $l_7 \geq 2$. Switching the roles of T_6 and T_7 if necessary, we may assume that $l_6 \geq l_7$ holds. We show that $l_7 \geq 3$ holds. Assume $l_7 = 2$. Then the bound 3.8.3.39 yields $b = 1$. Moreover we obtain and $c \leq 2$. The monomial m of g containing T_7 is of the form

$$m \ = \ T_1^{l_1} T_2^{l_2} T_3^{l_3} T_4^{l_4} T_7^2$$

with $l_3 + l_4 = 4$. Thus homogeneity of g yields

$$l_3 a + 4 - l_3 \ = \ l_3 a + l_4 b \ \leq \ \mu_1 = 2c \ \leq \ 4.$$

This yields $l_3 = 0$ and $l_4 = 4$. Moreover we obtain $c = 2$. Th relation g thus contains the monomial $m = T_4^4 T_7^2$. This violates condition (C2) from Setting 3.4.1. Thus $l_7 \geq 3$ holds. Note that by the bound 3.8.3.39 the T_3 appears in g with exponent at most one. Thus we have $l_6 \geq 5$. Moreover, due to the position of μ, the case $l_6 = l_7 = 6$ cannot occur. We thus distinguish the following six cases:

$$(l_5, l_6, l_7) \;=\; (2,5,3), \qquad (l_5, l_6, l_7) \;=\; (2,5,4), \qquad (l_5, l_6, l_7) \;=\; (2,5,5),$$

$$(l_5, l_6, l_7) \;=\; (2,6,3), \qquad (l_5, l_6, l_7) \;=\; (2,6,4), \qquad (l_5, l_6, l_7) \;=\; (2,6,5).$$

Case 3.8.3.5.4.3.1: $(l_5, l_6, l_7) = (2,5,3)$. The relation g is of the form

$$g \;=\; T_5^2 + T_1^{l_1} T_2^{l_2} T_3 T_6^5 + T_1^{m_1} T_2^{m_2} T_4^3 T_7^3,$$

where $l_1 m_1 = 0$ and $l_2 m_2 = 0$. Applying the bound 3.8.3.39 to the third monomial of g, we obtain

$$m_1 + m_2 + 3b \;\leq\; \mu_1 \;\leq\; 1 + b + c \;\leq\; 2b + 2.$$

This yields $b \leq 2$. Moreover, since $c \leq b + 1$ holds, we obtain the bound $c \leq 3$. In particular, this yields $mu_1 \leq 6$. Applying this to the second monomial of g, we obtain the bound $a \leq 6$. For each triple of possible values for a, b, c we determine all homogeneous trinomials g of degree $\deg(g) = \mu$ that satisfy conditions (C1) and (C2) from Setting 3.4.1 and filter by isomorphy. Depending on the values of a, b, c we obtain the following specifying data

(a,b,c)	$(3,2,3)$	$(4,2,3)$	$(5,2,3)$	$(6,2,3)$	$(3,1,2)$	$(4,1,2)$
ID	439-440	441-442	443	444	445	446

Case 3.8.3.5.4.3.2: $(l_5, l_6, l_7) = (2,5,4)$. The relation g is of the form

$$g \;=\; T_5^2 + T_1^{l_1} T_2^{l_2} T_3 T_6^5 + T_1^{m_1} T_2^{m_2} T_4^2 T_7^4,$$

where $l_1 m_1 = 0$ and $l_2 m_2 = 0$. To satisfy condition (C2) from Setting 3.4.1, the exponents m_1 and m_2 must both be positive and odd. Thus $l_1 = l_2 = 0$ holds. By homogeneity of g we obtain $a = 2c$. Moreover, the bound 3.8.3.39, together with the inequality $c \leq b + 1$, yield $m_1 = m_2 = 1$ and $b = c - 1$. Grading matrix and relation are thus given by

$$Q \;=\; \begin{bmatrix} 1 & 1 & 2c & c-1 & c & 0 & 0 \\ 0 & 0 & 1 & 1 & 3 & 1 & 1 \end{bmatrix}, \qquad g \;=\; T_5^2 + T_3 T_6^5 + T_1 T_2 T_4^2 T_7^4.$$

This is series S69.

Case 3.8.3.5.4.3.3: $(l_5, l_6, l_7) = (2,5,5)$. Switching roles of T_6 and T_7 if necessary, we can write g as

$$g \;=\; T_5^2 + T_1^{l_1} T_2^{l_2} T_3 T_6^5 + T_1^{m_1} T_2^{m_2} T_4 T_7^5,$$

where $l_1 m_1 = 0$ and $l_2 m_2 = 0$. Comparing the degrees of the second and third monomial of g shows that $m_1 + m_2 > 0$ holds. We may assume $m_2 > 0$. Thus we have $l_2 = 0$. If $l_1 = 0$ holds, then by homogeneity of g we obtain $a = 2c$ and $m_1 = 2c - b - m_2$. Grading matrix and relation are thus given by

$$Q = \begin{bmatrix} 1 & 1 & 2c & b & c & 0 & 0 \\ 0 & 0 & 1 & 1 & 3 & 1 & 1 \end{bmatrix}, \qquad g = T_5^2 + T_3 T_6^5 + T_1^{2c-b-l} T_2^l T_4 T_7^5.$$

This is series S70. If $l_1 > 0$ holds, then we have $m_1 = 0$. By homogeneity of g we obtain $l_1 = 2c - a$ and $l_2 = 2c - b$. Grading matrix and relation are thus given by

$$Q = \begin{bmatrix} 1 & 1 & a & b & c & 0 & 0 \\ 0 & 0 & 1 & 1 & 3 & 1 & 1 \end{bmatrix}, \qquad g = T_5^2 + T_1^{2c-a} T_3 T_6^5 + T_2^{2c-b} T_4 T_7^5.$$

This is series S71.

Case 3.8.3.5.4.3.4: $(l_5, l_6, l_7) = (2, 6, 3)$. Switching roles of T_1 and T_2 if necessary, we can write g as

$$g = T_5^2 + T_1^{l_1} T_2^{l_2} T_6^6 + T_2^{m_2} T_3^{m_3} T_4^{m_4} T_7^3,$$

where $l_1 > 0$, either $l_2 = 0$ or $m_2 = 0$ and $l_3 + l_4 = 3$ holds. To satisfy condition (C2) from Setting 3.4.1, the exponents l_1 and l_2 must both be positive and odd. Thus $m_2 = 0$ holds. Applying the bound 3.8.3.39 to the third monomial of g and having in mind that $c \leq b + 1$ holds, we obtain

$$3b \leq l_3 a + l_4 b \leq \mu_1 \leq 1 + b + c \leq 2b + 2.$$

This yields $b \leq 2$ and $c \leq 3$. Note the by the bound 3.8.3.39 the exponent m_3 is at most one. If $m_3 = 0$ holds, then by homogeneity of g we have $2c = 3b$. This yields $b = 2$ and $c = 3$. Grading matrix and relation are thus given by

$$Q = \begin{bmatrix} 1 & 1 & a & 2 & 3 & 0 & 0 \\ 0 & 0 & 1 & 1 & 3 & 1 & 1 \end{bmatrix}, \qquad g = T_5^2 + T_1^{l_1} T_2^{l_2} T_6^6 + T_4^3 T_7^3,$$

where $l_1 + l_2 = 6$. To satisfy condition (C2) from Setting 3.4.1, up to switching roles of T_1 and T_2 we have $l_1 = 5$ and $l_2 = 1$. This is series S72. If $m_3 = 1$ holds, then by homogeneity of g we obtain $a + 2b = 2c \leq 2b + 2$. Thus $a = 2$ holds. We also obtain $b = 1$ and $c = 2$. Grading matrix and relation are thus given by

$$Q = \begin{bmatrix} 1 & 1 & 2 & 1 & 2 & 0 & 0 \\ 0 & 0 & 1 & 1 & 3 & 1 & 1 \end{bmatrix}, \qquad g = T_5^2 + T_1^{l_1} T_2^{l_2} T_6^6 + T_3 T_4^2 T_7^3,$$

where $l_1 + l_2 = 4$. To satisfy condition (C2) from Setting 3.4.1, up to switching roles of T_1 and T_2, we have $l_1 = 3$ and $l_2 = 1$. This is specifying data no. 447.

Case 3.8.3.5.4.3.5: $(l_5, l_6, l_7) = (2, 6, 4)$. Switching roles of T_1 and T_2 if necessary, we can write g as

$$g = T_5^2 + T_1^{l_1} T_2^{l_2} T_6^6 + T_2^{m_2} T_3^{m_3} T_4^{m_4} T_7^4,$$

where $l_1 > 0$, either $l_2 = 0$ or $m_2 = 0$ and $m_3 + m_4 = 2$ holds. To satisfy condition (C2) from Setting 3.4.1, the exponents l_1 and l_2 must both be positive and odd. Thus $m_1 = m_2 = 0$ holds. With the same argument we obtain $l_3 = l_4 = 1$. Homogeneity of g together with the bound 3.8.3.39 yields $a = c + 1$ and $b = c - 1$. Grading matrix and relation are thus given by

$$Q = \begin{bmatrix} 1 & 1 & c+1 & c-1 & c & 0 & 0 \\ 0 & 0 & 1 & 1 & 3 & 1 & 1 \end{bmatrix}, \qquad g = T_5^2 + T_1^{l_1}T_2^{l_2}T_6^6 + T_3T_4T_7^4,$$

where $l_1 + l_2 = 2c$. This is series S73.

Case 3.8.3.5.4.3.6: $(l_5, l_6, l_7) = (2, 6, 5)$. Switching roles of T_1 and T_2 if necessary, we can write g as

$$g = T_5^2 + T_1^{l_1}T_2^{l_2}T_6^6 + T_1^{m_1}T_2^{m_2}T_3^{m_3}T_4^{m_4}T_7^5,$$

where $l_1 > 0$, either $l_2 = 0$ or $m_2 = 0$ and $m_3 + m_4 = 1$ holds. To satisfy condition (C2) from Setting 3.4.1, the exponents l_1 and l_2 must both be positive and odd. Thus $m_1 = m_2 = 0$ holds. If $m_3 = 1$, then we have $m_4 = 0$. By homogeneity of g we obtain $a = 2c$ and $l_1 = 2c - l_2$. Grading matrix and relation are given by

$$Q = \begin{bmatrix} 1 & 1 & 2c & b & c & 0 & 0 \\ 0 & 0 & 1 & 1 & 3 & 1 & 1 \end{bmatrix}, \qquad g = T_5^2 + T_1^{2c-l}T_2^lT_6^6 + T_3T_7^5.$$

This is series S74. If $m_4 = 1$, then we have $m_3 = 0$. By homogeneity of g we obtain $b = 2c$. Moreover we have $l_1 = 2c - l_2$. Grading matrix and relation are given by

$$Q = \begin{bmatrix} 1 & 1 & a & 2c & c & 0 & 0 \\ 0 & 0 & 1 & 1 & 3 & 1 & 1 \end{bmatrix}, \qquad g = T_5^2 + T_1^{2c-l}T_2^lT_6^6 + T_4T_7^5.$$

This leads to series S75. □

3.9 Proof of Theorem 3.1.1: Case $s = 6$

Setting 3.4.1 and Proposition 3.4.8 divide the proof of Theorem 3.1.1 into the five cases $s = 2, \dots, 6$, according to the number of rays spanned by the degrees $w_1, \dots, w_7$. In this section we treat the case $s = 6$.

Theorem 3.9.1. *The tables from 3.10.11 provide specifying data (Q, g) for 31 infinite series of locally factorial Fano fourfolds of Picard number $\rho = 2$ and complexity $c = 1$ with a hypersurface Cox ring and $s = 6$. Moreover, any locally factorial Fano fourfold with a hypersurface Cox ring and invariants $(\rho, c, s) = (2, 1, 6)$ is isomorphic to precisely one $X(Q, g)$ with (Q, g) from these tables.*

The proof of Theorem 3.9.1 splits into two parts. First, with the tools provided in Section 3.2 we verify that each specifying data (Q, g) from the tables 3.10.11 defines a locally factorial Fano fourfold $X(Q, g)$ with a hypersurface Cox ring and invariants $(\rho, c, s) = (2, 1, 6)$. Moreover, with the help of Remark 3.4.3 we verify that distinct specifying data from the tables in 3.10.11 define pairwise non-isomorphic varieties. The second part is to show that any locally factorial Fano fourfold with a hypersurface Cox ring and invariants $(\rho, c, s) = (2, 1, 6)$ is isomorphic to $X(Q, g)$ with (Q, g) from these tables. This is done in Proposition 3.9.3.

Lemma 3.9.2. *Let X as in Setting 3.4.1. If $s = 6$, then $\mu \notin \lambda$ holds.*

Proof. By Lemma 3.4.7 we have $\mu \in \mathrm{Eff}(R)^\circ$. Thus $\mu \in (\rho_2 + \rho_5)$ holds. Assume $\mu \in \lambda$. If $\mu \in \mathrm{Mov}(R)^\circ$, then by Proposition 3.3.2 the grading matrix $Q = (w_1, \dots, w_7)$ and the relation degree μ appear in the classification list of [45, Thm. 1.1]. However, there is no entry in that list with $s = 6$. Thus we must have $\mu \in \partial \mathrm{Mov}(R)$. By the definition of the moving cone, we have $(\rho_2 + \rho_5) \subseteq \mathrm{Mov}(R)$. This means that μ lies either on the ray ρ_2 or on the ray ρ_5. Reversing ordering of the generator degrees if necessary we achieve $\mu \in \rho_2$. As ρ_2 is a bounding ray of $\mathrm{Mov}(R)$, we have $n_1 = 1$. Moreover, by Remark 3.2.11, we have $n_2 = 2$. Thus the degree constellation of X is $(n_1, \dots, n_6) = (1, 2, 1, 1, 1, 1)$. The cone λ is contained in $\mathrm{Mov}(R)$ and has μ in it's bounding ray. By [45, Prop. 2.8] no generator degree lies in the interior of λ. This means that $\lambda = \rho_2 + \rho_3$ holds. Applying Lemma 3.2.8 to the triples (w_2, w_3, w_i) for $i = 4, \dots, 7$ shows that the cones $\rho_2 + \rho_i$ are all regular. We can thus apply a unimodular transformation to achieve

$$Q \;=\; \begin{bmatrix} a_1 & a_2 & a_3 & a_4 & a_5 & a_6 & 0 \\ -b_1 & 0 & 0 & 1 & 1 & 1 & 1 \end{bmatrix}, \qquad \mu \;=\; (\mu_2, 0).$$

Note that we have $\det(w_1, w_4) > 1$ as well as $\det(w_1, w_5) > 1$. Thus by Lemma 3.2.7, the relation g contains monomials of the form $T_1^{l_1} T_4^{l_4}$ and $T_1^{m_1} T_5^{m_5}$. By homogeneity of g and the position of μ, the exponents l_1, l_4, m_1, m_5 are all positive. This violates condition (C1) from Setting 3.4.1. Thus $\mu \notin \lambda$ holds. $\square$

Proposition 3.9.3. *Let X as in Setting 3.4.1 with $s = 6$. Then X is isomorphic to an $X(Q, g)$ with specifying data (Q, g) appearing in Classification list 3.10.11.*

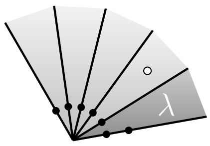
Proof. By Lemma 3.9.2 the relation degree μ is not contained in λ. We are thus in the situation of Lemma 3.4.6, which tells us that either λ^+ or λ^- is one-dimensional. By reversing the order of the variables if necessary, we may assume that $\lambda = \rho_1 + \rho_2$ holds. Moreover, we have $n_1 \geq 2$ and all generator degrees contained in ρ_1 are primitive. Since there are six rays and seven generator degrees, this already fixes the degree constellation of X to be $(n_1, \dots, n_6) = (2, 1, 1, 1, 1, 1)$. By Lemma 3.4.7, μ is contained in the interior of $\mathrm{Eff}(R)$. Thus applying Lemma 3.2.8 to the triple (w_1, w_2, w_7) shows that the cone $\mathrm{Eff}(R)$ is effective and that w_7 is primitive. We may thus assume that $\mathrm{Eff}(R)$ is the positive quadrant and that

$$w_1 \;=\; w_2 \;=\; (1, 0), \qquad w_7 \;=\; (0, 1)$$

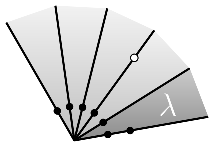
holds. By Remark 3.2.11 we have $\mu \in (\rho_2 + \rho_4) \backslash \rho_2$. Applying Lemma 3.2.8 to the triples (w_1, w_2, w_3) and (w_1, w_2, w_6) shows that the cones $\rho_1 + \rho_2$ and $\rho_1 + \rho_5$ are regular and that w_3 and w_6 are primitive. The grading matrix of X is thus of the form

$$Q = \begin{bmatrix} 1 & 1 & a & a_4 & a_5 & d & 0 \\ 0 & 0 & 1 & b_4 & b_5 & 1 & 1 \end{bmatrix}, \qquad a, a_4, a_5, b_4, b_5, d \in \mathbb{Z}_{\geq 1}.$$

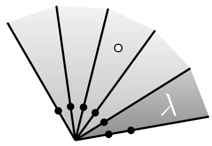
There are four possible positions of μ, displayed in the following pictures.



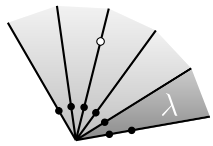
$\mu \in (\rho_2 + \rho_3)^\circ$



$\mu \in \rho_3$



$\mu \in (\rho_3 + \rho_4)^\circ$



$\mu \in \rho_4$

The black dots represent the generator degrees $w_1, \ldots, w_7$, the white circle represents the relation degree μ. We distinguish four cases, according to the position of μ.

Case 3.9.3.1: $\mu \in (\rho_2 + \rho_3)^\circ$. Applying Lemma 3.2.8 to the triples (w_1, w_2, w_4) and (w_1, w_2, w_5) shows that $b_4 = b_5 = 1$ holds. Set $b := a_4$ and $c := a_5$. Grading matrix and anticanonical class of X, due to Proposition 3.2.5, are given by

$$Q = \begin{bmatrix} 1 & 1 & a & b & c & d & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}, \qquad -\mathcal{K} = \begin{bmatrix} 2 + a + b + c + d - \mu_1 \\ 5 - \mu_2 \end{bmatrix}.$$

From X being Fano, ie. $\mu \in \lambda^\circ$, we infer the inequalities

$$\mu_2 \leq 4, \tag{3.9.3.1}$$
$$\mu_1 \leq 1 + (\mu_2 - 4)a + b + c + d. \tag{3.9.3.2}$$

The position of μ yields $\mu_2 b + 1 \leq \mu_1$. Combining this with Equation 3.9.3.2, we obtain

$$0 \leq (\mu_2 - 4)a + (1 - \mu_2)b + c + d.$$

By the ordering of the generator degrees, the right hand side of this inequality is negative for $\mu_2 \leq 4$. This is a contradiction to Equation 3.9.3.1. Thus the case $\mu \in (\rho_2 + \rho_3)^\circ$ does not occur.

Case 3.9.3.2: $\mu \in \rho_3$. Applying Lemma 3.2.8 to the triple (w_1, w_2, w_5) shows that $b_5 = 1$ holds. The relation degree μ and the generator degree w_4 lie on the same ray. If w_4 is not primitive, then Lemma 3.2.7 applied to the pair (w_1, w_4) shows that g contains a monomial of the form $T_4^{l_4}$. In particular, μ is a multiple of w_4. If w_4 is primitive, then clearly μ is a multiple of w_4. So in any case there is $k \geq 2$ with $\mu = k w_4$. Set $c := a_5$. Grading matrix and anticanonical class of X, due to Proposition 3.2.5, are given by

$$Q = \begin{bmatrix} 1 & 1 & a & a_4 & c & d & 0 \\ 0 & 0 & 1 & b_4 & 1 & 1 & 1 \end{bmatrix}, \qquad -\mathcal{K} = \begin{bmatrix} 2 + a + (1 - k)a_4 + c + d \\ 4 + (1 - k)b_4 \end{bmatrix}.$$

From X being Fano, ie. $\mu \in \lambda^{\circ}$, we infer the inequalities

$$\begin{aligned} (k-1)b_4 &\leq 3, && (3.9.3.3)\\ 0 &\leq 1 + ((k-1)b_4 - 3)a + (1-k)a_4 + c + d. && (3.9.3.4) \end{aligned}$$

By Equation 3.9.3.3 we have $b_4 \leq 3$. We distinguish three cases, depending on the value of b_4.

Case 3.9.3.2.1: $b_4 = 1$. Plugging the value for b_4 into Equation 3.9.3.4, we obtain

$$0 \leq 1 + (k-4)a + (1-k)a_4 + c + d.$$

By the ordering of the generator degrees, the right hand side of this inequality is negative for $k \leq 4$. This is a contradiction to Equation 3.9.3.3. Thus the case $b_4 = 1$ does not occur.

Case 3.9.3.2.2: $b_4 = 2$. By Equation 3.9.3.3 we have $k = 2$. Plugging the values for b_4 and k into Equation 3.9.3.4, we obtain

$$0 \leq 1 - a - a_4 + c + d.$$

By the ordering of the generator degrees, the right hand side of this inequality is negative. A contradiction. Thus the case $b_4 = 2$ does not occur.

Case 3.9.3.2.3: $b_4 = 3$. By Equation 3.9.3.3 we have $k = 2$. Plugging the values for b_4 and k into Equation 3.9.3.4, we obtain

$$0 \leq 1 - 2a_4 + c + d.$$

By the ordering of the generator degrees, the right hand side of this inequality is negative. A contradiction. Thus the case $b_4 = 3$ does not occur.

Case 3.9.3.3: $\mu \in (\rho_3 + \rho_4)^{\circ}$. Applying Lemma 3.2.8 to the triples (w_1, w_2, w_4) and (w_1, w_2, w_5) shows that $b_4 = b_5 = 1$ holds. Set $b := a_4$ and $c := a_5$. Grading matrix and anticanonical class of X, due to Proposition 3.2.5, are given by

$$Q = \begin{bmatrix} 1 & 1 & a & b & c & d & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}, \qquad -\mathcal{K} = \begin{bmatrix} 2 + a + b + c + d - \mu_1 \\ 5 - \mu_2 \end{bmatrix}.$$

From X being Fano, ie. $\mu \in \lambda^{\circ}$, we infer the inequalities

$$\begin{aligned} \mu_2 &\leq 4, && (3.9.3.5)\\ \mu_1 &\leq 1 + (\mu_2 - 4)a + b + c + d. && (3.9.3.6) \end{aligned}$$

The position of μ yields $\mu_2 c + 1 \leq \mu_1$. Combining this with Equation 3.9.3.6, we obtain

$$0 \leq (\mu_2 - 4)a + b + (1 - \mu_2)c + d.$$

The right hand side of the inequality is negative for $\mu_2 < 4$. Thus, by Equation 3.9.3.5, we obtain $\mu_2 = 4$. Plugging this into Equation 3.9.3.2, we obtain the bound

$$\mu_1 \ \leq \ 1 + b + c + d. \tag{3.9.3.7}$$

The relation g is a trinomial consisting of pairwise coprime monomials. Due to the position of μ, each monomial of g is thus divisible by precisely one of T_5, T_6, T_7. We establish bounds on the exponents l_5, l_6, l_7. Since $\mu_2 = 4$ holds, we immediately obtain $l_5, l_6, l_7 \leq 4$. Consider the monomial m of g containing T_5. It is of the form

$$m \ = \ T_1^{l_1} T_2^{l_2} T_3^{l_3} T_4^{l_4} T_5^{l_5}.$$

Homogeneity of g yields $l_3 + l_4 + l_5 = 4$ and $l_1 + l_2 + l_3 a + l_4 b + l_5 c = \mu_1$. Combining this with the bound 3.9.3.7, we obtain

$$0 \leq \ 1 + (l_5 - 3)b + (1 - l_5)c + d.$$

The right hand side of this inequality is negative for $l_5 \leq 3$. Thus $l_5 = 4$ holds. This also yields $l_3 = l_4 = 0$. The monomial m is therefore of the form $m = T_1^{l_1} T_2^{l_2} T_5^4$ with $l_1 + l_2 > 0$. We may assume $l_1 > 0$. So the variable T_1 does not appear in the other two monomials of g. Similarly we obtain $l_6 \geq 3$ and $l_7 \geq 2$. We show that $l_7 \geq 3$ holds. Assume $l_7 = 2$. For the monomial m of g containing T_7 we write

$$m \ = \ T_2^{l_2} T_3^{l_3} T_4^{l_4} T_7^2.$$

By homogeneity of g we have $l_3 + l_4 = 2$. With the bound 3.9.3.7 we obtain

$$2b \ \leq \ l_3 a + l_4 b \ \leq \ \mu_1 \ \leq \ 1 + b + c + d \ \leq \ b + 2c.$$

Thus we have $b \leq 2c$. Applying the bound 3.9.3.7 to the monomial of g containing T_5, we obtain

$$4c \ < \ l_1 + l_2 + 4c \ = \ \mu_1 \ \leq \ 1 + b + c + d \ \leq \ b + 2c.$$

Thus $b > 2c$ holds. A contradiction. Thus we must have $l_7 \geq 3$. Note that due to the location of μ, the case $l_5 = l_6 = l_7 = 4$ does not occur. We now distinguish the following three cases:

$$(l_5, l_6, l_7) \ = \ (4,3,3), \qquad (l_5, l_6, l_7) \ = \ (4,3,4), \qquad (l_5, l_6, l_7) \ = \ (4,4,3).$$

Case 3.9.3.3.1: $(l_5, l_6, l_7) = (4,3,3)$. The relation g is of the form

$$g \ = \ T_1^{l_1} T_2^{l_2} T_5^4 + T_2^{k_2} T_3^{k_3} T_4^{k_4} T_6^3 + T_2^{m_2} T_3^{m_3} T_4^{m_4} T_7^3,$$

where at most one of l_2, k_2, m_2 is non-zero and the other exponents in g satisfy the following conditions:

$$l_1 \ > \ 1, \qquad k_3 + k_4 \ = \ 1, \qquad m_3 + m_4 \ = \ 1, \qquad k_3 m_3 \ = \ 1, \qquad k_4 m_4 \ = \ 1.$$

We distinguish two cases, depending on the value of l_3.

Case 3.9.3.3.1.1: $l_3 = 1$. We have $m_3 = 0$, $l_4 = 0$ and $m_4 = 1$. Comparing degrees of the second and third monomial of g shows that $m_2 > 0$ holds. Thus we have $l_2 = k_2 = 0$. Grading matrix and relation are given by

$$Q \ = \ \begin{bmatrix} 1 & 1 & a & b & c & d & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}, \qquad g \ = \ T_1^{a-4c+3d}T_5^4 + T_3T_6^3 + T_2^{a-b+3d}T_4T_7^3.$$

This is series S76.

Case 3.9.3.3.1.2: $l_3 = 0$. We have $m_3 = 1$, $l_4 = 1$ and $m_4 = 0$. The relation g is of the form

$$g \ = \ T_1^{l_1}T_2^{l_2}T_5^4 + T_2^{k_2}T_4T_6^3 + T_2^{m_2}T_3T_7^3.$$

We further distinguish three cases, depending on the values of l_2, k_2, m_2.

Case 3.9.3.3.1.2.1: $l_2 > 0$. We have $k_2 = m_2 = 0$. Using homogeneity of g, we obtain $a = b + 3d$. Grading matrix and relation are given by

$$Q \ = \ \begin{bmatrix} 1 & 1 & b+3d & b & c & d & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}, \qquad g \ = \ T_1^{b-4c+3d-l}T_2^lT_5^4 + T_4T_6^3 + T_3T_7^3.$$

This is series S78.

Case 3.9.3.3.1.2.2: $k_2 > 0$. We have $l_2 = m_2 = 0$. Using homogeneity of g, we obtain $a = b + 3d$. Grading matrix and relation are given by

$$Q \ = \ \begin{bmatrix} 1 & 1 & a & b & c & d & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}, \qquad g \ = \ T_1^{a-4c}T_5^4 + T_2^{a-b-3d}T_4T_6^3 + T_3T_7^3.$$

This is series S79.

Case 3.9.3.3.1.2.3: $m_2 > 0$. We have $l_2 = k_2 = 0$. Using homogeneity of g, we obtain $a = b + 3d$. Grading matrix and relation are given by

$$Q \ = \ \begin{bmatrix} 1 & 1 & a & b & c & d & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}, \qquad g \ = \ T_1^{b-4c+3d}T_5^4 + T_4T_6^3 + T_2^{b+3d-a}T_3T_7^3.$$

This is series S81.

Case 3.9.3.3.2: $(l_5, l_6, l_7) = (4, 3, 4)$. The relation g is of the form

$$g \ = \ T_1^{l_1}T_5^4 + T_3^{l_3}T_4^{l_4}T_6^3 + T_2^{l_2}T_7^4,$$

where $l_3 + l_4 = 1$. We distinguish two cases, depending on the values of l_3 and l_4.

Case 3.9.3.3.2.1: $l_3 = 1$. We have $l_4 = 0$. Homogeneity of g yields $l_1 = a + 3d - 4c$ and $l_2 = a + 3d$. Grading matrix and relation are given by

$$Q = \begin{bmatrix} 1 & 1 & a & b & c & d & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}, \qquad g = T_1^{a-4c+3d}T_5^4 + T_3T_6^3 + T_2^{a+3d}T_7^4.$$

This is series S77.

Case 3.9.3.3.2.2: $l_4 = 1$. We have $l_3 = 0$. Homogeneity of g yields $l_1 = b + 3d - 4c$ and $l_2 = b + 3d$. Grading matrix and relation are given by

$$Q = \begin{bmatrix} 1 & 1 & a & b & c & d & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}, \qquad g = T_1^{b-4c+3d}T_5^4 + T_4T_6^3 + T_2^{b+3d}T_7^4.$$

This is series S82.

Case 3.9.3.3.3: $(l_5, l_6, l_7) = (4, 4, 3)$. The relation g is of the form

$$g = T_1^{l_1}T_5^4 + T_2^{l_2}T_6^4 + T_3^{l_3}T_4^{l_4}T_7^3,$$

where $l_3 + l_4 = 1$. We distinguish two cases, depending on the values of l_3 and l_4.

Case 3.9.3.3.3.1: $l_3 = 1$. We have $l_4 = 0$. Homogeneity of g yields $l_1 = a - 4c$ and $l_2 = a - 4d$. Grading matrix and relation are given by

$$Q = \begin{bmatrix} 1 & 1 & a & b & c & d & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}, \qquad g = T_1^{a-4c}T_5^4 + T_2^{a-4d}T_6^4 + T_3T_7^3.$$

This is series S80.

Case 3.9.3.3.3.2: $l_4 = 1$. We have $l_3 = 0$. Homogeneity of g yields $l_1 = b - 4c$ and $l_2 = b - 4d$. Grading matrix and relation are given by

$$Q = \begin{bmatrix} 1 & 1 & a & b & c & d & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}, \qquad g = T_1^{b-4c}T_5^4 + T_2^{b-4d}T_6^4 + T_4T_7^3.$$

This is series S83.

Case 3.9.3.4: $\mu \in \rho_4$. Applying Lemma 3.2.8 to the triple (w_1, w_2, w_4) shows that $b_4 = 1$ holds. The relation degree μ and the generator degree w_5 lie on the same ray. If w_5 is not primitive, then Lemma 3.2.7 applied to the pair (w_1, w_5) shows that g contains a monomial of the form $T_5^{l_5}$. In particular, μ is a multiple of w_5. If w_5 is primitive, then clearly μ is a multiple of w_5. So in any case there is $k \geq 2$ with $\mu = kw_5$. Set $b := a_4$. Grading matrix and anticanonical class of X, due to Proposition 3.2.5, are given by

$$Q = \begin{bmatrix} 1 & 1 & a & b & a_5 & d & 0 \\ 0 & 0 & 1 & 1 & b_5 & 1 & 1 \end{bmatrix}, \qquad -\mathcal{K} = \begin{bmatrix} 2 + a + b + (1-k)a_5 + d \\ 4 + (1-k)b_5 \end{bmatrix}.$$

From X being Fano, ie. $\mu \in \lambda^\circ$, we infer the inequalities

$$\begin{aligned} (k-1)b_5 &\leq 3, && (3.9.3.8) \\ 0 &\leq 1 + ((k-1)b_5 - 3)a + b - (k-1)a_5 + d. && (3.9.3.9) \end{aligned}$$

Equation 3.9.3.8 yields $b_5 \leq 3$. We distinguish three cases, depending on the value of b_5.

Case 3.9.3.4.1: $b_5 = 1$. Equation 3.9.3.8 yields $k \leq 4$. Plugging the value for b_5 into Equation 3.9.3.9, we obtain

$$0 \leq 1 + (k-4)a + b - (k-1)a_5 + d.$$

The right hand side is negative for $k \leq 3$. Thus $k = 4$ holds. Set $c := a_5$. Grading matrix and relation degree are given by

$$Q = \begin{bmatrix} 1 & 1 & a & b & c & d & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}, \qquad \mu = (4c, 4).$$

From Equation 3.9.3.9, we obtain the bound

$$\mu_1 \leq 1 + b + c + d. \qquad (3.9.3.10)$$

The relation g is a trinomial consisting of pairwise coprime monomials. Due to the position of μ, each monomial of g is thus divisible by precisely one of T_5, T_6, T_7. The relation g contains the monomial T_5^4. The other two monomials are each divisible by one of T_6, T_7. We establish bounds on the exponents l_6 and l_7. Since $\mu_2 = 4$ holds, we immediately obtain $l_6, l_7 \leq 4$. Consider the monomial m of g containing T_6. It is of the form

$$m = T_1^{l_1} T_2^{l_2} T_3^{l_3} T_4^{l_4} T_6^{l_6}.$$

By homogeneity of g we have $l_3 + l_4 + l_6 = 4$ and $l_1 + l_2 + l_3 a + l_4 b + l_6 d = \mu_1$. Combining this with the bound 3.9.3.10, we obtain

$$0 \leq 1 + (l_6 - 3)b + c + (1 - l_6)d.$$

The right hand side of this inequality is negative for $l_6 \leq 2$. Thus $l_6 \geq 3$ holds. Similarly we obtain $l_7 \geq 2$. We show that $l_7 \geq 3$ holds. Assume $l_7 = 2$. For the monomial m of g containing T_7 we have

$$m = T_1^{l_1} T_2^{l_2} T_3^{l_3} T_4^{l_4} T_7^{2},$$

where $l_3 + l_4 = 2$. Applying the bound 3.9.3.10 to the degree of m, we obtain $b \geq 2c$. On the other hand, the bound 3.9.3.10 applied to the degree of T_5^4, we obtain $b \geq 2c$. Thus $b = 2c$ holds. This yields $l_4 = 2$ and $l_1 = l_2 = l_3 = 0$. The relation g thus contains the monomials T_5^4 and $T_4^2 T_7^2$. This violates condition (C2) from 3.4.1. Thus $l_7 \geq 3$ holds. Note that due to condition (C2) from Setting 3.4.1 the case $l_5 = l_6 = l_7 = 4$ cannot occur. We thus distinguish the following three cases:

$$(l_5, l_6, l_7) = (4, 3, 3), \qquad (l_5, l_6, l_7) = (4, 3, 4), \qquad (l_5, l_6, l_7) = (4, 4, 3).$$

Case 3.9.3.4.1.1: $(l_5, l_6, l_7) = (4, 3, 3)$. The relation g is of the form

$$g \;=\; T_5^4 + T_1^{l_1} T_2^{l_2} T_3^{l_3} T_4^{l_4} T_6^3 + T_1^{m_1} T_2^{m_2} T_3^{m_3} T_4^{m_4} T_7^3,$$

where the exponents in g satisfy

$$l_1 m_1 = 0, \quad l_2 m_2 = 0, \quad l_3 m_3 = 0, \quad l_4 m_4 = 0, \quad l_3 + l_4 = 1, \quad m_3 + m_4 = 1.$$

We distinguish two cases, depending on the value of T_3.

Case 3.9.3.4.1.1.1: $l_3 = 1$. We have $l_4 = m_3 = 0$ and $m_4 = 1$. The relation g is of the form

$$g \;=\; T_5^4 + T_1^{l_1} T_2^{l_2} T_3 T_6^3 + T_1^{m_1} T_2^{m_2} T_4 T_7^3.$$

Comparing the degrees of the second and third monomial of g shows that $m_1 + m_2 > 0$ holds. We may assume $m_2 > 0$. So we have $l_2 = 0$. If $l_1 = 0$, then homogeneity of g yields $a = 4c - 3d$ and $m_1 = 4c - b - l_2$. Grading matrix and relation are given by

$$Q \;=\; \begin{bmatrix} 1 & 1 & 4c-3d & b & c & d & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}, \qquad g \;=\; T_5^4 + T_3 T_6^3 + T_1^{4c-b-l} T_2^l T_4 T_7^3.$$

This is series S84. If $l_1 > 0$, then $m_1 = 0$ holds. Homogeneity of g yields $l_1 = 4c - a - 3d$ and $m_2 = 4c - b$. Grading matrix and relation are given by

$$Q \;=\; \begin{bmatrix} 1 & 1 & a & b & c & d & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}, \qquad g \;=\; T_5^4 + T_1^{4c-a-3d} T_3 T_6^3 + T_2^{4c-b} T_4 T_7^3.$$

This is series S86.

Case 3.9.3.4.1.1.2: $l_3 = 0$. We have $l_4 = m_3 = 1$ and $m_4 = 0$. The relation g is of the form

$$g \;=\; T_5^4 + T_1^{l_1} T_2^{l_2} T_4 T_6^3 + T_1^{m_1} T_2^{m_2} T_3 T_7^3.$$

We distinguish four cases, depending on the values of $l_1 + l_2$ and $m_1 + m_2$.

Case 3.9.3.4.1.1.2.1: $l_1 = l_2 = m_1 = m_2 = 0$. Homogeneity of g yields $b = 4c - 3d$ and $a = 4c$. Grading matrix and relation are given by

$$Q \;=\; \begin{bmatrix} 1 & 1 & 4c & 4c-3d & c & d & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}, \qquad g \;=\; T_5^4 + T_4 T_6^3 + T_3 T_7^3.$$

This is series S88.

Case 3.9.3.4.1.1.2.2: $l_1 + l_2 > 0$, $m_1 = m_2 = 0$. Homogeneity of g yields $l_1 = 4c - b - 3d - l_2$ and $a = 4c$. Grading matrix and relation are given by

$$Q \;=\; \begin{bmatrix} 1 & 1 & 4c & b & c & d & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}, \qquad g \;=\; T_5^4 + T_1^{4c-b-3d-l} T_2^l T_4 T_6^3 + T_3 T_7^3.$$

This is series S89.

Case 3.9.3.4.1.1.2.3: $l_1 = l_2 = 0$, $m_1 + m_2 > 0$. Homogeneity of g yields $b = 4c - 3d$ and $m_1 = 4c - a - m_2$. Grading matrix and relation are given by

$$Q \;=\; \begin{bmatrix} 1 & 1 & a & 4c-3d & c & d & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}, \qquad g \;=\; T_5^4 + T_4 T_6^3 + T_1^{4c-a-l} T_2^l T_3 T_7^3.$$

This is series S91.

Case 3.9.3.4.1.1.2.4: $l_1 + l_2 > 0$, $m_1 + m_2 > 0$. Switching the roles of T_1 and T_2 if necessary, we may assume $l_1 > 0$ and $m_2 > 0$. So we have $l_2 = m_1 = 0$. Homogeneity of g yields $l_1 = 4c - b - 3d$ and $m_2 = 4c - a$. Grading matrix and relation are given by

$$Q \;=\; \begin{bmatrix} 1 & 1 & a & b & c & d & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}, \qquad g \;=\; T_5^4 + T_1^{4c-b-3d} T_4 T_6^3 + T_2^{4c-a} T_3 T_7^3.$$

This is series S87.

Case 3.9.3.4.1.2: $(l_5, l_6, l_7) = (4, 3, 4)$. The relation g is of the form

$$g \;=\; T_5^4 + T_1^{l_1} T_2^{l_2} T_3^{l_3} T_4^{l_4} T_6^3 + T_1^{m_1} T_2^{m_2} T_7^4,$$

where $l_1 m_1 = 0$, $l_2 m_2 = 0$ and $l_3 + l_4 = 1$. To satisfy condition (C2) from Setting 3.4.1, the exponents m_1 and m_2 must both be positive and odd. Thus $l_1 = l_2 = 0$ holds. If $l_3 = 1$, then $l_4 = 0$ holds. Homogeneity of g yields $a = 4c - 3d$ and $m_1 = 4c - m_2$. Grading matrix and relation are given by

$$Q \;=\; \begin{bmatrix} 1 & 1 & 4c-3d & b & c & d & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}, \qquad g \;=\; T_5^4 + T_3 T_6^3 + T_1^{4c-l} T_2^l T_7^4.$$

This is series S85. If $l_4 = 1$, then $l_3 = 0$ holds. Homogeneity of g yields $b = 4c - 3d$ and $m_1 = 4c - m_2$. Grading matrix and relation are given by

$$Q \;=\; \begin{bmatrix} 1 & 1 & a & 4c-3d & c & d & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}, \qquad g \;=\; T_5^4 + T_4 T_6^3 + T_1^{4c-l} T_2^l T_7^4.$$

This is series S92.

Case 3.9.3.4.1.3: $(l_5, l_6, l_7) = (4, 4, 3)$. The relation g is of the form

$$g \;=\; T_5^4 + T_1^{l_1} T_2^{l_2} T_6^4 + T_1^{m_1} T_2^{m_2} T_3^{m_3} T_4^{m_4} T_7^3,$$

where $l_1 m_1 = 0$, $l_2 m_2 = 0$ and $m_3 + m_4 = 1$. To satisfy condition (C2) from Setting 3.4.1, the exponents l_1 and l_2 must both be positive and odd. Thus $m_1 = m_2 = 0$ holds. If

$m_3 = 1$, then $m_4 = 0$ holds. Homogeneity of g yields $a = 4c$ and $l_1 = 4c - 4d - l_2$. Grading matrix and relation are given by

$$Q = \begin{bmatrix} 1 & 1 & 4c & b & c & d & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}, \qquad g = T_5^4 + T_1^{4c-4d-l} T_2^l T_6^4 + T_3 T_7^3.$$

This is series S90. If $m_4 = 1$, then $m_3 = 0$ holds. Homogeneity of g yields $b = 4c$ and $l_1 = 4c - 4d - l_2$. Grading matrix and relation are given by

$$Q = \begin{bmatrix} 1 & 1 & a & 4c & c & d & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}, \qquad g = T_5^4 + T_1^{4c-4d-l} T_2^l T_6^4 + T_4 T_7^3.$$

This is series S93.

Case 3.9.3.4.2: $b_5 = 2$. Equation 3.9.3.8 yields $k = 2$. Plugging the values for b_5 and k into Equation 3.9.3.9, we obtain

$$0 \leq 1 - a + b - a_5 + d$$

By the ordering of the generator degrees, the right hand side is negative. A contradiction. Thus the case $b_5 = 2$ does not occur.

Case 3.9.3.4.3: $b_5 = 3$. Equation 3.9.3.8 yields $k = 2$. Set $c := a_5$. Grading matrix and relation degree are given by

$$Q = \begin{bmatrix} 1 & 1 & a & b & c & d & 0 \\ 0 & 0 & 1 & 1 & 3 & 1 & 1 \end{bmatrix}, \qquad \mu = (2c, 6).$$

Plugging the values for b_5 and k into Equation 3.9.3.9, we obtain the bound

$$\mu_1 \leq 1 + b + c + d. \tag{3.9.3.11}$$

The relation g is a trinomial consisting of pairwise coprime monomials. Due to the position of μ, each monomial of g is thus divisible by precisely one of T_5, T_6, T_7. The relation g contains the monomial T_5^2. The other two monomials are each divisible by one of T_6, T_7. We establish bounds on the exponents l_6 and l_7. Since $\mu_2 = 6$ holds, we immediately obtain $l_6, l_7 \leq 6$. Consider the monomial m of g containing T_6. It is of the form

$$m = T_1^{l_1} T_2^{l_2} T_3^{l_3} T_4^{l_4} T_6^{l_6}.$$

Homogeneity of g yields $l_3 + l_4 + l_6 = 6$ and $l_1 + l_2 + l_3 a + l_4 b + l_6 d = \mu_1$. Combining this with the bound 3.9.3.11, we obtain

$$0 \leq 1 + (l_6 - 5)b + c + (1 - l_6)d.$$

The right hand side of this inequality is negative for $l_6 < 5$. Thus $l_6 \geq 5$ holds. Similarly we obtain $l_7 \geq 4$. Note that by condition (C2) from Setting 3.4.1 the case $l_6 = l_7 = 6$ cannot occur. We thus distinguish the followingh five cases:

$$(l_5, l_6, l_7) = (2, 5, 4), \qquad (l_5, l_6, l_7) = (2, 5, 5), \qquad (l_5, l_6, l_7) = (2, 5, 6),$$

$$(l_5, l_6, l_7) \;=\; (2,6,4), \qquad (l_5, l_6, l_7) \;=\; (2,6,5).$$

Case 3.9.3.4.3.1: $(l_5, l_6, l_7) = (2,5,4)$. The relation g is of the form

$$g \;=\; T_5^2 + T_1^{l_1}T_2^{l_2}T_3^{l_3}T_4^{l_4}T_6^5 + T_1^{m_1}T_2^{m_2}T_3^{m_3}T_4^{m_4}T_7^4,$$

where the exponents in g satisfy the following conditions:

$$l_3 + l_4 \;=\; 1, \quad m_3 + m_4 \;=\; 2, \quad l_1m_1 = 0, \quad l_2m_2 = 0, \quad l_3m_3 = 0, \quad l_4m_4 = 0.$$

Thus either $m_3 = 2$ or $m_4 = 2$ holds. To satisfy condition (C2) from Setting 3.4.1, the exponents m_1 and m_2 must be positive and odd. Thus $l_1 = l_2 = 0$ holds. We distinguish two cases, depending on the values of m_3 and m_4.

Case 3.9.3.4.3.1.1: $m_3 = 2$, $m_4 = 0$. We have $l_4 = 1$ and $l_3 = 0$. Homogeneity of g yields $b = 2c - 5d$ and $m_1 = 2c - 2a - m_2$. Grading matrix and relation are given by

$$Q \;=\; \begin{bmatrix} 1 & 1 & a & 2c-5d & c & d & 0 \\ 0 & 0 & 1 & 1 & 3 & 1 & 1 \end{bmatrix}, \qquad g \;=\; T_5^2 + T_4T_6^5 + T_1^{2c-2a-l}T_2^lT_3^2T_7^4.$$

This is series S94.

Case 3.9.3.4.3.1.2: $m_3 = 0$, $m_4 = 2$. We have $l_3 = 1$ and $l_4 = 0$. Homogeneity of g yields $a = 2c - 5d$ and $m_1 = 2c - 2b - m_2$. Grading matrix and relation are given by

$$Q \;=\; \begin{bmatrix} 1 & 1 & 2c-5d & b & c & d & 0 \\ 0 & 0 & 1 & 1 & 3 & 1 & 1 \end{bmatrix}, \qquad g \;=\; T_5^2 + T_3T_6^5 + T_1^{2c-2b-l}T_2^lT_4^2T_7^4.$$

This is series S97.

Case 3.9.3.4.3.2: $(l_5, l_6, l_7) = (2,5,5)$. The relation g is of the form

$$g \;=\; T_5^2 + T_1^{l_1}T_2^{l_2}T_3^{l_3}T_4^{l_4}T_6^5 + T_1^{m_1}T_2^{m_2}T_3^{m_3}T_4^{m_4}T_7^5,$$

where the exponents in g satisfy the following conditions:

$$l_3 + l_4 \;=\; 1, \quad m_3 + m_4 \;=\; 1, \quad l_1m_1 = 0, \quad l_2m_2 = 0, \quad l_3m_3 = 0, \quad l_4m_4 = 0.$$

We distinguish two cases, depending on the value of l_3.

Case 3.9.3.4.3.2.1: $l_3 = 1$. We have $l_4 = m_3 = 0$ and $m_4 = 1$. The relation g is of the form

$$g \;=\; T_5^2 + T_1^{l_1}T_2^{l_2}T_3T_6^5 + T_1^{m_1}T_2^{m_2}T_4T_7^5.$$

By homogeneity of g we have $m_1 + m_2 > 0$. We may assume that $m_2 > 0$ holds. So we have $l_2 = 0$. If $l_1 = 0$, then homogeneity of g yields $a = 2c - 5d$ and $m_1 = 2c - b - m_2$. Grading matrix and relation are thus given by

$$Q \;=\; \begin{bmatrix} 1 & 1 & 2c-5d & b & c & d & 0 \\ 0 & 0 & 1 & 1 & 3 & 1 & 1 \end{bmatrix}, \qquad g \;=\; T_5^2 + T_3T_6^5 + T_1^{2c-b-l}T_2^lT_4T_7^5.$$

This is series S98. If $l_1 > 0$, then we have $m_1 = 0$. Homogeneity of g yields $l_1 = 2c-a-5d$ and $m_2 = 2c-a$. Grading matrix and relation degree are thus given by

$$Q = \begin{bmatrix} 1 & 1 & a & b & c & d & 0 \\ 0 & 0 & 1 & 1 & 3 & 1 & 1 \end{bmatrix}, \qquad g = T_5^2 + T_1^{2c-a-5d}T_3T_6^5 + T_2^{2c-b}T_4T_7^5.$$

This is series S100.

Case 3.9.3.4.3.2.2: $l_3 = 0$. We have $l_4 = m_3 = 1$ and $m_4 = 0$. The relation g is of the form

$$g = T_5^2 + T_1^{l_1}T_2^{l_2}T_4T_6^5 + T_1^{m_1}T_2^{m_2}T_3T_7^5.$$

We distinguish four cases, depending on the values of $l_1 + l_2$ and $m_1 + m_2$.

Case 3.9.3.4.3.2.2.1: $l_1 = l_2 = m_1 = m_2 = 0$. Homogeneity of g yields $b = 2c-5d$ and $a = 2c$. Grading matrix and relation are given by

$$Q = \begin{bmatrix} 1 & 1 & 2c & 2c-5d & c & d & 0 \\ 0 & 0 & 1 & 1 & 3 & 1 & 1 \end{bmatrix}, \qquad g = T_5^2 + T_4T_6^5 + T_3T_7^5.$$

This is series S102.

Case 3.9.3.4.3.2.2.2: $l_1 + l_2 > 0$, $m_1 = m_2 = 0$. Homogeneity of g yields $l_2 = 2c - b - 5d - l_2$ and $a = 2c$. Grading matrix and relation are given by

$$Q = \begin{bmatrix} 1 & 1 & 2c & b & c & d & 0 \\ 0 & 0 & 1 & 1 & 3 & 1 & 1 \end{bmatrix}, \qquad g = T_5^2 + T_1^{2c-b-5d-l}T_2^lT_4T_6^5 + T_3T_7^5.$$

This is series S103.

Case 3.9.3.4.3.2.2.3: $l_1 = l_2 = 0$, $m_1 + m_2 > 0$. Homogeneity of g yields $b = 2c-5d$ and $m_1 = 2c - a - m_2$. Grading matrix and relation are given by

$$Q = \begin{bmatrix} 1 & 1 & a & 2c-5d & c & d & 0 \\ 0 & 0 & 1 & 1 & 3 & 1 & 1 \end{bmatrix}, \qquad g = T_5^2 + T_4T_6^5 + T_1^{2c-a-l}T_2^lT_3T_7^5.$$

This is series S95.

Case 3.9.3.4.3.2.2.4: $l_1 + l_2 > 0$, $m_1 + m_2 > 0$. We may assume $l_1 > 0$ and $m_2 > 0$. Then $l_2 = m_1 = 0$ holds. Homogeneity of g yields $l_1 = 2c - b - 5d$ and $m_2 = 2c - a$. Grading matrix and relation are given by

$$Q = \begin{bmatrix} 1 & 1 & a & b & c & d & 0 \\ 0 & 0 & 1 & 1 & 3 & 1 & 1 \end{bmatrix}, \qquad g = T_5^2 + T_1^{2c-b-5d}T_4T_6^5 + T_2^{2c-a}T_3T_7^5.$$

This is series S101.

Case 3.9.3.4.3.3: $(l_5, l_6, l_7) = (2, 5, 6)$. Switching roles of T_1 and T_2 if necessary, we may write g as

$$g \;=\; T_5^2 + T_1^{l_1} T_3^{l_3} T_4^{l_4} T_6^5 + T_1^{m_1} T_2^{m_2} T_7^6,$$

where $m_2 > 0$, at most one of l_1, m_1 is non-zero and $l_3 + l_4 = 1$. To satisfy condition (C2) from Setting 3.4.1, the exponents m_1 and m_2 must both be positive and odd. Thus we have $l_1 = 0$. If $l_3 = 1$, then $l_4 = 0$ holds. Homogeneity of g yields $a = 2c - 5d$ and $m_1 = 2c - m_2$. Grading matrix and relation are given by

$$Q \;=\; \begin{bmatrix} 1 & 1 & 2c-5d & b & c & d & 0 \\ 0 & 0 & 1 & 1 & 3 & 1 & 1 \end{bmatrix}, \qquad g \;=\; T_5^2 + T_3 T_6^5 + T_1^{2c-l} T_2^l T_7^6.$$

This is series S99. If $l_4 = 1$, then $l_3 = 0$ holds. Homogeneity of g yields $b = 2c - 5d$ and $m_1 = 2c - m_2$. Grading matrix and relation are given by

$$Q \;=\; \begin{bmatrix} 1 & 1 & a & 2c-5d & c & d & 0 \\ 0 & 0 & 1 & 1 & 3 & 1 & 1 \end{bmatrix}, \qquad g \;=\; T_5^2 + T_4 T_6^5 + T_1^{2c-l} T_2^l T_7^6.$$

This is series S96.

Case 3.9.3.4.3.4: $(l_5, l_6, l_7) = (2, 6, 4)$. Switching roles of T_1 and T_2 if necessary, we may write g as

$$g \;=\; T_5^2 + T_1^{l_1} T_2^{l_2} T_6^6 + T_2^{m_2} T_3^{m_3} T_4^{m_4} T_7^4,$$

where $l_1 > 0$, at most one of l_2, m_2 is non-zero and $m_3 + m_4 = 2$ holds. To satisfy condition (C2) from Setting 3.4.1, the exponents l_1 and l_2 must both be positive and odd. Thus we have $m_2 = 0$. With the same argument also we obtain $m_3 = m_4 = 1$. Homogeneity of g yields $l_1 = 2c - 6d - l_2$ and $a = 2c - b$. Grading matrix and relation are given by

$$Q \;=\; \begin{bmatrix} 1 & 1 & 2c-b & b & c & d & 0 \\ 0 & 0 & 1 & 1 & 3 & 1 & 1 \end{bmatrix}, \qquad g \;=\; T_5^2 + T_1^{2c-6d-l} T_2^l T_6^6 + T_3 T_4 T_7^4.$$

This is series S105.

Case 3.9.3.4.3.5: $(l_5, l_6, l_7) = (2, 6, 5)$. Switching roles of T_1 and T_2 if necessary, we can write g as

$$g \;=\; T_5^2 + T_1^{l_1} T_2^{l_2} T_6^6 + T_2^{m_2} T_3^{m_3} T_4^{m_4} T_7^5,$$

where $l_1 > 0$, at most one of l_2, m_2 is non-zero and $m_3 + m_4 = 1$ holds. To satisfy condition (C2) from Setting 3.4.1, the exponents l_1 and l_2 must both be positive and odd. Thus we have $m_2 = 0$. If $m_3 = 1$, then $m_4 = 0$ holds. Homogeneity of g yields $a = 2c$ and $l_1 = 2c - 6d - l_2$. Grading matrix and relation are given by

$$Q \;=\; \begin{bmatrix} 1 & 1 & 2c & b & c & d & 0 \\ 0 & 0 & 1 & 1 & 3 & 1 & 1 \end{bmatrix}, \qquad g \;=\; T_5^2 + T_1^{2c-6d-l} T_2^l T_6^6 + T_3 T_7^5.$$

This is series S104. If $m_4 = 1$, then $m_3 = 0$ holds. Homogeneity of g yields $b = 2c$ and $l_1 = 2c - 6d - l_2$. Grading matrix and relation are given by

$$Q = \begin{bmatrix} 1 & 1 & a & 2c & c & d & 0 \\ 0 & 0 & 1 & 1 & 3 & 1 & 1 \end{bmatrix}, \qquad g = T_5^2 + T_1^{2c-6d-l} T_2^l T_6^6 + T_3 T_7^5.$$

This is series S106. □

3.10 Classification lists

Here we provide the detailed presentation of our classification results. Let us briefly recall the background. Each locally factorial Fano fourfould X of Picard number two and complexity one with a hypersurface Cox ring can be encoded by the degree matrix Q, that means the list $[w_1, \dots, w_7]$ of degrees of the Cox ring generators in $\mathrm{Cl}(X) = \mathbb{Z}^2$, and the defining trinomial g. Each such variety X is isomorphic to precisely one $X(Q, g)$ with specifying data $Q = [w_1, \dots, w_7]$ and g appearing in the Classification lists 3.10.1 to 3.10.11. Here $X(Q, g) = X_g$ is the variety from Construction 3.2.2 associated with the $\mathbb{Z}^2$-graded $\mathbb{C}$-algebra R_g, where the grading is given by $\deg(T_i) = w_i$.

To make the classification easier to navigate, we split it into several lists, each one containing the specifying data for a given number s of rays generated by the degrees $w_1, \dots, w_7$, with either $\mu \in \lambda$ or $\mu \notin \lambda$. Moreover, in the case $\mu \notin \lambda$ we separate the sporadic cases from the infinite series of specifying data. Apart from the specifying data (Q, g), the classification lists also contain the relation degree $\mu = \deg(g)$, the anticanonical class $-\mathcal{K} \in \mathbb{Z}^2$ and, for the sporadic cases, also the anticanonical degree $\mathcal{K}^4$. A data file containing the complete classification data is also available at [18].

Classification list 3.10.1. *Locally factorial Fano fourfoulds of Picard number two with a hypersurface Cox ring and an effective three-torus action: Specifying data for the case with $s = 2$.*

ID	$[w_1, \dots, w_7]$	μ	$-\mathcal{K}$	$\mathcal{K}^4$	g
1	$\begin{bmatrix} 1&1&1&1&0&0&0 \\ 0&0&0&0&1&1&1 \end{bmatrix}$	$(1,1)$	$(3,2)$	432	$T_1T_6 + T_2T_5 + T_4T_7$
2 3	$\begin{bmatrix} 1&1&1&1&0&0&0 \\ 0&0&0&0&1&1&1 \end{bmatrix}$	$(2,1)$	$(2,2)$	256	$T_1^2T_6 + T_2T_3T_7 + T_4^2T_5$ $T_1^2T_6 + T_3^2T_7 + T_4^2T_5$
4 5	$\begin{bmatrix} 1&1&1&1&0&0&0 \\ 0&0&0&0&1&1&1 \end{bmatrix}$	$(3,1)$	$(1,2)$	80	$T_1^3T_7 + T_3^3T_5 + T_4^3T_6$ $T_1^3T_5 + T_2^2T_4T_6 + T_3^3T_7$
6	$\begin{bmatrix} 1&1&1&1&0&0&0 \\ 0&0&0&0&1&1&1 \end{bmatrix}$	$(1,2)$	$(3,1)$	270	$T_1T_6^2 + T_2T_7^2 + T_3T_5^2$
7 8	$\begin{bmatrix} 1&1&1&1&0&0&0 \\ 0&0&0&0&1&1&1 \end{bmatrix}$	$(3,2)$	$(1,1)$	26	$T_1^3T_7^2 + T_2^3T_6^2 + T_3^2T_4T_5^2$ $T_1^3T_5^2 + T_3^3T_6^2 + T_4^3T_7^2$

ID	$[w_1,\dots,w_7]$	μ	$-\mathcal{K}$	$\mathcal{K}^4$	g
9 10	$\begin{bmatrix} 1&1&1&1&3&0&0 \\ 0&0&0&0&0&1&1 \end{bmatrix}$	$(6,0)$	$(1,2)$	16	$T_1^3T_3^3+T_2^5T_4+T_5^2$ $T_1T_4^5+T_2^5T_3+T_5^2$
11 12 13	$\begin{bmatrix} 1&1&1&2&3&0&0 \\ 0&0&0&0&0&1&1 \end{bmatrix}$	$(6,0)$	$(2,2)$	64	$T_1^4T_2T_3+T_4^3+T_5^2$ $T_1^5T_3+T_4^3+T_5^2$ $T_1^3T_2^2T_3+T_4^3+T_5^2$
14	$\begin{bmatrix} 1&1&1&1&2&0&0 \\ 0&0&0&0&0&1&1 \end{bmatrix}$	$(4,0)$	$(2,2)$	128	$T_1T_3^3+T_2^3T_4+T_5^2$
15	$\begin{bmatrix} 1&1&1&1&1&0&0 \\ 0&0&0&0&0&1&1 \end{bmatrix}$	$(2,0)$	$(3,2)$	432	$T_1T_3+T_2T_5+T_4^2$
16	$\begin{bmatrix} 1&1&1&1&1&0&0 \\ 0&0&0&0&0&1&1 \end{bmatrix}$	$(3,0)$	$(2,2)$	192	$T_1^2T_4+T_2^2T_5+T_3^3$
17	$\begin{bmatrix} 1&1&1&1&1&0&0 \\ 0&0&0&0&0&1&1 \end{bmatrix}$	$(4,0)$	$(1,2)$	32	$T_1^3T_2+T_3^3T_4+T_5^4$
18	$\begin{bmatrix} 1&1&2&3&0&0&0 \\ 0&0&0&0&1&1&1 \end{bmatrix}$	$(6,0)$	$(1,3)$	54	$T_1^5T_2+T_3^3+T_4^2$

Classification list 3.10.2. *Locally factorial Fano fourfoulds of Picard number two with a hypersurface Cox ring and an effective three-torus action: Specifying data for the cases with $s=3$ and $\mu \in \lambda$.*

ID	$[w_1,\dots,w_7]$	μ	$-\mathcal{K}$	$\mathcal{K}^4$	g
19 20	$\begin{bmatrix} 1&1&1&1&0&0&-1 \\ 0&0&0&0&1&1&1 \end{bmatrix}$	$(1,1)$	$(2,2)$	416	$T_2^2T_7+T_1T_5+T_3T_6$ $T_3T_4T_7+T_1T_6+T_2T_5$
21 22	$\begin{bmatrix} 1&1&1&1&0&0&-1 \\ 0&0&0&0&1&1&1 \end{bmatrix}$	$(1,2)$	$(2,1)$	163	$T_2T_3^2T_7^2+T_1T_5^2+T_4T_6^2$ $T_3^3T_7^2+T_2T_5^2+T_4T_6^2$
26 27	$\begin{bmatrix} 1&1&1&1&0&0&-2 \\ 0&0&0&0&1&1&1 \end{bmatrix}$	$(1,1)$	$(1,2)$	464	$T_4^3T_7+T_2T_5+T_3T_6$ $T_3^2T_4T_7+T_1T_6+T_2T_5$
31 32	$\begin{bmatrix} 1&1&1&1&1&0&0 \\ 0&0&0&0&1&1&1 \end{bmatrix}$	$(2,1)$	$(3,2)$	352	$T_1T_4T_6+T_3^2T_7+T_2T_5$ $T_1^2T_7+T_4^2T_6+T_3T_5$
33 34	$\begin{bmatrix} 1&1&1&1&1&0&0 \\ 0&0&0&0&1&1&1 \end{bmatrix}$	$(3,2)$	$(2,1)$	81	$T_1^3T_7^2+T_2^2T_4T_6^2+T_3T_5^2$ $T_3^3T_6^2+T_4^3T_7^2+T_1T_5^2$
44 45	$\begin{bmatrix} 1&1&1&1&0&0&0 \\ -1&0&0&0&1&1&1 \end{bmatrix}$	$(1,1)$	$(3,1)$	432	$T_1T_5^2+T_3T_6+T_4T_7$ $T_2T_7+T_3T_5+T_4T_6$
46 47	$\begin{bmatrix} 1&1&1&1&0&0&0 \\ 0&0&1&1&1&1&1 \end{bmatrix}$	$(2,2)$	$(2,3)$	272	$T_1^2T_5T_7+T_2T_4T_6+T_3^2$ $T_1^2T_6T_7+T_2^2T_5^2+T_3T_4$
51 52	$\begin{bmatrix} 1&1&1&2&0&0&0 \\ 0&0&1&2&1&1&1 \end{bmatrix}$	$(4,4)$	$(1,2)$	34	$T_2^4T_6T_7^3+T_1T_3^3T_5+T_4^2$ $T_2^4T_5^3T_6+T_1^3T_3T_7^3+T_4^2$

ID	$[w_1, \dots, w_7]$	μ	$-\mathcal{K}$	$\mathcal{K}^4$	g
72	$\begin{bmatrix} 1&1&1&0&0&0&0 \\ 0&0&1&1&1&1&1 \end{bmatrix}$	$(2,2)$	$(1,3)$	216	$T_1^2T_4T_7 + T_2^2T_5T_6 + T_3^2$
73	$\begin{bmatrix} 1&1&1&0&0&0&0 \\ 0&0&2&1&1&1&1 \end{bmatrix}$	$(2,4)$	$(1,2)$	64	$T_1^2T_4T_6^3 + T_2^2T_5T_7^3 + T_3^2$
77	$\begin{bmatrix} 1&1&1&1&0&0&0 \\ 0&0&0&1&1&1&1 \end{bmatrix}$	$(2,2)$	$(2,2)$	192	$T_1T_3T_7^2 + T_2^2T_5T_6 + T_4^2$
78	$\begin{bmatrix} 1&1&1&1&0&0&0 \\ 0&0&0&1&1&1&1 \end{bmatrix}$	$(3,3)$	$(1,1)$	18	$T_1^3T_5T_6^2 + T_2T_3^2T_7^3 + T_4^3$
79	$\begin{bmatrix} 1&1&1&2&0&0&0 \\ 0&0&0&1&1&1&1 \end{bmatrix}$	$(4,2)$	$(1,2)$	48	$T_1^3T_2T_5^2 + T_3^4T_6T_7 + T_4^2$
80	$\begin{bmatrix} 1&1&1&2&0&0&0 \\ 0&0&0&2&1&1&1 \end{bmatrix}$	$(4,4)$	$(1,1)$	12	$T_1T_2^3T_7^4 + T_3^4T_5^3T_6 + T_4^2$
149	$\begin{bmatrix} 1&1&1&1&0&0&0 \\ 0&1&1&1&1&1&1 \end{bmatrix}$	$(2,2)$	$(2,4)$	352	$T_1^2T_5T_6 + T_2^2 + T_3T_4$
150 151	$\begin{bmatrix} 1&1&1&1&0&0&0 \\ 0&1&1&1&1&1&1 \end{bmatrix}$	$(3,3)$	$(1,3)$	99	$T_1^3T_5T_6^2 + T_2^3 + T_3T_4^2$ $T_1^3T_5T_6T_7 + T_2T_4^2 + T_3^3$
152 153	$\begin{bmatrix} 1&1&1&1&0&0&0 \\ 0&2&2&2&1&1&1 \end{bmatrix}$	$(2,4)$	$(2,5)$	304	$T_1^2T_5T_6^3 + T_2T_4 + T_3^2$ $T_1^2T_5T_6^2T_7 + T_2T_3 + T_4^2$
159 160	$\begin{bmatrix} 1&1&1&2&0&0&0 \\ 0&1&1&2&1&1&1 \end{bmatrix}$	$(4,4)$	$(1,3)$	66	$T_1^4T_5T_7^3 + T_2T_3^3 + T_4^2$ $T_1^4T_5T_6T_7^2 + T_2^3T_3 + T_4^2$
223	$\begin{bmatrix} 1&1&1&1&2&1&0 \\ 0&0&0&0&0&1&1 \end{bmatrix}$	$(4,0)$	$(3,2)$	160	$T_1T_3^3 + T_2^3T_4 + T_5^2$
224	$\begin{bmatrix} 1&1&1&1&1&1&0 \\ 0&0&0&0&0&1&1 \end{bmatrix}$	$(3,0)$	$(3,2)$	240	$T_1^2T_4 + T_2^2T_5 + T_3^3$
225	$\begin{bmatrix} 1&1&1&1&1&1&0 \\ 0&0&0&0&0&1&1 \end{bmatrix}$	$(2,0)$	$(4,2)$	480	$T_1T_3 + T_2T_5 + T_4^2$
226	$\begin{bmatrix} 1&1&1&1&1&2&0 \\ 0&0&0&0&0&1&1 \end{bmatrix}$	$(2,0)$	$(5,2)$	624	$T_1T_3 + T_2T_5 + T_4^2$

$$Q = \begin{bmatrix} 1&1&1&1&0&0&-1 \\ 0&0&0&0&1&1&1 \end{bmatrix} \qquad \mu = (2,1) \qquad -\mathcal{K} = (1,2) \qquad \mathcal{K}^4 = 224$$

ID	g	ID	g	ID	g
23	$T_1^2T_3T_7+T_2^2T_6+T_4^2T_5$	24	$T_3^3T_7+T_2^2T_5+T_4^2T_6$	25	$T_4^3T_7+T_1T_3T_5+T_2^2T_6$

$$Q = \begin{bmatrix} 1&1&1&1&0&0&-2 \\ 0&0&0&0&1&1&1 \end{bmatrix} \qquad \mu = (1,2) \qquad -\mathcal{K} = (1,1) \qquad \mathcal{K}^4 = 98$$

ID	g	ID	g	ID	g
28	$T_1^4T_3T_7^2+T_2T_5^2+T_4T_6^2$	29	$T_1^3T_2^2T_7^2+T_3T_5^2+T_4T_6^2$	30	$T_4^5T_7^2+T_2T_6^2+T_3T_5^2$

$$Q = \begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 1 & 1 & 1 \end{bmatrix} \quad \mu = (3,1) \quad -\mathcal{K} = (1,1) \quad \mathcal{K}^4 = 38$$

ID	g	ID	g	ID	g
35	$T_1^3T_7^4+T_2^3T_6+T_3T_4^2T_5$	36	$T_2^3T_5+T_3^3T_7+T_4^3T_6$	37	$T_1T_3^2T_5^2+T_2^3T_6+T_4^3T_7$
38	$T_1^3T_7^4+T_2^3T_5+T_4^3T_6$	39	$T_1^2T_3T_6^3+T_2^3T_7+T_4^3T_5$		

$$Q = \begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 1 & 1 & 1 \end{bmatrix} \quad \mu = (2,1) \quad -\mathcal{K} = (2,1) \quad \mathcal{K}^4 = 192$$

ID	g	ID	g	ID	g
40	$T_1T_4T_5^2+T_2^2T_7+T_3^2T_6$	41	$T_1^2T_7^3+T_2T_4T_6+T_3^2T_5$	42	$T_2^2T_7+T_3^2T_5+T_4^2T_6$
43	$T_1^2T_6^3+T_3^2T_7+T_4^2T_5$				

$$Q = \begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & 1 \end{bmatrix} \quad \mu = (3,3) \quad -\mathcal{K} = (1,2) \quad \mathcal{K}^4 = 51$$

ID	g	ID	g	ID	g
48	$T_2^3T_5T_6^2+T_1T_4^2T_7+T_3^3$	49	$T_2^3T_5^2T_7+T_1^2T_4T_6^2+T_3^3$	50	$T_1^3T_6T_7^2+T_2^3T_5^3+T_3T_4^2$

$$Q = \begin{bmatrix} 1 & 1 & 2 & 3 & 0 & 0 & 0 \\ 0 & 0 & 2 & 3 & 1 & 1 & 1 \end{bmatrix} \quad \mu = (6,6) \quad -\mathcal{K} = (1,2) \quad \mathcal{K}^4 = 17$$

ID	g	ID	g	ID	g
53	$T_1T_2^5T_5T_6T_7^4+T_3^3+T_4^2$	54	$T_1T_2^5T_5^3T_6T_7^2+T_3^3+T_4^2$	55	$T_1T_2^5T_5T_7^5+T_3^3+T_4^2$
56	$T_1^2T_2^4T_5T_6^5+T_3^3+T_4^2$	57	$T_1^6T_5^5T_6+T_3^3+T_4^2$	58	$T_1^2T_2^4T_5^3T_7^3+T_3^3+T_4^2$
59	$T_1^6T_5^2T_6T_7^3+T_3^3+T_4^2$	60	$T_1^6T_5^4T_6T_7+T_3^3+T_4^2$	61	$T_1^3T_2^3T_5^2T_6T_7^3+T_3^3+T_4^2$
62	$T_1^4T_2^2T_5^3T_6T_7^2+T_3^3+T_4^2$	63	$T_1^5T_2T_5^4T_6^2+T_3^3+T_4^2$	64	$T_1^5T_2T_5^2T_6^2T_7^2+T_3^3+T_4^2$
65	$T_1^3T_2^3T_5^5T_6+T_3^3+T_4^2$	66	$T_1^3T_2^3T_5^2T_7^4+T_3^3+T_4^2$	67	$T_1T_2^5T_7^6+T_3^3+T_4^2$
68	$T_1^2T_2^4T_5^4T_6T_7+T_3^3+T_4^2$	69	$T_1T_2^5T_5^3T_6^3+T_3^3+T_4^2$	70	$T_1^3T_2^3T_5T_6T_7^4+T_3^3+T_4^2$
71	$T_1^3T_2^3T_5^2T_6^2T_7^2+T_3^3+T_4^2$				

$$Q = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 3 & 1 & 1 & 1 & 1 \end{bmatrix} \quad \mu = (2,6) \quad -\mathcal{K} = (1,1) \quad \mathcal{K}^4 = 8$$

ID	g	ID	g	ID	g
74	$T_1^2T_5^5T_7+T_2^2T_4T_6^5+T_3^2$	75	$T_1^2T_5^5T_6+T_2^2T_4^3T_7^3+T_3^2$	76	$T_1^2T_6^3T_7^3+T_2^2T_4^3T_5^3+T_3^2$

$$Q = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix} \quad \mu = (2,2) \quad -\mathcal{K} = (3,4) \quad \mathcal{K}^4 = 378$$

ID	g	ID	g	ID	g
81	$T_1T_2T_7+T_3^2+T_4T_5$	82	$T_1^2T_6^2+T_2T_3+T_4T_5$	83	$T_1^2T_6T_7+T_3T_5+T_4^2$
84	$T_1^2T_6T_7+T_2T_4+T_3T_5$				

$$Q = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix} \quad \mu = (3,3) \quad -\mathcal{K} = (2,3) \quad \mathcal{K}^4 = 144$$

ID	g	ID	g	ID	g
85	$T_1^2T_5T_7^2+T_2^2T_4+T_3^3$	86	$T_1^3T_6^2T_7+T_2T_5^2+T_3T_4^2$	87	$T_1^3T_6T_7^2+T_3^3+T_4T_5^2$
88	$T_1T_5^2T_7+T_2^3+T_3T_4^2$	89	$T_1^3T_6^3+T_2^2T_4+T_3^2T_5$	90	$T_1^2T_4T_6T_7+T_2^2T_3+T_5^3$
91	$T_1^3T_6^2T_7+T_2T_3T_5+T_4^3$				

$Q = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}$ $\quad \mu = (4,4) \quad -\mathcal{K} = (1,2) \quad \mathcal{K}^4 = 20$

ID	g	ID	g	ID	g
92	$T_1^3T_4T_6^3+T_2T_5^3+T_3^4$	93	$T_1^3T_2T_6^2T_7+T_3^4+T_4T_5^3$	94	$T_1^4T_6T_7^3+T_2T_5^3+T_3^3T_4$
95	$T_1^4T_6^3T_7+T_2T_3^3+T_4^2T_5^2$	96	$T_1^2T_2^2T_6T_7+T_3^3T_5+T_4^4$	97	$T_1T_5^3T_7+T_2^3T_4+T_3^4$
98	$T_1^4T_6T_7^3+T_2T_5^3+T_4^4$	99	$T_1^4T_6^2T_7^2+T_2^3T_4+T_3T_5^3$	100	$T_1^4T_6^3T_7+T_2T_3^2T_4+T_5^4$
101	$T_1^4T_6^4+T_2T_3^3+T_4T_5^3$				

$Q = \begin{bmatrix} 1 & 1 & 1 & 1 & 2 & 0 & 0 \\ 0 & 1 & 1 & 1 & 2 & 1 & 1 \end{bmatrix}$ $\quad \mu = (4,4) \quad -\mathcal{K} = (2,3) \quad \mathcal{K}^4 = 96$

ID	g	ID	g	ID	g
102	$T_1^3T_4T_6T_7^2+T_2^3T_3+T_5^2$	103	$T_1T_2^3T_7+T_3^3T_4+T_5^2$	104	$T_1^4T_6^3T_7+T_2T_3^2T_4+T_5^2$
105	$T_1^4T_6T_7^3+T_3^3T_4+T_5^2$	106	$T_1^3T_4T_6^3+T_2^3T_3+T_5^2$	107	$T_1^2T_3^2T_6T_7+T_2T_4^3+T_5^2$

$Q = \begin{bmatrix} 1 & 1 & 1 & 1 & 3 & 0 & 0 \\ 0 & 1 & 1 & 1 & 3 & 1 & 1 \end{bmatrix}$ $\quad \mu = (6,6) \quad -\mathcal{K} = (1,2) \quad \mathcal{K}^4 = 10$

ID	g	ID	g	ID	g
108	$T_1^5T_2T_7^5+T_3T_4^5+T_5^2$	109	$T_1T_4^5T_7+T_2^5T_3+T_5^2$	110	$T_1^5T_3T_6^3T_7^2+T_2T_4^5+T_5^2$
111	$T_1^6T_6^5T_7+T_2^3T_3^2T_4+T_5^2$	112	$T_1^5T_3T_6^5+T_2^3T_4^3+T_5^2$	113	$T_1^4T_2^2T_6T_7^3+T_3^3T_4^3+T_5^2$
114	$T_1^3T_3^3T_6^2T_7+T_2^3T_4^3+T_5^2$	115	$T_1^6T_6^3T_7^3+T_2T_3T_4^4+T_5^2$	116	$T_1^5T_4T_6^4T_7+T_2^3T_3^3+T_5^2$
117	$T_1^4T_3^2T_6^3T_7+T_2T_4^5+T_5^2$	118	$T_1^6T_6^5T_7+T_3^5T_4+T_5^2$	119	$T_1^6T_6^3T_7^3+T_2T_4^5+T_5^2$
120	$T_1^2T_2^4T_6T_7+T_3^5T_4+T_5^2$	121	$T_1^3T_2^3T_7^3+T_3T_4^5+T_5^2$	122	$T_1^3T_4^3T_6T_7^2+T_2T_3^5+T_5^2$
123	$T_1^6T_6^5T_7+T_3^3T_4^3+T_5^2$	124	$T_1^5T_3T_6T_7^4+T_2T_4^5+T_5^2$	125	$T_1^6T_6^3T_7^3+T_2^2T_3^3T_4+T_5^2$
126	$T_1^6T_6T_7^5+T_2^4T_3T_4+T_5^2$	127	$T_1^5T_4T_6^2T_7^3+T_2^3T_3^3+T_5^2$	128	$T_1^2T_4^4T_6T_7+T_2^3T_3^3+T_5^2$
129	$T_1T_4^5T_7+T_2^3T_3^3+T_5^2$				

$Q = \begin{bmatrix} 1 & 1 & 1 & 2 & 3 & 0 & 0 \\ 0 & 1 & 1 & 2 & 3 & 1 & 1 \end{bmatrix}$ $\quad \mu = (6,6) \quad -\mathcal{K} = (2,3) \quad \mathcal{K}^4 = 48$

ID	g	ID	g	ID	g
130	$T_1^4T_3^2T_6T_7^3+T_4^3+T_5^2$	131	$T_1^4T_2T_3T_6T_7^3+T_4^3+T_5^2$	132	$T_1^3T_2^2T_3T_6^3+T_4^3+T_5^2$
133	$T_1^2T_2T_3^3T_6T_7+T_4^3+T_5^2$	134	$T_1T_2^4T_3T_6+T_4^3+T_5^2$	135	$T_1^4T_2T_3T_6^4+T_4^3+T_5^2$
136	$T_1T_2^2T_3^3T_7+T_4^3+T_5^2$	137	$T_1^2T_2^4T_6T_7+T_4^3+T_5^2$	138	$T_2^5T_3+T_4^3+T_5^2$
139	$T_1^2T_2T_3^3T_6^2+T_4^3+T_5^2$	140	$T_1^5T_3T_6T_7^4+T_4^3+T_5^2$	141	$T_1^3T_2^3T_6T_7^2+T_4^3+T_5^2$
142	$T_1^5T_3T_6^5+T_4^3+T_5^2$	143	$T_1^6T_6^5T_7+T_4^3+T_5^2$	144	$T_1^4T_2T_3T_6^2T_7^2+T_4^3+T_5^2$
145	$T_1^5T_2T_6^2T_7^3+T_4^3+T_5^2$	146	$T_1T_3^5T_6+T_4^3+T_5^2$	147	$T_1^3T_2T_3^2T_6^2T_7+T_4^3+T_5^2$
148	$T_1^2T_2^2T_3^2T_6T_7+T_4^3+T_5^2$				

$Q = \begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 2 & 2 & 2 & 1 & 1 & 1 \end{bmatrix}$ $\quad \mu = (3,6) \quad -\mathcal{K} = (1,3) \quad \mathcal{K}^4 = 54$

ID	g	ID	g	ID	g
154	$T_1^3T_5^2T_6T_7^3+T_2^2T_4+T_3^3$	155	$T_1^3T_5^2T_7^4+T_2^3+T_3^2T_4$	156	$T_1^3T_5^4T_6T_7+T_2^3+T_3^2T_4$
157	$T_1^3T_6T_7^5+T_2^2T_3+T_4^3$	158	$T_1^3T_5^2T_6^2T_7^2+T_2T_3^2+T_4^3$		

$Q = \begin{bmatrix} 1 & 1 & 1 & 2 & 0 & 0 & 0 \\ 0 & 2 & 2 & 4 & 1 & 1 & 1 \end{bmatrix}$ $\quad \mu = (4,8) \quad -\mathcal{K} = (1,3) \quad \mathcal{K}^4 = 36$

ID	g	ID	g	ID	g
161	$T_1^4T_5^5T_6T_7^2+T_2T_3^3+T_4^2$	162	$T_1^4T_5T_6^4T_7^3+T_2^3T_3+T_4^2$	163	$T_1^4T_5^3T_6^5+T_2T_3^3+T_4^2$

$$Q = \begin{bmatrix} 1 & 1 & 1 & 2 & 0 & 0 & 0 \\ 0 & 2 & 2 & 4 & 1 & 1 & 1 \end{bmatrix} \qquad \mu = (4,8) \qquad -\mathcal{K} = (1,3) \qquad \mathcal{K}^4 = 36$$

ID	g	ID	g	ID	g
164	$T_1^4T_5^2T_6^3T_7^3+T_2^3T_3+T_4^2$	165	$T_1^4T_5^7T_7+T_2T_3^3+T_4^2$	166	$T_1^4T_5^6T_6T_7+T_2T_3^3+T_4^2$

$$Q = \begin{bmatrix} 1 & 1 & 2 & 3 & 0 & 0 & 0 \\ 0 & 1 & 2 & 3 & 1 & 1 & 1 \end{bmatrix} \qquad \mu = (6,6) \qquad -\mathcal{K} = (1,3) \qquad \mathcal{K}^4 = 33$$

ID	g	ID	g	ID	g
167	$T_1^6T_5T_6^2T_7^3+T_3^3+T_4^2$	168	$T_1^6T_6^5T_7+T_3^3+T_4^2$	169	$T_1T_2^5T_7+T_3^3+T_4^2$
170	$T_1^3T_2^3T_6^2T_7+T_3^3+T_4^2$	171	$T_1^5T_2T_5^3T_6T_7+T_3^3+T_4^2$	172	$T_1^5T_2T_5^2T_6^3+T_3^3+T_4^2$
173	$T_1^2T_2^4T_5T_7+T_3^3+T_4^2$	174	$T_1^6T_5^4T_6T_7+T_3^3+T_4^2$	175	$T_1^5T_2T_5^4T_6+T_3^3+T_4^2$
176	$T_1^5T_2T_5^5+T_3^3+T_4^2$	177	$T_1^4T_2^2T_5T_6^3+T_3^3+T_4^2$	178	$T_1^5T_2T_5T_6^2T_7^2+T_3^3+T_4^2$
179	$T_1^4T_2^2T_5T_6T_7^2+T_3^3+T_4^2$	180	$T_1^3T_2^3T_5T_6T_7+T_3^3+T_4^2$		

$$Q = \begin{bmatrix} 1 & 1 & 2 & 3 & 0 & 0 & 0 \\ 0 & 2 & 4 & 6 & 1 & 1 & 1 \end{bmatrix} \qquad \mu = (6,12) \qquad -\mathcal{K} = (1,3) \qquad \mathcal{K}^4 = 18$$

ID	g	ID	g	ID	g
181	$T_1T_2^5T_5^2+T_3^3+T_4^2$	182	$T_1^6T_6^5T_7^7+T_3^3+T_4^2$	183	$T_1^6T_5^5T_6^2T_7^5+T_3^3+T_4^2$
184	$T_1^5T_2T_5^3T_6^5T_7^2+T_3^3+T_4^2$	185	$T_1^5T_2T_5^9T_7+T_3^3+T_4^2$	186	$T_1^5T_2T_5^3T_6^7+T_3^3+T_4^2$
187	$T_1^3T_2^3T_5^3T_6T_7^2+T_3^3+T_4^2$	188	$T_1^4T_2^2T_5T_6T_7^6+T_3^3+T_4^2$	189	$T_1^6T_5^4T_6^3T_7^5+T_3^3+T_4^2$
190	$T_1^3T_2^3T_5^4T_6T_7+T_3^3+T_4^2$	191	$T_1^4T_2^2T_5^5T_6^3+T_3^3+T_4^2$	192	$T_1^5T_2T_5^6T_6^2T_7^2+T_3^3+T_4^2$
193	$T_1^3T_2^3T_5^5T_7+T_3^3+T_4^2$	194	$T_1^6T_5^{11}T_7+T_3^3+T_4^2$	195	$T_1^6T_5^7T_6^3T_7^2+T_3^3+T_4^2$
196	$T_1^6T_5T_6^{10}T_7+T_3^3+T_4^2$	197	$T_1^5T_2T_5^7T_6T_7^2+T_3^3+T_4^2$	198	$T_1^2T_2^4T_5T_6^3+T_3^3+T_4^2$
199	$T_1^6T_5T_6^7T_7^4+T_3^3+T_4^2$	200	$T_1^6T_5^9T_6T_7^2+T_3^3+T_4^2$	201	$T_1^4T_2^2T_5^2T_6^3T_7^3+T_3^3+T_4^2$
202	$T_1^4T_2^2T_5^5T_6^2T_7+T_3^3+T_4^2$	203	$T_1^6T_5^5T_6T_7^6+T_3^3+T_4^2$	204	$T_1^5T_2T_5^3T_6^4T_7^3+T_3^3+T_4^2$
205	$T_1^5T_2T_5^8T_7^2+T_3^3+T_4^2$	206	$T_1^3T_2^3T_5^2T_7^4+T_3^3+T_4^2$	207	$T_1^5T_2T_5^4T_6^5T_7+T_3^3+T_4^2$
208	$T_1^5T_2T_7^{10}+T_3^3+T_4^2$	209	$T_1^5T_2T_5^2T_6^4T_7^4+T_3^3+T_4^2$	210	$T_1^5T_2T_5^3T_6^6T_7+T_3^3+T_4^2$
211	$T_1T_2^5T_6T_7+T_3^3+T_4^2$	212	$T_1^5T_2T_5^8T_6T_7+T_3^3+T_4^2$	213	$T_1^4T_2^2T_5T_6^3T_7^4+T_3^3+T_4^2$
214	$T_1^2T_2^4T_5T_6^2T_7+T_3^3+T_4^2$	215	$T_1^4T_2^2T_5T_6^7+T_3^3+T_4^2$	216	$T_1^5T_2T_6^5T_7^5+T_3^3+T_4^2$
217	$T_1^5T_2T_5^6T_6^4+T_3^3+T_4^2$	218	$T_1^6T_5T_6^8T_7^3+T_3^3+T_4^2$	219	$T_1^3T_2^3T_5^2T_6^2T_7^2+T_3^3+T_4^2$

$$Q = \begin{bmatrix} 1 & 1 & 1 & 2 & 3 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix} \qquad \mu = (6,0) \qquad -\mathcal{K} = (3,2) \qquad \mathcal{K}^4 = 80$$

ID	g	ID	g	ID	g
220	$T_1^4T_2T_3+T_4^3+T_5^2$	221	$T_1^5T_3+T_4^3+T_5^2$	222	$T_1^3T_2^2T_3+T_4^3+T_5^2$

Classification list 3.10.3. *Locally factorial Fano fourfoulds of Picard number two with a hypersurface Cox ring and an effective three-torus action: Specifying data for the sporadic cases with $s = 3$ and $\mu \notin \lambda$.*

ID	$[w_1, \ldots, w_7]$	μ	$-\mathcal{K}$	$\mathcal{K}^4$	g
227	$\begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 \end{bmatrix}$	$(1,2)$	$(3,2)$	352	$T_1T_5^2 + T_2T_6^2 + T_3T_7^2$
228					$T_1T_5^2 + T_2T_6^2 + T_4T_7$

ID	$[w_1,\dots,w_7]$	μ	$-\mathcal{K}$	$\mathcal{K}^4$	g
229	$\begin{bmatrix} 1&1&1&1&0&0&0 \\ 0&0&0&1&1&1&1 \end{bmatrix}$	$(1,3)$	$(3,1)$	140	$T_1T_5^3+T_2T_6^3+T_4T_7^2$
230 231 232	$\begin{bmatrix} 1&1&1&1&0&0&0 \\ 0&0&0&1&1&1&1 \end{bmatrix}$	$(2,3)$	$(2,1)$	65	$T_1^2T_5^3+T_2^2T_6^3+T_3T_4T_7^2$ $T_1^2T_5^3+T_2^2T_6^3+T_4^2T_7$ $T_1^2T_5^3+T_2T_3T_6^3+T_4^2T_7^2$
233 234	$\begin{bmatrix} 1&1&1&2&0&0&0 \\ 0&0&0&1&1&1&1 \end{bmatrix}$	$(2,3)$	$(3,1)$	122	$T_1^2T_5^3+T_2^2T_6^3+T_4T_7^2$ $T_1^2T_5^3+T_2T_3T_6^3+T_4T_7^2$
235 236	$\begin{bmatrix} 1&1&1&1&0&0&0 \\ 0&0&1&1&1&1&1 \end{bmatrix}$	$(1,3)$	$(3,2)$	208	$T_1T_5^3+T_2T_6^3+T_4T_7^2$ $T_1T_5^3+T_3T_6^2+T_4T_7^2$
237 238 239 240	$\begin{bmatrix} 1&1&1&0&0&0&0 \\ 0&0&1&1&1&1&1 \end{bmatrix}$	$(1,4)$	$(2,1)$	29	$T_1T_4^4+T_2T_6^4+T_3T_7^3$ $T_1T_4^3T_5+T_2T_6^4+T_3T_7^3$ $T_1T_4^2T_5^2+T_2T_6^4+T_3T_7^3$ $T_1T_4^4+T_2T_5^4+T_3T_6T_7^2$
241	$\begin{bmatrix} 1&1&1&0&0&0&0 \\ 0&0&1&1&1&2&3 \end{bmatrix}$	$(0,6)$	$(3,2)$	80	$T_4^5T_5+T_6^3+T_7^2$

Classification list 3.10.4. *Locally factorial Fano fourfoulds of Picard number two with a hypersurface Cox ring and an effective three-torus action: Specifying data for the series with $s=3$.*

ID	$[w_1,\dots,w_7]$	μ	$-\mathcal{K}$	g	
S1	$\begin{bmatrix} 1&1&1&a&0&0&0 \\ 0&0&0&1&1&1&1 \end{bmatrix}$	$(1,3)$	$(a{+}2,1)$	$T_1T_5^3+T_2T_6^3+T_3T_7^3$	$a\ge 1$
S2	$\begin{bmatrix} 1&1&1&a&0&0&0 \\ 0&0&0&1&1&1&1 \end{bmatrix}$	$(2,3)$	$(a{+}1,1)$	$T_1^2T_5^3+T_2^2T_6^3+T_3^2T_7^3$	$a\ge 1$
S3 S4	$\begin{bmatrix} 1&1&a&a&0&0&0 \\ 0&0&1&1&1&1&1 \end{bmatrix}$	$(a,4)$	$(a{+}2,1)$	$T_1^{a-l}T_2^lT_5^4+T_3T_6^3+T_4T_7^3$ $T_1^aT_5^4+T_2^aT_6^4+T_4T_7^3$	$a\ge 1,\ 0\le l\le a/2$ $a\ge 1,\ a$ odd

Classification list 3.10.5. *Locally factorial Fano fourfoulds of Picard number two with a hypersurface Cox ring and an effective three-torus action: Specifying data for the cases with $s=4$ and $\mu\in\lambda$.*

ID	$[w_1,\dots,w_7]$	μ	$-\mathcal{K}$	$\mathcal{K}^4$	g
242 243	$\begin{bmatrix} 1&1&1&1&1&0&0 \\ -1&0&0&0&1&1&1 \end{bmatrix}$	$(3,1)$	$(2,1)$	113	$T_2T_4^2T_7+T_3^3T_6+T_1T_5^2$ $T_3^3T_6+T_4^3T_7+T_1T_5^2$

ID	$[w_1,\dots,w_7]$	μ	$-\mathcal{K}$	$\mathcal{K}^4$	g
244	$\begin{bmatrix} 1&1&1&1&1&0&0 \\ 0&1&1&1&2&1&1 \end{bmatrix}$	$(2,2)$	$(3,5)$	433	$T_1T_5+T_2T_4+T_3^2$
245	$\begin{bmatrix} 1&1&1&1&1&0&0 \\ 0&2&2&2&3&1&1 \end{bmatrix}$	$(3,6)$	$(2,5)$	145	$T_1T_5^2+T_2^2T_3+T_4^3$
246 247	$\begin{bmatrix} 1&1&1&1&0&0&0 \\ 0&1&1&2&1&1&1 \end{bmatrix}$	$(2,4)$	$(2,3)$	144	$T_1^2T_6T_7^3+T_2T_3T_5^2+T_4^2$ $T_1T_3T_6^3+T_2^2T_5T_7+T_4^2$
248 249 250 251	$\begin{bmatrix} 1&1&1&2&0&0&0 \\ 0&1&1&3&1&1&1 \end{bmatrix}$	$(4,6)$	$(1,2)$	22	$T_1^4T_5T_7^5+T_2T_3^3T_6^2+T_4^2$ $T_1T_3^3T_6^3+T_2^4T_5T_7+T_4^2$ $T_1^3T_2T_7^5+T_3^4T_5T_6+T_4^2$ $T_1^4T_6^3T_7^3+T_2^3T_3T_5^2+T_4^2$
252 253	$\begin{bmatrix} 1&1&1&1&1&1&0 \\ -1&0&0&0&0&1&1 \end{bmatrix}$	$(2,0)$	$(4,1)$	431	$T_1T_6+T_2T_4+T_3^2$ $T_1T_6+T_2T_3+T_4T_5$
254	$\begin{bmatrix} 1&1&1&1&1&1&0 \\ -1&0&0&0&0&1&1 \end{bmatrix}$	$(4,0)$	$(2,1)$	62	$T_1^2T_6^2+T_2^3T_4+T_3^3T_5$
255 256 257	$\begin{bmatrix} 1&1&1&1&1&2&0 \\ -1&0&0&0&0&1&1 \end{bmatrix}$	$(3,0)$	$(4,1)$	376	$T_2^2T_3+T_4^3+T_1T_6$ $T_2T_5^2+T_3^2T_4+T_1T_6$ $T_2T_3T_5+T_4^3+T_1T_6$
258 259 260 261	$\begin{bmatrix} 1&1&1&1&1&3&0 \\ -1&0&0&0&0&1&1 \end{bmatrix}$	$(4,0)$	$(4,1)$	341	$T_2^3T_4+T_3^2T_5^2+T_1T_6$ $T_2^3T_3+T_4T_5^3+T_1T_6$ $T_2^4+T_3T_4^3+T_1T_6$ $T_2T_3^2T_5+T_4^4+T_1T_6$
262 263 264	$\begin{bmatrix} 1&1&1&1&3&1&0 \\ -1&0&0&0&0&1&1 \end{bmatrix}$	$(6,0)$	$(2,1)$	31	$T_1^3T_6^3+T_2T_3T_4^4+T_5^2$ $T_1^3T_6^3+T_3^5T_4+T_5^2$ $T_1^3T_6^3+T_2T_3^3T_4^2+T_5^2$

Classification list 3.10.6. *Locally factorial Fano fourfoulds of Picard number two with a hypersurface Cox ring and an effective three-torus action: Specifying data for the sporadic cases with $s = 4$ and $\mu \notin \lambda$.*

ID	$[w_1,\dots,w_7]$	μ	$-\mathcal{K}$	$\mathcal{K}^4$	g
279 280	$\begin{bmatrix} 1&1&1&4&2&3&0 \\ 0&0&0&1&2&3&1 \end{bmatrix}$	$(6,6)$	$(6,1)$	117	$T_1^2T_4T_7^5+T_5^3+T_6^2$ $T_1T_2T_4T_7^5+T_5^3+T_6^2$
281	$\begin{bmatrix} 1&1&1&5&2&3&0 \\ 0&0&0&1&2&3&1 \end{bmatrix}$	$(6,6)$	$(7,1)$	157	$T_1T_4T_7^5+T_5^3+T_6^2$
282	$\begin{bmatrix} 1&1&1&6&2&3&0 \\ 0&0&0&1&2&3&1 \end{bmatrix}$	$(6,6)$	$(8,1)$	203	$T_4T_7^5+T_5^3+T_6^2$
341 342	$\begin{bmatrix} 1&1&1&10&4&6&0 \\ 0&0&0&1&2&3&1 \end{bmatrix}$	$(12,6)$	$(11,1)$	322	$T_1^2T_4T_7^5+T_5^3+T_6^2$ $T_1T_2T_4T_7^5+T_5^3+T_6^2$

ID	$[w_1, \ldots, w_7]$	μ	$-\mathcal{K}$	$\mathcal{K}^4$	g
343	$\begin{bmatrix} 1 & 1 & 1 & 11 & 4 & 6 & 0 \\ 0 & 0 & 0 & 1 & 2 & 3 & 1 \end{bmatrix}$	$(12, 6)$	$(12, 1)$	387	$T_1T_4T_7^5 + T_5^3 + T_6^2$
344	$\begin{bmatrix} 1 & 1 & 1 & 12 & 4 & 6 & 0 \\ 0 & 0 & 0 & 1 & 2 & 3 & 1 \end{bmatrix}$	$(12, 6)$	$(13, 1)$	458	$T_4T_7^5 + T_5^3 + T_6^2$
345	$\begin{bmatrix} 1 & 1 & 1 & 2 & 2 & 0 & 0 \\ 0 & 0 & 0 & 1 & 2 & 1 & 1 \end{bmatrix}$	$(4, 4)$	$(3, 1)$	68	$T_1^3T_2T_6^4 + T_3^2T_4T_7^3 + T_5^2$
346	$\begin{bmatrix} 1 & 1 & 1 & 3 & 2 & 0 & 0 \\ 0 & 0 & 0 & 1 & 2 & 1 & 1 \end{bmatrix}$	$(4, 4)$	$(4, 1)$	114	$T_1^3T_2T_6^4 + T_3T_4T_7^3 + T_5^2$
347 348	$\begin{bmatrix} 1 & 1 & 1 & 4 & 2 & 0 & 0 \\ 0 & 0 & 0 & 1 & 2 & 1 & 1 \end{bmatrix}$	$(4, 4)$	$(5, 1)$	172	$T_1^3T_2T_6^4 + T_4T_7^3 + T_5^2$ $T_1^2T_2T_3T_6^4 + T_4T_7^3 + T_5^2$
349	$\begin{bmatrix} 1 & 1 & 1 & 2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 \end{bmatrix}$	$(3, 3)$	$(3, 1)$	102	$T_1^2T_2T_6^3 + T_3T_4T_7^2 + T_5^3$
350 351	$\begin{bmatrix} 1 & 1 & 1 & 3 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 \end{bmatrix}$	$(3, 3)$	$(4, 1)$	171	$T_1^2T_2T_6^3 + T_4T_7^2 + T_5^3$ $T_1T_2T_3T_6^3 + T_4T_7^2 + T_5^3$
352	$\begin{bmatrix} 1 & 1 & 2 & 2 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}$	$(5, 4)$	$(3, 1)$	29	$T_1T_6^4 + T_2T_4^2T_7^2 + T_3T_5^3$
353	$\begin{bmatrix} 1 & 1 & 3 & 3 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}$	$(6, 4)$	$(4, 1)$	38	$T_1T_2T_6^4 + T_3T_5^3 + T_4^2T_7^2$
364 365	$\begin{bmatrix} 1 & 1 & 4 & 4 & 2 & 3 & 0 \\ 0 & 0 & 1 & 1 & 2 & 3 & 1 \end{bmatrix}$	$(6, 6)$	$(9, 2)$	144	$T_1^2T_3T_7^5 + T_5^3 + T_6^2$ $T_1T_2T_3T_7^5 + T_5^3 + T_6^2$
366	$\begin{bmatrix} 1 & 1 & 5 & 5 & 2 & 3 & 0 \\ 0 & 0 & 1 & 1 & 2 & 3 & 1 \end{bmatrix}$	$(6, 6)$	$(11, 2)$	176	$T_1T_3T_7^5 + T_5^3 + T_6^2$
367	$\begin{bmatrix} 1 & 1 & 6 & 6 & 2 & 3 & 0 \\ 0 & 0 & 1 & 1 & 2 & 3 & 1 \end{bmatrix}$	$(6, 6)$	$(13, 2)$	208	$T_3T_7^5 + T_5^3 + T_6^2$
368	$\begin{bmatrix} 1 & 1 & 2 & 2 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}$	$(4, 4)$	$(3, 1)$	32	$T_1^3T_2T_6^4 + T_3T_4T_7^2 + T_5^4$
369	$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 2 & 1 & 1 \end{bmatrix}$	$(2, 4)$	$(3, 2)$	128	$T_1T_3T_6^3 + T_2T_4T_7^3 + T_5^2$
370	$\begin{bmatrix} 1 & 1 & 2 & 2 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 2 & 1 & 1 \end{bmatrix}$	$(2, 4)$	$(5, 2)$	192	$T_3T_6^3 + T_4T_7^3 + T_5^2$
378 379	$\begin{bmatrix} 1 & 1 & 2 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}$	$(4, 4)$	$(3, 1)$	32	$T_1^2T_3T_7^3 + T_4^4 + T_5^3T_6$ $T_1T_2T_3T_7^3 + T_4^4 + T_5^3T_6$
380	$\begin{bmatrix} 1 & 1 & 3 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}$	$(4, 4)$	$(4, 1)$	44	$T_1T_3T_7^3 + T_4^4 + T_5^3T_6$
381	$\begin{bmatrix} 1 & 1 & 4 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}$	$(4, 4)$	$(5, 1)$	56	$T_3T_7^3 + T_4^4 + T_5^3T_6$
399 400	$\begin{bmatrix} 1 & 1 & 5 & 1 & 1 & 3 & 0 \\ 0 & 0 & 1 & 1 & 1 & 3 & 1 \end{bmatrix}$	$(6, 6)$	$(6, 1)$	34	$T_1T_3T_7^5 + T_4^5T_5 + T_6^2$ $T_1T_3T_7^5 + T_4^3T_5^3 + T_6^2$
401 402	$\begin{bmatrix} 1 & 1 & 6 & 1 & 1 & 3 & 0 \\ 0 & 0 & 1 & 1 & 1 & 3 & 1 \end{bmatrix}$	$(6, 6)$	$(7, 1)$	40	$T_3T_7^5 + T_4^5T_5 + T_6^2$ $T_3T_7^5 + T_4^3T_5^3 + T_6^2$
403 404	$\begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 3 & 1 & 1 & 1 \end{bmatrix}$	$(2, 6)$	$(2, 1)$	14	$T_2^2T_5^5T_6 + T_1T_3T_7^5 + T_4^2$ $T_2^2T_5^3T_6^3 + T_1T_3T_7^5 + T_4^2$

$$Q = \begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 \end{bmatrix} \quad \mu = (1,3) \quad -\mathcal{K} = (2,1) \quad \mathcal{K}^4 = 83$$

ID	g	ID	g	ID	g
265	$T_3^2T_4T_7^2+T_1T_5^3+T_2T_6^3$	266	$T_2^4T_7^3+T_1T_5^3+T_4T_6^2$	267	$T_2^4T_7^3+T_1T_5^3+T_3T_6^3$
268	$T_2^3T_3T_7^3+T_1T_6^3+T_4T_5^2$	269	$T_1^2T_2^2T_7^3+T_3T_6^3+T_4T_5^2$	270	$T_1T_5^3+T_3T_6^3+T_4^2T_7$

$$Q = \begin{bmatrix} 1 & 1 & 1 & 2 & 2 & 3 & 0 \\ 0 & 0 & 0 & 1 & 2 & 3 & 1 \end{bmatrix} \quad \mu = (6,6) \quad -\mathcal{K} = (4,1) \quad \mathcal{K}^4 = 55$$

ID	g	ID	g	ID	g
271	$T_1T_2T_4^2T_7^4+T_5^3+T_6^2$	272	$T_1^4T_4T_7^5+T_5^3+T_6^2$	273	$T_1^3T_2T_4T_7^5+T_5^3+T_6^2$
274	$T_1^2T_2^2T_4T_7^5+T_5^3+T_6^2$	275	$T_1^2T_2T_3T_4T_7^5+T_5^3+T_6^2$		

$$Q = \begin{bmatrix} 1 & 1 & 1 & 3 & 2 & 3 & 0 \\ 0 & 0 & 0 & 1 & 2 & 3 & 1 \end{bmatrix} \quad \mu = (6,6) \quad -\mathcal{K} = (5,1) \quad \mathcal{K}^4 = 83$$

ID	g	ID	g	ID	g
276	$T_1^3T_4T_7^5+T_5^3+T_6^2$	277	$T_1^2T_2T_4T_7^5+T_5^3+T_6^2$	278	$T_1T_2T_3T_4T_7^5+T_5^3+T_6^2$

$$Q = \begin{bmatrix} 1 & 1 & 1 & 3 & 4 & 6 & 0 \\ 0 & 0 & 0 & 1 & 2 & 3 & 1 \end{bmatrix} \quad \mu = (12,6) \quad -\mathcal{K} = (4,1) \quad \mathcal{K}^4 = 35$$

ID	g	ID	g	ID	g
283	$T_1^2T_2T_4^3T_7^3+T_5^3+T_6^2$	284	$T_1T_2T_3T_4^3T_7^3+T_5^3+T_6^2$	285	$T_1^5T_2T_4^2T_7^4+T_5^3+T_6^2$
286	$T_1^3T_2^3T_4^2T_7^4+T_5^3+T_6^2$	287	$T_1^4T_2T_3T_4^2T_7^4+T_5^3+T_6^2$	288	$T_1^3T_2^2T_3T_4^2T_7^4+T_5^3+T_6^2$
289	$T_1^9T_4T_7^5+T_5^3+T_6^2$	290	$T_1^8T_2T_4T_7^5+T_5^3+T_6^2$	291	$T_1^7T_2^2T_4T_7^5+T_5^3+T_6^2$
292	$T_1^6T_2^3T_4T_7^5+T_5^3+T_6^2$	293	$T_1^5T_2^4T_4T_7^5+T_5^3+T_6^2$	294	$T_1^7T_2T_3T_4T_7^5+T_5^3+T_6^2$
295	$T_1^6T_2^2T_3T_4T_7^5+T_5^3+T_6^2$	296	$T_1^5T_2^3T_3T_4T_7^5+T_5^3+T_6^2$	297	$T_1^4T_2^4T_3T_4T_7^5+T_5^3+T_6^2$
298	$T_1^5T_2^2T_3^2T_4T_7^5+T_5^3+T_6^2$	299	$T_1^4T_2^3T_3^2T_4T_7^5+T_5^3+T_6^2$	300	$T_1^3T_2^3T_3^3T_4T_7^5+T_5^3+T_6^2$

$$Q = \begin{bmatrix} 1 & 1 & 1 & 4 & 4 & 6 & 0 \\ 0 & 0 & 0 & 1 & 2 & 3 & 1 \end{bmatrix} \quad \mu = (12,6) \quad -\mathcal{K} = (5,1) \quad \mathcal{K}^4 = 58$$

ID	g	ID	g	ID	g
301	$T_1^3T_2T_4^2T_7^4+T_5^3+T_6^2$	302	$T_1^2T_2T_3T_4^2T_7^4+T_5^3+T_6^2$	303	$T_1^8T_4T_7^5+T_5^3+T_6^2$
304	$T_1^7T_2T_4T_7^5+T_5^3+T_6^2$	305	$T_1^6T_2^2T_4T_7^5+T_5^3+T_6^2$	306	$T_1^5T_2^3T_4T_7^5+T_5^3+T_6^2$
307	$T_1^4T_2^4T_4T_7^5+T_5^3+T_6^2$	308	$T_1^6T_2T_3T_4T_7^5+T_5^3+T_6^2$	309	$T_1^5T_2^2T_3T_4T_7^5+T_5^3+T_6^2$
310	$T_1^4T_2^3T_3T_4T_7^5+T_5^3+T_6^2$	311	$T_1^4T_2^2T_3^2T_4T_7^5+T_5^3+T_6^2$	312	$T_1^3T_2^3T_3^2T_4T_7^5+T_5^3+T_6^2$

$$Q = \begin{bmatrix} 1 & 1 & 1 & 5 & 4 & 6 & 0 \\ 0 & 0 & 0 & 1 & 2 & 3 & 1 \end{bmatrix} \quad \mu = (12,6) \quad -\mathcal{K} = (6,1) \quad \mathcal{K}^4 = 87$$

ID	g	ID	g	ID	g
313	$T_1T_2T_4^2T_7^4+T_5^3+T_6^2$	314	$T_1^7T_4T_7^5+T_5^3+T_6^2$	315	$T_1^6T_2T_4T_7^5+T_5^3+T_6^2$
316	$T_1^5T_2^2T_4T_7^5+T_5^3+T_6^2$	317	$T_1^4T_2^3T_4T_7^5+T_5^3+T_6^2$	318	$T_1^5T_2T_3T_4T_7^5+T_5^3+T_6^2$
319	$T_1^4T_2^2T_3T_4T_7^5+T_5^3+T_6^2$	320	$T_1^3T_2^3T_3T_4T_7^5+T_5^3+T_6^2$	321	$T_1^3T_2^2T_3^2T_4T_7^5+T_5^3+T_6^2$

$$Q = \begin{bmatrix} 1 & 1 & 1 & 6 & 4 & 6 & 0 \\ 0 & 0 & 0 & 1 & 2 & 3 & 1 \end{bmatrix} \quad \mu = (12,6) \quad -\mathcal{K} = (7,1) \quad \mathcal{K}^4 = 122$$

ID	g	ID	g	ID	g
322	$T_1^6T_4T_7^5+T_5^3+T_6^2$	323	$T_1^5T_2T_4T_7^5+T_5^3+T_6^2$	324	$T_1^4T_2^2T_4T_7^5+T_5^3+T_6^2$

$$Q = \begin{bmatrix} 1 & 1 & 1 & 6 & 4 & 6 & 0 \\ 0 & 0 & 0 & 1 & 2 & 3 & 1 \end{bmatrix} \quad \mu = (12,6) \quad -\mathcal{K} = (7,1) \quad \mathcal{K}^4 = 122$$

ID	g	ID	g	ID	g
325	$T_1^3T_2^3T_4T_7^5+T_5^3+T_6^2$	326	$T_1^4T_2T_3T_4T_7^5+T_5^3+T_6^2$	327	$T_1^3T_2^2T_3T_4T_7^5+T_5^3+T_6^2$
328	$T_1^2T_2^2T_3^2T_4T_7^5+T_5^3+T_6^2$				

$$Q = \begin{bmatrix} 1 & 1 & 1 & 7 & 4 & 6 & 0 \\ 0 & 0 & 0 & 1 & 2 & 3 & 1 \end{bmatrix} \quad \mu = (12,6) \quad -\mathcal{K} = (8,1) \quad \mathcal{K}^4 = 163$$

ID	g	ID	g	ID	g
329	$T_1^5T_4T_7^5+T_5^3+T_6^2$	330	$T_1^4T_2T_4T_7^5+T_5^3+T_6^2$	331	$T_1^3T_2^2T_4T_7^5+T_5^3+T_6^2$
332	$T_1^3T_2T_3T_4T_7^5+T_5^3+T_6^2$	333	$T_1^2T_2^2T_3T_4T_7^5+T_5^3+T_6^2$		

$$Q = \begin{bmatrix} 1 & 1 & 1 & 8 & 4 & 6 & 0 \\ 0 & 0 & 0 & 1 & 2 & 3 & 1 \end{bmatrix} \quad \mu = (12,6) \quad -\mathcal{K} = (9,1) \quad \mathcal{K}^4 = 210$$

ID	g	ID	g	ID	g
334	$T_1^4T_4T_7^5+T_5^3+T_6^2$	335	$T_1^3T_2T_4T_7^5+T_5^3+T_6^2$	336	$T_1^2T_2^2T_4T_7^5+T_5^3+T_6^2$
337	$T_1^2T_2T_3T_4T_7^5+T_5^3+T_6^2$				

$$Q = \begin{bmatrix} 1 & 1 & 1 & 9 & 4 & 6 & 0 \\ 0 & 0 & 0 & 1 & 2 & 3 & 1 \end{bmatrix} \quad \mu = (12,6) \quad -\mathcal{K} = (10,1) \quad \mathcal{K}^4 = 263$$

ID	g	ID	g	ID	g
338	$T_1^3T_4T_7^5+T_5^3+T_6^2$	339	$T_1^2T_2T_4T_7^5+T_5^3+T_6^2$	340	$T_1T_2T_3T_4T_7^5+T_5^3+T_6^2$

$$Q = \begin{bmatrix} 1 & 1 & 2 & 2 & 2 & 3 & 0 \\ 0 & 0 & 1 & 1 & 2 & 3 & 1 \end{bmatrix} \quad \mu = (6,6) \quad -\mathcal{K} = (5,2) \quad \mathcal{K}^4 = 80$$

ID	g	ID	g	ID	g
354	$T_3^2T_4T_7^3+T_5^3+T_6^2$	355	$T_1T_2T_3^2T_7^4+T_5^3+T_6^2$	356	$T_1^2T_3T_4T_7^4+T_5^3+T_6^2$
357	$T_1T_2T_3T_4T_7^4+T_5^3+T_6^2$	358	$T_1^4T_3T_7^5+T_5^3+T_6^2$	359	$T_1^3T_2T_3T_7^5+T_5^3+T_6^2$
360	$T_1^2T_2^2T_3T_7^5+T_5^3+T_6^2$				

$$Q = \begin{bmatrix} 1 & 1 & 3 & 3 & 2 & 3 & 0 \\ 0 & 0 & 1 & 1 & 2 & 3 & 1 \end{bmatrix} \quad \mu = (6,6) \quad -\mathcal{K} = (7,2) \quad \mathcal{K}^4 = 112$$

ID	g	ID	g	ID	g
361	$T_3T_4T_7^4+T_5^3+T_6^2$	362	$T_1^3T_3T_7^5+T_5^3+T_6^2$	363	$T_1^2T_2T_3T_7^5+T_5^3+T_6^2$

$$Q = \begin{bmatrix} 1 & 1 & 1 & 1 & 2 & 0 & 0 \\ 0 & 0 & 1 & 1 & 3 & 1 & 1 \end{bmatrix} \quad \mu = (4,6) \quad -\mathcal{K} = (2,1) \quad \mathcal{K}^4 = 12$$

ID	g	ID	g	ID	g
371	$T_1^3T_2T_6^6+T_3^3T_4T_7^2+T_5^2$	372	$T_1T_3^3T_6^3+T_2T_4^3T_7^3+T_5^2$	373	$T_1^3T_3T_6^5+T_2T_4^3T_7^3+T_5^2$

$$Q = \begin{bmatrix} 1 & 1 & 2 & 2 & 3 & 0 & 0 \\ 0 & 0 & 1 & 1 & 3 & 1 & 1 \end{bmatrix} \quad \mu = (6,6) \quad -\mathcal{K} = (3,1) \quad \mathcal{K}^4 = 16$$

ID	g	ID	g	ID	g
374	$T_1T_2T_3^2T_6^4+T_4^3T_7^3+T_5^2$	375	$T_1^4T_3T_6^5+T_4^3T_7^3+T_5^2$	376	$T_1^3T_2T_3T_6^5+T_4^3T_7^3+T_5^2$
377	$T_1^2T_2^2T_3T_6^5+T_4^3T_7^3+T_5^2$				

$$Q = \begin{bmatrix} 1 & 1 & 2 & 1 & 1 & 3 & 0 \\ 0 & 0 & 1 & 1 & 1 & 3 & 1 \end{bmatrix} \quad \mu = (6,6) \quad -\mathcal{K} = (3,1) \quad \mathcal{K}^4 = 16$$

ID	g	ID	g	ID	g
382	$T_3^3T_7^3+T_4^5T_5+T_6^2$	383	$T_1T_2T_3^2T_7^4+T_4^5T_5+T_6^2$	384	$T_1T_2T_3^2T_7^4+T_4^3T_5^3+T_6^2$
385	$T_1^4T_3T_7^5+T_4^5T_5+T_6^2$	386	$T_1^4T_3T_7^5+T_4^3T_5^3+T_6^2$	387	$T_1^3T_2T_3T_7^5+T_4^5T_5+T_6^2$
388	$T_1^3T_2T_3T_7^5+T_4^3T_5^3+T_6^2$	389	$T_1^2T_2^2T_3T_7^5+T_4^5T_5+T_6^2$	390	$T_1^2T_2^2T_3T_7^5+T_4^3T_5^3+T_6^2$

$$Q = \begin{bmatrix} 1 & 1 & 3 & 1 & 1 & 3 & 0 \\ 0 & 0 & 1 & 1 & 1 & 3 & 1 \end{bmatrix} \quad \mu = (6,6) \quad -\mathcal{K} = (4,1) \quad \mathcal{K}^4 = 22$$

ID	g	ID	g	ID	g
391	$T_1^3T_3T_7^5+T_4^5T_5+T_6^2$	392	$T_1^3T_3T_7^5+T_4^3T_5^3+T_6^2$	393	$T_1^2T_2T_3T_7^5+T_4^5T_5+T_6^2$
394	$T_1^2T_2T_3T_7^5+T_4^3T_5^3+T_6^2$				

$$Q = \begin{bmatrix} 1 & 1 & 4 & 1 & 1 & 3 & 0 \\ 0 & 0 & 1 & 1 & 1 & 3 & 1 \end{bmatrix} \quad \mu = (6,6) \quad -\mathcal{K} = (5,1) \quad \mathcal{K}^4 = 28$$

ID	g	ID	g	ID	g
395	$T_1^2T_3T_7^5+T_4^5T_5+T_6^2$	396	$T_1^2T_3T_7^5+T_4^3T_5^3+T_6^2$	397	$T_1T_2T_3T_7^5+T_4^5T_5+T_6^2$
398	$T_1T_2T_3T_7^5+T_4^3T_5^3+T_6^2$				

$$Q = \begin{bmatrix} 1 & 1 & 2 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 3 & 1 & 1 & 1 \end{bmatrix} \quad \mu = (2,6) \quad -\mathcal{K} = (3,1) \quad \mathcal{K}^4 = 20$$

ID	g	ID	g	ID	g
405	$T_1^2T_5^5T_6+T_3T_7^5+T_4^2$	406	$T_1^2T_5^3T_6^3+T_3T_7^5+T_4^2$	407	$T_1T_2T_5^6+T_3T_7^5+T_4^2$
408	$T_1T_2T_5^5T_6+T_3T_7^5+T_4^2$	409	$T_1T_2T_5^4T_6^2+T_3T_7^5+T_4^2$	410	$T_1T_2T_5^3T_6^3+T_3T_7^5+T_4^2$

Classification list 3.10.7. *Locally factorial Fano fourfoulds of Picard number two with a hypersurface Cox ring and an effective three-torus action: Specifying data for the series with $s = 4$.*

ID	$[w_1, \dots, w_7]$	$\begin{bmatrix} \mu \\ -\mathcal{K} \end{bmatrix}$	g	
S18	$\begin{bmatrix} 1 & 1 & a & a & c & c & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}$	$\begin{matrix} (a,4) \\ (a+2c+2,1) \end{matrix}$	$T_1^{(a-4c)}T_5^4 + T_2^{(a-4c)}T_6^4 + T_4T_7^3$	$c \geq 1, a > 4c,$ a odd
S22	$\begin{bmatrix} 1 & 1 & a & a & 2 & 3 & 0 \\ 0 & 0 & 1 & 1 & 2 & 3 & 1 \end{bmatrix}$	$\begin{matrix} (6,6) \\ (2a+1,2) \end{matrix}$	$T_1^5T_2T_7^6 + T_5^3 + T_6^2$	$a \geq 2$
S25	$\begin{bmatrix} 1 & 1 & a & a & c & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}$	$\begin{matrix} (4c,4) \\ (2a-3c+2,1) \end{matrix}$	$T_1^{(a-4c)}T_3T_6^3 + T_2^{(a-4c)}T_4T_7^3 + T_5^4$	$c \geq 1, a > 4c$
S26	$\begin{bmatrix} 1 & 1 & 4a & 4a & a & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}$	$\begin{matrix} (4a,4) \\ (5a+2,1) \end{matrix}$	$T_1^{(4a-l)}T_2^lT_6^4 + T_4T_7^3 + T_5^4$	$a \geq 1, l$ odd, $0 \leq l < 2c$
S27	$\begin{bmatrix} 1 & 1 & a & a & c & 0 & 0 \\ 0 & 0 & 1 & 1 & 3 & 1 & 1 \end{bmatrix}$	$\begin{matrix} (2c,6) \\ (2a-c+2,1) \end{matrix}$	$T_1^{(2c-a)}T_3T_6^5 + T_2^{(2c-a)}T_4T_7^5 + T_5^2$	$a, c \geq 1,$ $c-1 \leq a < 2c$
S28	$\begin{bmatrix} 1 & 1 & a & a & a & 0 & 0 \\ 0 & 0 & 1 & 1 & 3 & 1 & 1 \end{bmatrix}$	$\begin{matrix} (2a,6) \\ (a+2,1) \end{matrix}$	$T_1^{(2a-l)}T_2^lT_6^6 + T_3T_4T_7^4 + T_5^2$	$a \geq 1, l$ odd, $1 \leq l \leq a$

ID	$[w_1,\dots,w_7]$	$\begin{bmatrix}\mu\\-\mathcal{K}\end{bmatrix}$	g	
S29	$\begin{bmatrix}1&1&2a&2a&a&0&0\\0&0&1&1&3&1&1\end{bmatrix}$	$(2a,6)$ $(3a+2,1)$	$T_5^2+T_1^{(2a-l)}T_2^lT_6^6+T_4T_7^5$	$a\geq 1, l$ odd, $1\leq l\leq a$
S30	$\begin{bmatrix}1&1&a&1&1&1&0\\0&0&1&1&1&1&1\end{bmatrix}$	$(4,4)$ $(a+1,1)$	$T_1^3T_2T_7^4+T_4^3T_5+T_6^4$	$a\geq 2$
S34	$\begin{bmatrix}1&1&a&c&0&0&0\\0&0&1&1&1&1&1\end{bmatrix}$	$(c,4)$ $(a+2,1)$	$T_1^cT_5^4+T_2^cT_6^4+T_4T_7^3$	$c\geq 1, a>c$, c odd

$$Q = \begin{bmatrix}1&1&1&a&2&3&0\\0&0&0&1&2&3&1\end{bmatrix} \qquad \mu = (6,6) \qquad -\mathcal{K} = (a+2,1)$$

ID	g	ID	g	ID	g
S5	$T_1^5T_2T_7^6+T_5^3+T_6^2$, $a\geq 2$	S6	$T_1^4T_2T_3T_7^6+T_5^3+T_6^2$, $a\geq 2$	S7	$T_1^3T_2^2T_3T_7^6+T_5^3+T_6^2$, $a\geq 2$

$$Q = \begin{bmatrix}1&1&1&a&4&6&0\\0&0&0&1&2&3&1\end{bmatrix} \qquad \mu = (12,6) \qquad -\mathcal{K} = (a+1,1)$$

ID	g	ID	g		
S8	$T_1^{11}T_2T_7^6+T_5^3+T_6^2$, $a\geq 3$	S9	$T_1^7T_2^5T_7^6+T_5^3+T_6^2$, $a\geq 3$	S10	$T_1^{10}T_2T_3T_7^6+T_5^3+T_6^2$, $a\geq 3$
S11	$T_1^9T_2^2T_3T_7^6+T_5^3+T_6^2$, $a\geq 3$	S12	$T_1^8T_2^3T_3T_7^6+T_5^3+T_6^2$, $a\geq 3$	S13	$T_1^7T_2^4T_3T_7^6+T_5^3+T_6^2$, $a\geq 3$
S14	$T_1^6T_2^5T_3T_7^6+T_5^3+T_6^2$, $a\geq 3$	S15	$T_1^7T_2^3T_3^2T_7^6+T_5^3+T_6^2$, $a\geq 3$	S16	$T_1^5T_2^5T_3^2T_7^6+T_5^3+T_6^2$, $a\geq 3$
S17	$T_1^5T_2^4T_3^3T_7^6+T_5^3+T_6^2$, $a\geq 3$				

$$Q = \begin{bmatrix}1&1&a&a&1&1&0\\0&0&1&1&1&1&1\end{bmatrix} \qquad \mu = (a+3,4) \qquad -\mathcal{K} = (a+1,1)$$

ID	g	ID	g
S19	$T_3T_5^3+T_4T_6^3+T_1^{(a+3-l)}T_2^lT_7^4$, $a\geq 2, 0\leq l\leq (a+3)/2$	S20	$T_1^{(a-1)}T_5^4+T_3T_6^3+T_2^{(a+3)}T_7^4$, $a\geq 2, a$ even
S21	$T_1^{(a-1)}T_5^4+T_3T_6^3+T_2^3T_4T_7^3$, $a\geq 2$		

$$Q = \begin{bmatrix}1&1&a&a&c&0&0\\0&0&1&1&1&1&1\end{bmatrix} \qquad \mu = (a,4) \qquad -\mathcal{K} = (a+c+2,1)$$

ID	g	ID	g
S23	$T_1^{(a-4c)}T_5^4+T_2^aT_6^4+T_4T_7^3$, $c\geq 1, a>4c$	S24	$T_1^{(a-l)}T_2^lT_5^4+T_3T_6^3+T_4T_7^3$, $c\geq 1, a>4c, 0\leq l\leq a/2$

$Q = \begin{bmatrix} 1 & 1 & a & 1 & 1 & 3 & 0 \\ 0 & 0 & 1 & 1 & 1 & 3 & 1 \end{bmatrix}$		$\mu = (6,6)$	$-\mathcal{K} = (a{+}1,1)$
ID	g	ID	g
S31	$T_1^5T_2T_7^6{+}T_4^5T_5{+}T_6^2$, $a \geq 2$	S32	$T_1^5T_2T_7^6{+}T_4^3T_5^3{+}T_6^2$, $a \geq 2$
S33	$T_1^3T_2^3T_7^6{+}T_4^5T_5{+}T_6^2$, $a \geq 2$		
$Q = \begin{bmatrix} 1 & 1 & a+1 & a & 0 & 0 & 0 \\ 0 & 0 & & 1 & 1 & 1 & 1 & 1 \end{bmatrix}$		$\mu = (a{+}1,4)$	$-\mathcal{K} = (a{+}2,1)$
ID	g	ID	g
S35	$T_1^{(a+1)}T_5^4{+}T_2T_4T_6^3{+}T_3T_7^3$, $a \geq 1$	S36	$T_1^{(a+1)}T_5^4{+}T_2^{(a+1)}T_6^4{+}T_3T_7^3$, $a \geq 1$, a even

Classification list 3.10.8. *Locally factorial Fano fourfoulds of Picard number two with a hypersurface Cox ring and an effective three-torus action: Specifying data for the cases with $s = 5$ and $\mu \in \lambda$.*

ID	$[w_1, \ldots, w_7]$	μ	$-\mathcal{K}$	$\mathcal{K}^4$	g
411	$\begin{bmatrix} 1 & 1 & 1 & 2 & 1 & 0 & 0 \\ 0 & 1 & 1 & 3 & 2 & 1 & 1 \end{bmatrix}$	$(4,6)$	$(2,3)$	65	$T_2^3T_3T_7^2 + T_1T_5^3 + T_4^2$
412					$T_2^3T_3T_6T_7 + T_1T_5^3 + T_4^2$
413					$T_2^4T_6T_7 + T_1T_5^3 + T_4^2$
414					$T_2^2T_3^2T_6T_7 + T_1T_5^3 + T_4^2$

Classification list 3.10.9. *Locally factorial Fano fourfoulds of Picard number two with a hypersurface Cox ring and an effective three-torus action: Specifying data for the sporadic cases with $s = 5$ and $\mu \notin \lambda$.*

ID	$[w_1, \ldots, w_7]$	μ	$-\mathcal{K}$	$\mathcal{K}^4$	g
415	$\begin{bmatrix} 1 & 1 & 1 & 2 & 3 & 1 & 0 \\ 0 & 0 & 0 & 1 & 2 & 1 & 1 \end{bmatrix}$	$(6,4)$	$(3,1)$	50	$T_3^4T_4T_7^3 + T_1T_2T_6^4 + T_5^2$
416					$T_1T_2T_6^4 + T_4^3T_7 + T_5^2$
417					$T_1^5T_2T_7^4 + T_3T_4T_6^3 + T_5^2$
418					$T_1^3T_2^3T_7^4 + T_3T_4T_6^3 + T_5^2$
419	$\begin{bmatrix} 1 & 1 & 1 & 3 & 3 & 1 & 0 \\ 0 & 0 & 0 & 1 & 2 & 1 & 1 \end{bmatrix}$	$(6,4)$	$(4,1)$	90	$T_3^3T_4T_7^3 + T_1T_2T_6^4 + T_5^2$
420					$T_1^5T_2T_7^4 + T_4T_6^3 + T_5^2$
421					$T_1^4T_2T_3T_7^4 + T_4T_6^3 + T_5^2$
422					$T_1^3T_2^2T_3T_7^4 + T_4T_6^3 + T_5^2$

ID	$[w_1,\ldots,w_7]$	μ	$-\mathcal{K}$	$\mathcal{K}^4$	g
423	$\begin{bmatrix} 1&1&1&4&3&1&0 \\ 0&0&0&1&2&1&1 \end{bmatrix}$	$(6,4)$	$(5,1)$	142	$T_1T_2T_6^4+T_3^2T_4T_7^3+T_5^2$
424	$\begin{bmatrix} 1&1&1&5&3&1&0 \\ 0&0&0&1&2&1&1 \end{bmatrix}$	$(6,4)$	$(6,1)$	206	$T_1T_2T_6^4+T_3T_4T_7^3+T_5^2$
425	$\begin{bmatrix} 1&1&1&6&3&1&0 \\ 0&0&0&1&2&1&1 \end{bmatrix}$	$(6,4)$	$(7,1)$	282	$T_1T_2T_6^4+T_4T_7^3+T_5^2$
426 427	$\begin{bmatrix} 1&1&2&2&4&1&0 \\ 0&0&1&1&3&1&1 \end{bmatrix}$	$(8,6)$	$(3,1)$	14	$T_1T_2T_6^6+T_3^3T_4T_7^2+T_5^2$ $T_2^2T_4^3T_7^3+T_1T_3T_6^5+T_5^2$
428	$\begin{bmatrix} 1&1&3&3&5&1&0 \\ 0&0&1&1&3&1&1 \end{bmatrix}$	$(10,6)$	$(4,1)$	18	$T_1^2T_3T_6^5+T_2T_4^3T_7^3+T_5^2$
429 430	$\begin{bmatrix} 1&1&4&4&6&1&0 \\ 0&0&1&1&3&1&1 \end{bmatrix}$	$(12,6)$	$(5,1)$	22	$T_1^3T_3T_6^5+T_4^3T_7^3+T_5^2$ $T_1^2T_2T_3T_6^5+T_4^3T_7^3+T_5^2$
431 432	$\begin{bmatrix} 1&1&4&4&7&2&0 \\ 0&0&1&1&3&1&1 \end{bmatrix}$	$(14,6)$	$(5,1)$	20	$T_2^2T_4^3T_7^3+T_3T_6^5+T_5^2$ $T_1T_2T_4^3T_7^3+T_3T_6^5+T_5^2$
433	$\begin{bmatrix} 1&1&5&5&8&2&0 \\ 0&0&1&1&3&1&1 \end{bmatrix}$	$(16,6)$	$(6,1)$	24	$T_1T_3T_6^5+T_2T_4^3T_7^3+T_5^2$
434 435	$\begin{bmatrix} 1&1&6&6&9&2&0 \\ 0&0&1&1&3&1&1 \end{bmatrix}$	$(18,6)$	$(7,1)$	28	$T_1^2T_3T_6^5+T_4^3T_7^3+T_5^2$ $T_1T_2T_3T_6^5+T_4^3T_7^3+T_5^2$
436	$\begin{bmatrix} 1&1&7&7&11&3&0 \\ 0&0&1&1&3&1&1 \end{bmatrix}$	$(22,6)$	$(8,1)$	30	$T_2T_4^3T_7^3+T_3T_6^5+T_5^2$
437	$\begin{bmatrix} 1&1&8&8&12&3&0 \\ 0&0&1&1&3&1&1 \end{bmatrix}$	$(24,6)$	$(9,1)$	34	$T_1T_3T_6^5+T_4^3T_7^3+T_5^2$
438	$\begin{bmatrix} 1&1&10&10&15&4&0 \\ 0&0&1&1&3&1&1 \end{bmatrix}$	$(30,6)$	$(11,1)$	40	$T_3T_6^5+T_4^3T_7^3+T_5^2$
439 440	$\begin{bmatrix} 1&1&3&2&3&0&0 \\ 0&0&1&1&3&1&1 \end{bmatrix}$	$(6,6)$	$(4,1)$	22	$T_1^3T_3T_6^5+T_4^3T_7^3+T_5^2$ $T_1^2T_2T_3T_6^5+T_4^3T_7^3+T_5^2$
441 442	$\begin{bmatrix} 1&1&4&2&3&0&0 \\ 0&0&1&1&3&1&1 \end{bmatrix}$	$(6,6)$	$(5,1)$	28	$T_1^2T_3T_6^5+T_4^3T_7^3+T_5^2$ $T_1T_2T_3T_6^5+T_4^3T_7^3+T_5^2$
443	$\begin{bmatrix} 1&1&5&2&3&0&0 \\ 0&0&1&1&3&1&1 \end{bmatrix}$	$(6,6)$	$(6,1)$	34	$T_1T_3T_6^5+T_4^3T_7^3+T_5^2$
444	$\begin{bmatrix} 1&1&6&2&3&0&0 \\ 0&0&1&1&3&1&1 \end{bmatrix}$	$(6,6)$	$(7,1)$	40	$T_3T_6^5+T_4^3T_7^3+T_5^2$
445	$\begin{bmatrix} 1&1&3&1&2&0&0 \\ 0&0&1&1&3&1&1 \end{bmatrix}$	$(4,6)$	$(4,1)$	24	$T_1T_3T_6^5+T_2T_4^3T_7^3+T_5^2$
446	$\begin{bmatrix} 1&1&4&1&2&0&0 \\ 0&0&1&1&3&1&1 \end{bmatrix}$	$(4,6)$	$(5,1)$	30	$T_2T_4^3T_7^3+T_3T_6^5+T_5^2$
447	$\begin{bmatrix} 1&1&2&1&2&0&0 \\ 0&0&1&1&3&1&1 \end{bmatrix}$	$(4,6)$	$(3,1)$	18	$T_1^3T_2T_6^6+T_3T_4^2T_7^3+T_5^2$

Classification list 3.10.10. *Locally factorial Fano fourfoulds of Picard number two with a hypersurface Cox ring and an effective three-torus action: Specifying data for the series with $s = 5$.*

ID	$[w_1,\ldots,w_7]$	$\begin{bmatrix}\mu\\-\mathcal{K}\end{bmatrix}$	g
S39	$\begin{bmatrix}1&1&a&a&b&c&0\\0&0&1&1&1&1&1\end{bmatrix}$	$(a,4)$ $(a+c+b+2,1)$	$T_1^{(a-4b)}T_5^4+T_2^{(a-4c)}T_6^4+T_4T_7^3$, $b>c\geq 1, a>4b, a$ odd
S42	$\begin{bmatrix}1&1&a&a&b&c&0\\0&0&1&1&1&1&1\end{bmatrix}$	$(4a,4)$ $(c+b-2a+2,1)$	$T_5^4+T_1^{(4b-a-3c)}T_3T_6^3+T_2^{(4b-a)}T_4T_7^3$, $b>c\geq 1, b>2c-1, 3b-c-1\leq a<4b-3c$
S43	$\begin{bmatrix}1&1&2a+2&2a+2&a+1&a&0\\0&0&1&1&1&1&1\end{bmatrix}$	$(4a+4,4)$ $(2a+3,1)$	$T_1^3T_2^2T_6^4+T_3T_4T_7^2+T_5^4$, $a\geq 1$
S44	$\begin{bmatrix}1&1&4a&4a&a&b&0\\0&0&1&1&1&1&1\end{bmatrix}$	$(4a,4)$ $(5a+b+2,1)$	$T_5^4+T_1^{(4a-4b-l)}T_2^lT_6^4+T_4T_7^3$, $a>b\geq 1, 0<l<2a+2b, l$ odd
S45	$\begin{bmatrix}1&1&a&a&a+2&1&0\\0&0&1&1&3&1&1\end{bmatrix}$	$(2a+4,6)$ $(a+1,1)$	$T_5^2+T_3T_4T_6^4+T_1^{(2a+4-l)}T_2^lT_7^6$, $a\geq 1, 0<l\leq a+2, l$ odd
S49	$\begin{bmatrix}1&1&a&a&b&c&0\\0&0&1&1&3&1&1\end{bmatrix}$	$(2b,6)$ $(2a-b+c+2,1)$	$T_5^2+T_1^{(2b-5c-a)}T_3T_6^5+T_2^{(2b-a)}T_4T_7^5$, $c\geq 1, b\geq 4c, b/3<a<2b-5c$
S53	$\begin{bmatrix}1&1&a&a&a&b&0\\0&0&1&1&3&1&1\end{bmatrix}$	$(2a,6)$ $(a+b+2,1)$	$T_5^2+T_1^{(2a-6b-l)}T_2^lT_6^6+T_3T_4T_7^4$, $b\geq 1, a>3b, 0<l\leq a-3b, l$ odd
S54	$\begin{bmatrix}1&1&2a&2a&a&b&0\\0&0&1&1&3&1&1\end{bmatrix}$	$(2a,6)$ $(b+3a+2,1)$	$T_5^2+T_1^{(2a-l)}T_2^lT_6^6+T_4T_7^5$, $b\geq 1, a>3b, 0<l\leq a, l$ odd
S55	$\begin{bmatrix}1&1&a+1&a&1&1&0\\0&0&1&1&1&1&1\end{bmatrix}$	$(a+3,4)$ $(a+2,1)$	$T_1^{(a-1)}T_5^4+T_4T_6^3+T_2^2T_3T_7^3$, $a\geq 2$
S56	$\begin{bmatrix}1&1&a+2&a&1&1&0\\0&0&1&1&1&1&1\end{bmatrix}$	$(a+3,4)$ $(a+3,1)$	$T_1^{(a-1)}T_5^4+T_4T_6^3+T_2T_3T_7^3$, $a\geq 2$
S57	$\begin{bmatrix}1&1&a+3&a&1&1&0\\0&0&1&1&1&1&1\end{bmatrix}$	$(a+3,4)$ $(a+4,1)$	$T_1^{(a-1-l)}T_2^lT_5^4+T_4T_6^3+T_3T_7^3$, $a\geq 2, 0\leq l\leq (a-1)/2$
S58	$\begin{bmatrix}1&1&a&b&1&1&0\\0&0&1&1&1&1&1\end{bmatrix}$	$(b+3,4)$ $(a+1,1)$	$T_1^{(b-1)}T_5^4+T_4T_6^3+T_2^{(b+3)}T_7^4$, $a>b\geq 2, b$ even
S59	$\begin{bmatrix}1&1&2a+1&2a&a&a&0\\0&0&1&1&1&1&1\end{bmatrix}$	$(4a+1,4)$ $(2a+2,1)$	$T_1T_5^4+T_2T_6^4+T_3T_4T_7^2$, $a\geq 1$
S60	$\begin{bmatrix}1&1&a&b&c&c&0\\0&0&1&1&1&1&1\end{bmatrix}$	$(a,4)$ $(2c+b+2,1)$	$T_1^{(a-4c)}T_5^4+T_2^{(a-4c)}T_6^4+T_3T_7^3$, $c\geq 1, b>2c-1, 4c<a\leq 1+b+2c, a$ odd
S61	$\begin{bmatrix}1&1&a&b&c&c&0\\0&0&1&1&1&1&1\end{bmatrix}$	$(b,4)$ $(a+2c+2,1)$	$T_1^{(b-4c)}T_5^4+T_2^{(b-4c)}T_6^4+T_4T_7^3$, $c\geq 1, b>4c, a>b, b$ odd
S64	$\begin{bmatrix}1&1&a&b&c&0&0\\0&0&1&1&1&1&1\end{bmatrix}$	$(b,4)$ $(a+c+2,1)$	$T_1^{(b-4c)}T_5^4+T_2^bT_6^4+T_4T_7^3$, $c\geq 1, b>4c, b<a\leq 1+b+c, b$ odd
S67	$\begin{bmatrix}1&1&a&b&c&0&0\\0&0&1&1&1&1&1\end{bmatrix}$	$(4c,4)$ $(a+b-3c+2,1)$	$T_5^4+T_1^{(4c-a)}T_3T_6^3+T_2^{(4c-b)}T_4T_7^3$, $c\geq 1, a>b>4c$
S68	$\begin{bmatrix}1&1&a&4b&b&0&0\\0&0&1&1&1&1&1\end{bmatrix}$	$(4b,4)$ $(a+b+2,1)$	$T_5^4+T_1^{(4b-l)}T_2^lT_6^4+T_4T_7^3$, $b\geq 1, 4b<a\leq 5b+1, 0<l<2b, l$ odd
S69	$\begin{bmatrix}1&1&2a&a-1&a&0&0\\0&0&1&1&3&1&1\end{bmatrix}$	$(2a,6)$ $(1+2a,1)$	$T_1T_2T_4^2T_7^4+T_3T_6^5+T_5^2$, $a\geq 2$
S70	$\begin{bmatrix}1&1&2b&a&b&0&0\\0&0&1&1&3&1&1\end{bmatrix}$	$(2a,6)$ $(3b-a+2,1)$	$T_5^2+T_3T_6^5+T_1^{(2b-a-l)}T_2^lT_4T_7^5$, $a,b\geq 1, b-1\leq a<2b$

ID	$[w_1,\dots,w_7]$	$\begin{bmatrix} \mu \\ -\mathcal{K} \end{bmatrix}$	g
S71	$\begin{bmatrix} 1 & 1 & a & b & c & 0 & 0 \\ 0 & 0 & 1 & 1 & 3 & 1 & 1 \end{bmatrix}$	$(2c,6)$ $(a+b-c+2,1)$	$T_5^2+T_1^{(2c-a)}T_3T_6^5+T_2^{(2c-b)}T_4T_7^5$, $c\geq 1, c-1\leq b<2c, b<a<2c$
S72	$\begin{bmatrix} 1 & 1 & a & 2 & 3 & 0 & 0 \\ 0 & 0 & 1 & 1 & 3 & 1 & 1 \end{bmatrix}$	$(6,6)$ $(a+1,1)$	$T_1^5T_2T_6^6+T_4^3T_7^3+T_5^2$, $a\geq 3$
S73	$\begin{bmatrix} 1 & 1 & a+1 & a-1 & a & 0 & 0 \\ 0 & 0 & 1 & 1 & 3 & 1 & 1 \end{bmatrix}$	$(2a,6)$ $(a+2,1)$	$T_1^{(2a-l)}T_2^lT_6^6+T_3T_4T_7^4+T_5^2$, $a\geq 2, 0<l\leq a$, l odd
S74	$\begin{bmatrix} 1 & 1 & 2b & a & b & 0 & 0 \\ 0 & 0 & 1 & 1 & 3 & 1 & 1 \end{bmatrix}$	$(2b,6)$ $(a+b+2,1)$	$T_5^2+T_1^{(2b-l)}T_2^lT_6^6+T_3T_7^5$, $a,b\geq 1, b-1\leq a<2b, 0<l\leq b$, l odd
S75	$\begin{bmatrix} 1 & 1 & a & 2b & b & 0 & 0 \\ 0 & 0 & 1 & 1 & 3 & 1 & 1 \end{bmatrix}$	$(2b,6)$ $(a+b+2,1)$	$T_5^2+T_1^{(2b-l)}T_2^lT_6^6+T_4T_7^5$, $b\geq 1, a>2b, 0<l\leq b$, l odd

$$Q = \begin{bmatrix} 1 & 1 & a & a & b & c & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & 1 \end{bmatrix} \quad \mu = (a+3c,4) \quad -\mathcal{K} = (a-2c+b+2,1)$$

ID	g	ID	g
S37	$T_1^{(a+3c-4b)}T_5^4+T_3T_6^3+T_2^{3c}T_4T_7^3$, $b>c\geq 1, b\geq 2c-1, a>4b-3c$	S38	$T_1^{(a+3c-4b)}T_5^4+T_3T_6^3+T_2^{(a+3c)}T_7^4$, $b>c\geq 1, b\geq 2c-1, a>4b-3c$, a or c odd

$$Q = \begin{bmatrix} 1 & 1 & 4a-3b & 4a-3b & a & b & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & 1 \end{bmatrix} \quad \mu = (4a,4) \quad -\mathcal{K} = (5a-5b+2,1)$$

ID	g	ID	g
S40	$T_5^4+T_3T_6^3+T_1^{(3b-l)}T_2^lT_4T_7^3$, $a>b\geq 1, a\geq 2b-1, 0\leq l\leq 3b/2$	S41	$T_5^4+T_3T_6^3+T_1^{(4a-l)}T_2^lT_7^4$, $a>b\geq 1, a\geq 2b-1, 0<l<2a$, l odd

$$Q = \begin{bmatrix} 1 & 1 & 2a-5b & 2a-5b & a & b & 0 \\ 0 & 0 & 1 & 1 & 3 & 1 & 1 \end{bmatrix} \quad \mu = (2a,6) \quad -\mathcal{K} = (3a-9b+2,1)$$

ID	g	ID	g
S46	$T_5^2+T_3T_6^5+T_1^{(10b-2a-l)}T_2^lT_4^2T_7^4$, $b\geq 1, 4b-1\leq a<5b$	S47	$T_5^2+T_3T_6^5+T_1^{(5b-l)}T_2^lT_4T_7^5$, $b\geq 1, a\geq 4b-1, 0\leq l\leq 5b/2$
S48	$T_5^2+T_3T_6^5+T_1^{(2a-l)}T_2^lT_7^6$, $b\geq 1, a\geq 4b-1, 0<l\leq a$, l odd		

$$Q = \begin{bmatrix} 1 & 1 & 2a+2 & 2a+2 & 3a+3 & a & 0 \\ 0 & 0 & 1 & 1 & 3 & 1 & 1 \end{bmatrix} \quad \mu = (6a+6,6) \quad -\mathcal{K} = (2a+3,1)$$

ID	g	ID	g
S50	$T_1^5T_2T_6^6+T_3^2T_4T_7^3+T_5^2$, $a\geq 1$	S51	$T_1^5T_2T_6^6+T_3^3T_7^3+T_5^2$, $a\geq 1$
S52	$T_1^3T_2^3T_6^6+T_3^2T_4T_7^3+T_5^2$, $a\geq 1$		

$Q = \left[\begin{smallmatrix} 1 & 1 & a & b & c & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & 1 \end{smallmatrix}\right]$	$\mu = (a,4)$	$-\mathcal{K} = (c+b+2,1)$	
ID	g	ID	g
S62	$T_1^{(a-4c)}T_5^4+T_3T_6^3+T_2^{(a-b)}T_4T_7^3$, $c \geq 1, b > 3c-1, 4c < a \leq 1+b+c$	S63	$T_1^{(a-4c)}T_5^4+T_2^aT_6^4+T_3T_7^3$, $c \geq 1, b > 3c-1, 4c < a \leq 1+b+c, a$ odd

$Q = \left[\begin{smallmatrix} 1 & 1 & 4b & a & b & 0 & 0 \\ 0 & 0 & & 1 & 1 & 1 & 1 \end{smallmatrix}\right]$	$\mu = (4b,4)$	$-\mathcal{K} = (a+b+2,1)$	
ID	g	ID	g
S65	$T_5^4+T_3T_6^3+T_1^{(4b-a-l)}T_2^lT_4T_7^3$, $b \geq 1, 3b-1 \leq a < 4b, 0 \leq l \leq (4b-a)/2$	S66	$T_5^4+T_1^{(4b-l)}T_2^lT_6^4+T_3T_7^3$, $b \geq 1, 3b-1 \leq a < 4b, 0 < l < 2b, l$ odd

Classification list 3.10.11. *Locally factorial Fano fourfoulds of Picard number two with a hypersurface Cox ring and an effective three-torus action: Specifying data for the series with $s = 6$.*

ID	$[w_1, \ldots, w_7]$	$\left[\begin{smallmatrix} \mu \\ -\mathcal{K} \end{smallmatrix}\right]$	g
S78	$\left[\begin{smallmatrix} 1 & 1 & b+3d & b & c & d & 0 \\ 0 & 0 & & 1 & 1 & 1 & 1 \end{smallmatrix}\right]$	$(b+3d,4)$ $(b+d+c+2,1)$	$T_1^{(b-4c+3d-l)}T_2^lT_5^4+T_4T_6^3+T_3T_7^3$, $c > d \geq 1, c \geq 2d-1, b > 4c-3d$, $0 \leq l \leq (b-4c+3d)/2$
S83	$\left[\begin{smallmatrix} 1 & 1 & a & b & c & d & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & 1 \end{smallmatrix}\right]$	$(b,4)$ $(d+c+a+2,1)$	$T_1^{(b-4c)}T_5^4+T_2^{(b-4d)}T_6^4+T_4T_7^3$, $c > d \geq 1, b > 4c, a > b, b$ odd
S88	$\left[\begin{smallmatrix} 1 & 1 & 4c & 4c-3d & c & d & 0 \\ 0 & 0 & 1 & & 1 & 1 & 1 & 1 \end{smallmatrix}\right]$	$(4c,4)$ $(5c-2d+2,1)$	$T_3T_7^3+T_4T_6^3+T_5^4$, $c > d \geq 1, c \geq 2d-1$
S93	$\left[\begin{smallmatrix} 1 & 1 & a & 4c & c & d & 0 \\ 0 & 0 & 1 & & 1 & 1 & 1 & 1 \end{smallmatrix}\right]$	$(4c,4)$ $(d+c+a+2,1)$	$T_5^4+T_1^{(4c-4d-l)}T_2^lT_6^4+T_4T_7^3$, $c > d \geq 1, a > 4c, 0 < l < 2c-2d, l$ odd
S102	$\left[\begin{smallmatrix} 1 & 1 & 2c & 2c-5d & c & d & 0 \\ 0 & 0 & 1 & & 1 & 3 & 1 & 1 \end{smallmatrix}\right]$	$(2c,4)$ $(3c-4d+2,3)$	$T_3T_7^5+T_4T_6^5+T_5^2$, $d \geq 1, c > 3, c \geq 4d-1$
S105	$\left[\begin{smallmatrix} 1 & 1 & 2c-b & b & c & d & 0 \\ 0 & 0 & & 1 & 1 & 3 & 1 & 1 \end{smallmatrix}\right]$	$(2c,4)$ $(c+d+2,3)$	$T_5^2+T_1^{(2c-6d-l)}T_2^lT_6^6+T_3T_4T_7^4$, $d \geq 1, c > 3d, b > 1, c-d-1 \leq b < c$, $0 < l \leq c-3d, l$ odd
S106	$\left[\begin{smallmatrix} 1 & 1 & a & 2c & c & d & 0 \\ 0 & 0 & 1 & & 1 & 3 & 1 & 1 \end{smallmatrix}\right]$	$(2c,4)$ $(d+c+a+2,3)$	$T_5^2+T_1^{(2c-6d-l)}T_2^lT_6^6+T_4T_7^5$, $d \geq 1, c > 3d, a > 2c, 0 < l \leq c-3d, l$ odd

$$Q = \begin{bmatrix} 1 & 1 & a & b & c & d & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & 1 \end{bmatrix} \quad \mu = (a+3d, 4) \quad -\mathcal{K} = (c+b-2d+2, 1)$$

ID	g	ID	g
S76	$T_1^{(a-4c+3d)}T_5^4+T_3T_6^3+T_2^{(a-b+3d)}T_4T_7^3$, $b>c>d\geq 1, a>4c-3d$, $a>b-3d, a\leq 1+b+c-2d$	S77	$T_1^{(a-4c+3d)}T_5^4+T_3T_6^3+T_2^{(a+3d)}T_7^4$, $b>c>d\geq 1, 4c-3d<a\leq 1+b+c-2d$, $a+3d$ odd

$$Q = \begin{bmatrix} 1 & 1 & a & b & c & d & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & 1 \end{bmatrix} \quad \mu = (a, 4) \quad -\mathcal{K} = (b+d+c+2, 1)$$

ID	g	ID	g
S79	$T_1^{(a-4c)}T_5^4+T_2^{(a-b-3d)}T_4T_6^3+T_3T_7^3$, $b>c>d\geq 1, a>4c, a>b+3d$, $a\leq 1+b+c+d$	S80	$T_1^{(a-4c)}T_5^4+T_2^{(a-4d)}T_6^4+T_3T_7^3$, $c>d\geq 1, b\geq 3c-d, 4c<a\leq 1+b+c+d$, a odd

$$Q = \begin{bmatrix} 1 & 1 & a & b & c & d & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & 1 \end{bmatrix} \quad \mu = (b+3d, 4) \quad -\mathcal{K} = (c+a-2d+2, 1)$$

ID	g	ID	g
S81	$T_1^{(b-4c+3d)}T_5^4+T_4T_6^3+T_2^{(b+3d-a)}T_3T_7^3$, $c>d\geq 1, b>4c-3d, b<a<b+3d$	S82	$T_1^{(b-4c+3d)}T_5^4+T_4T_6^3+T_2^{(b+3d)}T_7^4$, $c>d\geq 1, c\geq 2d-1, b>4c-3d$, $a>b, b+3d$ odd

$$Q = \begin{bmatrix} 1 & 1 & 4c-3d & b & c & d & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & 1 \end{bmatrix} \quad \mu = (4c, 4) \quad -\mathcal{K} = (c+b-2d+2, 1)$$

ID	g	ID	g
S84	$T_5^4+T_3T_6^3+T_1^{(4c-b-l)}T_2^lT_4T_7^3$, $c>d\geq 1, 3c-d-1\leq b<4c-3d$	S85	$T_5^4+T_3T_6^3+T_1^{(4c-l)}T_2^lT_7^4$, $d\geq 1, c>2d-1, 3c-d-1\leq b<4c-3d$, $0<l<2c, l$ odd

$$Q = \begin{bmatrix} 1 & 1 & a & b & c & d & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & 1 \end{bmatrix} \quad \mu = (4c, 4) \quad -\mathcal{K} = (d+b+a-3c+2, 1)$$

ID	g	ID	g
S86	$T_5^4+T_1^{(4c-a-3d)}T_3T_6^3+T_2^{(4c-b)}T_4T_7^3$, $c>d\geq 1, 3c-d-1\leq b<4c, a>4c-3d$	S87	$T_5^4+T_1^{(4c-b-3d)}T_4T_6^3+T_2^{(4c-a)}T_3T_7^3$, $d\geq 1, c>2d-1, 3c-d-1\leq b<4c-3d$, $b<a<4c$

$$Q = \begin{bmatrix} 1 & 1 & 4c & b & c & d & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & 1 \end{bmatrix} \quad \mu = (4c, 4) \quad -\mathcal{K} = (b+d+c+2, 1)$$

ID	g	ID	g
S89	$T_5^4+T_1^{(4c-b-3d-l)}T_2^lT_4T_6^3+T_3T_7^3$, $d\geq 1, c>2d-1, 3c-d-1\leq b<4c-3d$	S90	$T_5^4+T_1^{(4c-4d-l)}T_2^lT_6^4+T_3T_7^3$, $c>d\geq 1, 3c-d-1\leq b<4c$, $0<l<2c-2d, l$ odd

$$Q = \begin{bmatrix} 1 & 1 & a & 4c-3d & c & d & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & 1 \end{bmatrix} \quad \mu = (4c, 4) \quad -\mathcal{K} = (c+a-2d+2, 1)$$

ID	g	ID	g
S91	$T_5^4+T_4T_6^3+T_1^{(4c-a-l)}T_2^lT_3T_7^3$, $c>d\geq 1, c\geq 2d-1, 4c-3d<a<4c$	S92	$T_5^4+T_4T_6^3+T_1^{(4c-l)}T_2^lT_7^4$, $c, d\geq 1, c\geq 2d-1, a>4c-3d$, $0<l<2c, l$ odd

$$Q \;=\; \begin{bmatrix} 1 & 1 & a & 2c-5d & c & d & 0 \\ 0 & 0 & 1 & 1 & 3 & 1 & 1 \end{bmatrix} \quad \mu \;=\; (2c,4) \quad -\mathcal{K} \;=\; (c-4d+a+2,3)$$

ID	g	ID	g
S94	$T_5^2+T_4T_6^5+T_1^{(2c-2a-l)}T_2^lT_3^2T_7^4$, $d\geq 1, c>3d, 4d-1\leq c<5d$, $2c-5d<a<c, 0<l\leq c-a, l$ odd	S95	$T_5^2+T_4T_6^5+T_1^{(2c-a-l)}T_2^lT_3T_7^5$, $d\geq 1, c>3, c\geq 4d-1, 2c-5d<a<2c$, $0\leq l\leq (2c-a)/2$
S96	$T_5^2+T_4T_6^5+T_1^{(2c-l)}T_2^lT_7^6$, $d\geq 1, c>3, c\geq 4d-1, a>2c-5d$, $0<l\leq c, l$ odd		

$$Q \;=\; \begin{bmatrix} 1 & 1 & 2c-5d & b & c & d & 0 \\ 0 & 0 & 1 & 1 & 3 & 1 & 1 \end{bmatrix} \quad \mu \;=\; (2c,4) \quad -\mathcal{K} \;=\; (c-4d+b+2,3)$$

ID	g	ID	g
S97	$T_5^2+T_3T_6^5+T_1^{(2c-2b-l)}T_2^lT_4^2T_7^4$, $d\geq 1, c>4d-1, c-d-1\leq b<c$, $b<2c-5d, 0<l\leq c-b, l$ odd	S98	$T_5^2+T_3T_6^5+T_1^{(2c-b-l)}T_2^lT_4T_7^5$, $d\geq 1, c>3d, c-d-1\leq b<2c-5d$, $0\leq l\leq (2c-b)/2$
S99	$T_5^2+T_3T_6^5+T_1^{(2c-l)}T_2^lT_7^6$, $d\geq 1, c>4d-1, b>1, c-d-1\leq b<2c-5d$, $0<l\leq c$ odd		

$$Q \;=\; \begin{bmatrix} 1 & 1 & a & b & c & d & 0 \\ 0 & 0 & 1 & 1 & 3 & 1 & 1 \end{bmatrix} \quad \mu \;=\; (2c,4) \quad -\mathcal{K} \;=\; (d+b+a-c+2,3)$$

ID	g	ID	g
S100	$T_5^2+T_1^{(2c-a-5d)}T_3T_6^5+T_2^{(2c-b)}T_4T_7^5$, $d\geq 1, c>3d, c-d-1\leq b<a<2c-5d$	S101	$T_5^2+T_1^{(2c-b-5d)}T_4T_6^5+T_2^{(2c-a)}T_3T_7^5$, $d\geq 1, c>4d-1, b>1, c-d-1\leq b<2c-5d$, $b<a<2c$

$$Q \;=\; \begin{bmatrix} 1 & 1 & 2c & b & c & d & 0 \\ 0 & 0 & 1 & 1 & 3 & 1 & 1 \end{bmatrix} \quad \mu \;=\; (2c,4) \quad -\mathcal{K} \;=\; (b+d+c+2,3)$$

ID	g	ID	g
S103	$T_5^2+T_1^{(2c-b-5d-l)}T_2^lT_4T_6^5+T_3T_7^5$, $d\geq 1, c>4d-1, c-d-1\leq b<2c-5d$, $0\leq l\leq (2c-b-5d)/2$	S104	$T_5^2+T_1^{(2c-6d-l)}T_2^lT_6^6+T_3T_7^5$, $d\geq 1, c>3d, b>1, c-d-1\leq b<2c$, $0<l\leq c-3d, l$ odd

Finally, let us compare our results with existing classifications.

Remark 3.10.12. The 447 sporadic cases from Classification lists 3.10.1 to 3.10.11 encompass in particular the smooth Fano fourfolds with hypersurface Cox ring of Picard number two and torus action of complexity one. The following table translates their ID's in the present classification to the cases of [35, Thm. 1.2].

Theorem 1.2 in [35]	ID
1	84 (3.10.2)
2	20 (3.10.2)
4.A: $m=1$, $c=-1$	45 (3.10.2)
4.A: $m=1$, $c=0$	1 (3.10.1)
4.B: $m=1$	44 (3.10.2)
4.C: $m=1$	6 (3.10.1)

Theorem 1.2 in [35]	ID
5: $m=1$, $a=0$	228 (3.10.3)
7: $m=1$	253 (3.10.5)
10: $m=2$	244 (3.10.5)
11: $m=2$, $a_2=1$	225 (3.10.2)
11: $m=2$, $a_2=2$	226 (3.10.2)
12: $m=2$	15 (3.10.1)

Remark 3.10.13. At least 268 varieties from our classification admit a *one-parameter smoothing* to a smooth Fano fourfold of Picard number two. Here, by a one-parameter smoothing of X we mean a flat morphism of varieties $\varphi\colon \mathcal{X} \to \mathbb{C}$ such that $\mathcal{X}_0 := \varphi^{-1}(0)$ is isomorphic to X and there is a non-empty open subset $U \subseteq \mathbb{C}$ such that $\mathcal{X}_t := \varphi^{-1}(t)$ is smooth for all $t \in U$. The procedure to explicitly construct such a smoothing goes as follows: Let $X = X(Q, g)$ with (Q, g) from the lists 3.10.1 to 3.10.11 and assume that up to a unimodular transformation the data $Q = [w_1, \dots, w_7]$ and $\mu = \deg(g)$ appears in [45, Thm. 1.1]. Then there is a homogeneous spread polynomial h of degree $\deg(h) = \mu$ such that X_h is a smooth Fano fourfold with general hypersurface Cox ring. We extend the action of $H = (\mathbb{C}^*)^2$ on $\mathbb{C}^7$ given by the grading map Q to $\mathbb{C}^8$ by letting H act trivially on the last coordinate. We set

$$\bar{\mathcal{Z}} \;=\; \mathbb{C}^8, \qquad \widehat{\mathcal{Z}} \;=\; \bar{\mathcal{Z}}^{ss}(\tau), \qquad \mathcal{Z} \;=\; \widehat{\mathcal{Z}} /\!\!/ H,$$

where $\tau \in \Lambda(\mathbb{C}[T_1, \dots, T_7, T])$ is the unique GIT-cone that contains the anticanonical class $-\mathcal{K}_X$ in its interior. Moreover we set

$$\bar{\mathcal{X}} \;:=\; V((1-T)g + Th) \;\subseteq\; \mathbb{C}^8, \qquad \widehat{\mathcal{X}} \;=\; \bar{\mathcal{X}} \cap \widehat{\mathcal{Z}}, \qquad \mathcal{X} \;=\; \widehat{\mathcal{X}} /\!\!/ H.$$

The projection $\mathrm{pr}\colon \widehat{\mathcal{X}} \to \mathbb{C}$ to the last coordinate is H-invariant and thus factors through a morphism $\varphi\colon \mathcal{X} \to \mathbb{C}$. We have $X \cong \varphi^{-1}(0)$ and φ is a smoothing of X with fiber over $t = 1$ isomorphic to X_h. In the following table, for each entry (Q, μ) from the table in [45, Thm. 1.1] we list the IDs of the varieties $X(Q, g)$ from the present classification that admit such an explicit smoothing to a smooth Fano fourfold of Picard number two with a general hypersurface Cox ring and data (Q, μ).

[45, Thm. 1.1]	IDs
1	1
2	2, 3
3	4, 5
4	6
5	-
6	7, 8
7	19, 20
8	21, 22
9	23 - 25
10	-
11	26, 27
12	28 - 30
13	227, 228
14	230 - 232; S2: $a = 1$
15	265 - 270
16	31, 32
17	33, 34
18	35 - 39
19	40 - 43
20	44, 45
21	242, 243
22	46, 47
23	48 - 50
24	51, 52
25	53 - 71
26	72
27	73
28	74 - 76
29	77
30	78
31	79
32	80
33	415 - 418
34	81 - 84
35	85 - 91
36	92 - 101
37	102 - 107

[45, Thm. 1.1]	IDs
38	108 - 129
39	130 - 148
40	149
41	150, 151
42	152, 153
43	154 - 158
44	159, 160
45	161 - 166
46	167 - 180
47	181 - 219
48	244
49	245
50	246, 247
51	248 - 251
52	411 - 414

[45, Thm. 1.1]	IDs
53	252, 253
54	254
55	255 - 257
56	258 - 261
57	262 - 264
58	9, 10
59	11 - 13
60	220 - 222
61	14
62	223
63	16
64	224
65	15
66	225
67	226

With the smoothing procedure from above one obtains a one-parameter smoothing of the variety no. 17 in Classification list 3.10.1 to $X = Y \times \mathbb{P}^1$, where $Y \subseteq \mathbb{P}^4$ is a smooth quartic. The specifying data of X is missing from [45, Thm. 1.1].

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