Maximal hypersurfaces and other nonlinear geometric problems

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Ὦ Ἱππία φίλε, σὺ μὲν μακάριος εἶ, ὅτι τε οἶσθα ἃ χρὴ ἐπιτηδεύειν ἄνθρωπον, καὶ ἐπιτετήδευκας ἱκανῶς, ὡς φῄς· ἐμὲ δὲ δαιμονία τις τύχη, ὡς ἔοικε, κατέχει, ὅστις πλανῶμαι μὲν καὶ ἀπορῶ ἀεί, ἐπιδεικνὺς δὲ τὴν ἐμαυτοῦ ἀπορίαν ὑμῖν τοῖς σοφοῖς λόγῳ αὖ ὑπὸ ὑμῶν προπηλακίζομαι, ἐπειδὰν ἐπιδείξω. λέγετε γάρ με, ἅπερ καὶ σὺ νῦν λέγεις, ὡς ἠλίθιά τε καὶ σμικρὰ καὶ οὐδενὸς ἄξια πραγματεύομαι· ἐπειδὰν δὲ αὖ ἀναπεισθεὶς ὑπὸ ὑμῶν λέγω ἅπερ ὑμεῖς, ὡς πολὺ κράτιστόν ἐστιν οἷόν τ᾽ εἶναι λόγον εὖ καὶ καλῶς καταστησάμενον περαίνειν ἐν δικαστηρίῳ ἢ ἐν ἄλλῳ τινὶ συλλόγῳ, ὑπό τε ἄλλων τινῶν τῶν ἐνθάδε καὶ ὑπὸ τούτου τοῦ ἀνθρώπου τοῦ ἀεί με ἐλέγχοντος πάντα κακὰ ἀκούω. καὶ γάρ μοι τυγχάνει ἐγγύτατα γένους ὢν καὶ ἐν τῷ αὐτῷ οἰκῶν· ἐπειδὰν οὖν εἰσέλθω οἴκαδε εἰς ἐμαυτοῦ καί μου ἀκούσῃ ταῦτα λέγοντος, ἐρωτᾷ εἰ οὐκ αἰσχύνομαι τολμῶν περὶ καλῶν ἐπιτηδευμάτων διαλέγεσθαι, οὕτω φανερῶς ἐξελεγχόμενος περὶ τοῦ καλοῦ ὅτι οὐδ᾽ αὐτὸ τοῦτο ὅτι ποτέ ἐστιν οἶδα. ‟Καίτοι πῶς σὺ εἴσῃ,’ φησίν, ‟ἢ λόγον ὅστις καλῶς κατεστήσατο ἢ μή, ἢ ἄλλην πρᾶξιν ἡντινοῦν, τὸ καλὸν ἀγνοῶν· καὶ ὁπότε οὕτω διάκεισαι, οἴει σοι κρεῖττον εἶναι ζῆν μᾶλλον ἢ τεθνάναι·’ συμβέβηκε δή μοι, ὅπερ λέγω, κακῶς μὲν ὑπὸ ὑμῶν ἀκούειν καὶ ὀνειδίζεσθαι, κακῶς δὲ ὑπ᾽ ἐκείνου. ἀλλὰ γὰρ ἴσως ἀναγκαῖον ὑπομένειν ταῦτα πάντα· οὐδὲν γὰρ ἄτοπον εἰ ὠφελοίμην. ἐγὼ οὖν μοι δοκῶ, ὦ Ἱππία, ὠφελῆσθαι ἀπὸ τῆς ἀμφοτέρων ὑμῶν ὁμιλίας· τὴν γὰρ παροιμίαν ὅτι ποτὲ λέγει, τὸ ‟Χαλεπὰ τὰ καλά,’ δοκῶ μοι εἰδέναι.

Plato, *Hippias Major*, 304b-e

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Prelude

One of the central themes of geometric analysis and general relativity is the study of hypersurfaces which optimise natural geometric functionals and the way their geometry is related to the ambient space they reside in. Such hypersurfaces occur in many different guises: for example, as spacelike time slices in Lorentzian spacetimes, as constant curvature submanifolds in Riemannian geometry, or as evolving objects in geometric flows. This thesis examines three types of geometric problems: maximal hypersurfaces in the Schwarzschild spacetime, black hole and photon surface uniqueness, and rotational symmetry of ancient solutions to fully nonlinear curvature flows. All these results share the common thread of analysing curvature-driven equations and rigidity phenomena derived from the interplay of curvature, asymptotic behaviours and other geometric properties. Furthermore, they reveal deep interactions between analysis, geometry and physics. This reflects the broader truth that general relativity strictly relies on Riemannian and Lorentzian geometry, combined with the analysis of partial differential equations of different natures. In particular, in modern physics, cosmological models and other astrophysical phenomena such as black holes or star systems are modelled by Lorentzian manifolds solving the *Einstein field equations*

$$\mathfrak{Ric} - \Lambda \mathfrak{g} = \mathfrak{T} - \frac{\operatorname{tr}_{\mathfrak{g}} \mathfrak{T}}{2} \mathfrak{g}, \qquad \mathfrak{div}\, \mathfrak{T} = 0\,. \tag{1.0.1}$$

These coupled, nonlinear, second-order equations encode gravity in the intrinsic curvature of the Lorentzian manifold, prescribing the Ricci tensor $\mathfrak{Ric}$ in terms of a suitable matter field $\mathfrak{T}$. We then claim that the techniques developed in this thesis –such as the construction of barriers to control solutions to nonlinear partial differential equations and monotonic quantities– could find application in more general geometric and physical contexts.

Boosted maximal hypersurfaces in Schwarzschild. In the *initial value formulation* of (1.0.1), the equations drastically simplify under the condition that the spacetime is foliated by spacelike Riemannian hypersurfaces with vanishing *mean curvature* $\mathrm{H} = 0$. Such hypersurfaces, called *maximal* have thereby played a key role in the dynamic aspects –Hamiltonian approach– of general relativity and in numerical computation of asymptotically flat spacetimes. For simplicity, we describe the variational problem and the partial differential equation in the case of Minkowski $\mathbb{R}^{n,1} = \mathbb{R}^n \times \mathbb{R}$, with flat metric $\eta = -dt^2 + dx_i^2$. The graph of a function $u : \Omega \subset \mathbb{R}^n \to \mathbb{R}$ is *weakly spacelike* if $|\mathrm{D}u| \leq 1$. If $\varphi : \partial\Omega \to \mathbb{R}$ is a bounded function, then graph u is said to be an *area maximizing surface* if it maximizes the area functional among the admissible functions $\mathscr{K}(\Omega, \varphi) = \{v \in C^{0,1}(\Omega) : |\mathrm{D}v| \leq 1, v|_{\partial\Omega} = \varphi\}$. The corresponding Euler-Lagrange equation is the following *quasi-linear* partial differential equation

$$0 = \mathrm{H}(u) = a^{ij}(\mathrm{D}u)\,\mathrm{D}^2_{ij}u, \qquad a^{ij}(\mathrm{D}u) = \frac{1}{\sqrt{1-|\mathrm{D}u|^2}}\left(\delta^{ij} + \frac{\mathrm{D}^i u \mathrm{D}^j u}{1-|\mathrm{D}u|^2}\right). \tag{1.0.2}$$

reading as the vanishing of the mean curvature $\mathrm{H}(u)$ of the graph of u. It can be shown that equation (1.0.2) is *elliptic*, provided that u is *strictly spacelike*, namely $|\mathrm{D}u| < 1$ holds uniformly. Classical solutions to (1.0.2) are called *maximal.* The existence of solutions to the variational problem and to (1.0.2) on bounded domains in Minkowski was proven by Bartnik–Simon in [BS82]. Right before these results, in analogy with *minimal graphs* in the Euclidean space, Calabi and Chen–Yau [Cal70; CY76] proved a *Calabi–Bernstein theorem*: every entire spacelike maximal surface, namely any *spacelike* graph of a smooth function $u : \mathbb{R}^n \to \mathbb{R}$ solving the *quasilinear elliptic equation* (1.0.2) is an affine plane. On the other hand, in the recent work [HY21], Hong–Yuan considered solutions over exterior domains in Minkowski approaching affine planes – *boosts*– at infinity; moreover, they provided an asymptotic analysis of exterior maximal surfaces, showing that they always approach boosts, again resembling a Bernstein-type result at infinity. Some of the notions above can be suitably extended to other spacetimes, with a much more involved analysis due to the presence of ambient singularities and the absence of a standard choice of coordinates. In [Bar84], Bartnik showed the existence of maximal hypersurfaces in asymptotically flat spacetimes. In that regard, he proved that under the *Bartnik's time-dependent conditions* on tensorial quantities depending on the choice of time function, an a priori height estimate gives an a priori gradient estimate. This reduces the problem to finding barriers and computing integral estimates. In turn, once the existence of barriers is shown, an a priori height estimate relies again on the *Bartnik's time-dependent conditions* and on the so-called *Bartnik's uniform interior conditions.*

We highlight that the aforementioned *Bernstein problem* mirrors the observation that entire solutions to elliptic equations tend to exhibit rigidity properties. With the long-term goal of establishing such a theorem in curved Loretzian manifolds, we first address the problem in the *maximally extended Schwarzschild* spacetime, which is an explicit non-trivial solution to (1.0.1) in vacuum ($\mathfrak{T} = 0$). We prove the existence of *entire maximal hypersurfaces* in this spacetime with prescribed asymptotics corresponding to boosts in the asymptotically flat ends. While in Minkowski spacetime the problem reduces to the trivial observation that affine planes are maximal hypersurfaces, the situation in the Schwarzschild spacetime is profoundly challenging. In fact, we first have to find suitable coordinate-dependent boosted hypersurfaces admitting barriers at the asymptotically flat ends. This is a non-trivial task, as the natural candidates –hypersurfaces that resemble affine planes in Minkowski– are inadequate for constructing barriers due to their mean curvature not decaying fast enough. Complicating matters further, the Kruskal–Szekeres time coordinate fails to meet *Bartnik's time-dependent conditions*, obstructing a direct application of existing results by [Bar84]. To circumvent this, we introduce a new time function in the maximally extended Schwarzschild spacetime. Notably, this function is a *Cauchy time function*, and its properties are of independent interest. Finally, we also show that a relaxed version of the *Bartnik's uniform interior conditions* is satisfied with our choice of time function. This verification requires a delicate analysis involving both the use of the crushing nature of the Schwarzschild singularity and a proof of monotonicity for the relevant geometric quantities involved. As a consequence of the above construction, we can generate a foliation of maximal hypersurfaces in exterior Schwarzschild, corresponding to a geometric choice of time function associated to any boost of parameter $\mathrm{b} \in [0,1)$. Moreover, we remark that the tools developed to tackle this problem can also be employed to show the existence of exterior solutions in exterior Schwarzschild for a natural class of interior boundaries, extending some of the results in [HY21].

The analysis of this problem covers the whole Chapter 2; the structure thereof is described in Section 2.1.2. Here, we recall where to find the most important results. In the main Theorem 2.4.4, we establish the existence of an entire maximal hypersurface in the maximally extended Schwarzschild. The construction of barriers in the asymptotically flat end is the content of Section 2.6 and, in particular, of Proposition 2.6.1. The new time function is defined and characterized in Proposition 2.5.2 and, more generally, discussed in Section 2.5. Finally, the Bartnik's uniform interior condition for the new time function is addressed in Section 2.5.1.

Black hole and equipotential photon surface uniqueness. The aforementioned *maximally extended Schwarzschild* spacetime was constructed by Kruskal–Szekeres [Kru60; Sze60] as an extension of the first known explicit non-trivial solution to (1.0.1) in vacuum, which we will call *(exterior) Schwarzschild solution*. This was introduced by Schwarzschild [Sch16], imposing rotational symmetry. It consists of a family, parametrized by $m > 0$, of *static* spacetimes $(\mathfrak{L}, \mathfrak{g})$, namely of the form

$$\mathfrak{L} = \mathbb{R} \times M, \qquad \mathfrak{g} = -N^2 dt^2 + g \tag{1.0.3}$$

being (M, g) a 3-dimensional Riemannian manifold and $N > 0$ a smooth function on M. They present a coordinate singularity at the so-called *Schwarzschild radius* $\{r = 2m\}$. By employing different coordinates on the Riemannian time slices to smoothly extend the metric, it is easy to see that the hypersurface $\{r = 2m\} \cap M$ is minimal in M, namely has vanishing mean curvature $\mathrm{H}_g(\{r = 2m\}) = 0$ and physically corresponds to the notion of *static black hole*. The renowned *Black Hole Uniqueness theorem* first proven by Israel [Isr67] states that the exterior Schwarzschild is the only asymptotically flat static spacetime in vacuum containing a black hole as a boundary. This statement has been generalised to other boundary conditions related to *equipotential photon surfaces* since the work of Cederbaum [Ced15]. Both boundary conditions represent not only extremely relevant physical objects, but also mathematically allow to show the above rigidity statements, employing, among others, their constant mean curvature.

In the last decades, these theorems have also been proven in different matter models (e.g. in electro-vacuum) using various techniques with the scope of removing restrictive assumptions. We tackled the above problem in vacuum via diverse methods in [CCF24] (based, among others, on the master thesis of the author [Cog20]) and in the paper *Uniqueness of static vacuum asymptotically flat black holes and equipotential photon surfaces in* $n+1$ *dimensions à la Robinson* [Ced+], included in this dissertation. In particular, in the latter we adapted to higher dimensions the *divergence technique* by Robinson [Rob77], employing tensors inspired by the analysis of Ricci solitons, see [Cao13]. Moreover, we found an explicit relation between this method and the functionals of Agostiniani–Mazzieri [AM17], employed in the other works [CCF24] and [BCC24]. On the other hand, in the joint work *Black Hole and Equipotential Photon Surface Uniqueness in 4-dimensional Asymptotically Flat Electrostatic Electro-Vacuum Spacetimes* [BCC24], also included in this dissertation, we address the same problem in the more general *electro-vacuum* setting. To give an idea of the equations involved, we will briefly outline the mathematical formulation in this latter case.

A family of exact rotationally symmetric solutions to the *Einstein-Maxwell field equations* (1.0.1) with matter model $\mathfrak{T}$ associated to an electric potential Ψ is given by the *Reissner–Nordström solutions* for fixed parameters $m, q \in \mathbb{R}$. If $q = 0$, they correspond to the family of exterior Schwarzschild solutions (with positive and negative mass m). These solutions are of the form (1.0.3) and satisfy the system

$$N \,\mathrm{Ric} = \mathrm{D}^2 N - \frac{2}{N}\, d\Psi^2 + \frac{|\mathrm{D}\Psi|^2}{N}\, g, \qquad \Delta N = \frac{|\mathrm{D}\Psi|^2}{N}, \qquad \Delta \Psi = \frac{\mathrm{D}N(d\Psi)}{N}, \tag{1.0.4}$$

which translates (1.0.1) into equations involving only tensors and functions on the Riemannian manifold (M, g). In vacuum, the same equations hold with $\Psi \equiv 0$. Another feature of the Reissner–Nordström solution is the *asymptotic flatness*, namely as $r \to \infty$, the Riemannian part of the metric approaches the Euclidean metric and $N \to 1$. Finally, we notice that for $m^2 < q^2$ the maximal interval of definition for N to be well-defined is $r > r_{m,q} := 0$, while for $m^2 \geq q^2$, N vanishes on $\{r = m + \sqrt{m^2 - q^2} =: r_{m,q}\}$, which is a *static black hole* in the sense of having vanishing mean curvature. Also, all Reissner–Nordström spacetimes have spherically symmetric *totally umbilic* timelike hypersurfaces going through $\{r = r_1\}$, for each radius $r_1 > r_{m,q}$. These are *equipotential photon surfaces* and on these hypersurfaces N is constant, see [CJV23]. We show that

Reissner–Nordström spacetimes are the unique solutions of (1.0.4) with suitable asymptotic conditions and boundary being a black hole ($N = 0$ on ∂M) or an equipotential photon surface ($N = N_0 > 0$ constant on ∂M), recovering the aforementioned uniqueness theorems in electro-vacuum, without assuming that N regularly foliates M and under weaker asymptotic assumptions. Weakening these decay conditions corresponds to physically realistic conditions, but on the other hand, it makes the analysis complicated. In particular, we provide, up to restricting the range of their radii, the uniqueness result for connected photon surfaces in the super-extremal case ($m^2 < q^2$), not yet treated in the literature. Our proof is based on a cylindrical ansatz inspired by the work by Agostiniani–Mazzieri [AM17]: the rigidity statements are attained by studying suitable monotone quantities along the level sets of N, relying on the Bochner formula and the divergence theorem, so that the problem can be transferred to the asymptotics and to the boundary conditions.

This is the content of the first and second co-authored papers presented in Chapter 3.

Uniqueness of ancient solutions to fully nonlinear flows. With opposite signs (depending on the setting being Lorentzian or Riemannian), the right-hand side of (1.0.2) is the local expression of the mean curvature of a hypersurface in $\mathbb{R}^{n+1}$, computed with respect to the normal ν (e.g. outward pointing). The motion of a family of hypersurfaces with speed $-\mathrm{H}\,\nu$, is the canonical example of an extrinsic curvature flow, the *mean curvature flow (MCF).* Geometrically, the mean curvature is the sum of the principal curvatures $\mathrm{H} = \lambda_1 + \cdots + \lambda_n$. Other scalar geometric quantities can be obtained via symmetric convex or concave functions γ of the principal curvatures, increasing in its arguments, and give rise to parabolic flows for a family of embeddings $F(t, \cdot) = x$ in the form

$$\frac{\partial}{\partial t}F = -G\,\nu\,, \qquad G(x) = \gamma(\lambda_1(x), \ldots, \lambda_n(x))\,. \tag{1.0.5}$$

An example of these was introduced in [BH17] for deep classification results of Riemannian manifolds and has speed $\gamma(\lambda_1, \ldots, \lambda_n) = \left(\sum_{i<j} \frac{1}{\lambda_i+\lambda_j}\right)^{-1}$. While, as observed for (1.0.2), the mean curvature H reads locally as a quasi-linear elliptic second-order operator, the corresponding partial differential equations for all the other speeds are *fully nonlinear.*

Ancient solutions to parabolic equations –existing on $(-\infty, T]$– are analogues of entire solutions to elliptic partial differential equations and tend to exhibit rigidity properties as well. Moreover, ancient solutions to geometric flows arise as dilations –*blow-up limits*– of singularities. Therefore, their classification results can be essential for studying singularities, which in turn are crucial to exploiting the flows for results in geometry and topology. In the joint work *Rotational symmetry of ancient solutions to fully nonlinear curvature flows* [CLM23], included in this dissertation, we address the classification of ancient solutions to such nonlinear flows, under natural conditions on the speed (including, in particular, the aforementioned speed). We show that every convex, non-collapsing, uniformly 2-convex ancient, noncompact solution is either a self-similar shrinking cylinder or a rotationally symmetric translating soliton. The analogous result was obtained for MCF by Brendle–Choi [BC19], [BC21]. In the fully nonlinear case, overcoming the non-variational origin of the flows and the consequent absence of Huisken's monotonicity formula is particularly technical. In particular, to replace the role of the monotonicity formula for sharp asymptotic estimates, we have to construct a family of barriers as convex, rotationally symmetric shrinking solutions, satisfying for each $a > 0$ the corresponding fully nonlinear ODE

$$\gamma\left(\frac{\psi_{\rho\rho}}{1+\psi_\rho^2}, \frac{\psi_\rho}{\rho}, \dots, \frac{\psi_\rho}{\rho}\right) = \frac{1}{2} + \frac{1}{2a^2}(\rho\psi_\rho - \psi)\,, \qquad \psi(0) = 0, \quad \psi_\rho(0) = 0$$

and carefully analyse their asymptotic behaviour.

This is the content of the third co-authored paper presented in Chapter 3.

1.1 Overview and declarations

This thesis consists of three chapters.

The single author's work is presented in Chapter 2. The specific question as addressed in the work, all the analysis and all the writing have been done independently by the author. I am indebted to Prof. Gerhard Huisken for proposing the study of this problem and for helpful discussions. I am grateful to Prof. Piotr Chruściel for interesting remarks on the problem. Finally, I am thankful to the thesis examiners for reviewing the work and in particular, to Prof. Carla Cederbaum for suggesting some implementations.

Chapter 3 comprises three articles co-authored with collaborators, reproduced in their original form as published papers or preprints on ArXiv. As is usual in mathematics, the authors of the manuscripts are listed alphabetically. The papers are, in order of appearance:

1. Stefano Borghini, Carla Cederbaum and Albachiara Cogo, *Black Hole and Equipotential Photon Surface Uniqueness in 4-dimensional Asymptotically Flat Electrostatic Electro-Vacuum Spacetimes*, published in: Annales Henri Poincaré (December 2024).
2. Carla Cederbaum, Albachiara Cogo, Benedito Leandro and João Paulo dos Santos, *Uniqueness of static vacuum asymptotically flat black holes and equipotential photon surfaces in (n+1) dimensions à la Robinson*, arXiv:math/2403.14422, 2024, under revision after submission to a journal.
3. Albachiara Cogo, Stephen Lynch and Olivia Vičánek Martínez, *Rotational symmetry of ancient solutions to fully nonlinear curvature flows*, arXiv:math/2310.08301, 2023, under revision after submission to a journal.

Entire boosted maximal hypersurfaces in the Schwarzschild spacetime

2.1 Introduction

Before setting the problem of constructing complete, non-compact boosted maximal hypersurfaces in the maximally extended Schwarzschild spacetime, we will give a brief historical overview of the main objects and results that motivate this work in Section 2.1.1. An outline of the main results and the structure of this work is then displayed in Section 2.1.2. The mathematically rigorous formulation of the problem and the strategy of the proof can be instead found in Section 2.4, making use of the notations, definitions and basic results provided in Section 2.2.

2.1.1 Historical synopsis and motivation

Here we mostly aim to highlight, in perspective of the formulation of our problem, the interplay between the mathematical analytical and geometric tools such as *quasi-linear elliptic partial differential equations* arising from geometric objects and the theories of *Special* and *General Relativity* with their striking impact in physics.

Special Relativity and the Bernstein problem

The theory of *special relativity* introduced by Einstein [Ein05a; Ein05b] in 1905 and modelled by the Minkowski spacetime [Min08]

$$\mathbb{R}^{1,n} := \mathbb{R} \times \mathbb{R}^n\,, \qquad \eta := -\mathrm{d}t^2 + \sum_{i=1}^{n} \mathrm{d}x_i^2\,, \tag{2.1.1}$$

allowed to solve the pre-relativistic problem of combining the *principle of relativity*, universally accepted since the works of [Gal53] and Newton [New53] in the 17^{th} -18^{th} centuries and stating that laws of nature have the same form in every inertial frame, with the *electromagnetism* as described by the Maxwell equations [Max73], introduced in 1861. The solution was based on rejecting the idea of absolute time and on the postulate that the speed of light is equal to the same universal constant in every inertial frame. In the following, we will refer to an *inertial frame* as a choice of coordinates in which Newton's law of dynamics holds. The transformations between inertial frames, which respect the postulate of absolute speed of light, are given by the isometries of (2.1.1) and *boosts* are the ones involving both space and time –for a more extensive discussion see Section 2.2.2. Fixing an inertial frame $(t, x_1, \ldots, x_n)$ and considering another $(\widetilde{t}, \widetilde{x}_1, \ldots, \widetilde{x}_n)$ via the aforementioned transformation, we obtain that the *time slices* $\{\widetilde{t} = \widetilde{t}_o\}$ for fixed constants $\widetilde{t}_o \in \mathbb{R}$ are affine planes and in particular *entire (linear) graphs* with respect to the original fixed frame. Since affine planes are totally geodesic, they have vanishing extrinsic curvature or *mean curvature*, namely they are *maximal hypersurfaces*.

In analogy with *minimal graphs* in the Euclidean space, first Calabi [Cal70] in 1968 for the dimension $n + 1 = 3$ and then Cheng–Yau [CY76] in 1976 for any dimension $n + 1 \geq 3$ proved a *Calabi–Bernstein theorem*: every entire spacelike maximal surface, namely any *spacelike* graph of a smooth function $u : \mathbb{R}^n \to \mathbb{R}$ solving the *quasilinear elliptic equation*

$$0 = \mathrm{div}_{\mathbb{R}^n}\left(\frac{\mathrm{D}u}{\sqrt{1 - |\mathrm{D}u|^2}}\right) = \frac{1}{\sqrt{1 - |\mathrm{D}u|^2}}\left(\Delta u + \frac{\mathrm{D}^2 u\,(\mathrm{D}u, \mathrm{D}u)}{1 - |\mathrm{D}u|^2}\right), \qquad |\mathrm{D}u| < 1 \tag{2.1.2}$$

is an affine plane. The *Bernstein problem* mirrors the observation that entire solutions to elliptic equations tend to exhibit rigidity properties. In fact, entire bounded solutions to the *Laplace equation* $\Delta u = 0$, the easiest elliptic second-order equation, are constant by the *Liouville theorem*. Following the observation that the Laplace equation is the linearized version for small $\mathrm{D}u$ of (2.1.2) (and of the *minimal surface equation*, corresponding to (2.1.2) with opposite signs), in 1915 Bernstein [Ber15] showed that entire solutions to the minimal surface equation are affine planes for $n = 2$; we highlight that the nonlinearity of the equation somehow enforces the rigidity, as the boundedness assumption is not required –see, as a counterexample, the harmonic function $\mathrm{e}^x \sin y$. The conjecture extending this to higher dimensions, was proven true in dimension $3 \leq n \leq 7$ by the combination of the work of Fleming [Fle62], De Giorgi [De 65], Almgren [Alm66] and Simons [Sim68] and false in dimension $n \geq 8$ due to the *Simons cone*, which serves as a counterexample, as shown by Bombieri–De Giorgi–Giusti [BDG69]. While the case $n = 2$, the so-called *Calabi correspondence* [Cal70] allows to transfer the results for minimal graphs in $\mathbb{R}^3$ to maximal graphs in $\mathbb{R}^{1,2}$ and vice-versa, in higher dimensions these objects behave differently; in fact, there is no limitation of dimension in the Cheng–Yau proof [CY76]. Their result follows from proving that the induced Lorentz metric on a spacelike maximal surface is a complete Riemannian metric and that the norm of the second fundamental form is bounded by a multiple of its trace.

We remark that while any rigidity statement on entire solutions to elliptic partial differential equations is of striking mathematical interest, the Bernstein theorem for the maximal surface equation has also a deep physical impact as it established a one-to-one correspondence in Minkowski spacetime between entire *maximal hypersurfaces*, *affine planes*, *boosts* as isometries and *inertial frames*.

General Relativity and the Cauchy problem

In parallel to the works on the Bernstein problem, starting from the '50s, much progress was made in the context of *general relativity* introduced by Einstein [Ein15a; Ein15b] in 1915. Since the formulation of this theory and in modern physics, cosmological models and other astrophysical phenomena are modelled by *spacetimes* $(\mathfrak{L}^{n+1}, \mathfrak{g})$ – (smooth) manifolds endowed with a Lorentzian metric (semi-Riemannian of index 1)– solving the *Einstein field equations* reading:

$$\mathfrak{Ric} - \Lambda\mathfrak{g} = \mathfrak{T} - \frac{\mathrm{tr}_{\mathfrak{g}}\,\mathfrak{T}}{2}\mathfrak{g}, \qquad \mathfrak{div}\,\mathfrak{T} = 0\,. \tag{2.1.3}$$

These equations encode gravity in the intrinsic curvature of the Lorentzian manifold, prescribing the Ricci tensor $\mathfrak{Ric}$ in terms of a suitable matter field $\mathfrak{T}$. In particular, while locally, physics is sufficiently well approximated by special relativity (the existence of normal coordinates gives a locally inertial system, where, up to second order, physics is Minkowskian), effects due to gravitation are codified in the curvature of space-time. Due to the complicated and non-linear nature of the *Einstein field equations* (EE), explicit solutions could be analytically found (starting with the *Schwarzschild solution* [Sch16] in 1916), but only appealing to idealised highly symmetrical hypotheses. Thus, different perspectives were developed to study generic and global properties of *spacetimes* as solutions to the EE, their stability against perturbations and, jointly, to generate systematic methods of producing more general solutions. On the one hand, approaches based on physical arguments and Loretzian geometry shed light on the *causal structures* of spacetimes, starting with the pioneering works on *singularity theorems* by Hawking and Penrose [Pen65; Haw67; HP70]. On the other hand, the mathematical community shifted their focus to an *Cauchy Problem* reformulation of the equations, with initial data given by a Riemannian manifold (M, g) and a symmetric $(0, 2)$-tensor K. Roughly speaking, the initial position and velocity of the gravitational field given by g and K are evolved in time into a Lorentzian manifold which contains (M, g) as a hypersurface with extrinsic curvature (*second fundamental form*) K. In particular, the resulting Lorentzian manifold is isometric to

$$\mathbb{R} \times M, \qquad -N^2(t, \cdot)\, dt^2 + g(t, \cdot)\,, \tag{2.1.4}$$

where the *time function* t is the natural projection into the first factor, N is a smooth function and g restricts to a Riemannian metric on the *time-slices* $\{t_0\} \times M$. This is not restrictive as well-behaved Lorenztian manifolds in terms of causality are of this form –see [BS05a]– and is the reason why this *Initial Value* formulation is also called $3+1$ *Splitting*. This *Cauchy Problem* formulation was built on ideas by Lichnerowicz [Lic39; Lic44] and the proof of well-posedness and existence of local solutions by Choquet-Bruhat [Fou52] in

1952 (and by Geroch and Choquet-Bruhat [CG69] for a global version), which exploits the harmonic gauge introduced by [Lan22] in 1922 and the recent developments in hyperbolic partial differential equations. The idea of evolving the so-called *Initial Data Sets* into a solution of the EE fits perfectly with the physical paradigm of predicting the future development of a system. In fact, it served as a foundation of the *Hamiltonian formulations* of general relativity by Dirac [Dir58], and Arnowitt–Deser–Misner [ADM08]. Moreover, the $3+1$ *Splitting* provided the basic tools to the noteworthy branch of *numerical relativity* starting from the '70s and relying on the formalizations by Wheeler [Whe64] and York [Yor73; Yor79].

Solutions to the EE provided the modern framework for *cosmology*, as well as models for astrophysical phenomena such as gravitational lensing, gravitational waves detection and the description of Black Holes, stellar systems and other compact objects. Regarding the latter, since distances between astrophysical bodies tend to be much larger than the diameter of each of them, typically any exterior influence can be neglected; this is why the exterior gravitational fields of *isolated systems* or isolated massive objects are modeled by the so-called *asymptotically flat spacetimes*, which outside a compact spacial region approach the Minkowski metric. The understanding of isolated system and hence the study of the maximal globally hyperbolic evolution of asymptotically flat initial data ensured by [CG69], is the core of the so-called *Boost problem in general relativity.* Due to the development of the theory of Weighted Sobolev Spaces especially by Choquet-Bruhat and Christodoulou [CC81], in 1981 Christodoulou–O'Murchadha [CO81] showed that given an initial data in suitable Weighted Sobolev spaces, its development region contains asymptotically boosted slices relative to it and the evolution equations preserved the asymptotic decay.

Maximal surfaces role in the Initial Value formulation. In most of the mathematical and physical challenges related to the *Initial Value Formulation* of the Einstein Equations and in their applications to numerical problems, both *maximal surfaces* and the so-called *Boost problem in general relativity* often play a crucial role; we will survey here some of them. A more complete review specifically on *maximal surfaces* will instead be presented in Section 2.2.1.

– ***Solving the Constraint Equations.*** Due to the Gauss-Codazzi equations, (M, g) and K have to satisfy the so-called *Einstein constraint equations*

$$\mathrm{R}_g - |\,\mathrm{K}\,|_g^2 + (\mathrm{tr}_g\,\mathrm{K})^2 = 2\,(\boldsymbol{\mu} + \Lambda) \tag{2.1.5}$$

$$\mathrm{div}_g\,\mathrm{K} - \mathrm{d}(\mathrm{tr}_g\,\mathrm{K}) = \boldsymbol{J} \tag{2.1.6}$$

where $\boldsymbol{\mu}, \boldsymbol{J}$ are given data and respectively correspond to the *energy density* $\boldsymbol{\mu} = \mathfrak{T}(\nu, \nu)$ and the *momentum density* $\boldsymbol{J}(\cdot) = -\mathfrak{T}(\nu, \cdot)$, being ν the future-directed unit normal to (M, g, K) inside a spacetime solving (2.1.3). Since the constraint equations, combined with the so-called *Einstein evolution equations* produce solutions to (2.1.3), they not only describe the admissible initial data, but they also effectively determine the space of maximally globally hyperbolic solutions of the theory. To solve (2.1.5)–(2.1.6), which is a highly underdetermined elliptic system, the most effective technique is the so-called *conformal method.* This consists in decomposing g and K into prescribed data (a conformal class for g, a prescribed trace for K and a transverse-traceless tensor contributing to the traceless part for K) and unknowns (a conformal factor for g and the traceless part for K) so that (2.1.5)–(2.1.6) turn into a determined elliptic system. The original conformal method was initiated by Lichnerowicz [Lic44] and later extended by O'Murchadha–York [Yor73; OY73], constituting a still active research field –see e.g. [HMM22] for relatively recent results. In the *vacuum* case $\boldsymbol{\mu} = 0, \boldsymbol{J} = 0$, on a *maximal initial submanifold*, namely under the prescription $\operatorname{tr}_g \mathrm{K} = 0$, the constraints can be split into a decoupled system consisting in the linear prescription of divergence freeness for the traceless part of K and a non-linear equation for the conformal factor, the so-called *Lichnerowicz equation.* This had extensive applications in numerical integration schemes for Einstein equations, see e.g. the work of Estabrook–Wahlquist–Christensen–DeWitt–Smarr–Tsiang [Est+73]. For completeness, we remark that since D. Maxwell [Max14] proved that the so-called *conformal thin sandwich approach* coincides with the conformal method, other ways to tackle the constraint equations –see [AP14, Chapter 16], [BI21], [Max21] and references therein for more information– at the moment involve only four additional approaches which enable to enrich the picture by admitting different scenarios. The first, proposed by Baierlein–Sharp–Wheeler and studied mostly in the '60, is the *thin sandwich construction.* The second consists in solving the *prescribed scalar curvature equation* employing a semi-linear parabolic problem, initiated by the works of Bartnik [Bar93], Shi–Tam [ST02] and Smith–Weinstein [SW00]. The third consists of perturbation techniques –see [Hua10]. The latter is the *gluing procedure*, which produces new solutions to (2.1.5)–(2.1.6) by combining existing ones. The *connected sum gluing*, pioneered by Isenberg, Mazzeo and Pollack [IMP02] and then by Chruściel–Delay [CD03], is extremely valuable for applications such as modelling the physics of a system consisting of N astrophysical bodies interacting gravitationally, adding a Black Hole or a Warmhole to spacetimes and removing topological obstructions from Initial Data Sets. On the other hand, the *asymptotic exterior gluing*, first developed by Corvino–Schoen [Cor00; CS06], is effective for constructing spacetimes with regular asymptotic structure. In a recent joint work with Chrusciel and Nützi [CCN24], the author constructed a solution operator for the linearized constant scalar curvature equation at hyperbolic space, extending the recently introduced Mao-Oh-Tao [MOT23] technique to the hyperbolic setting and providing a new method to perform the Corvino–Schoen hyperbolic gluing.

–Positive Mass Conjecture. The existence of maximal submanifolds was also assumed in the first proofs of the *Positive Mass Conjecture* for asymptotically flat spacetimes, which states that for a nontrivial isolated physical system satisfying the so-called

Dominant Energy Condition, the total (ADM) mass –defined in the aforementioned work [ADM08]– is positive. The role of maximal surfaces was crucial foremost in the works of Brill–Deser [BD68] in 1968, Choquet-Bruhat and Marsden [CM76] in 1976, both considering data close enough to the flat ones in the $n = 3$ case, and then in the most general groundbreaking results by Schoen–Yau [SY79b; SY79a] in 1979, up to dimension $3 \leq n < 8$. The maximality condition $\operatorname{tr}_g \mathrm{K} = 0$ could then be removed, but initially only in terms of reduction of the proof to this case. In particular, in [BD68; CM76] the reduction to $\operatorname{tr}_g \mathrm{K} = 0$ could be performed by asserting the existence of a maximal surface in asymptotically flat spacetimes, only proven much later in 1984 by Bartnik [Bar84]; on the other hand, the reduction to the maximal case in dimension $n = 3$ was approached by Schoen–Yau in [SY81] by establishing and employing the technical proof of the existence of a solution to the quasilinear *Jang's equation*, first introduced in [Jan78]. We must mention that solving the *Jang's equation* actually allows one to reduce the problem to the so-called *Time-symmetric* or *Riemannian* case, namely with $\mathrm{K} \equiv 0$, even in higher dimensions dimension $3 \leq n < 8$, as shown by Eichmair [Eic13]. We also recall that a self-contained proof, avoiding the reduction to the time-symmetric case, is provided by Eichmair–Huang–Lee–Schoen [Eic+16]. More specifically, they showed the positivity of the (ADM) energy E and then recovered the positivity of the (ADM) mass $m = \sqrt{E^2 - |P|^2}$, where P is the (ADM) momentum, by using results related to the aforementioned *Boost problem in General Relativity* [CO81]: spacetime developments of the asymptotically flat end of Initial Data Sets contain boosted slices with computable energy related to E and P and the boosting factor. To conclude, we highlight that the assumption of *Dominant Energy Condition* for the positive mass conjecture, expressed in terms of $\boldsymbol{\mu}$ and $\boldsymbol{J}$ –see (2.1.5),(2.1.6)– as $\boldsymbol{\mu} \geq |\boldsymbol{J}|$, is strongly physically motivated, as it corresponds to the fact that the local energy density measured by any observer is always non-negative and never propagates faster than light –see [HE73, Chapter 4.3]–. Now, getting back to the maximality condition assumed in the first approaches to the Positive Mass Conjecture, we remark that for zero cosmological constant $\Lambda = 0$, the equation (2.1.5) translates the *Dominant Energy Condition* on an Initial Data with $\operatorname{tr}_g \mathrm{K} = 0$ into the intrinsic geometric condition $\mathrm{R}_g \geq 0$, which is essential to conclude the proofs via analytical and geometrical arguments.

–Stability of the Minkowski Space. Finally, the role of maximal surfaces was crucial in the renowned work on *The Global Nonlinear Stability of the Minkowski Space* by Christodoulou–Klainerman [CK93], which established a constructive proof of global, smooth, nontrivial, solutions to the EE in vacuum starting with generic Initial Data, sufficiently close to the flat ones. This groundbreaking result fits in the study of the maximal globally hyperbolic evolution of asymptotically flat initial data ensured by [CG69] and finds its foundations in the boost problem [CO81] and in the consequent *boosted energy estimates* [Kla85]. The proof relies on an involved comparison argument with the Minkowski spacetime at the level of the geometric structures given by the canonical space-like foliation of Minkowski, its null structure and its conformal group structure. In particular, the first feature is mirrored in the generic solution by prescribing a maximal space-like foliation. This is translated analytically into an elliptic equation for the scalar quantity called *lapse*, which ensures that the condition $\mathrm{tr}_g \mathrm{K} = 0$ is preserved under evolution.

Boosted maximal surfaces in Schwarzschild: motivation

The notion of free-falling observer in Special and General Relativity. In this paragraph, we will mostly refer to [PW14, Chapter 5] and [Cho15, Chapter III].

The root of Einstein's revolutionary formulation of *General relativity* formalized in the EE (2.1.3) lies in the identification of gravitation with inertia. More precisely, besides appealing to certain conservation postulates, the theory is based on the so-called *equivalence principles*. The *weak equivalence principle* relies on the equivalence of gravitational and inertial mass, physically tested since the times of Galileo: *test particles* (massive pointlike objects not modifying the gravitational field) move with the same acceleration in a gravitational field, irrespective of their mass or internal composition. The *strong* version of Einstein requires the weak principle to hold everywhere and that in sufficiently small neighbourhoods of spacetime points, the effects of gravitation are cancelled out by moving to a *locally inertial coordinate system*, where the laws of nature are described by those of special relativity. In particular, the locally inertial coordinate system is given in a neighbourhood of a point of the spacetime by *normal coordinates* –in which at the point the metric is $\{\eta_{ij}\}_{ij}$ and its derivatives vanish–. In such a neighbourhood, up to second order, physics looks Minkowskian. On the other hand, we also keep in mind that the Minkowski spacetime is a trivial (because flat) solution to the EE (2.1.3) in vacuum $\mathfrak{T} = 0$ with $\Lambda = 0$ and in that sense, special relativity is also a special case of general relativity.

A timelike curve in spacetime gives a suitable definition of *observer*. In Minkowski, an observer is matched to a cartesian coordinate system and can observe events in the whole spacetime, in mathematical and physical terms. This is a consequence of the mathematical fact that the tangent space at each point of Minkowski is isomorphic to the spacetime itself. In any other curved spacetime, an observer carries only local coordinates with itself and can not globally describe the spacetime nor the motion of other observers with respect to its own.

A special case of observer is a *free-falling particle* –which we will interchangeably call *free-falling observer*–. As a consequence of the *equivalence principles*, it is mathematically described by a timelike geodesic in the spacetime $(\mathfrak{L}^{n+1}, \mathfrak{g})$; this is a parametrised curve with timelike tangent vector with vanishing acceleration, namely solving the geodesic equation

$$\gamma : I \subset \mathbb{R} \to \mathfrak{L}^{n+1}\,, \qquad \mathfrak{g}(\dot\gamma, \dot\gamma) < 0\,, \qquad \ddot\gamma^\mu + \overline{\Gamma}^\mu_{\nu\lambda}\dot\gamma^\nu\dot\gamma^\lambda = 0\,, \tag{2.1.7}$$

where $\overline{\Gamma}^{\mu}_{\nu\lambda}$ are the Christoffel symbols of $\mathfrak{g}$, which expressly encode gravitation and inertia. In fact, normal coordinates follow exactly timelike geodesics in the timelike direction at the point where they are centred. Now, if we consider Minkowski spacetime as a solution to EE (2.1.3), a free-falling observer is a straight line in timelike direction and it uniquely (up to rotations) corresponds to a (global) orthonormal frame, which is in particular an *inertial frame*. As described in Section 2.1.1, the free-falling observer then gives a foliation of time-slices, normal to its direction of motion, which are affine planes and hence a *foliation by maximal surfaces*.

The notion of a *foliation by maximal surfaces* in any Lorentzian manifold is independent of the choice of coordinates, as the vanishing of the mean curvature is a geometric condition. Moreover, given the problems outlined in Section 2.1.1, it seems to be a very natural choice in the study of spacetime. Finally, as we will discuss later in Section 2.2.1 and as abstracted in [BI21], the approach of specifying the mean extrinsic curvature is geometrically natural, since it leads to a quasilinear elliptic equation with strictly spacelike and smooth solutions, at least to the extent permitted by the regularity of the ambient spacetime and boundary conditions; although there are examples of spacetimes not admitting maximal hypersurfaces, such non-existence behaviour is caused by very special global topological obstructions.

It is clear, by its geometric nature, that a foliation by maximal surfaces corresponds to a geometric and hence preferred choice of time function. Moreover, in view of the above discussion on the properties of maximal surfaces –often reflecting features of the spacetime they are embedded in– and in analogy with what happens in Minkowski, it is legit to ask ourselves whether they capture fundamental physical information. For example, as it is shown in [ONe83, Chapter 13, Lemma 9], the observers moving in the normal direction with respect to the (totally geodesic and hence maximal) time slices in exterior Schwarzschild are at rest, but they do not corresponds to free-falling particles. More precisely, their acceleration is non-vanishing –so they do not solve (2.1.7)– and equals to $\frac{m}{r^2}$ in radial direction, being m the Schwarzschild mass and r the Schwarzschild radius; this shows that the foliation by maximal surface given by the static time not only explicitely encodes the mass of the object generating the gravitational field but also, for large radii –where the metric approaches the Minkowski metric–, captures the Newtonian gravity. Moreover, it corresponds to observers that have acceleration vanishing at infinity and hence, in a way, get closer and closer to be free-falling. In particular, for our purposes, one expects the foliation to approach, in a suitable sense, the foliation by affine planes of Minkowski in regions of the spacetime where the metric is close to the Minkowski metric –notice that this notion of closeness is given in terms of a choice of coordinates–. This will be clarified in the next paragraph.

Asymptotic symmetries in asymptotically flat spacetimes. We already mentioned that *asymptotically flat spacetimes*, which outside a compact spatial region approach the

Minkowski metric, model the exterior gravitational fields of *isolated systems* or isolated massive objects. Physically speaking, it is expected that an observer far from the sources of the gravitational field should experience phenomena almost as in a flat spacetime and, in particular, its symmetries, which are given by the *Poincaré group* being a semi-direct product of the *Lorentz group* with translations –see Section 2.2.2 for a more extensive discussion. Nevertheless, in 1962 Bondi–Metzner–Sachs [BVM62; Sac62] discovered that general relativity actually does not reduce to special relativity in the limit of large distances from gravitational sources and that gravity leaves a footprint. Motivated by the investigation of the flow of energy at null infinity due to propagating gravitational waves, they employed the *Penrose's conformal compactification* [Pen11] of spacetimes which identifies the *future null infinity* –in literature denoted by $\mathscr{I}^+$ and collecting the ends of the future directed null geodesics (light rays)– with $\mathbb{S}^{n-1} \times \mathbb{R}$ and investigated the asymptotic symmetries in terms of transformations keeping $\mathscr{I}^+$ invariant. They showed that the resulting asymptotic transformations form an infinite-dimensional group, the *BMS group*, with a structure independent of the particular gravitational field and given by the semi-direct product of the *Lorentz group* and an infinite-dimensional abelian group –*supertranslations*– instead of the finite-dimensional group of translations. In particular, as the translation form a normal subgroup of the Poincaré group, so the supertranslations form a normal subgroup of the BMS group; the quotient group is the Lorentz group in both cases. We refer to [Wan23] and references therein for more precise information and a discussion on how this affects the charges (conserved quantities) in the ADM formalism.

Given the above discussion, we have an asymptotic notion of Lorentz transformation (in equivalence classes of supertranslations), which can be reduced to boosts in rotationally symmetric spacetimes. This justifies the idea at the end of the previous paragraph of expecting a foliation by maximal surfaces, if admissible, to approach the foliation by affine planes of Minkowski in a suitable sense.

Outline of the problem With the final goal of having a (unique) foliation of maximal surfaces for any element of the asymptotic Lorentz group for asymptotically flat spacetimes, one first has to show its existence. In the flat case of Minkowski, this simply coincides with the trivial problem of identifying the affine planes as foliations of maximal hypersurfaces. In curved asymptotically flat spacetimes, the problem consists of solving a complicated *quasilinear elliptic equation* and presents significant difficulties mainly in terms of constructing suitable barriers at the asymptotically flat end and in controlling the interior of the spacetime. Relying on his work [Bar84] on maximal surfaces in asymptotically flat spacetimes, Bartnik in collaboration with Chruściel and O'Murchadha addressed the above problem in [BCO90], but imposed conditions for the interior of the spacetime, not allowing to cover, for example, the case where the spacetime possesses black and white holes delimited by a bifurcate Killing horizon including a bifurcation surface. To shed some light on these questions, we address the problem on the *Schwarzschild* *spacetime* [Sch16], since it is the prototype of a nontrivial asymptotically flat solution to the Einstein Equations and a model of spacetime with those features. Considering the problem in Schwarzschild spacetime is a natural choice and, in fact, new insights on spacelike constant (non-zero) mean curvature (CMC) have been provided by recent works by Tam [Tam23] and Li–Shi–Tam [LST24] exactly on this spacetime.

In this work, we show the existence of an entire (complete, non-compact, without boundary) smooth maximal hypersurface in the *maximally extended Schwarzschild*, approaching a suitable coordinate-dependent boosted hypersurface in the asymptotically flat end; this corresponds to approaching an element of the Lorentz group identified by a boost parameter $\mathrm{b} \in [0, 1)$ –see Section 2.2.2 for boosts in Minkowski–.

To tackle the problem, we need to find suitable coordinate-dependent boosted hypersurfaces admitting barriers at spatial infinity. Moreover, we introduce a new time function in the maximally extended Schwarzschild spacetime, which allows us to employ the main results by [Bar84]; this is also interesting on its own as it is a *Cauchy time function*.

As future directions, we plan to continue investigating this problem by addressing the *uniqueness* of the maximal hypersurface constructed, after showing that it can be written as a graph and hence regarded as *entire*. This is a *Bernstein-type problem* and would complete the analogy with the Minkowski spacetime presented in Section 2.1.1, answering some of the open problems highlighted by Bartnik himself [Bar21]. We believe that we will need first to show that maximal hypersurfaces do not go null towards the asymptotically flat end, using similar ideas to [HY21], but also [Sim87] and [CL03] on exterior solutions to the minimal surface and the Monge-Ampere equations, respectively. Afterwards, it will probably be useful the use of *calibrations* and of a careful *blow-down analysis*, which then allows to employ the classification of the variational solutions in Minkowski provided by [Eck86] and [Hon20b]. Other results that can be employed are, for example, the ones on the asymptotics of maximal foliations provided in [BCO90] and the convergence of surfaces moving along *mean curvature flow* towards maximal ones in asymptotically flat spacetimes [Eck93, Theorem 3.1].

2.1.2 Outline of the results and structure of the work

In this Section, we will provide an idea of the main results and outline the structure of this work. The mathematically rigorous formulation of the problem and the strategy of the proof can be found instead in Section 2.4, making use of the notations, definitions and basic results provided in Section 2.2.

In the whole chapter, the results of the author are carried in dimension $n = 3$, although they should be easily extendible to any dimension.

Main results. Consider the maximally extended Schwarzschild spacetime $\left(\mathfrak{L}^K_m, \mathfrak{g}^K_m\right)$ with the Kruskal–Szekeres coordinates $\mathsf{T}, \mathsf{X}, \theta_I$ and let $\{\mathsf{X} = \mathsf{T} = 0\}$ the so-called *bifurcation* *sphere* –see Section 2.3.2 for details–. Let $\mathrm{H}_{\mathfrak{g}_m^K}(M)$ indicate the mean curvature of a hypersurface M. We denote by $\mathscr{B}_\mathrm{b}$ the class of a suitable notion of *coordinate boosts* of parameter $\mathrm{b} \in [0,1)$ in the asymptotically flat ends of exterior Schwarzschild –see Definition 2.4.1 for a precise characterization–: these correspond to complete, non-compact, smooth spacelike hypersurfaces $\mathscr{S}_\mathrm{b}$ which, for large radii, can be written as the graph of the form $\{t = \mathrm{b}\, x_1 + O_\infty(\log(|\,\boldsymbol{x}\,|))\}$, being t the static time and $\boldsymbol{x} = (x_1, x_2, x_3)$ the spatial isotropic coordinates –see Section 2.3.1–. The main result of this work asserts the existence of an *entire* (complete, non-compact, without boundary) smooth, maximal hypersurface M_b in $\left(\mathfrak{L}_m^K, \mathfrak{g}_m^K\right)$, which is asymptotic in a suitable sense to a sensibly chosen *coordinate boost* $\mathscr{S}_\mathrm{b} \in \mathscr{B}_\mathrm{b}$, for any boost parameter $\mathrm{b} \in [0,1)$.

Theorem. 2.4.4 (Main Theorem, simplified version) *For any boost parameter* $\mathrm{b} \in [0,1)$*, in the maximally extended Schwarzschild spacetime* $\left(\mathfrak{L}_m^K, \mathfrak{g}_m^K\right)$*, there exists an entire smooth maximal hypersurface*

$$M_\mathrm{b} \hookrightarrow \left(\mathfrak{L}_m^K, \mathfrak{g}_m^K\right)\,, \qquad \mathrm{H}_{\mathfrak{g}_m^K}(M_\mathrm{b}) = 0\,, \qquad \{\mathsf{X} = \mathsf{T} = 0\} \subset M_\mathrm{b}$$

which is asymptotic to a coordinate boost $\mathscr{S}_\mathrm{b} \in \mathscr{B}_\mathrm{b}$ *of parameter* b.

As an immediate consequence, we can generate a foliation of maximal surfaces in exterior Schwarzschild, corresponding to a geometric choice of time function associated to any boost of parameter $\mathrm{b} \in [0,1)$.

Corollary. 2.4.6 *For any* $\mathrm{b} \in [0,1)$*, the exterior Schwarzchild spacetime* $\left((\mathfrak{L}_{ext}^I)_m, \mathfrak{g}_m^I\right)$ *admits a foliation by (boosted) maximal spacelike hypersurfaces asymptotic to hypersurfaces in the class* $\mathscr{B}_\mathrm{b}$*. The foliation is generated by an entire maximal surface in the maximally extended Schwarzschild* $\left(\mathfrak{L}_m^K, \mathfrak{g}_m^K\right)$*. This provides a geometric choice of time function associated to the asymptotic boost of parameter* $\mathrm{b} \in [0,1)$.

The proof of Theorem 2.4.4 is based on considering the boundaries $\{\Sigma_k^2\}_{k=1}^\infty$, $\Sigma_k^2 \hookrightarrow \mathscr{S}_\mathrm{b}$ of an exhaustion of a properly chosen coordinate-dependent spacelike hypersurface $\mathscr{S}_\mathrm{b} \in \mathscr{B}_\mathrm{b}$ and solving the (compact) Dirichlet problem associated to the maximal surface equation with boundary data Σ_k^2. We then aim to show that the sequence of such compact hypersurfaces $\{M_k\}_{k=1}^\infty$ converges to the desired hypersurface M_b, which asymptotically approaches the chosen coordinate boost $\mathscr{S}_\mathrm{b} \in \mathscr{B}_\mathrm{b}$.

The first problem that needs to be addressed consists in wisely choosing the spacelike hypersurface $\mathscr{S}_\mathrm{b} \in \mathscr{B}_\mathrm{b}$, which is coordinate-dependent, so that upper and lower barriers can be constructed. This is, in fact, necessary to ensure that M_b is asymptotic to the chosen $\mathscr{S}_{\rm b} \in \mathscr{B}_{\rm b}$, by showing that all the hypersurfaces M_k lie between these barriers. The natural candidate $\{t = f_{\rm b}(\boldsymbol{x}) = {\rm b}\, x_1\}$ in isotropic coordinates, for example, does not allow for the construction of such barriers by adapting barriers from the Minkowski spacetime. We will thus provide the following result, relying on [Mey63] and [BCO90].

Proposition. 2.6.1 *For any* $\mathrm{b} \in [0,1)$*, there exists a coordinate boost* $\mathscr{S}_{\rm b} \in \mathscr{B}_{\rm b}$ *and a function* $\bar{f}_{\rm b} \in C^\infty(\mathbb{R}^3 \setminus \overline{B}_1)$ *satisfying*

$$\sup_{\boldsymbol{y}\in\mathbb{R}^3\setminus\overline{B}_1} \left(\frac{|\bar{f}_{\rm b}(\boldsymbol{y})|}{\log(|\,\boldsymbol{y}\,|)} + |\,\boldsymbol{y}\,| \sum_{i=1}^{3} |\partial_i \bar{f}_{\rm b}(\boldsymbol{y})| + |\,\boldsymbol{y}\,|^2 \sum_{i,j=1}^{3} |\partial^2_{ij} \bar{f}_{\rm b}(\boldsymbol{y})| + |\,\boldsymbol{y}\,|^3 \sum_{i,j,k=1}^{3} |\partial^3_{ijk} \bar{f}_{\rm b}(\boldsymbol{y})| \right) = C_{\bar{f}_{\rm b}}$$

for some constant $0 < C_{\bar{f}_{\rm b}} < \infty$ *such that, in isotropic coordinates,*

$$\mathscr{S}_{\rm b} \cap \{|\,\boldsymbol{x}\,| > \bar{K}m\} = \{t = u_{\rm b}(\boldsymbol{x})\}\,, \qquad u_{\rm b}(\boldsymbol{x}) \coloneqq {\rm b}\, x_1 + m\bar{f}_{\rm b}\left(\frac{\boldsymbol{x}}{m}\right)$$

for some constant $\bar{K} \geq 1$ *and such that it admits upper and lower barriers. More precisely, for any* $0 < \varepsilon < 1$*, there exists* $K_{\rm b} = K_{\rm b}({\rm b}, \varepsilon, C_{\bar{f}_{\rm b}}) > \bar{K}$ *such that for any* $K \geq K_{\rm b}$ *there exist hypersurfaces* $\mathscr{W}^{\pm}_{{\rm b},K,\varepsilon}$ *satisfying*

$$\mathscr{W}^{\pm}_{{\rm b},K,\varepsilon} = \{t = w^{\pm}_{{\rm b},K,\varepsilon}(\boldsymbol{x})\} \;\; \textit{spacelike on} \;\; \{|\,\boldsymbol{x}\,| \geq Km\}$$

$$\mathrm{H}_{\mathfrak{g}^I_m}(\mathscr{W}^{+}_{{\rm b},K,\varepsilon}) < 0\,,\;\; 0 < w^{+}_{{\rm b},K,\varepsilon} - u_{\rm b} = O(|\,\boldsymbol{x}\,|^{-\varepsilon})\,,\;\; (w^{+}_{{\rm b},K,\varepsilon} - u_{\rm b})\big|_{\{|\,\boldsymbol{x}\,|=Km\}} \geq K^{\varepsilon}\, m\sqrt{1-{\rm b}^2}$$
$$\mathrm{H}_{\mathfrak{g}^I_m}(\mathscr{W}^{-}_{{\rm b},K,\varepsilon}) > 0\,,\;\; 0 < u_{\rm b} - w^{-}_{{\rm b},K,\varepsilon} = O(|\,\boldsymbol{x}\,|^{-\varepsilon})\,,\;\; (u_{\rm b} - w^{-}_{{\rm b},K,\varepsilon})\big|_{\{|\,\boldsymbol{x}\,|=Km\}} \geq K^{\varepsilon}\, m\sqrt{1-{\rm b}^2}$$

with $\left|\mathrm{H}_{\mathfrak{g}^I_m}(w^{\pm}_{{\rm b},K,\varepsilon}) \pm \frac{K^{2\varepsilon}\varepsilon(1-\varepsilon)m^{1+\varepsilon}}{|\,\boldsymbol{z}(\boldsymbol{x})|^{2+\varepsilon}}\right| \leq \frac{C_{\rm b}(1+K^{2\varepsilon})}{|\,\boldsymbol{x}\,|^3}$ *as* $|\,\boldsymbol{x}\,| \to \infty$*, for* $C_{\rm b}$ *is independent of* K*.*

Now, once this result provides the choice of $\mathscr{S}_{\rm b}$ and the existence of barriers, the fact that all the solutions M_k lie between the barriers $\mathscr{W}^{\pm}_{{\rm b},K,\varepsilon}$ for some large K can be ensured by showing a uniform height estimate, presented in Theorem 2.8.1. As in [Bar84, Theorem 5.3], this can be proven in asymptotically flat spacetimes by combining the so-called *Bartnik's uniform interior condition* with the *pointwise gradient estimate* in [Bar84, Theorem 3.1 (iv)]. The latter holds under the so-called *time-dependent Bartnik's conditions*, which have then to be guaranteed for a suitable choice of time function on the whole spacetime $(\mathfrak{L}^K_m, \mathfrak{g}^K_m)$. On the other hand, once a uniform height estimate is established, [Bar84, Theorem 3.1 (iii)] allows to get directly a gradient estimate. Again, the *time-dependent Bartnik's conditions* have to hold and moreover, the boundary data $\{\Sigma^2_k\}_{k=1}^\infty$ need to lie in a constant time slice. In particular, we require the zero time-slice of the desired time function to coincide with the previously chosen $\mathscr{S}_{\rm b}$. A uniform gradient estimate is, in fact, necessary to ensure that the sequence M_k sub-converges in a

$C^{2,\alpha}$ sense, after combining it with the De Giorgi–Nash theory and the interior Schauder estimate.

In view of these observations, we will show how the time function T of Kruskal-Szekeres coordinates does not satisfy the crucial *time-dependent Bartnik's conditions* on non-compact sets of $\mathfrak{L}^K_m$. We will then introduce a new time function $\widetilde{\mathsf{T}}$.

Proposition. 2.5.2 (Simplified version) *The function*

$$\widetilde{\mathsf{T}} := 4m \operatorname{artanh}\left(\frac{\mathsf{T}}{\sqrt{1+\mathsf{X}^2}}\right) \in (-\infty, +\infty)$$

is a well-defined time function in the maximally extended Schwarzschild $\left(\mathfrak{L}^K_m, \mathfrak{g}^K_m\right)$*. In particular, it is a Cauchy time function.*

Moreover, the time-dependent Bartnik's conditions *hold in a region* $\{r > \kappa_0 m\}$ *for any* $\kappa_0 > 0$*, where* r *denotes the Schwarzschildean radius.*

The fact that the *time-dependent Bartnik's conditions* can be guaranteed on the region $\{r > \kappa_0 m\}$ is not restrictive due to the *crushing singularity* nature of the spacetime singularity of Schwarzschild, as discussed in Lemma 2.3.1–Proposition 2.3.2. We highlight that the function $\widetilde{\mathsf{T}}$ is also interesting on its own as it is a Cauchy time function in the maximally extended Schwarzschild spacetime. In addition, to complete the list of conditions guaranteeing the validity of the uniform height estimate of Theorem 2.8.1, we show in Proposition 2.5.3 that a (weaker version) of *Bartnik's uniform interior condition* holds. This result makes use of the crushing singularity and of the fact that the quantity involved obeys a monotonicity principle with respect to the new time function $\widetilde{\mathsf{T}}$.

Finally, to combine all the needs outlined above, the new time function $\widetilde{\mathsf{T}}$ has to be glued to a time function generated via staticity by the hypersurface $\mathscr{S}_{\mathrm{b}}$, as in Theorem 2.7.1.

Structure of the work. In Section 2.2 we introduce the basic notations, definitions and preparatory results. In particular, in Section 2.2.1 and Section 2.2.2 we will combine preliminary results with short surveys on *maximal surfaces* and *boosts in Minkowski*, respectively, since these are crucial objects of the work.

We dedicate the whole Section 2.3 to an extensive overview of the *Schwarzschild spacetime*, recalling the *Schwarzschildean* and *isotropic coordinates* in Section 2.3.1 and the *Kruskal–Szekeres coordinates* in Section 2.3.2. We then rigorously show the existence of barriers at the *crushing singularities* in Section 2.3.3. Finally, we present a *blow-down analysis* for the exterior Schwarzschild spacetime in relation to the Minkowski spacetime in Section 2.3.4.

In Section 2.4, after introducing the important Definition 2.4.1 of *Schwarschild coordinate-dependent boosts*, we state the main Theorem 2.4.4, specifically in Section 2.4.1 and we outline the strategy of the proof in Section 2.4.2.

In Section 2.5, we first discuss the *Bartnik's time-dependent conditions* for the three coordinates in Schwarzschild introduced in Section 2.3. In these regards, Proposition 2.5.1 is particularly relevant for later applications. The rest of the Section is dedicated to defining and proving properties of the new Cauchy time function on the maximally extended Schwarzschild, specifically addressed in Proposition 2.5.2. The monotonic quantity related to the *Bartnik's uniform interior condition* is then the content of Section 2.5.1 and, more precisely, of Proposition 2.5.3.

In Section 2.6, we largely discuss how and under which conditions we are able to construct barriers in the asymptotically flat end, given a wise choice of Schwarzschild coordinate-dependent boost. We then show the existence of a suitable coordinate-dependent boost admitting barriers in Proposition 2.6.1, proving the result in Section 2.6.1.

In Section 2.7 the time functions provided by Proposition 2.5.2 and Proposition 2.6.1 are glued.

In Section 2.8 we provide a *uniform height estimate* via Theorem 2.8.1.

Section 2.9 concludes the work by completing the proof of the main Theorem 2.4.4.

The Appendix A.1 is dedicated to the long computations necessary for the proof of Proposition 2.5.2 in Section 2.5.

2.2 Preliminaries

When not differently indicated, we will use notations and basic definitions mostly consistent with our main reference [Bar84]. We will then introduce the main tools regarding *maximal surfaces* (Section 2.2.1) and *boosts* (Section 2.2.2) in combination with short, but exhaustive surveys.

In this work, a *spacetime* $(\mathfrak{L}^4, \mathfrak{g})$ is a 4-dimensional smooth manifold endowed with a smooth metric $\mathfrak{g}$ of signature $(-1, 1, 1, 1)$. We denote its *Levi Civita connection* by $\overline{\nabla}$, the *Christoffel symbols* by $\overline{\Gamma}^{\lambda}_{\mu\nu}$, where μ, ν, λ are indexes with respect to a local frame and the metric pairing induced by $\mathfrak{g}$ on tensor fields by $\langle \cdot, \cdot \rangle$. Given a (a, b)-tensor $\boldsymbol{B} \in \Gamma(T^b_a \mathfrak{L}^4)$, where $\Gamma(T^b_a \mathfrak{L}^4)$ denotes the space of sections of the (a, b)-tensor bundle $T^b_a \mathfrak{L}^4$, we write $\overline{\nabla}^k \boldsymbol{B}$ for the k-th tensor covariant derivative. For simplicity, we denote $T\mathfrak{L}^4 := T^1_0 \mathfrak{L}^4$ and $T^*\mathfrak{L}^4 := T^0_1 \mathfrak{L}^4$. The *Lie derivative* in direction $\mathfrak{X} \in \Gamma(T\mathfrak{L}^4)$ will be denoted by $\mathscr{L}_{\mathfrak{X}}$. Analogous conventions will be used for any other (semi)-Riemannian manifold.

At any point $\mathrm{p} \in \mathfrak{L}^4$, a tangent vector $X_\mathrm{p} \in T_\mathrm{p}\mathfrak{L}^4$ is said to be

$$\begin{aligned} &\textit{spacelike} && \text{if } \mathfrak{g}_\mathrm{p}(X_\mathrm{p}, X_\mathrm{p}) > 0\,, \\ &\textit{null / lightlike} && \text{if } \mathfrak{g}_\mathrm{p}(X_\mathrm{p}, X_\mathrm{p}) = 0\,, \\ &\textit{timelike} && \text{if } \mathfrak{g}_\mathrm{p}(X_\mathrm{p}, X_\mathrm{p}) < 0\,, \\ &\textit{causal} && \text{if } \mathfrak{g}_\mathrm{p}(X_\mathrm{p}, X_\mathrm{p}) \leq 0\,. \end{aligned}$$

This definition of the *causal character* of vectors extends to curves $\gamma : I \subset \mathbb{R} \to \mathfrak{L}^4$, which are said to be *spacelike, null, timelike* or *causal* if the tangent vector $\dot{\gamma}$ at each point is spacelike, null, timelike or causal, respectively. In particular, *geodesics*, namely curves satisfying the *geodesic equation*

$$\ddot{\gamma}^\mu + \overline{\Gamma}^\mu_{\nu\lambda}\dot{\gamma}^\nu\dot{\gamma}^\lambda = 0\,,$$

have an unique causal character. Given a hypersurface as an embedded smooth submanifold, we highlight the following definition, which is essential for our analysis.

Definition 2.2.1. *A codimension-one submanifold $M \subset \mathfrak{L}^4$ is said to be a* spacelike hypersurface *if at each point* $\mathrm{p} \in M$*, the unit normal to M is a timelike vector.*

The above definition is equivalent to the induced metric g on M being Riemannian, that is, positive definite.

Throughout this work, we will use the *Einstein summation convention* with indexes. The greek indexes will refer to spatial and time components ranging $1 \leq \lambda, \mu, \nu \leq 4$, the latin indexes will refer to spatial components ranging $1 \leq i, j, k, a, b, c \leq 3$ (in particular, we will use a, b, c for local frames on spacelike hypersurfaces and i, j, k specifically for frames on time slices), while the capital indexes will range $1 \leq I, J, K, L \leq 2$ and often refer to components on submanifolds of a spatial slice.

Time function and time-dependent Riemannian norm. Since spacetimes are motivated as physical models, to encode the concepts of *past* and *future*, the following definitions are needed.

Definition 2.2.2. *A smooth function $t : \mathfrak{L}^4 \to \mathbb{R}$ is called* time function *if* $\mathrm{d}t|_\mathrm{p} \neq 0$ *and its gradient* $\mathrm{grad}_\mathfrak{g}\, t|_\mathrm{p}$ *is a timelike vector for any* $\mathrm{p} \in \mathfrak{L}^4$.

Its level sets $\mathcal{S}_{t_o} = \{t = t_o\}$ are smooth hypersurfaces called time slices.

The function t is called a Cauchy time function *if all the time slices $\mathcal{S}_{t_o}$ are* Cauchy surfaces*, namely they are intersected exactly once by any inextendible timelike curve.*

Notice that the timelike condition on $\mathrm{grad}_{\mathfrak{g}}\, t$ coincides with the level sets $\mathcal{S}_t$ being spacelike. A coordinate frame on a time slice $\mathcal{S}_t$ will be often represented by $\{\partial_i\}_{i=1}^3$ and the induced metric by ${}^t g$ with components ${}^t g_{ij}$. The Levi Civita connection is denoted by ${}^t\mathrm{D}$ and the Christoffel symbols by ${}^t\Gamma$. Given a function $f \in C^\infty(\mathfrak{L}^4)$, we write its gradient with respect to ${}^t g$ as

$$ {}^t\mathrm{D}f \coloneqq \mathrm{grad}_{{}^t g}\, f = {}^t g^{ij}\partial_j f\, \partial_i = {}^t\mathrm{D}^i f\, \partial_i \tag{2.2.1} $$

and by ${}^t\mathrm{D}^i f = {}^t g^{ij}\partial_j f$ its components. The *Hessian* will be denoted by ${}^t\mathrm{D}^2 f$, reading in coordinates

$$ {}^t\mathrm{D}^2_{ij} f = {}^t\mathrm{D}^2 f(\partial_i, \partial_j) = \langle {}^t\mathrm{D}_{\partial_i}({}^t\mathrm{D}f), \partial_j \rangle \tag{2.2.2} $$

and its trace with respect to ${}^t g$, namely the Laplace–Beltrami operator of $\mathcal{S}_t$, by $\Delta_t f$.

The vector field $-\,\mathrm{grad}_{\mathfrak{g}}\, t$ defines a *time orientation* on $(\mathfrak{L}^4, \mathfrak{g})$ in the sense that for any point $\mathrm{p} \in \mathfrak{L}^4$, a causal vector $X \in T_{\mathrm{p}}\mathfrak{L}^4$ is said to be *future-pointing* if $\mathfrak{g}(X, -\,\mathrm{grad}_{\mathfrak{g}}\, t|_{\mathrm{p}}) < 0$ and *past-pointing* if $\mathfrak{g}(X, -\,\mathrm{grad}_{\mathfrak{g}}\, t|_{\mathrm{p}}) > 0$. A causal curve γ is then said to be *future-directed* or *past-directed* according to $\dot\gamma$ being future-pointing or past-pointing along the whole curve. It is then easy to see that the function t is strictly increasing on any future-directed causal curve.

Given a time function t, the vector field $-\,\mathrm{grad}_{\mathfrak{g}}\, t$ is also a future-pointing (with respect to the time orientation defined by itself) normal vector field to all the time slices $\mathcal{S}_t$. We define the *lapse* of the time function t to be

$$ \alpha_t^2 \coloneqq -\langle\, \mathrm{grad}_{\mathfrak{g}}\, t,\ \mathrm{grad}_{\mathfrak{g}}\, t\,\rangle^{-1} \tag{2.2.3} $$

and this implies that

$$ \mathscr{T} \coloneqq -\alpha_t\ \mathrm{grad}_{\mathfrak{g}}\, t \tag{2.2.4} $$

is the *future-pointing unit normal* to the time slices $\mathcal{S}_t$.

A time function also enables the definition of a *time-dependent Riemannian norm* as follows. Considering a local orthonormal frame $\{\mathrm{E}_i\}_{i=1}^3$ on the time slices $\mathcal{S}_t$ and the future-pointing unit normal given by (2.2.4), we get an *adapted local orthonormal frame* $\{\mathrm{E}_\mu\}_{\mu=1}^4$ just by setting $\mathrm{E}_4 \coloneqq \mathscr{T}$. Let us then denote by E^*_μ the elements of the dual frame. Given $\boldsymbol{B} \in \Gamma(T^m_l\mathfrak{L}^4)$, we define

$$ \begin{aligned} {}^t\|\boldsymbol{B}\| &\coloneqq \sup_{\mathrm{p}\in\mathfrak{L}^4} \sqrt{\sum_{\lambda_1,\dots,\lambda_l,\mu_1,\dots,\mu_m} \Big|\boldsymbol{B}|_{\mathrm{p}}\left(\mathrm{E}_{\lambda_1},\dots,\mathrm{E}_{\lambda_l},\mathrm{E}^*_{\mu_1},\dots,\mathrm{E}^*_{\mu_m}\right)\Big|^2} \\ {}^t\|\boldsymbol{B}\|_N &\coloneqq \sum_{k=0}^N {}^t\|\,\overline{\nabla}^k \boldsymbol{B}\|\,. \end{aligned} \tag{2.2.5} $$

This is a computationally convenient definition for our purposes. It is easy to see that this corresponds to the norm induced by the *time-dependent Riemannian metric*

$$^{t}\mathfrak{g}_R \coloneqq \mathfrak{g} + 2\,\alpha_t^2\,\mathrm{d}t^2\,.$$

Geometric quantities on spacelike hypersurfaces. Consider a spacelike hypersurface $M \hookrightarrow (\mathfrak{L}^4, \mathfrak{g})$ and the induced metric ${}^{M}g$ with the associated Levi-Civita connection ∇. Let $\{X_a\}_{a=1}^3$ a local frame on M and $\{\mathbf{e}_a\}_{a=1}^3$ a local ${}^{M}g$-orthonormal frame on M. Consider a vector field $Y \in \Gamma(T\mathfrak{L}^4)$ and smooth function $f \in C^\infty(M)$ and let

$$\mathrm{grad}_M\, f = {}^{M}g^{ab} X_b(f)\, X_a = \sum_{a=1}^{3} \mathbf{e}_a(f)\,\mathbf{e}_a$$

the gradient of f with respect to ${}^{M}g$. We use the following convention for the operators div_M, Δ_M of divergence and Laplacian on the hypersurface, which are provided in a coordinate-free notation, followed by a coordinate-frame notation and an orthonormal-frame notation:

$$\begin{aligned}\mathrm{div}_M\, Y &\coloneqq \mathrm{tr}_M(\overline{\nabla}\, Y) = {}^{M}g^{ab}\,\langle \overline{\nabla}_{X_a}\, Y,\, X_b\rangle = \sum_{a=1}^{3} \langle \overline{\nabla}_{\mathbf{e}_a}\, Y,\, \mathbf{e}_a\rangle\,,\\ \Delta_M f &\coloneqq \mathrm{div}_M(\mathrm{grad}_M\, f) = {}^{M}g^{ab}\,\langle \overline{\nabla}_{X_a}\, \mathrm{grad}_M\, f,\, X_b\rangle = \sum_{a=1}^{3} \mathbf{e}_a\left(\mathbf{e}_a(f)\right) - \left(\nabla_{\mathbf{e}_a}\mathbf{e}_a\right)(f).\end{aligned}$$

where tr_M is the trace operator with respect to the metric ${}^{M}g$.

Consider now the future-pointing unit normal N to the hypersurface $M \hookrightarrow (\mathfrak{L}^4, \mathfrak{g})$. We define the *second fundamental form* –also called *extrinsic curvature*– at any point $p \in M$ as the bilinear form

$$\begin{aligned}\mathrm{K}_p : T_pM \times T_pM &\to \mathbb{R}\\ (X, Y) &\mapsto \mathrm{K}_p(X, Y) \coloneqq \langle X, \overline{\nabla}_Y\, \mathrm{N}\rangle|_p = -\langle \overline{\nabla}_X\, Y,\, \mathrm{N}\rangle|_p\end{aligned}$$

and the *mean curvature* of $M \hookrightarrow (\mathfrak{L}^4, \mathfrak{g})$ as

$$\begin{aligned}\mathrm{H}_{\mathfrak{g}}(M) &\coloneqq \mathrm{tr}_M\, \mathrm{K} = \mathrm{div}_M\, \mathrm{N}\\ &= {}^{M}g^{ab}\,\langle \overline{\nabla}_{X_a}\, \mathrm{N},\, X_b\rangle = \sum_{a=1}^{3} \mathrm{K}(\mathbf{e}_a,\, \mathbf{e}_a)\,,\end{aligned} \tag{2.2.6}$$

where this notation keeps the dependence on the ambient metric $\mathfrak{g}$ explicit. In particular, the above definitions apply to the time slices $\mathcal{S}_t$, which are spacelike hypersurfaces. Given local coordinates (t, x^i), where t is a time function, and the associated coordinate frame

$\{\partial_t, \partial_i\}_{i=1}^3$ the metric reads

$$\mathfrak{g} = -(\alpha_t^2 - |\beta|_g^2)\mathrm{d}t^2 + 2\beta_i \mathrm{d}x^i \mathrm{d}t + {}^t g_{ij} \mathrm{d}x^i \mathrm{d}x^j \,, \tag{2.2.7}$$

where α_t is the lapse function defined in (2.2.3) and $\beta = g^{ij}\beta_i\partial_j$ is the so-called *shift vector*. The shift vector indicates the deviation from ∂_t of the direction of the unit normal $\mathscr{T}$ given in (2.2.4) to the time slices $\mathscr{S}_t$, as there holds $\mathscr{T} = \alpha_t^{-1}(\partial_t - \beta)$. We denote by div^o the divergence on the slices $\mathscr{S}_t$ and by

$$\mathrm{K}^o_{ij} = \langle \overline{\nabla}_{\partial_i} \mathscr{T}, \partial_j \rangle = \frac{\alpha_t^{-1}}{2}\left(\partial_t({}^t g_{ij}) - \mathscr{L}_\beta({}^t g_{ij})\right) \tag{2.2.8}$$

$$\mathrm{H}^o = \mathrm{div}^o(\mathscr{T}) = {}^t g^{ij}\mathrm{K}^o_{ij} = \frac{\alpha_t^{-1}}{2}\, {}^t g^{ij}\partial_t({}^t g_{ij}) - \alpha_t^{-1}\,\mathrm{div}^o(\beta) \tag{2.2.9}$$

their second fundamental form and mean curvature, respectively.

To analyze the mean curvature as a partial differential equation, it is essential to introduce the *height function* $u \in C^\infty(M)$ as the restriction of the time function to the spacelike surface, namely

$$\begin{aligned} u : M &\to \mathbb{R} \\ p &\mapsto u(p) := t|_M(p)\,. \end{aligned} \tag{2.2.10}$$

We then often use the notation $\mathrm{H}_\mathfrak{g}(u) = \mathrm{H}_\mathfrak{g}(M)$. In particular, when the time slices $\mathscr{S}_t$ admit a global coordinate chart $\{x^i\}_{i=1}^3$ –as it holds in Minkowski or in exterior Schwarzschild with standard coordinates– and the hypersurface M can be written as a graph

$$M = \left\{ \mathrm{p} \in \mathfrak{L}^{n+1} \,:\, t(\mathrm{p}) = \bar{u}(x^i(\mathrm{p}))\right\}$$

of a smooth function $\bar{u} : \mathscr{S}_0 \to \mathbb{R}$ over the zero-time slice $\mathscr{S}_0$, the height function coincides with $u = \bar{u} \circ x^i$. For simplicity, in the absence of ambiguity, in the case of a graph, we will directly say that $M = \operatorname{graph} u$ and u is also the height function of M; accordingly, $\mathrm{H}_\mathfrak{g}(u) = \mathrm{H}_\mathfrak{g}(M)$ denotes the mean curvature of M.

A generalization of the concept of mean curvature for hypersurfaces can be defined for (smooth) submanifolds $\Sigma = \Sigma^2 \hookrightarrow (\mathfrak{L}^4, \mathfrak{g})$ of codimension-2 and called *spacetime mean curvature vector* as follows. Let us first consider the *vector-valued second fundamental form* of Σ in $(\mathfrak{L}^4, \mathfrak{g})$ defined as

$$\begin{aligned} \vec{\mathscr{K}} : \Gamma(T\Sigma) \times \Gamma(T\Sigma) &\to \Gamma(T^\perp\Sigma) \\ (X, Y) &\mapsto \vec{\mathscr{K}}(X,Y) := \left(\overline{\nabla}_X Y\right)^\perp \,, \end{aligned}$$

where for a vector $\mathfrak{X} \in T_p\mathfrak{L}^4$, $\mathfrak{X}^\perp$ indicates the orthogonal projection onto $T_p^\perp\Sigma$. Then the *spacetime mean curvature vector* of Σ is given by the trace tr_Σ with respect to the induced metric on Σ, namely

$$\vec{\mathscr{H}}_{\mathfrak{g}}(\Sigma) := \operatorname{tr}_{\Sigma} \vec{\mathscr{K}}\,, \tag{2.2.11}$$

which is a section of the 2-dimensional normal bundle of Σ in $(\mathfrak{L}^4, \mathfrak{g})$.

A geometric quantity that will play a crucial role is

$$\mathscr{v}_{t}(M) := -\langle \mathrm{N}, \mathscr{T} \rangle\,, \tag{2.2.12}$$

which at each point indicates the angle between the hypersurface M having unit normal N and the time slice $\mathscr{S}_t$ which the point belongs to. As for the mean curvature, if $M = \operatorname{graph} u$, we write $\mathscr{v}_t(u) = \mathscr{v}_t(M)$. Once the time function is chosen, this quantity identifies the gradient of the height function in the following sense

$$\begin{aligned} \operatorname{grad}_M u &= \operatorname{grad}_{\mathfrak{g}} t + \langle \operatorname{grad}_{\mathfrak{g}} t, \mathrm{N}\rangle\, \mathrm{N} \\ &= \operatorname{grad}_{\mathfrak{g}} t - \alpha_t^{-1} \langle \mathscr{T}, \mathrm{N}\rangle\, \mathrm{N} \\ &= \operatorname{grad}_{\mathfrak{g}} t + \alpha_t^{-1}\, \mathscr{v}_t(M)\, \mathrm{N} \end{aligned} \tag{2.2.13}$$

where we used (2.2.4). It immediately follows that

$$|\operatorname{grad}_M u|^2_{\mathfrak{g}} = \alpha_t^{-2} \left(\mathscr{v}_t^2(M) - 1 \right) \tag{2.2.14}$$

and thus, in particular, $\mathscr{v}_t(M) \geq 1$. Moreover, one can rewrite the mean curvature defined in (2.2.6) as

$$\mathrm{H}_{\mathfrak{g}}(M)\, \mathscr{v}_t(M) = \operatorname{div}_M(\alpha_t \; \operatorname{grad}_M u) + \operatorname{div}_M \mathscr{T}\,, \tag{2.2.15}$$

where one can compute

$$\operatorname{div}_M \mathscr{T} = \mathrm{H}^o + (\mathscr{v}_t^2(M) - 1) \left(|\operatorname{grad}_M u|^{-2} \; \mathrm{K}^o(\operatorname{grad}_M u, \operatorname{grad}_M u) - \alpha_t^{-1} \mathscr{T}(\alpha_t) \right) \tag{2.2.16}$$

Finally, as a consequence of [Bar84, Proposition 2.1], on a surface with $\mathrm{H}_{\mathfrak{g}}(M) = 0$, there holds

$$\begin{aligned} \Delta_M \mathscr{v} = \mathscr{v} \left(|\mathrm{K}|^2_{\mathfrak{g}} + \mathfrak{Ric}(\mathrm{N},\mathrm{N}) \right) + \\ + \frac{1}{2} (\overline{\nabla}_{\mathrm{N}}\, \mathscr{L}_{\mathscr{T}} \mathfrak{g})(\mathbf{e}_a, \mathbf{e}_a) - (\overline{\nabla}_{\mathbf{e}_a}\, \mathscr{L}_{\mathscr{T}} \mathfrak{g})(\mathrm{N}, \mathbf{e}_a) - \mathscr{L}_{\mathscr{T}} \mathfrak{g}(\mathbf{e}_a, \mathbf{e}_b)\, \mathrm{K}(\mathbf{e}_a, \mathbf{e}_b)\,, \end{aligned} \tag{2.2.17}$$

where $\{\mathbf{e}_a\}_{a=1}^3$ a local ${}^M g$-orthonormal frame on M and for simplicity we wrote $\mathscr{v} = \mathscr{v}_t(M)$. As we will highlight, this is a crucial equation in the so-called a priori gradient estimate.

Finally, it is useful to write the mean curvature $\mathrm{H}_{\mathfrak{g}}(M)$ of a hypersurface M in terms of geometric quantities of the time slices. Let us still consider local coordinates (t, x^i)

and the form of the metric (2.2.7). Extending the height function $u \in C^\infty(M)$ to $\mathfrak{L}^4$ by $\partial_t u = 0$ so that $M = \{t - u = 0\}$ is a level set and its future-pointing unit normal vector field is the normalization of the vector field $-\operatorname{grad}_{\mathfrak{g}}(t-u) = \alpha_t^{-1}\mathscr{T} + \mathfrak{g}^{\mu i}\partial_i u\,\partial_\mu = \alpha_t^{-1}\mathscr{T} + {}^t\mathrm{D}u + \alpha_t^{-1}\langle {}^t\mathrm{D}u, \beta\rangle\mathscr{T}$. The unit normal is then given by

$$\mathrm{N} = \upsilon_t(M)\,(\mathrm{U} + \mathscr{T})\,, \qquad \text{where } \mathrm{U} = \frac{\alpha_t\,{}^t\mathrm{D}u}{1 + \langle \beta, {}^t\mathrm{D}u\rangle}, \quad \upsilon_t(M) = \frac{1}{\sqrt{1 - |\mathrm{U}|^2}} \tag{2.2.18}$$

and hence the mean curvature of M reads

$$\mathrm{H}_{\mathfrak{g}}(M) = \operatorname{div}^o(\upsilon\,\mathrm{U}) + \upsilon\,\mathrm{H}^o + \upsilon\langle \mathrm{U}, \overline{\nabla}_{\mathscr{T}}\mathscr{T}\rangle + \frac{\upsilon^3}{2}\mathscr{T}(|\mathrm{U}|^2)\,, \tag{2.2.19}$$

where again for simplicity we wrote υ for $\upsilon_t(M)$. The extended computations for the above formula can be found in [Bar84, (2.17)].

Asymptotically flat spacetimes. Before providing the definition of asymptotically flat (of order 1) spacetime, we declare our use of the *big-O* notation.

Writing $f = O_k(|\,\boldsymbol{y}\,|^{-\alpha})$ for some $k \in \mathbb{N}$, $\alpha \in \mathbb{R}$ means that

$$\sum_{0 \le |J| \le k} |\,\boldsymbol{y}\,|^{\alpha + |J|}|\partial^J f| = O(1)\,, \qquad \text{as } |\,\boldsymbol{y}\,| \to \infty.$$

In analogy, $f = O_k(\log(|\,\boldsymbol{y}\,|)$ for some $k \in \mathbb{N}$ means

$$\frac{|f|}{\log(|\,\boldsymbol{y}\,|)} + \sum_{1 \le |J| \le k} |\,\boldsymbol{y}\,|^{|J|}|\partial^J f| = O(1)\,, \qquad \text{as } |\,\boldsymbol{y}\,| \to \infty.$$

The notation $f = O_\infty(|\,\boldsymbol{y}\,|^{-\alpha})$ means that $f = O_k(|\,\boldsymbol{y}\,|^{-\alpha})$ for any $k \ge 0$. Since throughout the work we will mostly encounter functions in such a class, we will usually omit the subscript in O_∞ and simply write $f = O(|\,\boldsymbol{y}\,|^{-\alpha})$.

Definition 2.2.3. *A spacetime* $(\mathfrak{L}^4, \mathfrak{g})$ *is said to be* asymptotically flat *(of order $\alpha = 1$) if it admits a chart $(\boldsymbol{y}, t)$ with $\boldsymbol{y} = (y^1, y^2, y^3)$ on an open set $\mathfrak{L}_E$ such that $\mathfrak{L}_E = \{\mathrm{p} \in \mathfrak{L}^4 : |\boldsymbol{y}(\mathrm{p})| > R_0\}$ for a constant $R_0 \ge 1$ and $|\boldsymbol{y}(\mathrm{p})| = \sqrt{\sum_{i=1}^n y^i(\mathrm{p})}$ such that t is a time function on $\mathfrak{L}_E$ and*

$$\mathfrak{g}_{\mu\nu} - \eta_{\mu\nu} = O_k(|\,\boldsymbol{y}\,|^{-1}) \tag{2.2.20}$$

with $k \ge 2$. Here $\mathfrak{g}_{\mu\nu}$ are the components of the metric in the given chart $(\boldsymbol{y}, t)$ and the notation O_k refers to derivatives with respect to all the coordinates of the chart $(\boldsymbol{y}, t)$.

We call $\mathfrak{L}_E$ the exterior asymptotically flat region(s) *or* asymptotically flat end(s)*, while $\mathfrak{L}_I := \mathfrak{L}^4 \setminus \mathfrak{L}_E$ the* interior region.

2.2.1 Maximal hypersurfaces

After defining maximal hypersurfaces, we will describe them analytically in the flat case of Minkowski, trying to motivate some crucial concepts introduced as preliminaries. Finally, while providing a short survey on related results –for results until the 80's, we refer to the survey in [Bar87]–, we will recall the main theorems of [Bar84], which are going to be essential for this work.

Definition 2.2.4. *A hypersurface $M \hookrightarrow (\mathfrak{L}^4, \mathfrak{g})$ is said to be* maximal *if it is spacelike and its mean curvature defined in* (2.2.6) *vanishes, providing the geometric identity* $\mathrm{H}_M = 0$*.*

As for minimal surfaces, their name is motivated by a variational problem, which we briefly illustrate in the flat case. We first notice that in Minkowski spacetime $(\mathbb{R}^{1,3}, \eta)$ defined in (2.1.1), employing the notation of the previous section, the vector field defining the time orientation $-\operatorname{grad}_\eta t = \partial_t$ and the lapse (2.2.3) is $\alpha_t \equiv 1$, which thus gives as future pointing normal to the time slices $\mathscr{T} = \partial_t$. Consistently with (2.2.1), we denote the gradient of a function in Minkowski by D, omitting the indication of the choice of time, as it is the standard one.

Let us consider *weakly spacelike hypersurfaces* in Minkowski, namely Lipschitz hypersurfaces with causal normal vector when defined. It is easy to check that the weakly spacelike condition combined with the inverse function theorem ensures that they can be written as the graphs of Lipschitz functions $v \in C^{0,1}$ over the zero-time slice $\mathbb{R}^3$, with gradient $|\mathrm{D}v| \leq 1$. Considering now a bounded domain $\Omega \subset \mathbb{R}^3$ and a bounded function $\varphi : \partial\Omega \to \mathbb{R}$, given the concave nature of the area functional of graphs of functions over Ω, one can consider its maximization among the admissible candidates $\mathscr{K}$, establishing the problem of maximizing

$$\mathscr{A}(u) = \int_\Omega \sqrt{1 - |\mathrm{D}u|^2} \qquad u \in \mathscr{K}(\Omega, \varphi) = \{v \in C^{0,1}(\Omega) \,:\, |\mathrm{D}v| \leq 1,\, v|_{\partial\Omega} = \varphi\}\,. \tag{2.2.21}$$

A function attaining the maximum of this variational problem is called *area maximizing.* Computing the Euler-Lagrange equation of (2.2.21) for sufficiently regular functions gives rise to the following *quasi-linear second order partial differential equation*

$$0 = a^{ij}(\mathrm{D}u)\,\mathrm{D}^2_{ij}u, \qquad a^{ij}(\mathrm{D}u) = \frac{1}{\sqrt{1 - |\mathrm{D}u|^2}}\left(\delta^{ij} + \frac{\mathrm{D}^i u\,\mathrm{D}^j u}{1 - |\mathrm{D}u|^2}\right) \tag{2.2.22}$$

and the corresponding Dirichlet problem. It is easy to see that a tangent frame for the spacelike hypersurface $M = \operatorname{graph} u$ is given by $X_i = \partial_i + \partial_i u\,\partial_t$ and the induced metric in this frame reads ${}^M g_{ij} = {}^M g(X_i, X_j) = \delta_{ij} - \partial_i u\,\partial_j u$ with inverse ${}^M g^{ij} = \delta^{ij} + \frac{\mathrm{D}^i u\,\mathrm{D}^j u}{1-|\mathrm{D}u|^2}$. Besides, the future-pointing unit normal to M and the corresponding quantity (2.2.12) read

$$\mathrm{N} = \frac{\partial_t + \mathrm{D}u}{\sqrt{1-|\mathrm{D}u|^2}}\,, \qquad \upsilon_t(u) = -\eta\,(\mathrm{N}, \partial_t) = \frac{1}{\sqrt{1-|\mathrm{D}u|^2}}\,. \tag{2.2.23}$$

Its mean curvature as defined in (2.2.6), by direct computations via the above objects and consistently with (2.2.19) is given by

$$\begin{aligned} \mathrm{H}_\eta(u) &= \mathrm{div}_M\,\mathrm{N} = \frac{1}{\sqrt{1-|\mathrm{D}u|^2}}\ \mathrm{div}_M\,(\partial_t + \mathrm{D}u) \\ &= \mathrm{div}_{\mathbb{R}^n}\left(\frac{\mathrm{D}u}{\sqrt{1-|\mathrm{D}u|^2}}\right) = \left(\delta^{ij} + \frac{\mathrm{D}^i u\,\mathrm{D}^j u}{1-|\mathrm{D}u|^2}\right)\frac{\mathrm{D}^2_{ij}u}{\sqrt{1-|\mathrm{D}u|^2}} \end{aligned} \tag{2.2.24}$$

where $\mathrm{D}^2_{ij}u = \partial^2_{ij}u$ is its Hessian. This shows that the equation (2.2.22) is actually a geometric partial differential equation, reading $\mathrm{H}_\eta(u) = 0$ and corresponding to the vanishing of the mean curvature of the graph of u. Notably, it already exhibits the main features that appear in the general non-flat case. In particular, the operator degenerates at the points where $|\mathrm{D}u| = 1$, namely where the hypersurface *goes null*, containing pieces of *light cones* $\{t = \pm|x| + c\}$ for some $c \in \mathbb{R}$. That is why, to find solutions to (2.2.22) one has to restrict to functions with $|\mathrm{D}u| < 1$ or equivalently to spacelike hypersurfaces as in Definition 2.2.1. Now, if one can ensure a uniform bound on the gradient

$$|\mathrm{D}u| < 1 - \theta^2\,, \quad \text{or equivalently} \quad \upsilon_t(u) < \theta^{-1}$$

for $\theta \in (0,1)$, the quasi-linear second order operator in (2.2.22) is *uniformly elliptic*, as there holds

$$|\zeta|^2 \leq \left(\delta^{ij} + \frac{\mathrm{D}^i u\,\mathrm{D}^j u}{1-|\mathrm{D}u|^2}\right)\frac{\zeta_i\,\zeta_j}{\sqrt{1-|\mathrm{D}u|^2}} \leq \theta^{-3}\,|\zeta|^2 \qquad \text{for any } \zeta \in \mathbb{R}^3\,.$$

In analogy with minimal graphs in Euclidean space –see [MS73]–, to find solutions to the Dirichlet problem associated with (2.2.22) via a *Leray–Shauder fixed point theorem*, relying on the linear elliptic Schauder theory, it is necessary to have an *a priori $C^{1,\alpha}$ estimate*. The *De Giorgi–Nash theorem* allows us to reduce it to a C^1-a priori estimate, namely on an *a priori height estimate* and on an a priori bound for υ. More details on the solvability of a large class of quasi-linear elliptic problems are provided in the dedicated paragraph at the end of this Section.

Motivated by the physically and numerically relevant problems discussed in Section 2.1.1, the study of the existence of *maximal hypersurfaces* was pioneered by Choquet-Bruhat [Cho75], Brill–Flaherty [BF78], Bancel [Ban78], Eardley–Smarr [ES79], Marsden–Tipler [MT80], Eardley–Moncrief [ME81] and Isemberg–Moncrief [IM82]. These results covered mainly the existence of maximal surfaces in specific spacetimes admitting compact slices and of the solution to the Dirichlet problem under restrictive boundary data.

Relying on the analogy with the thriving research on *minimal surfaces*, Bartnik–Simon [BS82] in 1982 provided existence theorems and shed complete light on the analytical properties of maximal surfaces in Minkowski (2.2.22) with prescribed boundary data, finding the link with the solution to the variational problem (2.2.21) already investigated by [Ave63]. Other results in Minkowski on variational solutions with isolated singularities have been forged by Ecker [Eck86] and Hong–Yuan [HY21] and related implications for maximal surfaces over exterior domains can be found in the recent work by Hong [Hon20a].

Adapting the analysis by Bartnik–Simon to curved spacetimes, Gerhardt [Ger83] studied the solvability of the Dirichlet problem in nonflat environment and the existence of constant mean curvature (CMC) hypersurfaces in cosmological models. In parallel, Bartnik [Bar84] developed more sophisticated techniques to tackle the Dirichlet problem in curved spaces and, most importantly, to show the existence of maximal surfaces in asymptotically flat spacetimes with control on the asymptotic, completing the preliminary and restrictive attempts by Cantor–Fisher–Mardsen–O'Murchadha–York [Can+76] in vacuum. To highlight the difficulty of the problem and, at the same time, the persistence of the main features underlined in the Minkowski case, we will present the expression in (local) coordinates of the geometric equation $\mathrm{H}_{M,\mathfrak{g}} = 0$, where the role of $\boldsymbol{v}$ in relation to the elliptic nature of the equation becomes explicit –as in (2.2.19) this is not directly evident. Consider coordinates (t, x^i) with associated local frame $\{\partial_t, \partial_i\}_{i=1}^3$, giving a metric of the form (2.2.7) with $\beta \equiv 0$ –this is locally always possible. Then using (2.2.18), we get

$$\mathrm{N} = \boldsymbol{v}_t(u)(\mathrm{U} + \mathscr{T}) = \boldsymbol{v}_t(u)\left(\alpha_t\, {}^t\mathrm{D}u + \alpha_t^{-1}\partial_t\right), \qquad \boldsymbol{v}_t(u) = \frac{1}{\sqrt{1 - \alpha_t^2\, |{}^t\mathrm{D}u|_{{}^tg}}}\,. \tag{2.2.25}$$

Since the tangent vectors to M are given by $X_i = \partial_i + \partial_i u\, \partial_t$, we can compute the induced metric on M as

$${}^M g_{ij} = \mathfrak{g}(X_i, X_j) = {}^t g_{ij} - \alpha_t^2\, \partial_i u\, \partial_j u\,, \qquad {}^M g^{ij} = {}^t g^{ij} + \alpha_t^2 \boldsymbol{v}_t^2(u)\, {}^t\mathrm{D}^i u\, {}^t\mathrm{D}^j u$$

and consequently compute the mean curvature as

$$\begin{aligned} \mathrm{H}_{\mathfrak{g}}(M) &= \mathrm{div}_M\left(\boldsymbol{v}_t(u)(\mathrm{U} + \mathscr{T})\right) = -{}^M g^{ij} \boldsymbol{v}_t(u)\left\langle \alpha_t\, {}^t\mathrm{D}u + \alpha_t^{-1}\partial_t,\, \overline{\nabla}_{X_i} X_j \right\rangle \\ &= a^{ij}\left(u, {}^t\mathrm{D}u\right)\, \partial^2_{ij} u + \mathscr{F}\left(u, {}^t\mathrm{D}u\right) \end{aligned} \tag{2.2.26}$$

where $a^{ij}\left(u, {}^t\mathrm{D}u\right) = \alpha_t \boldsymbol{v}_t(u)\left({}^t g^{ij} + \alpha_t^2 \boldsymbol{v}_t^2(u)\, {}^t\mathrm{D}^i u\, {}^t\mathrm{D}^j u\right)$ and $\mathscr{F}$ a smooth function depending on the geometry of the time slices; more precisely it holds

$$\mathscr{F}\left(u, {}^t\mathrm{D}u\right) = \boldsymbol{v}_t(u)\, \mathrm{H}^o + \boldsymbol{v}_t(u)(1 + \boldsymbol{v}_t^2(u))\langle {}^t\mathrm{D}u, {}^t\mathrm{D}\alpha_t\rangle \tag{2.2.27}$$

$$+ \boldsymbol{v}_t^3(u)\left(\partial_t \alpha_t |{}^t\mathrm{D}u|_{{}^tg} - \alpha_t^2\, \mathrm{K}^o_{ij}\, {}^t\mathrm{D}^i u\, {}^t\mathrm{D}^j u\right). \tag{2.2.28}$$

For any $\zeta \in \mathbb{R}^3$, we have $a^{ij}\zeta_i\zeta_j = \alpha_t \boldsymbol{v}_t(u)\left(|\zeta|^2_{{}^t\!g} + \alpha_t^2 \boldsymbol{v}_t^2(u)({}^t g^{km}\partial_k u\, \zeta_m)^2\right)$ and hence, just by the Cauchy–Schwarz inequality, we have

$$\alpha_t|\zeta|^2_{{}^t\!g} \le a^{ij}\zeta_i\zeta_j \le \alpha_t\, \boldsymbol{v}_t^3(u)|\zeta|^2_{{}^t\!g} \tag{2.2.29}$$

which shows that, up to ensuring a uniform bound on $0 < \alpha_t < \infty$, the quantity $\boldsymbol{v}_t(M)$ actually controls the ellipticity of the second-order quasi-linear operator. As in the Minkowski case, a crucial step towards finding a hypersurface M solving $\mathrm{H}_{\mathfrak{g}}(M) = 0$ consists of an a priori gradient estimate equivalent to controlling the quantity $\boldsymbol{v}_t(M)$. Bartnik was able to show that the gradient estimate directly follows from an a priori height estimate, provided the existence of a time function t with respect to which the *Bartnik's spacetime conditions* are satisfied: there exists a constant $0 \le C_o < \infty$ such that

$${}^t\|\mathfrak{Ric}\|\,,\ {}^t\|\log\alpha_t\|_1\,,\ {}^t\|\,\overline{\nabla}\,\mathscr{T}\|_1 \le C_o \tag{2.2.30}$$

where ${}^t\|\cdot\|$ is the time-dependent Riemannian norm defined in (2.2.5) and $\mathfrak{Ric}$ is the Ricci tensor of $(\mathfrak{L}^4, \mathfrak{g})$. These conditions are essential to estimate the terms involving $\mathfrak{Ric}$ and the Lie derivatives on the right-hand side of (2.2.17), which is the main identity employed for an a priori gradient estimate –see [Bar84, proof of Theorem 3.1]. It is easy to see that in Minkowski they all vanish, but in non-flat spacetime, they becomes difficult to control even if we can compute them explicitly, whence the necessity of introducing the conditions (2.2.30). In particular, it holds

Theorem 2.2.5 (Gradient estimate [Bar84, Theorem 3.1 (iii-iv)]). *Let* $(\mathfrak{L}^4, \mathfrak{g})$ *be a spacetime and let* t *be a time function which satisfy the conditions* (2.2.30). *Consider a compact spacelike (as in Definition 2.2.1) hypersurface* $M \hookrightarrow (\mathfrak{L}^4, \mathfrak{g})$ *with vanishing mean curvature* $\mathrm{H}_{\mathfrak{g}}(M) = 0$, *height function* u *(as in* (2.2.10)*) and with boundary* $\partial M \neq \emptyset$.

(i) Assume the spacetime mean curvature vector of ∂M *(as in* (2.2.11)*) satisfies*

$${}^t\|\mathscr{H}_{\partial M,\mathfrak{g}}\| \le C_o \tag{2.2.31}$$

and that $u|_{\partial M} = c$ *for a constant* $c \in \mathbb{R}$. *Then in* M *there holds*

$$\boldsymbol{v}_t(M) \le 2\exp\left(\bar{C}\,\sup_M |u|\right) \tag{2.2.32}$$

for a constant $\bar{C} = \bar{C}(C_o)$.

(ii) For $p \in M$ *such that*

$$u(p) - \sup_{\partial M} u \ge \varepsilon > 0\,, \tag{2.2.33}$$

the following pointwise estimate holds

$$v_t(M)\big|_p \leq 2\bar{C}\, \exp\left(\sup_M u - u(p)\right) \tag{2.2.34}$$

for a constant $\bar{C} = \bar{C}(C_o, \varepsilon^{-1})$.

It follows that if the conditions (2.2.30) are ensured, the main theorem [Bar84, Theorem 5.4] on the existence of maximal surfaces in asymptotically flat spacetimes depends fully on the *uniform height estimate* [Bar84, Theorem 5.3]. In turn, a uniform height estimate can be recovered via integral estimates relying on the existence of suitable barriers in the asymptotically flat end, on the pointwise estimate in Theorem 2.2.5-(ii) and on the *Bartnik's uniform interior condition* reading

$$\begin{aligned} &\sup_{\mathrm{q}\in\{|x|=R_0\}} \; \sup_{\mathrm{p}\in\mathfrak{L}_I\setminus I^+(\mathrm{q})} \; t(\mathrm{p}) - t(\mathrm{q}) \leq C_I \qquad \text{if } t(\mathrm{q}) \geq 0\,, \\ &\sup_{\mathrm{q}\in\{|x|=R_0\}} \; \sup_{\mathrm{p}\in\mathfrak{L}_I\setminus I^-(\mathrm{q})} \; t(\mathrm{q}) - t(\mathrm{p}) \leq C_I \qquad \text{if } t(\mathrm{q}) \leq 0 \end{aligned} \tag{2.2.35}$$

where $\mathfrak{L}_I$ is the interior region defined in Definition 2.2.3. Notice that the *Bartnik's uniform interior condition* was imposed to avoid pathological cases, such as the construction by Brill [BF76] of a spacetime not admitting maximal hypersurfaces. We signal other analogous non-existence results for CMC surfaces in [Bar88b], [CIP05], [LO24].

In 1988, Bartnik [Bar88a] also defined and studied the corresponding variational problem to (2.2.21) in nonflat spaces and found the relations between the variational solutions *area maximizing surfaces* and the classical solutions to the Dirichlet problem, extending his previous results in Minkowski. As aforementioned, in 1990 Barnik–Chruściel–O'Murchadha [BCO90], motivated by the results on the *boost problem in general relativity* [CO81], studied maximal surfaces approaching boosted slices under the *Bartnik's interior condition* and proved the existence of maximal slices in strictly stationary, asymptotically flat spacetimes. The assumption of strict stationarity was then relaxed by Chruściel–Wald [CW94], who proved the analogous result for spacetimes admitting an asymptotic timelike Killing Vector Field.

Other asymptotic were considered by [Aku89], who proved the existence of maximal hypersurfaces in asymptotically anti-de Sitter spacetimes. In addition, a relevant amount of research addressed the study of non-zero CMC surfaces, starting with [Stu81] and [Tre82] and extended by [AI99] in asymptotic Schwarzschildean spacetimes; in view of the discussion in Section 2.1.1, those hypersurfaces are particularly suitable for studying the propagation of gravitational radiation, as they become asymptotically null and in fact in some spacetimes they can be indexed by the BMS supertranslation subgroup. Non-zero CMC surfaces remain an active subject of study due to their role in the *spacetimes singularity theorems* and in the *Bartnik's splitting conjecture* for cosmological spaces, see e.g. [Gal19; GL18] for more information.

Finally, specifically regarding the Schwarzschild spacetime, we highlight the very recent works by Tam [Tam23] and Li–Shi–Tam [LST24] on non-zero CMC surface approaching elements of the BMS supertranslation subgroup at null infinity $\mathscr{I}^+$ and the construction of rotationally symmetric CMC surfaces by Lee–Lee [Lee16; LL16], who simplified and extended the foliation results by Beig–O'Murchadha [BO98]. We remark that those hypersurfaces are essential to have a complete picture of maximal surfaces in Schwarzschild and to identify the differences with Minkowski, as for example, they do not have any singularity, while in the flat case they do.

Solving Quasilinear Elliptic Dirichlet problems The solvability of the *maximal surface equation* relies on the Leray-Shauder theory developed [LS34; Ler38], see also [CL72]. The theory makes use of the so-called *Leray-Schauder fixed point theorem* –see [GT01, Chapter 11]–, which allows to use the linear theory for second-order elliptic equations, up to ensuring a suitable a priori $C^{1,\alpha}$ uniform estimate on the set of solutions. For our specific purposes, we will refer to the version in [Bar84, Theorem 4.1].

We first recall one of the fundamental results from the linear theory, which can provide a uniform $C^{2,\alpha}$ estimate on the solution by employing a uniformly elliptic partial differential equation.

Theorem 2.2.6. (Schauder's interior estimate, [GT01, Theorem 6.2]) *Let $u \in C^{2,\alpha}(\Omega)$, for Ω open set in $\mathbb{R}^n$, be a solution to the equation*

$$Lu := a^{ij}(x)\mathrm{D}^2_{ij}u(x) + b^k(x)\mathrm{D}_k u(x) + c(x)u(x) = f(x), \quad \text{in } \Omega,$$

where $a^{ij}, f \in C^{0,\alpha}(\Omega)$, $b^k \in C^{1,\alpha}(\Omega)$, $c \in C^{2,\alpha}(\Omega)$ and L is uniformly elliptic, namely there exists $\lambda > 0$ such that

$$\lambda|\zeta|^2 \leq a^{ij}(x)\zeta_i\zeta_j \leq \lambda^{-1}|\zeta|^2, \quad \forall \zeta \in \mathbb{R}^n,\ \forall x \in \Omega. \tag{2.2.36}$$

Then there exists a constant $C = C(n, \lambda, \alpha)$, such that on $\Omega' \subset\subset \Omega$

$$\|u\|_{C^{2,\alpha}(\Omega')} \leq C \left(\|u\|_{C^0(\Omega)} + \|f\|_{C^{0,\alpha}(\Omega)}\right).$$

The following is an essential tool that allows to get a $C^{1,\alpha}$ uniform estimate just by ensuring a C^1 estimate and is based on the celebrated *De Giorgi – Nash Theorem* [GT01, Theorem 8.22].

Theorem 2.2.7. ([GT01, Theorem 13.1]) *Let $\Omega \subset \mathbb{R}^n$ be open and bounded and let $u \in H^1_{loc}(\Omega)$ be a weak solution of* $\mathrm{div}(A(x)\mathrm{D}u) = 0$. *Assume that the coefficients of the* *matrix $A(x) = (a^{ij}(x))$ satisfy the uniform ellipticity condition* (2.2.36) *and are bounded measurable functions. Then u is locally Hölder continuous in Ω.*

2.2.2 Boosts in Minkowski spacetime

In the following, we will quickly recall well-known notions of Special Relativity, mostly referring to [Cho15, Chapter II]; other extensive discussions can be found e.g. in [Møl52, Chapter 2] or in [Nab12]. We will, in particular, highlight the role of *boosts* by outlining the change of paradigm from Galilean–Newtonian physics to Einstein's theory.

In both Galilean–Newtonian and Einstein theories, the notion of *Inertial Frame* is strictly related to the *Principle of Relativity*, which asserts that physics laws are independent of the inertial observers (or frames) and hence defines those as the frames in which the physics laws hold true. In the Galilean–Newtonian regime, the physic laws that define the inertial frame consist basically in the *fundamental law of Newtonian dynamics* $F = ma$, stating that the applied force F to a particle of mass m is proportional to its acceleration a. In addition to this law of mechanics, Einstein's relativity principle aims to include other physical laws and in particular electromagnetism. This was possible by a deep conceptual jump involving the notions of time and of light as an electromagnetic phenomenon: while for Galileo and Newton *time* was absolute and the simultaneity of two events did not depend on the inertial observer, Einstein discarded this idea and substituted it with the postulate that the *speed of light* in vacuum was absolute, namely independent of the inertial observer. This postulate, already implicit in Maxwell's equations, was actually also experimentally confirmed by Michelson and Morley in 1887. The difference between the two theories is reflected in the mathematical model employed and in the transformations which keep the above postulates holding.

The Galilean group. The *Galileo–Newton spacetime* can be conceived as the direct product of $(\mathbb{R}, \mathrm{d}t^2)$ and the Euclidean space $(\mathbb{R}^3, \delta_{\mathbb{R}^3})$, where $\delta_{\mathbb{R}^3} = \sum_{i=1}^{3} \mathrm{d}x_i^2$. The coordinates transformations which keep Newton's law invariant and time absolute are given by the so-called *Galilean group* transforming $(t, \boldsymbol{x})$ with $\boldsymbol{x} = (x_1, x_2, x_3)$ into $(\widetilde{t}, \widetilde{\boldsymbol{x}})$ with $\widetilde{\boldsymbol{x}} = (\widetilde{x}_1, \widetilde{x}_2, \widetilde{x}_3)$ via

$$\widetilde{t} = t_0 + t, \qquad \widetilde{\boldsymbol{x}} = \boldsymbol{x}_0 + \mathrm{A}\,\boldsymbol{x} + \mathrm{v}t$$

for fixed $t_0 \in \mathbb{R}$, $\boldsymbol{x}_0 \in \mathbb{R}^3$, representing translations. The parameter $\mathrm{v} \in \mathbb{R}^3$ is the constant speed of motion with respect to the original frame of coordinates and $\mathrm{A} \in SO(3)$ is a space-rotation. We remark that transformations of the form $\widetilde{\boldsymbol{x}} = \boldsymbol{x}_0 + \mathrm{A}\,\boldsymbol{x}$ constitute a subgroup of the so-called *Euclidean group*, which are isometries of the Euclidean space preserving the orientation. On the other hand, we notice that the Galilean group does not provide the isometries of the direct product $(\mathbb{R} \times \mathbb{R}^3, \mathrm{d}t^2 + \delta_{\mathbb{R}^3})$, unless $\mathrm{v} = 0$.

The Poincaré group. The revolutionary intuition of Einstein in his *special relativity theory* [Ein05a] was mathematically translated into substituting the above spacetime with the Minkowski spacetime (2.1.1), introduced first just as a technical tool by Poincaré [Poi05] in 1905 and then formalized by Minkowski [Min08] in 1908. Since the Maxwell equations were not invariant under the Galilean group, starting from 1887 many physicists, including Hendrik Lorentz, investigated the transformations which kept them in the same form. Then Poincaré recognized that the transformations found by Lorentz and others formed a group, the so-called *Poincaré group* and constituted isometries of the Minkowski space. Normalizing the universal constant c of speed of light to $c = 1$, they transform $(t, \boldsymbol{x})$ with $\boldsymbol{x} = (x_1, x_2, x_3)$ into $(\widetilde{t}, \widetilde{\boldsymbol{x}})$ with $\widetilde{\boldsymbol{x}} = (\widetilde{x}_1, \widetilde{x}_2, \widetilde{x}_3)$ via the combinations of translations $t \mapsto t_0 + t$, $\boldsymbol{x} \mapsto \boldsymbol{x}_0 + \boldsymbol{x}$ for fixed $t_0 \in \mathbb{R}$, $\boldsymbol{x}_0 \in \mathbb{R}^3$, space-rotations $\boldsymbol{x} \mapsto \mathrm{A}\,\boldsymbol{x}$ given an element of the special orthogonal group $\mathrm{A} \in SO(3)$ and the so-called *boosts*, given by

$$\widetilde{t} = \frac{t - \mathrm{b}x_1}{\sqrt{1-\mathrm{b}^2}}\,, \qquad \widetilde{x}_1 = \frac{x_1 - \mathrm{b}t}{\sqrt{1-\mathrm{b}^2}}\,, \qquad \widetilde{x}_i = x_i \quad \text{for } i = 2, 3 \tag{2.2.37}$$

for a fixed $\mathrm{b} \in \mathbb{R}$, $|\mathrm{b}| < 1$ standing for the constant relative speed of motion. Due to the rotational symmetry, we will always consider $\mathrm{b} \in [0, 1)$, without loss of generality. Clearly, this transformation gives an isometry, as

$$\begin{aligned} -\mathrm{d}\widetilde{t}^2 + \sum_{i=1}^{3} \mathrm{d}\widetilde{x}_i^2 &= -\frac{1}{1-\mathrm{b}^2}\left(\mathrm{d}t^2 + \mathrm{b}^2\mathrm{d}x_1^2\right) + \sum_{i=2}^{3} \mathrm{d}x_i^2 + \frac{1}{1-\mathrm{b}^2}\left(\mathrm{d}x_1^2 + \mathrm{b}^2\mathrm{d}t^2\right) \\ &= -\mathrm{d}t^2 + \sum_{i=1}^{3} \mathrm{d}x_i^2\,. \end{aligned}$$

We remark that the transformations of the Poincaré group not involving translations, namely the compositions of boosts and spatial rotations, give the so-called *Lorentz group*. As aforementioned, we notice that the time slices of $\widetilde{t}$ are given for fixed $\widetilde{t}_o \in \mathbb{R}$ by

$$\{\widetilde{t} = \widetilde{t}_o\} = \left\{t = \sqrt{1-\mathrm{b}^2}\,\widetilde{t}_o + \mathrm{b}x_1\right\}$$

namely by affine planes given, up to a constant, by the graph of a linear function

$$u_{\mathrm{b},\widetilde{t}_o} : \mathbb{R}^3 \to \mathbb{R}, \qquad u_{\mathrm{b},\widetilde{t}_o}(\boldsymbol{x}) = \sqrt{1-\mathrm{b}^2}\,\widetilde{t}_o + \mathrm{b}x_1 \tag{2.2.38}$$

of slope b with $\mathrm{b} \in [0, 1)$. Moreover, it is clear that since $\partial_i u_{\mathrm{b},\widetilde{t}_o} = \mathrm{b}\,\delta_{i1}$ and $\partial^2_{ij} u_{\mathrm{b},\widetilde{t}_o} = 0$ for $i, j = 1, 2, 3$, we have

$$\mathrm{H}(u_{\mathrm{b},\widetilde{t}_o}) = \mathrm{div}_{\mathbb{R}^3}\left(\frac{\mathrm{D}u_{\mathrm{b},\widetilde{t}_o}}{\sqrt{1-|\mathrm{D}u_{\mathrm{b},\widetilde{t}_o}|^2}}\right) = \left(\delta_{ij} + \frac{\partial_i u_{\mathrm{b},\widetilde{t}_o}\ \partial_j u_{\mathrm{b},\widetilde{t}_o}}{1-|\mathrm{D}u_{\mathrm{b},\widetilde{t}_o}|^2}\right)\frac{\partial^2_{ij} u_{\mathrm{b},\widetilde{t}_o}}{\sqrt{1-|\mathrm{D}u_{\mathrm{b},\widetilde{t}_o}|^2}} = 0$$

directly by the computation (2.2.24).

2.3 The Schwarzschild spacetime

The *Schwarzschild spacetimes* are a 1-parameter family of first exact non-trivial (non-flat) solution to the Einstein Equations (2.1.3) in *vacuum* $\mathfrak{T} = 0$ with zero cosmological constant $\Lambda = 0$, reading

$$\mathfrak{Ric} = 0\,. \tag{2.3.1}$$

They were found in 1916 by Karl Schwarzschild [Sch16] right after the introduction of the field equations by Einstein, by imposing rotational symmetry and hence reducing (2.3.1) to an ordinary differential equation.

The Schwarzschild solution depicts a valid approximation of the geometry of the spacetime of our solar system. In addition, the features of the Schwarzschild spacetime have been crucial to feeding the research in general relativity in many directions. In particular, the first coordinates employed showed an inextendibility of the metric components across the so-called *Schwarzschild radius*. Starting with the work of Finkelstein [Fin58], this phenomenon motivated the study of *black holes*, first from a mathematical and theoretical point of view and then as physical and observable objects –in 2019, the Event Horizon Telescope (EHT) provided the first image of a black hole.

In this work, we will consider three different coordinates describing the (positive mass) Schwarzschild spacetime. We will employ their diverse features according to our needs: the first two are asymptotically flat, while the third cover the maximally extended spacetime.

2.3.1 Schwarzschildean and Isotropic coordinates

Schwarzschildean coordinates. The *exterior Schwarzschild* solution, as found by Schwarzschild himself, is given in the so-called *Schwarzschildean coordinates* as the Lorentzian manifold

$$\begin{aligned} (\mathfrak{L}^S_{ext})_m &= \mathbb{R} \times (2m, \infty) \times \mathbb{S}^2 \ni (t, r, \vartheta)\,, \\ \mathfrak{g}^S_m &= -h_m(r)\mathrm{d}t^2 + \frac{1}{h_m(r)}\mathrm{d}r^2 + r^2 g_{\mathbb{S}^2}\,, \qquad h_m(r) = 1 - \frac{2m}{r} > 0 \end{aligned} \tag{2.3.2}$$

where $m > 0$ is the *mass parameter* and $g_{\mathbb{S}^2}$ is the standard round sphere. Taking $m = 0$ it simply reduces to the Minkowski space (2.1.1). This spacetime is *rotationally symmetric* and *(standard)-static*, since it is a warped product with a time-independent warped function. In particular, the *lapse function* as defined in (2.2.3) is given by ${}^t\alpha_m = \sqrt{h_m(r)}$. Notice that $\big((\mathfrak{L}^S_{ext})_m, \mathfrak{g}^S_m\big)$ is asymptotically flat in the sense of Definition 2.2.3, as a consequence of

$$h_m(r) = 1 + O_\infty(r^{-1}) \qquad \text{as } r \to +\infty\,.$$

Now, when considering

$$(\mathfrak{L}^S_{int})_m = \mathbb{R} \times (0, 2m) \times \mathbb{S}^2 \tag{2.3.3}$$

still endowed with the metric $\mathfrak{g}^S_m$ in (2.3.2), we find a smooth Lorentzian manifold. Nevertheless, since $h_m < 0$, the causal nature of ∂_t and ∂_r is swapped and the spacetime is no longer rotationally symmetric nor static. We will call $((\mathfrak{L}^S_{int})_m, \mathfrak{g}^S_m)$ *interior Schwarzschild* solution of mass $m > 0$ in *Schwarzschildean coordinates.*

For the sake of computations, we recall some of the *Christoffel symbols* in these coordinates, for $I = 1, 2$

$$\overline{\Gamma}^t_{tr} = \frac{(\mathfrak{g}^S_m)^{tt}}{2} \frac{\partial (\mathfrak{g}^S_m)_{tt}}{\partial r} = \frac{h'_m}{2h_m} = \frac{m}{r^2}\left(1 - \frac{2m}{r}\right)^{-1}, \tag{2.3.4}$$

$$\overline{\Gamma}^I_{Ir} = \frac{(\mathfrak{g}^S_m)^{II}}{2} \frac{\partial (\mathfrak{g}^S_m)_{II}}{\partial r} = \frac{1}{r}, \tag{2.3.5}$$

$$\overline{\Gamma}^r_{It} = \overline{\Gamma}^t_{It} = \overline{\Gamma}^J_{It} = \overline{\Gamma}^t_{IJ} = 0 \tag{2.3.6}$$

$$\overline{\Gamma}^r_{IJ} = -\frac{(g^S_m)^{rr}}{2} \frac{\partial (g^S_m)_{IJ}}{\partial r} = -r h_m (g_{\mathbb{S}^2})_{IJ} \tag{2.3.7}$$

$$\overline{\Gamma}^K_{IJ} = \frac{(g_{\mathbb{S}^2})^{KL}}{2}\left(\frac{\partial (g_{\mathbb{S}^2})_{IL}}{\partial \theta_J} + \frac{\partial (g_{\mathbb{S}^2})_{JL}}{\partial \theta_I} - \frac{\partial (g_{\mathbb{S}^2})_{IJ}}{\partial \theta_L}\right) \tag{2.3.8}$$

As aforementioned, it is easy to see that in both cases, some metric coefficients degenerate at the *Schwarzschild radius* $\{r = 2m\}$ since $(\mathfrak{g}^S_m)_{tt} = -h_m(r) \to 0^\mp$ and $(\mathfrak{g}^S_m)_{rr} = h_m(r)^{-1} \to \pm\infty$ as $r \to 2m^\pm$. It will be clear –see Section 2.3.2– that this *singularity* is related to the choice of coordinates, in the sense that both the spacetimes $\big((\mathfrak{L}^S_{ext})_m, \mathfrak{g}^S_m\big)$ and $\big((\mathfrak{L}^S_{int})_m, \mathfrak{g}^S_m\big)$ can be isometrically embedded in a larger spacetime, containing the so-called *(event) horizon* $\{r = 2m\}$. On the other hand, in $(\mathfrak{L}^S_{int})_m$, the metric also degenerates when approaching $r \to 0^+$. This singularity is instead coordinate independent and the spacetime can not be extended beyond it; in particular, one can find incomplete geodesics defined on $\mathbb{R}$ not extending as $r \to 0^+$ and, concurrently, geometric quantities such as the *Kretschman scalar* blowing up when approaching this *spacetime singularity.*

Isotropic coordinates. Starting from the exterior Schwarzschild in Schwarzschildian coordinates, we consider the change of coordinates

$$r = |\,\boldsymbol{x}\,|\left(1 + \frac{m}{2|\,\boldsymbol{x}\,|}\right)^2 \tag{2.3.9}$$

where $|\,\boldsymbol{x}\,| = \sqrt{\sum_{i=1}^3 (x^i)^2}$ is the new radial coordinate. Noticing that there are two roots of (2.3.9), we consider $|\,\boldsymbol{x}\,| > \frac{m}{2}$. We obtain the static spacetime

$$\begin{aligned}(\mathfrak{L}^I_{ext})_m &= \mathbb{R}\times(\mathbb{R}^3\setminus\overline{B}_{\frac{m}{2}}) \ \ni (t,\boldsymbol{x})\\ \mathfrak{g}^I_m &= -N_m(|\,\boldsymbol{x}\,|)^2\mathrm{d}t^2+\varphi_m(|\,\boldsymbol{x}\,|)^4\delta_{\mathbb{R}^3}\,,\\ N_m(|\,\boldsymbol{x}\,|) &= \frac{1-\frac{m}{2|\,\boldsymbol{x}\,|}}{1+\frac{m}{2|\,\boldsymbol{x}\,|}}\,,\quad \varphi_m(|\,\boldsymbol{x}\,|)=1+\frac{m}{2|\,\boldsymbol{x}\,|}\end{aligned} \tag{2.3.10}$$

where every spatial slice is conformal to the Euclidean metric $\delta_{\mathbb{R}^3}$. As in Schwarzschildean coordinates, we consider the mass parameters $m>0$; taking $m=0$ reduces it to the Minkowski space (2.1.1). In these coordinates, the Schwarzschild spacetime is asymptotically flat as well, since

$$N_m(|\,\boldsymbol{x}\,|)=1+O_\infty(|\,\boldsymbol{x}\,|^{-1})\,,\quad \varphi_m(|\,\boldsymbol{x}\,|)=1+O_\infty(|\,\boldsymbol{x}\,|^{-1}) \qquad \text{as } |\,\boldsymbol{x}\,|\to+\infty\,.$$

The horizon now corresponds to $\{r=2m\}=\big\{|\,\boldsymbol{x}\,|=\frac{m}{2}\big\}$. Since $N_m(|\,\boldsymbol{x}\,|)\to 0$ as $|\,\boldsymbol{x}\,|\to\frac{m}{2}$, the metric degenerates and ceases to be Lorentzian across it. Nevertheless, observing that $\varphi_m(|\,\boldsymbol{x}\,|)^4\to 2^4$ as $|\,\boldsymbol{x}\,|\to\frac{m}{2}$, the metric of each time slice can be smoothly extended to $\mathbb{R}^3\setminus\{0\}$ with the same metric. Moreover, after this extension, each of such time slice consists of two isometric copies $\big\{0<|\,\boldsymbol{x}\,|<\frac{m}{2}\big\}$ and $\big\{|\,\boldsymbol{x}\,|>\frac{m}{2}\big\}$ of the corresponding time slice in the exterior Schwarzschild $\big((\mathfrak{L}^S_{ext})_m,\mathfrak{g}^S_m\big)$ glued at $\big\{|\,\boldsymbol{x}\,|=\frac{m}{2}\big\}$. The isometry is given by the map $|\,\boldsymbol{x}\,|\mapsto\frac{m^2}{4|\,\boldsymbol{x}\,|}$ and shows that the region $\big\{0<|\,\boldsymbol{x}\,|<\frac{m}{2}\big\}$ is also asymptotically flat.

Spacelike hypersurfaces and their mean curvature on exterior Schwarzschild. In the following, we work in Schwarzschildean coordinates, in particular exploiting its staticity, but analogous results hold for the exterior Schwarzschild in isotropic coordinates $\big((\mathfrak{L}^I_{ext})_m,\mathfrak{g}^I_m\big)$.

Consider a smooth spacelike hypersurface $M\hookrightarrow\big((\mathfrak{L}^S_{ext})_m,\mathfrak{g}^S_m\big)$ given as a graph of a smooth function $u:\Omega\subset\mathbb{R}^3\setminus\overline{B}_{2m}\to\mathbb{R}$

$$M=\{t=u(r,\theta_I)\}\,.$$

The spacelike condition on $\{r>2m\}$ then reads

$$(\partial_r u)^2+\frac{g^{II}_{\mathbb{S}^2}}{r^2\,h_m}(\partial_I u)^2<\frac{1}{h_m^2}\,. \tag{2.3.11}$$

To compute its mean curvature by using (2.2.19), we first observe that the metric (2.3.2) has vanishing shift $\beta\equiv 0$. Moreover, by staticity, it is easy to see that every time slice has vanishing second fundamental form and hence $\mathrm{H}^o=0$ and that the coefficients of the metric do not depend on t, so in particular $\mathscr{T}(|\mathrm{U}|)=0$. Hence, recalling that the unit normal to the time slices is $\mathscr{T} = \frac{1}{\sqrt{h_m(r)}}\partial_t$, we compute $\boldsymbol{v}_t(u) = \frac{1}{\sqrt{1-h_m|^t\mathrm{D}u|^2_{^tg}}}$ and the mean curvature

$$\begin{aligned}
\mathrm{H}_{\mathfrak{g}^S_m}(u) &= \mathrm{div}^o\left(\frac{\sqrt{h_m}\,{}^t\mathrm{D}u}{\sqrt{1-h_m|^t\mathrm{D}u|^2_{^tg}}}\right) + \frac{\boldsymbol{v}_t(u)}{\sqrt{h_m}}\left\langle {}^t\mathrm{D}u,\, \overline{\nabla}_{\partial_t}\partial_t\right\rangle \\
&= \mathrm{div}^o\left(\frac{\sqrt{h_m}\,{}^t\mathrm{D}u}{\sqrt{1-h_m|^t\mathrm{D}u|^2_{^tg}}}\right) + \boldsymbol{v}_t(u)\left\langle {}^t\mathrm{D}u,\, {}^t\mathrm{D}\sqrt{h_m}\right\rangle \qquad (2.3.12)\\
&= \sqrt{h_m}\boldsymbol{v}_t(u)\Big({}^tg^{ij} + h_m\boldsymbol{v}_t^2(u)\;{}^t\mathrm{D}^iu\,{}^t\mathrm{D}^ju\Big)\,{}^t\mathrm{D}^2_{ij}u \\
&\quad + \boldsymbol{v}_t(u)(1+\boldsymbol{v}_t^2(u))\left\langle {}^t\mathrm{D}u,\, {}^t\mathrm{D}\sqrt{h_m}\right\rangle \qquad (2.3.13)
\end{aligned}$$

where we used that $\overline{\nabla}_{\partial_t}\partial_t = \overline{\Gamma}^r_{tt}\partial_r = \frac{(\mathfrak{g}^S_m)^{rr}}{2}\frac{\partial(\mathfrak{g}^S_m)_{tt}}{\partial r}\,\partial_r = \frac{h_m}{2}h'_m(r)\,\partial_r = \sqrt{h_m}\;{}^t\mathrm{D}\sqrt{h_m}$ and we expanded the divergence as follows

$$\begin{aligned}
\mathrm{div}^o\left(\sqrt{h_m}\boldsymbol{v}_t(u)\,{}^t\mathrm{D}u\right) &= \sqrt{h_m}\boldsymbol{v}_t(u)\Delta_{^tg}u + \boldsymbol{v}_t(u)\left\langle {}^t\mathrm{D}u,\, {}^t\mathrm{D}\sqrt{h_m}\right\rangle \\
&\quad + \sqrt{h_m}\frac{\boldsymbol{v}_t^3(u)}{2}\left\langle {}^t\mathrm{D}u,\, {}^t\mathrm{D}\left(h_m|^t\mathrm{D}u|^2_{^tg}\right)\right\rangle \\
&= \sqrt{h_m}\boldsymbol{v}_t(u)\Delta_{^tg}u + \boldsymbol{v}_t(u)\left(1+\boldsymbol{v}_t^2(u)h_m|^t\mathrm{D}u|^2_{^tg}\right)\left\langle {}^t\mathrm{D}u,\, {}^t\mathrm{D}\sqrt{h_m}\right\rangle \\
&\quad + h_m^{\frac{3}{2}}\boldsymbol{v}_t^3(u)\;{}^t\mathrm{D}^2u({}^t\mathrm{D}u, {}^t\mathrm{D}u) \\
&= \sqrt{h_m}\boldsymbol{v}_t(u)\left(\Delta_{^tg}u + h_m\boldsymbol{v}_t^2(u)\;{}^t\mathrm{D}^2u({}^t\mathrm{D}u, {}^t\mathrm{D}u)\right) \\
&\quad + \boldsymbol{v}_t^3(u)\left\langle {}^t\mathrm{D}u,\, {}^t\mathrm{D}\sqrt{h_m}\right\rangle .
\end{aligned}$$

There we used $\boldsymbol{v}_t^2(u)h_m|^t\mathrm{D}u|^2_{^tg} = \boldsymbol{v}_t^2(u) - 1$ and that

$$ {}^t\mathrm{D}^2u({}^t\mathrm{D}u, {}^t\mathrm{D}u) = \langle {}^t\mathrm{D}_{^t\mathrm{D}u}{}^t\mathrm{D}u, {}^t\mathrm{D}u\rangle = \tfrac{1}{2}\langle {}^t\mathrm{D}|{}^t\mathrm{D}u|^2_{^tg}, {}^t\mathrm{D}u\rangle\,. $$

As aforementioned, the computations above actually hold on any static spacetime. In particular, in isotropic coordinates, the spacelike condition on a graph of a smooth function $u : \Omega \subset \mathbb{R}^3 \setminus \overline{B}_{\frac{m}{2}} \to \mathbb{R}$ reads

$$\sum_{i=1}^{3}(\partial_i u(\boldsymbol{x}))^2 < \frac{\varphi_m^4(|\,\boldsymbol{x}\,|)}{N_m^2(|\,\boldsymbol{x}\,|)} \qquad (2.3.14)$$

and the mean curvature is given by

$$\mathrm{H}_{\mathfrak{g}^I_m} = N_m\boldsymbol{v}_t(u)\Big({}^tg^{ij} + N_m^2\boldsymbol{v}_t^2(u)\;{}^t\mathrm{D}^iu\,{}^t\mathrm{D}^ju\Big)\,{}^t\mathrm{D}^2_{ij}u + \boldsymbol{v}_t(u)(1+\boldsymbol{v}_t^2(u))\left\langle {}^t\mathrm{D}u,\, {}^t\mathrm{D}N_m\right\rangle \qquad (2.3.15)$$

where $\boldsymbol{v}_t(u) = \frac{1}{\sqrt{1-N_m^2|^t\mathrm{D}u|^2_{^tg}}}$ and ${}^tg = \varphi_m^4\delta_{\mathbb{R}^3}$.

2.3.2 Kruskal–Szekeres coordinates

In 1960 Kruskal and Szekeres [Kru60; Sze60] showed that $\big((\mathfrak{L}^S_{ext})_m, \mathfrak{g}^S_m\big)$ and $\big((\mathfrak{L}^S_{int})_m, \mathfrak{g}^S_m\big)$ can be isometrically embedded in *the maximally extended Schwarzschild spacetime*, also called *Kruskal extension* and described by the *Kruskal–Szekeres coordinates*.

As described in [Wal84, Section 6.4], the main idea employed to construct the extension subsists on considering radial null geodesics in $\big((\mathfrak{L}^S_{ext})_m, \mathfrak{g}^S_m\big)$ and $\big((\mathfrak{L}^S_{int})_m, \mathfrak{g}^S_m\big)$, which satisfy $t = \pm\left(r + 2m\log\left|\frac{r}{2m} - 1\right|\right)$ and parameterizing the two spacetimes by those geodesics, defining the so-called *null coordinates* and then the actual *Kruskal–Szekeres coordinates*, which then extend across $\{r = 2m\}$. Instead of showing the steps of the construction, we rather consider the resulting spacetime and describe how some open sets are (isometrically) related to the spacetimes above.

The *maximally extended Schwarzschild spacetime* is the Loretzian manifold

$$\begin{aligned} \mathfrak{L}^K_m &= \left\{(\mathsf{T},\mathsf{X}) \in \mathbb{R}^2 \,:\, \mathsf{T}^2 - \mathsf{X}^2 < 1\right\} \times \mathbb{S}^2 \\ \mathfrak{g}^K_m &= \xi_m(r)\left(-\mathrm{d}\mathsf{T}^2 + \mathrm{d}\mathsf{X}^2\right) + r^2 g_{\mathbb{S}^2}\,, \qquad \xi_m(r) = \frac{32m^3}{r}e^{-\frac{r}{2m}} > 0 \end{aligned} \tag{2.3.16}$$

where the coordinate $r \in (0, +\infty)$ is implicitly given by

$$\mathsf{X}^2 - \mathsf{T}^2 = \rho_m(r) := \left(\frac{r}{2m} - 1\right) e^{\frac{r}{2m}} \in (-1, +\infty)\,. \tag{2.3.17}$$

Notice that it is well defined as $\rho_m(r)$ is a strictly increasing function in r and hence it admits an inverse $r(\mathsf{T},\mathsf{X})$ with values in $(0, +\infty)$. Let us define the following open sets

$$\begin{aligned} &\textit{region I} && \left\{(\mathsf{T},\mathsf{X}) \in \mathfrak{L}^K_m \,:\, \mathsf{T}^2 < \mathsf{X}^2,\, \mathsf{X} > 0\right\}, \\ \textit{black hole region / }&\textit{region II} && \left\{(\mathsf{T},\mathsf{X}) \in \mathfrak{L}^K_m \,:\, \mathsf{T}^2 > \mathsf{X}^2,\, \mathsf{T} > 0\right\}, \\ &\textit{region I}' && \left\{(\mathsf{T},\mathsf{X}) \in \mathfrak{L}^K_m \,:\, \mathsf{T}^2 < \mathsf{X}^2,\, \mathsf{X} < 0\right\}, \\ \textit{white hole region / }&\textit{region II}' && \left\{(\mathsf{T},\mathsf{X}) \in \mathfrak{L}^K_m \,:\, \mathsf{T}^2 > \mathsf{X}^2,\, \mathsf{T} < 0\right\}. \end{aligned}$$

It is easy to see that the *region* I is isometric to the *region* I′ and the *region* II is isometric to the *region* II′ via the map $(\mathsf{T},\mathsf{X}) \mapsto (-\mathsf{T},-\mathsf{X})$. Now, let us consider the additional change of coordinates on $\mathfrak{L}^K \setminus \{\mathsf{T}^2 = \mathsf{X}^2\}$ given by

$$t = 2m\log\left|\frac{\mathsf{X}+\mathsf{T}}{\mathsf{X}-\mathsf{T}}\right| = \begin{cases} 4m\,\mathrm{artanh}\,\frac{\mathsf{T}}{\mathsf{X}} & \text{in } \textit{regions } \mathrm{I}, \mathrm{I}' \\ 4m\,\mathrm{artanh}\,\frac{\mathsf{X}}{\mathsf{T}} & \text{in } \textit{regions } \mathrm{II}, \mathrm{II}' \end{cases} \tag{2.3.18}$$

One can check that *region* I and *region* I′ are both isometric to the exterior Schwarzschild spacetime $\big((\mathfrak{L}^S_{ext})_m, \mathfrak{g}^S_m\big)$, while *region* II and *region* II′ are both isometric to the interior Schwarzschild spacetime $\big((\mathfrak{L}^S_{int})_m, \mathfrak{g}^S_m\big)$, for r and t being the radial Schwarzschildean coordinate and the static time, respectively. In particular, the spacetime singularity corresponds to the disjoint union of the hyperboloids

$$\{r=0\}=\{\mathsf{T}^2-\mathsf{X}^2=1\}=\left\{\mathsf{T}=\sqrt{1+\mathsf{X}^2}\right\}\sqcup\left\{\mathsf{T}=-\sqrt{1+\mathsf{X}^2}\right\} \tag{2.3.19}$$

in the *regions* II and II′. On the other hand, the points at the Schwarzschildean radius correspond to

$$\{r=2m\}=\{\mathsf{T}^2=\mathsf{X}^2\}\,,$$

which is called *bifurcate horizon*, as it is given by the union of the null hypersurfaces $\{\mathsf{T}=\mathsf{X}\neq 0\}$ and $\{\mathsf{T}=-\mathsf{X}\neq 0\}$ and the so-called *bifurcation sphere* $\{\mathsf{T}=\mathsf{X}=0\}$. We notice that, due to (2.3.17), the hypersurfaces given by constant radii in $\big((\mathfrak{L}^S_{ext})_m,\mathfrak{g}^S_m\big)$ and $\big((\mathfrak{L}^S_{int})_m,\mathfrak{g}^S_m\big)$ correspond to the hyperbolas

$$\{r=r_0\}=\left\{\mathsf{X}^2-\mathsf{T}^2=\rho_0:=\rho_m(r_0)\right\} \tag{2.3.20}$$

for ρ_m as in (2.3.17). On the other hand, the hypersurfaces given by constant time in $\big((\mathfrak{L}^S_{ext})_m,\mathfrak{g}^S_m\big)$ and $\big((\mathfrak{L}^S_{int})_m,\mathfrak{g}^S_m\big)$ correspond to the half-lines

$$\{t=t_0\}=\begin{cases}\left\{\mathsf{T}=\tanh\left(\frac{t_0}{4m}\right)\mathsf{X}\right\} & \text{in } \textit{regions } \mathrm{I},\mathrm{I}'\,,\\ \left\{\mathsf{X}=\tanh\left(\frac{t_0}{4m}\right)\mathsf{T}\right\} & \text{in } \textit{regions } \mathrm{II},\mathrm{II}'\,.\end{cases} \tag{2.3.21}$$

Despite providing a description of the maximal extension of $\big((\mathfrak{L}^S_{ext})_m,\mathfrak{g}^S_m\big)$, the Kruskal coordinates do not capture its asymptotically flat nature. In fact, for any fixed time $\mathsf{T}=\mathsf{T}_0$, the identity (2.3.17) can provide a reparametrized radial coordinate X with the property $\mathsf{X}^2\to+\infty$ if and only if $r\to+\infty$. Now, considering the metric coefficients in (2.3.16), we have

$$\xi_m(r)\to 0\,,\qquad \text{as } r\to+\infty$$

and hence, in these coordinates, the Schwarzschild spacetime does not manifest the property of being asymptotically flat. Consequently, throughout the work, we will use both the Kruskal and the Schwarzschildean (or the isotropic) coordinates, according to the portion of spacetime and the properties we need to exploit.

For the sake of the computations, we recall the inverse transformations, given (2.3.17) and (2.3.18), between the Kruskal coordinates and the Schwarzschildean coordinates:

$$\begin{aligned}&\mathsf{T}=\sqrt{\frac{r}{2m}-1}\,e^{\frac{r}{4m}}\sinh\left(\frac{t}{4m}\right), && \mathsf{X}=\sqrt{\frac{r}{2m}-1}\,e^{\frac{r}{4m}}\cosh\left(\frac{t}{4m}\right) && \text{in } \textit{region } \mathrm{I}\\ &\mathsf{T}=\sqrt{\frac{r}{2m}-1}\,e^{\frac{r}{4m}}\cosh\left(\frac{t}{4m}\right), && \mathsf{X}=\sqrt{\frac{r}{2m}-1}\,e^{\frac{r}{4m}}\sinh\left(\frac{t}{4m}\right) && \text{in } \textit{region } \mathrm{II}\end{aligned} \tag{2.3.22}$$

and the usual map $(\mathsf{T},\mathsf{X})\mapsto(-\mathsf{T},-\mathsf{X})$ provides the change of coordinates in the *regions* I′ and II′. Moreover, using the identity (2.3.17) we can express the differential of the Schwarzschildean radial function in terms of the Kruskal coordinates, computing $\mathsf{T}d\mathsf{T}-$

$\mathsf{X}\mathrm{d}\mathsf{X} = -\frac{r}{8m^2}\, e^{\frac{r}{2m}}\, \mathrm{d}r$, which gives

$$\mathrm{d}r = \frac{8m^2}{r} e^{-\frac{r}{2m}} (\mathsf{X}\mathrm{d}\mathsf{X} - \mathsf{T}\mathrm{d}\mathsf{T}) = \frac{\xi_m}{4m} (\mathsf{X}\mathrm{d}\mathsf{X} - \mathsf{T}\mathrm{d}\mathsf{T})\,. \tag{2.3.23}$$

Now, we compute

$$\frac{d\xi_m}{dr} = -\frac{\xi_m}{2m}\left(1 + \frac{2m}{r}\right) \tag{2.3.24}$$

and comparing (2.3.23) with $\mathrm{d}r = \frac{\partial r}{\partial \mathsf{T}}\mathrm{d}\mathsf{T} + \frac{\partial r}{\partial \mathsf{X}}\mathrm{d}\mathsf{X}$, we find

$$\frac{\partial r}{\partial \mathsf{T}} = -\frac{8m^2}{r} e^{-\frac{r}{2m}}\, \mathsf{T} = -\frac{\xi_m}{4m}\mathsf{T}\,, \qquad \frac{\partial r}{\partial \mathsf{X}} = \frac{8m^2}{r} e^{-\frac{r}{2m}}\, \mathsf{X} = \frac{\xi_m}{4m}\mathsf{X}\,. \tag{2.3.25}$$

Using the above, we can compute the *Christoffel symbols* of the Kruskal metric (2.3.16) as follows

$$\begin{aligned}
\overline{\Gamma}^{\mathsf{T}}_{\mathsf{TT}} = \overline{\Gamma}^{\mathsf{X}}_{\mathsf{TX}} = \overline{\Gamma}^{\mathsf{T}}_{\mathsf{XX}} &= \frac{\xi_m^{-1}}{2}\frac{\partial \xi}{\partial r}\frac{\partial r}{\partial \mathsf{T}} = \frac{\mathsf{T}}{16m^2}\left(1 + \frac{2m}{r}\right)\xi_m \\
\overline{\Gamma}^{\mathsf{X}}_{\mathsf{TT}} = \overline{\Gamma}^{\mathsf{T}}_{\mathsf{TX}} = \overline{\Gamma}^{\mathsf{X}}_{\mathsf{XX}} &= \frac{\xi_m^{-1}}{2}\frac{\partial \xi}{\partial r}\frac{\partial r}{\partial \mathsf{X}} = -\frac{\mathsf{X}}{16m^2}\left(1 + \frac{2m}{r}\right)\xi_m \\
\overline{\Gamma}^{\mathsf{T}}_{\mathsf{T}I} = \overline{\Gamma}^{\mathsf{X}}_{\mathsf{T}I} = \overline{\Gamma}^{\mathsf{T}}_{\mathsf{X}I} = \overline{\Gamma}^{\mathsf{X}}_{\mathsf{X}I} &= 0 \\
\overline{\Gamma}^{I}_{\mathsf{XX}} = \overline{\Gamma}^{I}_{\mathsf{TT}} = \overline{\Gamma}^{I}_{\mathsf{XT}} &= 0 \\
\overline{\Gamma}^{\mathsf{T}}_{IJ} &= \frac{\xi_m^{-1}}{2} 2r\, \frac{\partial r}{\partial \mathsf{T}} (g_{\mathbb{S}^{n-1}})_{IJ} = -\frac{\mathsf{T}}{4m\, r}\bar{\mathcal{g}}_{IJ}\, \delta_{IJ} \\
\overline{\Gamma}^{\mathsf{X}}_{IJ} &= -\frac{\xi_m^{-1}}{2} 2r\, \frac{\partial r}{\partial \mathsf{X}} (g_{\mathbb{S}^{n-1}})_{IJ} = -\frac{\mathsf{X}}{4m\, r}\bar{\mathcal{g}}_{IJ}\, \delta_{IJ} \\
\overline{\Gamma}^{J}_{I\mathsf{T}} &= -\,\delta^J_I\, \frac{8m^2}{r^2}\, e^{-\frac{r}{2m}}\, \mathsf{T} = -\delta^J_I \frac{\xi}{4m\, r}\mathsf{T} \\
\overline{\Gamma}^{J}_{I\mathsf{X}} &= \delta^J_I\, \frac{8m^2}{r^2}\, e^{-\frac{r}{2m}}\, \mathsf{X} = \delta^J_I \frac{\xi}{4m\, r}\mathsf{X}\,.
\end{aligned} \tag{2.3.26}$$

We conclude by underlining that the vector field $-\,\mathrm{grad}_{\mathfrak{g}^K_m}\mathsf{T} = \xi_m^{-1}\partial_{\mathsf{T}}$ gives the lapse

$$\alpha_{\mathsf{T}}^2 = \xi_m \tag{2.3.27}$$

as defined in (2.2.3). Consequently, the vector field

$$\mathscr{T}^K_m = \xi_m^{-\frac{1}{2}}\partial_{\mathsf{T}} \tag{2.3.28}$$

is the future-pointing unit normal to the time slices as in definition (2.2.4). Observe that, in view of the change of coordinates (2.3.22) and of the positivity of $\xi_m > 0$, the vector field $\mathscr{T}^K_m$ points in the direction of ∂_t in the *region* I, but in the direction of $-\partial_t$ in the *region* I′. Similarly, it points in the direction of $-\partial_r$ in the *region* II and in the direction of ∂_r in the *region* II′.

To conclude, we highlight that the maximally extended Schwarzschild spacetime is *globally hyperbolic*, with the hypersurface $\{\mathsf{T}=0\}$ serving as a *Cauchy surface*.

2.3.3 Barriers at the crushing singularities

The definition of *crushing singularity* was first given by Eardley–Smarr [ES79] as a singularity admitting a neighbourhood foliated by hypersurfaces whose mean curvature tends to infinity as the singularity itself is approached. They had been specifically inspired by the known behaviour of the spacelike hypersurfaces (2.3.20) in the black and white holes regions of Schwarzschild.

For completeness, we explicitly carry out the computations in Schwarzschildean coordinates (2.3.3) with metric defined in (2.3.2).

Lemma 2.3.1. *In the* maximally extended Schwarzschild $\left(\mathfrak{L}^K_m, \mathfrak{g}^K_m\right)$*, the mean curvature of the spacelike hypersurfaces* $\mathrm{B}_{r_0} \coloneqq \{r=r_0\}$*,* $0<r_0<2m$ *is constant, given by*

$$\mathrm{H}_{\mathfrak{g}^S_m}(\mathrm{B}_{r_0}) = \begin{cases} \frac{2r_0-3m}{\sqrt{r_0^3(2m-r_0)}} & \textit{in } \text{region II} \\ -\frac{2r_0-3m}{\sqrt{r_0^3(2m-r_0)}} & \textit{in } \text{region II}' \end{cases} \tag{2.3.29}$$

In region II *for any* $0<r_0<\frac{3}{2}m$*, their mean curvature is* $\mathrm{H}_{\mathfrak{g}^S_m}(\mathrm{B}_{r_0})<0$ *and they provide a crushing singularity as* $\mathrm{H}_{\mathfrak{g}^S_m}(\mathrm{B}_{r_0}) \searrow -\infty$ *as* $r_0 \searrow 0^+$*, namely as they approach the future spacetime singularity* $\left\{\mathsf{T}=\sqrt{1+\mathsf{X}^2}\right\}$*.*

In region II$'$ *for any* $0<r_0<\frac{3}{2}m$*, their mean curvature is* $\mathrm{H}_{\mathfrak{g}^S_m}(\mathrm{B}_{r_0})>0$ *and they provide also a crushing singularity as* $\mathrm{H}_{\mathfrak{g}^S_m}(\mathrm{B}_{r_0}) \nearrow +\infty$ *as* $r_0 \searrow 0^+$*, namely as they approach the past spacetime singularity* $\left\{\mathsf{T}=-\sqrt{1+\mathsf{X}^2}\right\}$*.*

Proof. The future-pointing unit normal to the hypersurfaces $\mathrm{B}_{r_0} \coloneqq \{r=r_0\}$ with $0<r_0<2m$ is given by

$$\mathrm{N}_{r_0} = \begin{cases} -\sqrt{-h_m}\,\partial_r & \text{in } \textit{region } \text{II} \\ \sqrt{-h_m}\,\partial_r & \text{in } \textit{region } \text{II}', \end{cases} \tag{2.3.30}$$

where we are using that $h_m<0$. In both cases, tangent vectors are given by $\partial_t, \partial_{\theta^I}$ with $I=1,2$ and in particular, the induced metric is diagonal with coefficients $(\mathfrak{g}^S_m)_{tt}$ and $r_0^2(g_{\mathbb{S}^2})_{II}$. We then just need to compute the mean curvature of B_{r_0} in *region* II and the result in *region* II$'$ will differ just by a sign, obtaining (2.3.29). According to (2.2.6), we compute

$$\mathrm{div}_{\mathrm{B}_{r_0}}\left(-\sqrt{-h_m}\,\partial_r\right) = -\left(\sqrt{-h_m}\,(\mathfrak{g}^S_m)^{tt}\langle\overline{\nabla}_{\partial_t}\partial_r, \partial_t\rangle - \sqrt{-h_m}\,(\mathfrak{g}^S_m)^{II}\langle\overline{\nabla}_{\partial_{\theta^I}}\partial_r, \partial_{\theta^I}\rangle\right)\Big|_{r=r_0}$$

$$= -\sqrt{-h_m(r_0)}\left(\overline{\Gamma}^t_{tr} + \overline{\Gamma}^I_{Ir}\right)\big|_{r=r_0} = \sqrt{-h_m(r_0)}\left(\frac{h'_m(r_0)}{-2h_m(r_0)} - \frac{2}{r_0}\right)$$
$$= \frac{2r_0 - 3m}{\sqrt{r_0^3(2m - r_0)}}\,,$$

where we used (2.3.4) and (2.3.5). $\square$

We will label the hypersurfaces in *region* II by $\mathrm{B}^+_{r_0}$, as they are *upper barriers* and in *region* II′ by $\mathrm{B}^-_{r_0}$, as they are *lower barriers.*

Proposition 2.3.2. *In the* maximally extended Schwarzschild $\left(\mathfrak{L}^K_m, \mathfrak{g}^K_m\right)$*, the hypersurfaces* $\{\mathrm{B}^+_{r_0}\}_{0<r_0<\frac{3}{2}m}$ *in* region II *and* $\{\mathrm{B}^-_{r_0}\}_{0<r_0<\frac{3}{2}m}$ region II′ *are a foliation of* upper barriers *and* lower barriers*, respectively. This means that if a compact maximal hypersurface M has boundary $\partial M \subset \{r > r_0\}$ for some $0 < r_0 < \frac{3}{2}m$, then $M \subset \{r > r_0\}$.*

Proof. The proof relies on a standard maximum principle argument. Assuming by contradiction that $M \cap \{r \leq r_0\} \neq \emptyset$, then there exist $0 < \overline{r}_0 \leq r_0$ and a point $q \in M$ such that $r(q) = \overline{r}_0 = \min_{p\in M} r(p)$. In a neighborhood $\mathscr{U}_q$ of q, the hypersurface $M \cap \mathscr{U}_q$, the coordinate r is a time function as it fulfills Definition 2.2.2, with lapse function $\alpha_r^2 = -h_m^{-1}(r) = \frac{1}{\frac{2m}{r}-1}$ and shift vector $\beta \equiv 0$. Since the future-pointing unit normal to $\mathscr{S}_r$ as in (2.3.30) points in the direction of increasing r in *region* II′, we will assume q lies in this region. The same argument, with opposite signs, applies to *region* II.

Considering the height function u on $M\cap\mathscr{U}_q$, since q is a minimum, we have ${}^r\mathrm{D}u|_q = 0$ (and consequently $\upsilon|_q = 1$) and ${}^r g^{ij}\partial^2_{ij}u|_q \geq 0$, where ${}^r g^{ij}$ is the metric induced on the time slices $\mathscr{S}_r$. Using the maximality of M, the identity (2.2.19) at the point q reads

$$0 = \alpha_r(q)\,{}^r g^{ij}\partial^2_{ij}u|_q + \mathrm{H}^o\,|_q \geq -\frac{2\overline{r}_0 - 3m}{\sqrt{\overline{r}_0^3(2m - \overline{r}_0)}} > 0$$

where we used that the time slices $\mathscr{S}_r = \mathrm{B}^-_r$ coincide with the foliation of constant positive mean curvature of Lemma 2.3.1. This is absurd and concludes the proof. $\square$

2.3.4 Blow-down analysis

Since the goal of this work is to construct a geometric notion of boost in Schwarzschild, motivated by its asymptotic flatness, it is interesting to explore the relation between exterior Schwarzschild and Minkowski in terms of a blow-down analysis. We will work in isotropic coordinates (2.3.10), but the same results hold in Schwarzschildian coordinates (2.3.2). Consider $\left((\mathfrak{L}^I_{ext})_m, \mathfrak{g}^I_m\right)$ and for simplicity let us call $\mathfrak{L}_m = (\mathfrak{L}^I_{ext})_m = \mathbb{R}\times(\mathbb{R}^3\setminus\overline{B}_{m/2})$ and $\mathfrak{g}_m = \mathfrak{g}^I_m$.

Definition 2.3.3. *Given a sequence of increasing numbers $\{k_i\}_{i\in\mathbb{N}}$, with $k_i \to +\infty$ as $i \to +\infty$, we define a* blow-down sequence *of the exterior Schwarzschild spacetimes by*

$$\begin{aligned}\mathfrak{L}_m^{k_i} &:= \left\{(\bar t, \bar{\boldsymbol{x}}) \in \mathbb{R} \times (\mathbb{R}^3 \setminus \{0\}) \,:\, (k_i \bar t, k_i \bar{\boldsymbol{x}}) \in \mathfrak{L}_m\right\} = \mathbb{R} \times (\mathbb{R}^3 \setminus \overline{B}_{m/2k_i}) \\ \mathfrak{g}_m^{k_i}(\bar t, \bar{\boldsymbol{x}}) &:= k_i^{-2}\, \mathfrak{g}_m(k_i \bar t, k_i \bar{\boldsymbol{x}}) \,.\end{aligned}$$

Considering a spacelike hypersurface $M \hookrightarrow (\mathfrak{L}_m, \mathfrak{g}_m)$ that is the graph of a function $u : \mathbb{R}^3 \setminus \overline{B}_{m/2} \to \mathbb{R}$, in consistency of the above, we define its blow-down sequence *as*

$$M_{k_i} := \left\{(\bar t, \bar{\boldsymbol{x}}) \in \mathfrak{L}_m^{k_i} \,:\, k_i \bar t = u(k_i \bar{\boldsymbol{x}})\right\}$$

and accordingly, we get a blow-down sequence *of functions*

$$u_{k_i} : \mathbb{R}^3 \setminus \overline{B}_{m/2k_i} \to \mathbb{R} \qquad u_{k_i}(\bar{\boldsymbol{x}}) := \frac{u(k_i\, \boldsymbol{x})}{k_i} \,.$$

Given a sequence $\{k_i\}_i$, we call $(\bar{\mathfrak{g}}, \bar u)$ a blow-down limit *of $\{\mathfrak{g}_m^{k_i}, u_{k_i}\}_i$ if $\bar{\mathfrak{g}}$ is a smooth Lorentzian metric on $\mathbb{R} \times (\mathbb{R}^3 \setminus \{0\})$ and $\bar u : \mathbb{R}^3 \setminus \{0\} \to \mathbb{R}$ is a (continuous) function such that $\mathfrak{g}_m^{k_i} \to \bar{\mathfrak{g}}$ in $C^\infty_{loc}(\mathbb{R}^3 \setminus \{0\})$ and $u_{k_i} \rightrightarrows \bar u$ uniformly on compact sets in $\mathbb{R}^3 \setminus \{0\}$ as $k_i \to +\infty$.*

Using the definition (2.3.10), we can immediately compute

$$\begin{aligned}\mathfrak{g}_m^{k_i}(\bar t, \bar{\boldsymbol{x}}) &= -N_m^2(k_i|\bar{\boldsymbol{x}}|)\mathrm{d}\bar t^2 + \varphi_m^4(k_i|\bar{\boldsymbol{x}}|)\delta_{\mathbb{R}^3} \\ &= -N_{m/k_i}^2(|\bar{\boldsymbol{x}}|)\mathrm{d}\bar t^2 + \varphi_{m/k_i}^4(|\bar{\boldsymbol{x}}|)\delta_{\mathbb{R}^3} = \mathfrak{g}_{m/k_i}(\bar t, \bar{\boldsymbol{x}})\end{aligned}$$

and this shows that Definition 2.3.3 is a sequence of isometries in case $m = 0$, namely in Minkowski. On the other hand, as the following convergences hold in $C^\infty_{loc}(\mathbb{R}^n \setminus \{0\})$

$$N_{m/k_i}^2(|\bar{\boldsymbol{x}}|) = \frac{1 - \frac{m}{2k_i|\bar{\boldsymbol{x}}|}}{1 + \frac{m}{2k_i|\bar{\boldsymbol{x}}|}} \to 1\,, \quad \varphi_{m/k_i}^2(|\bar{\boldsymbol{x}}|) = 1 + \frac{m}{2k_i|\bar{\boldsymbol{x}}|} \to 1 \qquad \text{as } k_i \to +\infty$$

for any sequence $\{k_i\}_i$, then any blow-down limit of $\mathfrak{g}_m$ is the Minkowski metric (2.1.1).

2.4 Statement of the main Theorem and strategy

Before stating the main theorem, we provide a definition of coordinate-dependent boost in Schwarzschild, suited to our purposes. Due to the rotational symmetry of Schwarzschild, without loss of generality, we will consider the boost parameter $\mathrm{b} \in [0, 1)$ instead of $\mathrm{b} \in (-1, 1)$, as outlined in Section 2.2.2 for the Minkowski spacetime.

Definition 2.4.1. *An (isotropic)* coordinate boost *of parameter* $\mathrm{b} \in [0,1)$ *in the maximally extended Schwarzschild* (2.3.16) *is a complete, non-compact, smooth spacelike hypersurface* $\mathscr{S}_{\mathrm{b}} \hookrightarrow \left(\mathfrak{L}^K_m, \mathfrak{g}^K_m\right)$ *such that, outside a spatial compact set, it is fully contained in* region I *and* region I$'$. *Moreover, in* region I, *in isotropic coordinates it reads*

$$\mathscr{S}_{\mathrm{b}} \cap \{|\,\boldsymbol{x}\,| > \widetilde{K}m\} = \left\{(t, \boldsymbol{x}) \in (\mathfrak{L}^I_{ext})_m \,:\, t = u_{\mathrm{b}}(\boldsymbol{x})\right\}$$

for some $\widetilde{K} > \frac{1}{2}$ *and a smooth function* $u_{\mathrm{b}} : \mathbb{R}^3 \setminus B_{\widetilde{K}m} \to \mathbb{R}$ *satisfying*

$$u_{\mathrm{b}}(\boldsymbol{x}) = \mathrm{b}\, x_1 + O_\infty \left(\log(|\,\boldsymbol{x}\,|)\right) \,. \tag{2.4.1}$$

The same property has to hold for the portion of $\mathscr{S}_{\mathrm{b}}$ *contained in* region I$'$, *after the isometry* $(\mathsf{X}, \mathsf{T}) \mapsto (-\mathsf{X}, -\mathsf{T})$ *has been applied.*

We denote the class of (isotropic) coordinate boosts of parameter $\mathrm{b} \in [0,1)$ *by* $\mathscr{B}_{\mathrm{b}}$.

As always, the subscript in O_∞ will be omitted.

Moreover, we highlight that throughout the work we will often deal with *coordinate boosts* just considering their representation as graphs in the asymptotically flat end of *region* I. In fact, once we have the graph of a function of the form (2.4.1) on large radii of *region* I, we can construct a complete, non-compact hypersurface $\mathscr{S}_{\mathrm{b}} \in \mathscr{B}_{\mathrm{b}}$ via the isometry $(\mathsf{X}, \mathsf{T}) \mapsto (-\mathsf{X}, -\mathsf{T})$, which produces a symmetric portion of hypersurface in *region* I$'$ and by gluing them in the same way as described in Section 2.7.

Remark 2.4.2. *Notice that, up to rotations of* $\mathbb{R}^3$ *(producing isometries in Schwarzschild), the function* (2.4.1) *formally corresponds to a coordinate boost in Minkowski* (2.2.38) *with a freedom of the order* $O\left(\log(|\,\boldsymbol{x}\,|)\right)$. *The necessity of allowing for additional terms becomes evident already if, for example, we consider in Schwarzschildean coordinates* (2.3.2) *the spacelike hypersurface*

$$\left\{(t, r, \theta_I) \in (\mathfrak{L}^S_{ext})_m \,:\, t = f_{\mathrm{b}}(r, \theta_I) = \mathrm{b}\, r \cos\theta_1\right\} \,.$$

One legitimately expects that, written in isotropic coordinates, it fits the Definition 2.4.1. After applying the change of coordinates (2.3.9) *and using that* $\cos\theta_1 = \frac{x_1}{|\,\boldsymbol{x}\,|}$, *the function* $f_{\mathrm{b}}((r(\boldsymbol{x}), \theta_I))$ *reads*

$$f_{\mathrm{b}}(r(\boldsymbol{x}), \theta_I) = \mathrm{b}\, x_1 + \mathrm{b}\, m \frac{x_1}{|\,\boldsymbol{x}\,|} + \frac{\mathrm{b}\, m^2}{4} \frac{x_1}{|\,\boldsymbol{x}\,|^2}$$

which belongs to the class of functions of the form (2.4.1), *defining then a hypersurface in* $\mathscr{B}_{\mathrm{b}}$. *On the other hand, the choice of the order* $O\left(\log(|\,\boldsymbol{x}\,|)\right)$ *is particularly suitable for our purposes, as it will be clarified later.*

A precise examination of this issue and its relation to ADM-charges [ADM08] lies beyond the scope of this work. Nonetheless, we wish to emphasize that, in light of the discussion in Section 2.1.1, this pertains to the fact that a boost in asymptotically flat spacetimes should be regarded as an equivalence class up to supertranslations *(which are related to the infinite choices of asymptotically flat coordinates). In conclusion, in Definition 2.4.1, we defined a notion of boost in isotropic coordinates, allowing freedom within an equivalence class of supertranslations. According to ongoing work by Cederbaum, the ADM energy remains invariant across all elements of such a class.*

Remark 2.4.3. *It is easy to see that, in terms of Definition 2.3.3, any spacelike hypersurface given as a graph of a function of the type* $\mathrm{b}\,x_1 + O_0(\log|\boldsymbol{x}|)$ *admits as unique blow-down limit the boosted (maximal) hypersurface* $\{t = \mathrm{b}x_1\}$ *in Minkowski spacetime, as in* (2.2.37) *with* $\widetilde{t}_0 = 0$. *In fact, given any sequence* $\{k_i\}_i$, *there exists an uniform constant* $C > 0$ *such that* $|u_i(\boldsymbol{x}) - \mathrm{b}\,x_1| \leq C k_i^{-1} \log(k_i|\boldsymbol{x}|) \to 0$ *as* $k_i \to +\infty$ *uniformly on compact sets.*

2.4.1 The main Theorem

Given the definition of the class $\mathscr{B}_{\mathrm{b}}$ and the above observations, we can finally state the main result of this work.

Theorem 2.4.4 (Main Theorem). *In the maximally extended Schwarzschild spacetime* (2.3.16), *for any* $\mathrm{b} \in [0, 1)$, *there exists an entire smooth maximal hypersurface*

$$M_{\mathrm{b}} \hookrightarrow \left(\mathfrak{L}^K_m, \mathfrak{g}^K_m\right), \qquad \mathrm{H}_{\mathfrak{g}^K_m}(M_{\mathrm{b}}) = 0\,, \qquad \{\mathsf{X} = \mathsf{T} = 0\} \subset M_{\mathrm{b}}$$

and a coordinate boost $\mathscr{S}_{\mathrm{b}} \in \mathscr{B}_{\mathrm{b}}$ *of parameter* b *as in Definition 2.4.1, invariant under the isometry* $(\mathsf{X}, \mathsf{T}) \mapsto (-\mathsf{X}, -\mathsf{T})$, *such that* M_{b} *is asymptotic to* $\mathscr{S}_{\mathrm{b}}$ *in both* region I *and* region I′. *More precisely, given* $\mathscr{S}_{\mathrm{b}} = \{t = u_{\mathrm{b}}(\boldsymbol{x})\}$, *in isotropic coordinates* (2.3.10) *in* region I, *for any fixed* $0 < \varepsilon < 1$, *there holds*

$$\left| t|_{M_{\mathrm{b}}} - u_{\mathrm{b}} \right| = O(|\boldsymbol{x}|^{-\varepsilon}) \qquad \text{as } |\boldsymbol{x}| \to +\infty \tag{2.4.2}$$

where $t|_{M_{\mathrm{b}}}$ *the restriction of the static time* t *to* M_{b}.

In particular, M_{b} *is entirely contained in the union of the bifurcation sphere* $\{\mathsf{X} = \mathsf{T} = 0\}$ *with the* regions I *and* I′.

Any blow-down limit of M_{b} *corresponds to the boosted (maximal) linear hypersurface of slope* b *passing through the origin in Minkowski spacetime, as in* (2.2.37).

Remark 2.4.5. *By* entire *hypersurface we mean that it is complete, non-compact and without boundary. In particular, by construction, it will be a smooth entire graph over* $\{\mathsf{T}=0\}=\mathbb{R}\times\mathbb{S}^2$.

We emphasise that, in the above theorem, we do not claim uniqueness of the solution; this question will be addressed in future work. Due to the absence of a uniqueness statement in Lemma 2.6.2, even the function u_{b} *defining the coordinate boost* $\mathcal{S}_{\mathrm{b}}$ *cannot be asserted to be unique, as will become evident in Proposition 2.6.1. In particular, by the characterisation of the class* $\mathscr{B}_{\mathrm{b}}$ *in Definition 2.4.1, aligned with the construction in Proposition 2.6.1, such different coordinate boosts* $\mathcal{S}_{\mathrm{b}}$ *can differ by a function proportional (up to)* $\log(|\boldsymbol{x}|)$*. On the other hand, we remark that, as a consequence of this observation, the maximal hypersurface* M_{b} *of Theorem 2.4.4 is not asymptotic to any coordinate boost* $\mathcal{S}_{\mathrm{b}'}$ *for a boost parameter* $\mathrm{b}'\in[0,1)$ *with* $\mathrm{b}'\neq\mathrm{b}$.

Last but not least, we underline that in Theorem 2.4.4 we construct a maximal hypersurface M_{b}*, which, for large radii, is given by a function of the form* $\mathrm{b}\,x_1+O_0(\log(|\boldsymbol{x}|)$*. By employing the maximal surface equation and scaling arguments as in [Bar84, Theorem 5.4], one should be able to control all higher-order derivatives, showing that* M_{b} *indeed takes the form described in*(2.4.1)*, namely* $M_{\mathrm{b}}\in\mathscr{B}_{\mathrm{b}}$*. This will be addressed in future work and would imply that the considerations on the ADM energy in Remark 2.4.2apply, in particular, to* M_{b}*.*

Before outlining the strategy of the proof of the main Theorem, which will be presented in detail in Section 2.9, we establish a consequence of it.

Corollary 2.4.6. *For any* $\mathrm{b}\in[0,1)$*, the exterior Schwarzchild spacetime* $\big((\mathfrak{L}^I_{ext})_m,\mathfrak{g}^I_m\big)$ *admits a foliation by (boosted) maximal spacelike hypersurfaces asymptotic to hypersurfaces in the class* $\mathscr{B}_{\mathrm{b}}$*, generated by an entire maximal surface in the maximally extended Schwarzschild* $\big(\mathfrak{L}^K_m,\mathfrak{g}^K_m\big)$*. This provides a geometric choice of time function associated to the asymptotic boost of parameter* $\mathrm{b}\in[0,1)$*.*

Proof. Given the existence of the hypersurface M_{b} in Theorem 2.4.4, we can consider its intersection to *region* I, corresponding to the exterior Schwarzschild endowed with the static time t. We have that the spacelike hypersurfaces

$$M^c_{\mathrm{b}}\coloneqq\{(\boldsymbol{x},t+c)\in(\mathfrak{L}^I_{ext})_m\,:\,(\boldsymbol{x},t)\in M_{\mathrm{b}}\}$$

provide a foliation by (boosted) maximal hypersurfaces of the whole *region* I. In fact, it is easy to see that $\mathrm{H}_{\mathfrak{g}^I_m}(M^c_{\mathrm{b}})=0$ since the foliation flows in the direction of the Killing vector field ∂_t or one can explicitly check it by using the formula (2.3.13). The function determined by the parameter $c\in\mathbb{R}$ is then a time function, since all the hypersurfaces M^c_{b} are spacelike. It can be conceived as a geometric choice of time function associated to the asymptotic boost of parameter b as all its level sets are maximal. □

2.4.2 Strategy of the proof

We outline the strategy of the proof of Theorem 2.4.4, presented in detail in Section 2.9.

As in [Bar84, proof of Theorem 5.4], we approach the problem by considering the boundaries $\{\Sigma_k^2\}_{k=1}^{\infty}$, $\Sigma_k^2 \hookrightarrow \mathscr{S}_{\mathrm{b}}$ of an exhaustion of a complete, non-compact spacelike hypersurface $\mathscr{S}_{\mathrm{b}} \in \mathscr{B}_{\mathrm{b}}$ –properly chosen, see discussion below– and solving the (compact) Dirichlet problem associated to the maximal surface equation with boundary data Σ_k^2. The existence of these solutions is ensured by the presence of the barriers at the crushing singularity as in Lemma 2.3.1–Proposition 2.3.2 and by [Bar84, Theorem 4.3]. We point out that, alternatively, one can argue directly by using the same a priori height estimate and the consequent a priori gradient estimate given by Theorem 2.2.5-(i) that need to be established for the next steps.

Given the above sequence of maximal hypersurfaces $\{M_k\}_{k=1}^{\infty}$ with $\partial M_k = \Sigma_k^2$ and $\mathrm{H}_{\mathfrak{g}_m^K}(M_k) = 0$, we need to show how to construct the desired hypersurface M_{b}, asymptotically approaching the chosen coordinate boost $\mathscr{S}_{\mathrm{b}} \in \mathscr{B}_{\mathrm{b}}$, as a (sub)-limit of $\{M_k\}_{k=1}^{\infty}$. To this end, we aim to show a height estimate uniform in k with respect to a wisely chosen time function, which satisfies the hypotheses of Theorem 2.2.5. This then provides a uniform gradient estimate and control on the ellipticity of the maximal surface equation. In turn, a uniform $C^{2,\alpha}$ estimate on compact sets follows from the De Giorgi–Nash theory and the interior Schauder estimate, see Theorem 2.2.7–Theorem 2.2.6. By the compactness of $C^{2,\beta}$ in $C^{2,\alpha}$ for any $0 < \beta < \alpha < 1$ and by a standard diagonal argument, we can finally construct as a sub-limit of $\{M_k\}_{k=1}^{\infty}$, a complete, non-compact, smooth maximal hypersurface $M_{\mathrm{b}} \hookrightarrow (\mathfrak{L}_m^K, \mathfrak{g}_m^K)$.

In this regard, we now highlight the main issues that need to be addressed.

- **Barriers in the asymptotically flat end**: to ensure that M_{b} is suitably asymptotic to an element of the class $\mathscr{B}_{\mathrm{b}}$ as in Definition 2.4.1, we first need to choose $\mathscr{S}_{\mathrm{b}} \in \mathscr{B}_{\mathrm{b}}$ as spacelike hypersurface where to consider the aforementioned sequence of boundary data $\{\Sigma_k^2\}_{k=1}^{\infty}$. It is then necessary to show that we can construct upper and lower barriers –by adapting barriers in the Minkowski spacetime– suitably converging to $\mathscr{S}_{\mathrm{b}}$ as $|\,\boldsymbol{x}\,| \to +\infty$ and demonstrating that all the solutions M_k lie between them. It will be clear by the discussion in Section 2.6 that not every hypersurface $\mathscr{S}_{\mathrm{b}} \in \mathscr{B}_{\mathrm{b}}$ allows for the construction of such barriers, as the mean curvature of the chosen $\mathscr{S}_{\mathrm{b}}$ has to decay fast enough. The natural candidate $\{t = f_{\mathrm{b}}(\boldsymbol{x}) = \mathrm{b}\, x_1\}$ in isotropic coordinates, for example, does not satisfy the required decay condition. It is thereby necessary to choose the coordinate-dependent boost $\mathscr{S}_{\mathrm{b}}$ wisely, recalling that Definition 2.4.1 allows to move in a range of the order $O(\log(|\,\boldsymbol{x}\,|))$. Due to a result in [Mey63], it will be possible to perturb the function f_{b} by adding a solution to a Laplace equation and obtain a hypersurface with the desired properties. This construction of coordinate-dependent boost $\mathscr{S}_{\mathrm{b}}$ and of the corresponding barriers will be the content of Section 2.6.

- **Height estimate, gradient estimate and the choice of time function**: ensuring an a priori height estimate on solutions with boundaries on the chosen $\mathscr{S}_{\mathrm{b}}$ is essential not only to construct M_{b} as a limiting hypersurface, as discussed above, but also to ensure that all the solutions $\{M_k\}_{k=1}^{\infty}$ and consequently M_{b} lie between the barriers mentioned in the previous point. We also highlight that the height estimate can be established with respect to a chosen time function. As in [Bar84, Theorem 5.3], an a priori height estimate can be obtained in asymptotically flat spacetimes by combining the *Bartnik's uniform interior condition* (2.2.35) –this is addressed in the next point– with the *pointwise gradient estimate* of Theorem 2.2.5-(*ii*). The latter holds under the *time-dependent Bartnik's conditions* (2.2.30), which have then to be ensured for a suitable choice of time function on the whole spacetime $(\mathfrak{L}_m^K, \mathfrak{g}_m^K)$.

  On the other hand, once a uniform height estimate is established, Theorem 2.2.5-(*i*) allows to get directly a gradient estimate, under the time-dependent hypotheses therein: again, we have to ensure the *time-dependent Bartnik's conditions* (2.2.30); moreover, the boundary data $\{\Sigma_k^2\}_{k=1}^{\infty}$ need to lie in a constant time slice and have a uniform bound on the time-dependent Riemannian norm of their spacetime mean curvature (2.2.31). In particular, we require the zero time-slice of the desired time function in the asymptotically flat ends to coincide with $\mathscr{S}_{\mathrm{b}}$, as constructed in the previous point.

  In view of the above, in Section 2.5 we will briefly discuss how the time function T of Kruskal-Szekeres coordinates (2.3.16) does not satisfy (2.2.30) on non-compact sets of $\mathfrak{L}_m^K$. We will then introduce a new time function $\widetilde{\mathsf{T}}$ and prove that it serves to satisfy the *time-dependent Bartnik's conditions* on $\{r \geq r_0\} \subset \mathfrak{L}_m^K$, which is sufficient for our purposes due to the presence of barriers at the crushing singularity as in Lemma 2.3.1–Proposition 2.6.1. The function $\widetilde{\mathsf{T}}$ is also interesting on its own as it is a Cauchy time function in the maximally extended Schwarzschild spacetime. A comprehensive discussion on this new time function is addressed in Section 2.5, while the tedious computations on the time-dependent Bartnik's conditions are performed in Appendix A.1.

  Finally, to combine all the above needs, in the exterior regions I and I′, we glue the new time function $\widetilde{\mathsf{T}}$ with a time function generated via staticity by the hypersurface $\mathscr{S}_{\mathrm{b}}$ admitting upper and lower barriers. The gluing procedure is examined in Section 2.7.

- **Bartnik's interior condition in presence of a crushing singularity**: as introduced in Section 2.2.1 and in the previous point, a uniform height a priori estimate relies, among others, on the *Bartnik's uniform interior condition* (2.2.35). The spacetime singularity (2.3.19), which corresponds to the timelike incompleteness of the spacetime, is a clear obstruction. Nevertheless, since it is a crushing singularity

–see Section 2.3.3–, the existence of the barriers of Lemma 2.3.1–Proposition 2.6.1 can help recover those conditions given a suitable choice of time function. It is non-trivial but possible to show that (2.2.35) are satisfied with respect to the new time function $\widetilde{\mathsf{T}}$ –introduced in the previous point– by verifying that the quantity involved obeys a monotonicity principle. The computations are handled in Section 2.5.1.

After all the previous problems have been addressed, a uniform height estimate is provided in Section 2.8 and the final proof of the main Theorem 2.4.4 in Section 2.9.

2.5 A Cauchy time function and the time-dependent Bartnik's conditions

As aforementioned, the need to introduce a new time function in the maximally extended Schwarzschild spacetime is rooted in the necessity of satisfying the time-dependent conditions (2.2.30). We will briefly discuss under which requirements these are attained by the other classical coordinates, and then define the new time function with its properties. Beyond the purposes of this work, we remark that this time function is interesting on its own as it is a *Cauchy time function*, while the Kruskal time T is clearly not – since this observation is not essential for the main goals of this work, we simply refer to [BS05b] for definitions and properties of Cauchy time functions.

We notice that, in general, since the Schwarzschild spacetime solves the Vacuum Einstein Equation (2.3.1), we have $\mathfrak{Ric} = 0$ and the conditions (2.2.35) reduce to estimate ${}^{t}\| \log \alpha_t \|_1$ and ${}^{t}\| \overline{\nabla}\, \mathscr{T} \|_1$.

Bartnik's time-dependent conditions on static time in exterior Schwarzschild. Regarding the static time t, as outlined in Section 2.3, it is not defined on the horizon –reading $\{r = 2m\} = \{|\,\boldsymbol{x}\,| = \frac{m}{2}\}$ in Schwarzschildean (2.3.2) and isotropic (2.3.10) coordinates, respectively– and many geometric quantities degenerate as approaching it, starting from the lapse function. Nevertheless, it is easy to see that when strictly far from the horizon, all the conditions (2.2.30) are satisfied.

In the following proposition, we present the time-dependent Bartnik's conditions for time functions defined by suitable spacelike hypersurfaces on exterior Schwarzschild, which in particular includes the case of the static slices. We present this more general version as it is going to be useful later and by setting the function f of the Proposition $f \equiv 0$, we recover the desired bounds for the static time. The result is stated in terms of isotropic coordinates, but clearly holds also in the Schwarzschildean ones.

Proposition 2.5.1. *Consider a spacelike hypersurface $\mathscr{S} = \{t = f(\boldsymbol{x})\}$ in the* region I *of exterior Schwarzschild, strictly far from the horizon $\{|\,\boldsymbol{x}\,| > \bar{\kappa}m > \frac{m}{2}\} \subset (\mathfrak{L}_{ext})^I_m$. Assume that $f \in C^\infty(\mathbb{R}^3 \setminus \overline{B}_{\bar{\kappa}m})$ and that there exists an uniform constant $C_f > 0$ bounding $\boldsymbol{v}_t(f)$*

and its higher derivatives

$$\sup_{\{|\,\boldsymbol{x}\,|>\bar{\kappa}m\}} \boldsymbol{v}_t(f) + \sum_{i,j=1}^{3} |\partial^2_{ij} f|\, m + \sum_{i,j,k=1}^{3} |\partial^3_{ijk} f|\, m^2 \leq C_f\,. \tag{2.5.1}$$

Then the function

$$\tau := t - f(\boldsymbol{x})$$

is a time function on $\{|\,\boldsymbol{x}\,| > \bar{\kappa}m\}$ *satisfying the conditions* (2.2.30), *namely there exists a constant* $C_o = C_o(C_f, \bar{\kappa})$ *such that*

$$^{\tau}\|\log \alpha_\tau\| + {}^{\tau}\|\mathrm{d}\log \alpha_\tau\| m + {}^{\tau}\|\,\overline{\nabla}\,\mathscr{T}_f\| m + {}^{\tau}\|\,\overline{\nabla}^2\,\mathscr{T}_f\| m^2 \leq C_o(C_f, \bar{\kappa})$$

on $\{|\,\boldsymbol{x}\,| \geq \bar{\kappa}m\}$, *where* α_τ *is the lapse function of* τ *as defined in* (2.2.3) *and* $\mathscr{T}_f$ *is the future-pointing unit normal to the time slices of* τ.

Proof. We first recall that $\boldsymbol{v}_t(f)^2 = \frac{1}{1-\frac{N_m^2}{\varphi_m^4}\sum_{i=1}^3(\partial_i f)^2}$ and hence the bound in the hypotheses ensures, in particular, that

$$\frac{N_m^2}{\varphi_m^4}\sum_{i=1}^{3}(\partial_i f)^2 < 1 - \frac{1}{C_f^2}\,.$$

The lapse for the time function τ reads

$$\alpha_\tau^2 = -\bar{g}(\mathrm{d}\tau, \mathrm{d}\tau)^{-1} = \frac{N_m^2}{1-\frac{N_m^2}{\varphi_m^4}\sum_{i=1}^3(\partial_i f)^2} = N_m^2\, \boldsymbol{v}_t(f)^2$$

and thus we can estimate

$$\log\left(\frac{1-\frac{1}{2\bar{\kappa}}}{1+\frac{1}{2\bar{\kappa}}}\right) < \log \alpha_\tau < \log C_f\,. \tag{2.5.2}$$

To estimate the other quantities in terms of the norm (2.2.5), we have to find an adapted orthonormal frame. We consider

$$\{\mathscr{T}_f, V_1, V_2, V_3\}\,,$$

where $\mathscr{T}_f$ is the unit normal to the time slices and $\{V_1, V_2, V_3\}$ are tangent to the hypersurface $\mathscr{S} = \{t = f(\boldsymbol{x})\}$. We can compute

$$-\alpha_\tau \mathrm{d}\tau = -N_m \boldsymbol{v}_t(f)\left(\mathrm{d}t - \partial_i f\, \mathrm{d}x^i\right)$$

and consequently

$$\mathscr{T}_f = N_m \boldsymbol{v}_t(f)\left(N_m^{-2}\,\partial_t + \varphi_m^{-4}\sum_{i=1}^{3}\partial_i f\,\partial_{x^i}\right) = \alpha_\tau\left(N_m^{-2}\,\partial_t + \varphi_m^{-4}\sum_{i=1}^{3}\partial_i f\,\partial_{x^i}\right)\,. \tag{2.5.3}$$

On the other hand, the tangential frame can be obtained by applying the Gram–Schmidt process to the frame

$$\widetilde{V}_i = \partial_{x^i} + \partial_i f\, \partial_t \,. \tag{2.5.4}$$

In particular, we can set

$$V_1 = \frac{1}{\sqrt{\varphi_m^4 - N_m^2(\partial_1 f)^2}}\, \widetilde{V}_1 \tag{2.5.5}$$

and compute

$$\mathfrak{g}_m^I\left(\widetilde{V}_2, V_1\right) = \frac{-N_m^2(\partial_1 f)(\partial_2 f)}{\sqrt{\varphi_m^4 - N_m^2(\partial_1 f)^2}}\,, \qquad \mathfrak{g}_m^I\left(\widetilde{V}_3, V_1\right) = \frac{-N_m^2(\partial_1 f)(\partial_3 f)}{\sqrt{\varphi_m^4 - N_m^2(\partial_1 f)^2}}\,.$$

A tangent vector field to $\mathscr{S} = \{t = f(\boldsymbol{x})\}$ that is orthogonal to V_1 is then given by $\widetilde{V}_2 - \mathfrak{g}_m^I\left(\widetilde{V}_2, V_1\right) V_1$. Its normalisation gives

$$\begin{aligned} V_2 = & \sqrt{\frac{\varphi_m^4 - N_m^2(\partial_1 f)^2}{\varphi_m^4\left[\varphi_m^4 - N_m^2\left((\partial_1 f)^2 + (\partial_1 f)^2\right)\right]}}\, \widetilde{V}_2 \\ & + \frac{N_m^2(\partial_1 f)(\partial_2 f)}{\sqrt{\varphi_m^4\left[\varphi_m^4 - N_m^2\left((\partial_1 f)^2 + (\partial_1 f)^2\right)\right]\left[\varphi_m^4 - N_m^2(\partial_1 f)^2\right]}}\, \widetilde{V}_1\,. \end{aligned} \tag{2.5.6}$$

Finally, we compute

$$\begin{aligned} \mathfrak{g}_m^I\left(\widetilde{V}_3, V_2\right) = & \sqrt{\frac{\varphi_m^4 - N_m^2(\partial_1 f)^2}{\varphi_m^4\left[\varphi_m^4 - N_m^2\left((\partial_1 f)^2 + (\partial_2 f)^2\right)\right]}}\, \mathfrak{g}_m^I\left(\widetilde{V}_3, \widetilde{V}_2\right) \\ & + \frac{N_m^2(\partial_1 f)(\partial_2 f)}{\sqrt{\varphi_m^4\left[\varphi_m^4 - N_m^2\left((\partial_1 f)^2 + (\partial_2 f)^2\right)\right]\left[\varphi_m^4 - N_m^2(\partial_1 f)^2\right]}}\, \mathfrak{g}_m^I\left(\widetilde{V}_3, \widetilde{V}_1\right) \\ = & -\frac{N_m^2\,(\partial_2 f)(\partial_3 f)\ \left(\varphi_m^4 - N_m^2(\partial_1 f)^2 + N_m^2(\partial_1 f)^2\right)}{\varphi_m^2\sqrt{\left[\varphi_m^4 - N_m^2\left((\partial_1 f)^2 + (\partial_2 f)^2\right)\right]\left[\varphi_m^4 - N_m^2(\partial_1 f)^2\right]}} \\ = & -\frac{\varphi_m^2 N_m^2\,(\partial_2 f)(\partial_3 f)}{\sqrt{\left[\varphi_m^4 - N_m^2\left((\partial_1 f)^2 + (\partial_2 f)^2\right)\right]\left[\varphi_m^4 - N_m^2(\partial_1 f)^2\right]}} \end{aligned}$$

and we consider the tangent vector field to $\mathscr{S} = \{t = f(\boldsymbol{x})\}$ that is orthogonal to V_1 and V_2, given by $\widetilde{V}_3 - \mathfrak{g}_m^I\left(\widetilde{V}_3, V_2\right) V_2 - \mathfrak{g}_m^I\left(\widetilde{V}_3, V_1\right) V_1$. Its normalisation gives

$$\begin{aligned} V_3 = & \sqrt{\frac{\varphi_m^4 - N_m^2\left((\partial_1 f)^2 + (\partial_2 f)^2\right)}{\varphi_m^4\left[\varphi_m^4 - N_m^2\sum_{i=1}^3(\partial_i f)^2\right]}}\, \widetilde{V}_3 \\ & + \frac{N_m^2\,(\partial_2 f)(\partial_3 f)}{\sqrt{\varphi_m^4\left[\varphi_m^4 - N_m^2\sum_{i=1}^3(\partial_i f)^2\right]\left[\varphi_m^4 - N_m^2\left((\partial_1 f)^2 + (\partial_2 f)^2\right)\right]}}\, \widetilde{V}_2 \\ & + \frac{N_m^2(\partial_1 f)(\partial_3 f)}{\sqrt{\varphi_m^4\left[\varphi_m^4 - N_m^2\sum_{i=1}^3(\partial_i f)^2\right]\left[\varphi_m^4 - N_m^2\left((\partial_1 f)^2 + (\partial_2 f)^2\right)\right]}}\, \widetilde{V}_1 \end{aligned} \tag{2.5.7}$$

The frame defined by (2.5.3), (2.5.5), (2.5.6) and (2.5.7) is orthonormal. Since V_1, V_2, V_3 are linear combinations of $\widetilde{V}_1, \widetilde{V}_2, \widetilde{V}_3$, as defined in (2.5.4), with coefficients bounded by the hypothesis (2.5.1), it will be sufficient to verify the desired conditions for the latter.

Observe now that $2\alpha^{-1}\mathrm{d}\alpha = \alpha^{-2}\mathrm{d}\alpha^2$, so we compute

$$\alpha_\tau^{-2}\mathrm{d}\alpha_\tau^2 = \frac{2}{N_m}\,\mathrm{d}N_m + \frac{\alpha_\tau^2}{N_m^2}\left(\frac{2N_m\sum_{i=1}^3(\partial_i f)^2}{\varphi_m^4}\mathrm{d}N_m - \frac{4N_m^2\sum_{i=1}^3(\partial_i f)^2}{\varphi_m^5}\mathrm{d}\varphi_m + \frac{2N_m^2}{\varphi_m^4}\partial_i f\,\partial_{ij}^2 f\,dx^j\right)$$

where

$$\mathrm{d}N_m = \frac{\partial N_m}{\partial x^i}\mathrm{d}x^i = \frac{x^i}{|\,\boldsymbol{x}\,|}N_m'(|\,\boldsymbol{x}\,|)\,\mathrm{d}x^i = \frac{x^i}{|\,\boldsymbol{x}\,|}\frac{m}{|\,\boldsymbol{x}\,|^2\left(1+\frac{m}{2|\,\boldsymbol{x}\,|}\right)^2}\,\mathrm{d}x^i = \frac{x^i}{|\,\boldsymbol{x}\,|}\frac{4m}{(2|\,\boldsymbol{x}\,|+m)^2}\,\mathrm{d}x^i$$

and

$$\mathrm{d}\varphi_m = \frac{\partial\varphi_m}{\partial x^i}\mathrm{d}x^i = \frac{x^i}{|\,\boldsymbol{x}\,|}\varphi_m'(|\,\boldsymbol{x}\,|)\,\mathrm{d}x^i = -\frac{x^i}{|\,\boldsymbol{x}\,|}\frac{m}{2|\,\boldsymbol{x}\,|^2}\,\mathrm{d}x^i$$

and we need to estimate the following

$$\begin{aligned}
|\mathfrak{g}_m^I\left(\alpha_\tau^{-2}\mathrm{d}\alpha_\tau^2, -\alpha_\tau\mathrm{d}\tau\right)| &= N_m\boldsymbol{v}_t(f)\,\varphi_m^{-4}\,\partial_i f\left|\frac{2}{N_m}\frac{x^i}{|\,\boldsymbol{x}\,|}\frac{4m}{(2|\,\boldsymbol{x}\,|+m)^2}+\right.\\
&+\frac{\alpha_\tau^2}{N_m^2}\left(\frac{2N_m\sum_{i=1}^3(\partial_i f)^2}{\varphi_m^4}\frac{x^i}{|\,\boldsymbol{x}\,|}\frac{4m}{(2|\,\boldsymbol{x}\,|+m)^2}+\frac{4N_m^2\sum_{i=1}^3(\partial_i f)^2}{\varphi_m^5}\frac{x^i}{|\,\boldsymbol{x}\,|}\frac{m}{2|\,\boldsymbol{x}\,|^2}\right.\\
&\left.\left.+\frac{2N_m^2}{\varphi_m^4}\partial^j f\,\partial_{ij}^2 f\right)\right|\\
&\le C_f\sqrt{1-\frac{1}{C_f}}\left(\frac{8}{(4\bar\kappa^2-1)\,m}+\frac{C_f^3(2\bar\kappa+1)^2}{(2\bar\kappa-1)^3 m}+\frac{2C_f}{\bar\kappa^2 m}+\frac{2C_f^2}{m}\right)
\end{aligned}\tag{2.5.8}$$

and similarly, we compute

$$\begin{aligned}
|\alpha_\tau^{-2}\mathrm{d}\alpha_\tau^2(\widetilde{V}_i)| &\le \left|\frac{2}{N_m}\frac{x^i}{|\,\boldsymbol{x}\,|}\frac{4m}{(2|\,\boldsymbol{x}\,|+m)^2}+\frac{\alpha_\tau^2}{N_m^2}\left(\frac{2N_m\sum_{i=1}^3(\partial_i f)^2}{\varphi_m^4}\frac{x^i}{|\,\boldsymbol{x}\,|}\frac{4m}{(2|\,\boldsymbol{x}\,|+m)^2}\right.\right.\\
&\left.\left.+\frac{4N_m^2\sum_{i=1}^3(\partial_i f)^2}{\varphi_m^5}\frac{x^i}{|\,\boldsymbol{x}\,|}\frac{m}{2|\,\boldsymbol{x}\,|^2}+\frac{2N_m^2}{\varphi_m^4}\partial^j f\,\partial_{ij}^2 f\right)\right|\\
&\le\frac{8}{(4\bar\kappa^2-1)\,m}+\frac{4C_f^3}{(2\bar\kappa-1)(2\bar\kappa+1)m}+\frac{2C_f}{\bar\kappa^2 m}+\frac{2C_f^2}{m}\,,
\end{aligned}\tag{2.5.9}$$

Combining (2.5.2), (2.5.8), (2.5.9) and the bounds on the coefficients of the tangential frame V_1, V_2, V_3 with respect to $\widetilde{V}_1, \widetilde{V}_2, \widetilde{V}_3$ we obtain

$${}^\tau\|\log\alpha_\tau\| + {}^\tau\|\,\mathrm{d}\log\alpha_\tau\|\,m \le C_o(C_f,\bar\kappa)\,.$$

For the sake of the next computations, we first observe that

$$|\partial_i \alpha_\tau| = |\mathrm{d}\alpha_\tau(\partial_{x^i})| = |\alpha_\tau||\alpha_\tau^{-1}\mathrm{d}\alpha_\tau(\widetilde{V}_i)| \leq \frac{\widetilde{C}_o(C_f, \bar{\kappa})}{m}$$

We then compute

$$\begin{aligned}
\overline{\nabla}_{\partial_{x^i}} \mathscr{T}_f &= \partial_i \alpha_\tau \left(N_m^{-2}\, \partial_t + \varphi_m^{-4} \partial_j f\, \partial_{x^j}\right) + \alpha_\tau \Bigg(- \frac{2\partial_i N_m}{N_m^3}\partial_t - \frac{4\partial_i \varphi_m\, \partial_j f}{\varphi_m^5}\partial_{x^j} + \frac{\partial^2_{ij} f}{\varphi_m^4}\partial_{x^j} \\
&\qquad + N_m^{-2}\left(\overline{\Gamma}^k_{it}\partial_{x^k} + \overline{\Gamma}^t_{it}\partial_t\right) + \varphi_m^{-4}\partial_j f\left(\overline{\Gamma}^k_{ij}\partial_{x^k} + \overline{\Gamma}^t_{ij}\partial_t\right)\Bigg) \\
&= \left(\partial_i \alpha_\tau N_m^{-2} - \alpha_\tau \frac{2\partial_i N_m}{N_m^3} + \alpha_\tau N_m^{-2}\overline{\Gamma}^t_{it} + \alpha_\tau \varphi_m^{-4}\partial_j f \overline{\Gamma}^t_{ij}\right)\partial_t \\
&\quad + \left(\partial_i \alpha_\tau \varphi_m^{-4}\partial_k f - \alpha_\tau \frac{4\partial_i \varphi_m\, \partial_k f}{\varphi_m^5} - \frac{\partial^2_{ik} f}{\varphi_m^4}\alpha_\tau N_m^{-2}\overline{\Gamma}^k_{it} + \alpha_\tau \varphi_m^{-4}\partial_j f \overline{\Gamma}^k_{ij}\right)\partial_{x^k}
\end{aligned}$$

and similarly

$$\begin{aligned}
\overline{\nabla}_{\partial_t} \mathscr{T}_f &= \alpha_\tau \left(N_m^{-2}\left(\overline{\Gamma}^k_{tt}\partial_{x^k} + \overline{\Gamma}^t_{tt}\partial_t\right) + \varphi_m^{-4}\partial_j f\left(\overline{\Gamma}^k_{tj}\partial_{x^k} + \overline{\Gamma}^t_{tj}\partial_t\right)\right) \\
&= \alpha_\tau \left(N_m^{-2}\overline{\Gamma}^t_{tt} + \varphi_m^{-4}\partial_j f \overline{\Gamma}^t_{tj}\right)\partial_t + \alpha_\tau \left(N_m^{-2}\overline{\Gamma}^k_{tt} + \varphi_m^{-4}\partial_j f \overline{\Gamma}^k_{tj}\right)\partial_{x^k}
\end{aligned}$$

Since it is now easy to see that all the objects appearing in the coefficients are bounded, as well as their derivatives, we conclude that

$${}^{\tau}\|\, \overline{\nabla}\, \mathscr{T}_f \|\, m + {}^{\tau}\|\, \overline{\nabla}^2\, \mathscr{T}_f \|\, m^2 \leq C_o(C_f, \bar{\kappa})\,.$$

□

Bartnik's time-dependent conditions on the Kruskal-Szekeres time. Consider now the time function T of Kruskal-Szekeres coordinates (2.3.16) and the associated time-dependent Riemannian norm as defined in (2.2.5). A local orthonormal frame to compute the norm, given the future-pointing unit normal (2.3.28) to the time slices, is given by

$$\left\{\mathscr{T}_m^K = \xi_m^{-\frac{1}{2}}\partial_{\mathsf{T}},\ \xi_m^{-\frac{1}{2}}\partial_{\mathsf{X}},\ r^{-1}\sqrt{g^{II}_{\mathbb{S}^2}}\,\partial_I\right\}, \quad I = 1, 2\,.$$

The lapse function (2.3.27) is given by

$$\alpha_{\mathsf{T}} = \sqrt{\xi_m} = (2m)^{\frac{3}{2}}\,\frac{e^{-\frac{r}{4m}}}{\sqrt{r}}$$

and it is possible to bound it uniformly $0 < \alpha_{\mathsf{T}} < \infty$ provided that we restrict to $\{r > \kappa_0 m\}$ for some $\kappa_0 > 0$. This is not a restrictive condition, as we can employ the barriers at the crushing singularity Lemma 2.3.1 for a lower bound on the radius. On the other hand, the norm ${}^{\mathsf{T}}\|\log\alpha_{\mathsf{T}}\|_1$ does not have an uniform bound in the region $\{r > \kappa_0 m\}$. In fact, computing

$$\mathrm{d}\log\alpha_{\mathsf{T}} = \alpha_{\mathsf{T}}^{-1}\frac{\partial\alpha_{\mathsf{T}}}{\partial r}\left(\frac{\partial r}{\partial\mathsf{T}}\mathrm{d}\mathsf{T} + \frac{\partial r}{\partial\mathsf{X}}\mathrm{d}\mathsf{X}\right) = \frac{\xi_m^{\frac{3}{2}}}{8m^2}\left(1+\frac{2m}{r}\right)(\mathsf{T}\mathrm{d}\mathsf{T} - \mathsf{X}\mathrm{d}\mathsf{X})$$

where we used (2.3.24) and (2.3.25). Thus, we obtain

$$\begin{aligned}{}^{\mathsf{T}}\|\,\overline{\nabla}\log\alpha_{\mathsf{T}}\|^2 &= \sup_{\{r>\kappa_0 m\}}\left[|\mathrm{d}\log\alpha_{\mathsf{T}}(\xi_m^{-\frac{1}{2}}\partial_{\mathsf{T}})|^2 + |\mathrm{d}\log\alpha_{\mathsf{T}}(\xi_m^{-\frac{1}{2}}\partial_{\mathsf{X}})|^2\right]\\ &= \sup_{\{r>\kappa_0 m\}}\frac{\xi_m^2(r)}{16m^4}\left(1+\frac{2m}{r}\right)^2(\mathsf{T}^2+\mathsf{X}^2)\end{aligned}$$

which fails to be bounded, unless computed on a compact region of $\{r > \kappa_0 m\}$, namely imposing bounds $|\mathsf{X}| \leq \mathsf{X}_0$ and $|\mathsf{T}| \leq \mathsf{T}_0$.

The new time function. As discussed in Section 2.3, while the Kruskal-Szekeres coordinates (2.3.16) are necessary to cover the maximally extended Schwarzschild spacetime, they do not capture the asymptotic flatness of the spacetime, which is instead evident in Schwarzschildean (2.3.2) and isotropic (2.3.10) coordinates. Motivated by this observation, we consider a foliation of the Kruskal extension $\left(\mathfrak{L}_m^K, \mathfrak{g}_m^K\right)$ by the family of (spacelike) hypersurfaces

$$\left\{\mathsf{T} = a\sqrt{1+\mathsf{X}^2}, \quad |a| < 1\right\}. \tag{2.5.10}$$

Since $\lim_{\mathsf{X}\to+\infty}\frac{a\sqrt{1+\mathsf{X}^2}}{\mathsf{X}} = a$ and $\lim_{\mathsf{X}\to-\infty}\frac{a\sqrt{1+\mathsf{X}^2}}{\mathsf{X}} = -a$, they are asymptotic to the surfaces

$$\begin{aligned}\{\mathsf{T} = a\mathsf{X}, \quad |a|<1\} &= \{t = 4m\,\mathrm{artanh}\,a, \quad |a|<1\}\,, \text{ if } \mathsf{X}>0\\ \{\mathsf{T} = -a\mathsf{X}, \quad |a|<1\} &= \{t = 4m\,\mathrm{artanh}\,(-a), \quad |a|<1\}\,, \text{ if } \mathsf{X}<0\end{aligned}$$

representing time slices of the static time. The time function having (2.5.10) as time slices turns out to be a Cauchy function and to satisfy the conditions (2.2.35) on $\{r > r_0\}$.

Proposition 2.5.2. *The function*

$$\widetilde{\mathsf{T}} := 4m\,\mathrm{artanh}\left(\frac{\mathsf{T}}{\sqrt{1+\mathsf{X}^2}}\right) \in (-\infty, +\infty) \tag{2.5.11}$$

is a well-defined time function in the maximally extended Schwarzschild $\left(\mathfrak{L}_m^K, \mathfrak{g}_m^K\right)$. *In particular, it is a Cauchy time function.*

Moreover, the spacetime conditions (2.2.30) *are satisfied by the time-dependent Riemannian norm* (2.2.5) *defined with respect to* $\widetilde{\mathsf{T}}$ *in a region* $\{r > \kappa_0 m\}$ *for any* $\kappa_0 > 0$, *with a constant $C_o = C_o(\kappa_0)$, that is,*

$$^{\widetilde{\mathsf{T}}}\|\log\alpha_{\widetilde{\mathsf{T}}}\| + {}^{\widetilde{\mathsf{T}}}\|\mathrm{d}\log\alpha_{\widetilde{\mathsf{T}}}\|m + {}^{\widetilde{\mathsf{T}}}\|\overline{\nabla}\,\widetilde{\mathscr{T}}\|m + {}^{\widetilde{\mathsf{T}}}\|\overline{\nabla}^2\,\widetilde{\mathscr{T}}\|m^2 \le C_o(\kappa_0) \tag{2.5.12}$$

where $\alpha_{\widetilde{\mathsf{T}}}$ is the lapse function of $\widetilde{\mathsf{T}}$ as defined in (2.2.3) *and $\widetilde{\mathscr{T}}$ is the future-pointing unit normal to the $\widetilde{\mathsf{T}}$-time slices.*

Proof. By definition, $\widetilde{\mathsf{T}}$ is a smooth function of the previous coordinates (T,X) and is well defined in the whole maximally extended Schwarzschild spacetime (2.3.16), as $\mathsf{T}^2 < 1+\mathsf{X}^2$ and the function artanh is defined on $(-1,1)$. It is a time function, as its differential reads

$$\mathrm{d}\widetilde{\mathsf{T}} = \frac{4m}{1-\frac{\mathsf{T}^2}{1+\mathsf{X}^2}}\frac{1}{\sqrt{1+\mathsf{X}^2}}\left(\mathrm{d}\mathsf{T} - \frac{\mathsf{X}\mathsf{T}}{1+\mathsf{X}^2}\mathrm{d}\mathsf{X}\right)$$

and consequently

$$\begin{aligned}\mathfrak{g}_m^K\left(\mathrm{d}\widetilde{\mathsf{T}},\mathrm{d}\widetilde{\mathsf{T}}\right) &= 16m^2\,\xi_m^{-1}\frac{1+\mathsf{X}^2}{(1+\mathsf{X}^2-\mathsf{T}^2)^2}\left(-1+\frac{\mathsf{X}^2\mathsf{T}^2}{(1+\mathsf{X}^2)^2}\right)\\ &= 16m^2\,\xi_m^{-1}\frac{\mathsf{X}^2\mathsf{T}^2-(1+\mathsf{X}^2)^2}{(1+\mathsf{X}^2-\mathsf{T}^2)^2\,(1+\mathsf{X}^2)}\\ &= -\frac{16m^2\,\xi_m^{-1}}{1+\mathsf{X}^2-\mathsf{T}^2}\left(1+\frac{\mathsf{T}^2}{(1+\mathsf{X}^2-\mathsf{T}^2)(1+\mathsf{X}^2)}\right) < 0\end{aligned} \tag{2.5.13}$$

where we used that

$$(1+\mathsf{X}^2-\mathsf{T}^2)(1+\mathsf{X}^2) = (1+\mathsf{X}^2)^2 - \mathsf{X}^2\mathsf{T}^2 - \mathsf{T}^2 \tag{2.5.14}$$

and the fact that $\xi_m > 0$ and $1+\mathsf{X}^2-\mathsf{T}^2 > 0$.

The time function $\widetilde{\mathsf{T}}$ is Cauchy since each time slice $\mathscr{S}_{\widetilde{\mathsf{T}}}$ is a Cauchy surface: every inextendible timelike curve in $\left(\mathfrak{L}_m^K,\mathfrak{g}_m^K\right)$ intersects $\mathscr{S}_{\widetilde{\mathsf{T}}}$ exactly once. It is well-known that $\{\mathsf{T}=0\}$ is Cauchy. Now, for each timelike (future-directed) inextendible curve $s\mapsto\gamma(s)\in\mathfrak{L}_m^K$ passing through $\{\mathsf{T}=0\}$ at $\mathrm{p}_\gamma = \gamma\cap\{\mathsf{T}=0\}$, we know that γ is contained in the regions between the null hypersurfaces $\{\mathsf{T}=\mathsf{X}-\mathsf{X}(\mathrm{p}_\gamma)\}$ and $\{\mathsf{T}=-(\mathsf{X}-\mathsf{X}(\mathrm{p}_\gamma))\}$ which both intersect each of the time slices. Hence γ intersects $\mathscr{S}_{\widetilde{\mathsf{T}}}$ and only once, since as a general result for time functions, $\widetilde{\mathsf{T}}$ is strictly increasing along it:

$$\frac{d}{ds}(\widetilde{\mathsf{T}}\circ\gamma(s)) = \left\langle \mathrm{grad}_{\mathfrak{g}_m^K}\widetilde{\mathsf{T}},\dot\gamma\right\rangle > 0\,,$$

where the last inequality follows from the fact that γ is future directed with respect to the orientation provided by $\mathrm{grad}_{\mathfrak{g}_m^K}\mathsf{T}$ so, in particular, $\left\langle\mathrm{grad}_{\mathfrak{g}_m^K}\mathsf{T},\dot\gamma\right\rangle > 0$, the fact that the gradients of T and $\widetilde{\mathsf{T}}$ are mutually future directed $\left\langle -\mathrm{grad}_{\mathfrak{g}_m^K}\widetilde{\mathsf{T}}, -\mathrm{grad}_{\mathfrak{g}_m^K}\mathsf{T}\right\rangle < 0$ and that $\dot\gamma$ and $\mathrm{grad}_{\mathfrak{g}_m^K}\widetilde{\mathsf{T}}$ are timelike, so they can be projected into $\mathscr{T}$ and an adapted orthonormal frame on the time slices of T.

Finally, let us compute the lapse $\alpha_{\widetilde{\mathsf{T}}}$, the future-pointing unit normal $\widetilde{\mathscr{T}}$ and the associated orthonormal frame. From the computation (2.5.13), we directly obtain

$$\alpha_{\widetilde{\mathsf{T}}}^2 = -\mathfrak{g}_m^K\left(\mathrm{d}\widetilde{\mathsf{T}},\mathrm{d}\widetilde{\mathsf{T}}\right)^{-1} = \frac{\xi_m(r)}{16m^2}\,\frac{1+\mathsf{X}^2-\mathsf{T}^2}{1+f(\mathsf{X},\mathsf{T})}\,, \tag{2.5.15}$$

$$f(\mathsf{X},\mathsf{T}) \coloneqq \frac{\mathsf{T}^2}{(1+\mathsf{X}^2-\mathsf{T}^2)(1+\mathsf{X}^2)} \geq 0\,. \tag{2.5.16}$$

Now, we can compute

$$\begin{aligned}
-\alpha_{\widetilde{\mathsf{T}}}\,\mathrm{d}\widetilde{\mathsf{T}} &= -\alpha_{\widetilde{\mathsf{T}}}\,\frac{4m\sqrt{1+\mathsf{X}^2}}{1+\mathsf{X}^2-\mathsf{T}^2}\left(\mathrm{d}\mathsf{T}-\frac{\mathsf{X}\mathsf{T}}{1+\mathsf{X}^2}\mathrm{d}\mathsf{X}\right)\\
&= -\frac{\sqrt{\xi_m}}{4m}\frac{\sqrt{1+\mathsf{X}^2-\mathsf{T}^2}}{\sqrt{1+f}}\frac{4m\sqrt{1+\mathsf{X}^2}}{1+\mathsf{X}^2-\mathsf{T}^2}\left(\mathrm{d}\mathsf{T}-\frac{\mathsf{X}\mathsf{T}}{1+\mathsf{X}^2}\mathrm{d}\mathsf{X}\right)\\
&= -\sqrt{\frac{\xi_m}{1+f}}\,\frac{1}{\sqrt{1+\mathsf{X}^2-\mathsf{T}^2}}\left(\sqrt{1+\mathsf{X}^2}\,\mathrm{d}\mathsf{T}-\frac{\mathsf{X}\mathsf{T}}{\sqrt{1+\mathsf{X}^2}}\mathrm{d}\mathsf{X}\right)
\end{aligned}$$

and this provides the unitary future-directed normal to the time slices

$$\widetilde{\mathscr{T}} = \frac{1}{\sqrt{\xi_m(1+f)}}\,\frac{1}{\sqrt{1+\mathsf{X}^2-\mathsf{T}^2}}\left(\sqrt{1+\mathsf{X}^2}\,\partial_{\mathsf{T}}+\frac{\mathsf{X}\mathsf{T}}{\sqrt{1+\mathsf{X}^2}}\partial_{\mathsf{X}}\right)\,, \tag{2.5.17}$$

for f defined in (2.5.16). To find the other elements of the adapted frame, we consider a time slice $\{\mathsf{T}=a\sqrt{1+\mathsf{X}^2}\,,\ |a|<1\}$ and its tangent vectors given by $\partial_{\mathsf{X}}+\frac{a\mathsf{X}}{\sqrt{1+\mathsf{X}^2}}\partial_{\mathsf{T}} = \partial_{\mathsf{X}}+\frac{\mathsf{T}\mathsf{X}}{1+\mathsf{X}^2}\partial_{\mathsf{T}}, \partial_I$. Computing their norm

$$\begin{aligned}
\mathfrak{g}_m^K\left(\partial_{\mathsf{X}}+\tfrac{\mathsf{T}\mathsf{X}}{1+\mathsf{X}^2}\partial_{\mathsf{T}},\partial_{\mathsf{X}}+\tfrac{\mathsf{T}\mathsf{X}}{1+\mathsf{X}^2}\partial_{\mathsf{T}}\right) &= \xi_m\left(-\frac{\mathsf{T}^2\mathsf{X}^2}{(1+\mathsf{X}^2)^2}+1\right) = \xi_m\,\frac{(1+\mathsf{X}^2)^2-\mathsf{T}^2\mathsf{X}^2}{(1+\mathsf{X}^2)^2}\\
&= \xi_m\,\frac{(1+\mathsf{X}^2-\mathsf{T}^2)(1+\mathsf{X}^2)+\mathsf{T}^2}{(1+\mathsf{X}^2)^2}\\
&= \xi_m\,\frac{1+\mathsf{X}^2-\mathsf{T}^2}{1+\mathsf{X}^2}\left(1+\frac{\mathsf{T}^2}{(1+\mathsf{X}^2-\mathsf{T}^2)(1+\mathsf{X}^2)}\right)\\
&= \xi_m\,\frac{1+\mathsf{X}^2-\mathsf{T}^2}{1+\mathsf{X}^2}\,(1+f)\,,
\end{aligned}$$

we obtain that an adapted frame is given by $\{\widetilde{\mathscr{T}},\widetilde{V}_{\mathsf{X}},\widetilde{V}_I\}$, with $I=1,2$, $\widetilde{\mathscr{T}}$ as in (2.5.17) and the remaining vector fields given by

$$\widetilde{V}_{\mathsf{X}} = \frac{1}{\sqrt{\xi_m(1+f)}}\,\frac{1}{\sqrt{1+\mathsf{X}^2-\mathsf{T}^2}}\left(\frac{\mathsf{X}\mathsf{T}}{\sqrt{1+\mathsf{X}^2}}\partial_{\mathsf{T}}+\sqrt{1+\mathsf{X}^2}\,\partial_{\mathsf{X}}\right)\,, \tag{2.5.18}$$

$$\widetilde{V}_I = \frac{\sqrt{g_{\mathbb{S}^2}^{II}}}{r}\,\partial_I\,. \tag{2.5.19}$$

Now, given any $\kappa_0>0$, there exists a constant $C_o=C_o(\kappa_0)$ such that (2.5.12) are satisfied.

This is shown in the estimate (A.1.4), (A.1.10), (A.1.22) and (A.1.46), following from the long computations displayed in the Appendix A.1. □

2.5.1 Bartnik's uniform interior condition via monotonicity

The region $\{\kappa_0 m < r < K_0 m\} \subset \mathfrak{L}^K_m$ of the maximally extended Schwarzschild spacetime is non-compact and, a priori, both T and the time function $\widetilde{\mathsf{T}}$ defined in Proposition 2.5.2 are unbounded. Hence, the following proposition, relying on a monotonicity argument, is necessary to ensure a relaxed version of Bartnik's uniform interior condition (2.2.35). Our weaker condition replaces the set of points $\{\kappa_0 m < r < K_0 m\} \setminus I^{\pm}(\mathrm{p})$ with $\{\kappa_0 m < r < K_0 m\} \setminus I^{\pm}_{\mathrm{p}}$ for $I^{\pm}_{\mathrm{p}}$ given in (2.5.21). It will be clear in the proof of Theorem 2.8.1 that this is the right condition to ensure the a priori height estimate.

Proposition 2.5.3. *For any radius $\kappa_0 m < 2m$ and any radius $K_0 m > 2m$, there exists a constant $C_{\kappa_0,K_0} = C_{\kappa_0,K_0}(\kappa_0, K_0)$ such that for any $\mathrm{p} \in \{r = K_0 m\} \subset \mathfrak{L}^K_m$, the following holds in the region $\{r > \kappa_0 m\} \subset \mathfrak{L}^K_m$ of the maximally extended Schwarzschild spacetime:*

$$\begin{aligned} \sup_{\mathrm{q}\in\{\kappa_0 m<r<K_0 m\}\setminus I^+_{\mathrm{p}}} \widetilde{\mathsf{T}}(\mathrm{q}) - \widetilde{\mathsf{T}}(\mathrm{p}) &\leq C_{\kappa_0,K_0} m \qquad \text{if } \widetilde{\mathsf{T}}(\mathrm{p}) \geq 0\,, \\ \sup_{\mathrm{p}\in\{\kappa_0 m<r<K_0 m\}\setminus I^-_{\mathrm{p}}} \widetilde{\mathsf{T}}(\mathrm{p}) - \widetilde{\mathsf{T}}(\mathrm{q}) &\leq C_{\kappa_0,K_0} m \qquad \text{if } \widetilde{\mathsf{T}}(\mathrm{p}) \leq 0\,. \end{aligned} \tag{2.5.20}$$

where $\widetilde{\mathsf{T}}$ is the time function defined in Proposition 2.5.2 and where we denoted

$$I^+_{\mathrm{p}} := \bigcup_{\overline{\mathrm{p}}\in\{r=K_0 m\}\cap\left\{\widetilde{\mathsf{T}}=\widetilde{\mathsf{T}}(\mathrm{p})\right\}} I^+(\overline{\mathrm{p}})\,, \qquad I^-_{\mathrm{p}} := \bigcup_{\overline{\mathrm{p}}\in\{r=K_0 m\}\cap\left\{\widetilde{\mathsf{T}}=\widetilde{\mathsf{T}}(\mathrm{p})\right\}} I^-(\overline{\mathrm{p}})\,. \tag{2.5.21}$$

Proof. For simplicity, we will work in *region I* as the computations are symmetric for *region I'*. Let us consider $\mathrm{p} \in \{r = K_0 m\} \cap \{\mathsf{X} > 0\}$ and the function ρ_m defined in (2.3.17). We will denote the constants

$$\rho_{K_0} := \left(\tfrac{K_0}{2} - 1\right) e^{\frac{K_0}{2}} = \left(\tfrac{K_0 m}{2m} - 1\right) e^{\frac{K_0 m}{2m}} = \rho_m(K_0 m) > 0\,, \qquad \rho_{\kappa_0} = \rho_m(\kappa_0 m) \in (-1, 0)$$

for $0 < \kappa_0 < \frac{3}{2}$. Let $\mathsf{T}_{\mathrm{p}} := \mathsf{T}(\mathrm{p})$, then (2.3.17) yields

$$\mathsf{X}_{\mathrm{p}} := \mathsf{X}(\mathrm{p}) = \sqrt{\mathsf{T}^2_{\mathrm{p}} + \rho_{K_0}}\,. \tag{2.5.22}$$

As a consequence, we get

$$\widetilde{\mathsf{T}}_{\mathrm{p}} := \widetilde{\mathsf{T}}(\mathrm{p}) = 4m \,\mathrm{artanh}\left(\frac{\mathsf{T}_{\mathrm{p}}}{\sqrt{1 + \mathsf{T}^2_{\mathrm{p}} + \rho_{K_0}}}\right)\,.$$

Consider then the hypersurface $\{\mathsf{T} = (\mathsf{T}(\mathrm{p}) + \mathsf{X}(\mathrm{p})) - \mathsf{X}\}$ representing the (inwards with respect to $\{r = K_0\}$) null hypersurface arising from $\{r = K_0 m\} \cap \{\widetilde{\mathsf{T}} = \widetilde{\mathsf{T}}_{\mathrm{p}}\}$. It is easy to see that the supremum of (2.5.20) is attained at the points in the intersection

$$\{\mathsf{T} = (\mathsf{T}(\mathrm{p}) + \mathsf{X}(\mathrm{p})) - \mathsf{X}\} \cap \{\mathsf{T}^2 = \mathsf{X}^2 - \rho_{\kappa_0}\}$$

for $0 < \kappa_0 < \frac{3}{2}$. We recall that $\{\mathsf{X}^2 = \mathsf{T}^2 + \rho_{\kappa_0}\} = \{r = \kappa_0 m\} = \mathrm{B}_{\kappa_0 m}$ is a barrier as in Lemma 2.3.1. Let $\overline{\mathrm{q}} \in \{\mathsf{T}^2 = \mathsf{X}^2 - \rho_{\kappa_0}\} \cap \{\mathsf{T} = (\mathsf{T}_{\mathrm{p}} + \mathsf{X}_{\mathrm{p}}) - \mathsf{X}\}$, then it follows

$$\mathsf{X}(\overline{\mathrm{q}}) = \frac{\mathsf{T}_{\mathrm{p}} + \mathsf{X}_{\mathrm{p}}}{2} + \frac{\rho_{\kappa_0}}{2\,(\mathsf{T}_{\mathrm{p}} + \mathsf{X}_{\mathrm{p}})}\,, \qquad \mathsf{T}(\overline{\mathrm{q}}) = (\mathsf{T}_{\mathrm{p}} + \mathsf{X}_{\mathrm{p}}) - \mathsf{X}(\overline{\mathrm{q}}) = \frac{\mathsf{T}_{\mathrm{p}} + \mathsf{X}_{\mathrm{p}}}{2} - \frac{\rho_{\kappa_0}}{2\,(\mathsf{T}_{\mathrm{p}} + \mathsf{X}_{\mathrm{p}})}$$

and consequently, by the definition of $\widetilde{\mathsf{T}}$ (2.5.11) and by using (2.5.22), we get

$$\begin{aligned}\widetilde{\mathsf{T}}_{\overline{\mathrm{q}}} \coloneqq \widetilde{\mathsf{T}}(\overline{\mathrm{q}}) = 4m \,\mathrm{artanh}\left(\frac{\mathsf{T}(\overline{\mathrm{q}})}{\sqrt{1 + \mathsf{X}(\overline{\mathrm{q}})^2}}\right) &= 4m\,\mathrm{artanh}\left(\frac{\mathsf{T}_{\mathrm{p}} + \mathsf{X}_{\mathrm{p}} - \frac{\rho_{\kappa_0}}{\mathsf{T}_{\mathrm{p}} + \mathsf{X}_{\mathrm{p}}}}{\sqrt{4 + \left(\mathsf{T}_{\mathrm{p}} + \mathsf{X}_{\mathrm{p}} + \frac{\rho_{\kappa_0}}{\mathsf{T}_{\mathrm{p}} + \mathsf{X}_{\mathrm{p}}}\right)^2}}\right) \\ &= 4m\,\mathrm{artanh}\left(\frac{\mathsf{T}_{\mathrm{p}} + \sqrt{\mathsf{T}_{\mathrm{p}}^2 + \rho_{K_0}} - \frac{\rho_{\kappa_0}}{\mathsf{T}_{\mathrm{p}} + \sqrt{\mathsf{T}_{\mathrm{p}}^2 + \rho_{K_0}}}}{\sqrt{4 + \left(\mathsf{T}_{\mathrm{p}} + \sqrt{\mathsf{T}_{\mathrm{p}}^2 + \rho_{K_0}} + \frac{\rho_{\kappa_0}}{\mathsf{T}_{\mathrm{p}} + \sqrt{\mathsf{T}_{\mathrm{p}}^2 + \rho_{K_0}}}\right)^2}}\right).\end{aligned}$$

Thus, the quantity we want to estimate reads

$$\begin{aligned}\widetilde{\mathsf{T}}_{\overline{\mathrm{q}}} - \widetilde{\mathsf{T}}_{\mathrm{p}} = 4m\,\mathrm{artanh}&\left(\frac{\mathsf{T}_{\mathrm{p}} + \sqrt{\mathsf{T}_{\mathrm{p}}^2 + \rho_{K_0}} - \frac{\rho_{\kappa_0}}{\mathsf{T}_{\mathrm{p}} + \sqrt{\mathsf{T}_{\mathrm{p}}^2 + \rho_{K_0}}}}{\sqrt{4 + \left(\mathsf{T}_{\mathrm{p}} + \sqrt{\mathsf{T}_{\mathrm{p}}^2 + \rho_{K_0}} + \frac{\rho_{\kappa_0}}{\mathsf{T}_{\mathrm{p}} + \sqrt{\mathsf{T}_{\mathrm{p}}^2 + \rho_{K_0}}}\right)^2}}\right) \\ &- 4m\,\mathrm{artanh}\left(\frac{\mathsf{T}_{\mathrm{p}}}{\sqrt{1 + \mathsf{T}_{\mathrm{p}}^2 + \rho_{K_0}}}\right),\end{aligned}$$

which depends only on T_{p} for fixed κ_0, K_0, corresponding to fixed constants ρ_{κ_0} and ρ_{K_0}. Let us consider the function

$$h_{\kappa_0,K_0}(x) \coloneqq \mathrm{artanh}\left(\frac{x + \sqrt{x^2 + \rho_{K_0}} - \frac{\rho_{\kappa_0}}{x + \sqrt{x^2 + \rho_{K_0}}}}{\sqrt{4 + \left(x + \sqrt{x^2 + \rho_{K_0}} + \frac{\rho_{\kappa_0}}{x + \sqrt{x^2 + \rho_{K_0}}}\right)^2}}\right) - \mathrm{artanh}\left(\frac{x}{\sqrt{1 + x^2 + \rho_{K_0}}}\right)$$

and let us show that it is monotonically decreasing. Recalling that $\frac{d}{dx}\,\mathrm{artanh}(x) = \frac{1}{1-x^2}$, we compute

$$
\begin{aligned}
&\frac{d}{dx}h_{\kappa_0,K_0}(x) = \\
&= \frac{\left(1+\frac{x}{\sqrt{x^2+\rho_{K_0}}}\right)}{1-\frac{\left(x+\sqrt{x^2+\rho_{K_0}}-\frac{\rho_{\kappa_0}}{x+\sqrt{x^2+\rho_{K_0}}}\right)^2}{4+\left(x+\sqrt{x^2+\rho_{K_0}}+\frac{\rho_{\kappa_0}}{x+\sqrt{x^2+\rho_{K_0}}}\right)^2}} \left(\frac{1+\frac{\rho_{\kappa_0}}{\left(x+\sqrt{x^2+\rho_{K_0}}\right)^2}}{\sqrt{4+\left(x+\sqrt{x^2+\rho_{K_0}}+\frac{\rho_{\kappa_0}}{x+\sqrt{x^2+\rho_{K_0}}}\right)^2}}\right. \qquad (2.5.23)\\
&\qquad \left. -\frac{\left(\left(x+\sqrt{x^2+\rho_{K_0}}\right)^2-\frac{\rho_{\kappa_0}^2}{\left(x+\sqrt{x^2+\rho_{K_0}}\right)^2}\right)\left(1-\frac{\rho_{\kappa_0}}{\left(x+\sqrt{x^2+\rho_{K_0}}\right)^2}\right)}{\left(\sqrt{4+\left(x+\sqrt{x^2+\rho_{K_0}}+\frac{\rho_{\kappa_0}}{x+\sqrt{x^2+\rho_{K_0}}}\right)^2}\right)^3}\right) \\
&\qquad -\frac{1}{1-\frac{x^2}{1+x^2+\rho_{K_0}}}\left(\frac{1}{\sqrt{1+x^2+\rho_{K_0}}}-\frac{x^2}{\left(\sqrt{1+x^2+\rho_{K_0}}\right)^3}\right) \\
&= \frac{\left(1+\frac{x}{\sqrt{x^2+\rho_{K_0}}}\right)4\left(1+\rho_{\kappa_0}\right)\left(1+\frac{\rho_{\kappa_0}}{\left(x+\sqrt{x^2+\rho_{K_0}}\right)^2}\right)}{4\left(1+\rho_{\kappa_0}\right)\sqrt{4+\left(x+\sqrt{x^2+\rho_{K_0}}+\frac{\rho_{\kappa_0}}{x+\sqrt{x^2+\rho_{K_0}}}\right)^2}} \\
&\qquad -\frac{\left(1+x^2+\rho_{K_0}\right)-x^2}{\left(1+\rho_{K_0}\right)\sqrt{1+x^2+\rho_{K_0}}} \\
&= \frac{\left(1+\frac{x}{\sqrt{x^2+\rho_{K_0}}}\right)\left(1+\frac{\rho_{\kappa_0}}{\left(x+\sqrt{x^2+\rho_{K_0}}\right)^2}\right)}{\sqrt{4+\left(x+\sqrt{x^2+\rho_{K_0}}+\frac{\rho_{\kappa_0}}{x+\sqrt{x^2+\rho_{K_0}}}\right)^2}} - \frac{1}{\sqrt{1+x^2+\rho_{K_0}}} \leq 0\,, \qquad (2.5.24)
\end{aligned}
$$

where we have used the following computations

$$
\begin{aligned}
4+&\left(x+\sqrt{x^2+\rho_{K_0}}+\frac{\rho_{\kappa_0}}{x+\sqrt{x^2+\rho_{K_0}}}\right)^2-\left(x+\sqrt{x^2+\rho_{K_0}}-\frac{\rho_{\kappa_0}}{x+\sqrt{x^2+\rho_{K_0}}}\right)^2 = \\
&= 4+\left(x+\sqrt{x^2+\rho_{K_0}}-\frac{\rho_{\kappa_0}}{x+\sqrt{x^2+\rho_{K_0}}}+2\frac{\rho_{\kappa_0}}{x+\sqrt{x^2+\rho_{K_0}}}\right)^2 \\
&\quad -\left(x+\sqrt{x^2+\rho_{K_0}}-\frac{\rho_{\kappa_0}}{x+\sqrt{x^2+\rho_{K_0}}}\right)^2
\end{aligned}
$$

$$= 4 + \left(x + \sqrt{x^2 + \rho_{K_0}} - \frac{\rho_{\kappa_0}}{x + \sqrt{x^2 + \rho_{K_0}}} \right)^2 + \frac{4\rho_{\kappa_0}^2}{\left(x + \sqrt{x^2 + \rho_{K_0}} \right)^2}$$
$$+ \frac{4\rho_{\kappa_0}}{x + \sqrt{x^2 + \rho_{K_0}}} \left(x + \sqrt{x^2 + \rho_{K_0}} - \frac{\rho_{\kappa_0}}{x + \sqrt{x^2 + \rho_{K_0}}} \right)$$
$$- \left(x + \sqrt{x^2 + \rho_{K_0}} - \frac{\rho_{\kappa_0}}{x + \sqrt{x^2 + \rho_{K_0}}} \right)^2 = 4\left(1 + \rho_{\kappa_0}\right)$$

and

$$\left(1 + \frac{\rho_{\kappa_0}}{\left(x + \sqrt{x^2 + \rho_{K_0}} \right)^2} \right) \left(x + \sqrt{x^2 + \rho_{K_0}} + \frac{\rho_{\kappa_0}}{x + \sqrt{x^2 + \rho_{K_0}}} \right)^2$$
$$- \left(1 - \frac{\rho_{\kappa_0}}{\left(x + \sqrt{x^2 + \rho_{K_0}} \right)^2} \right) \left(\left(x + \sqrt{x^2 + \rho_{K_0}} \right)^2 - \frac{\rho_{\kappa_0}^2}{\left(x + \sqrt{x^2 + \rho_{K_0}} \right)^2} \right)$$
$$= \frac{2\rho_{\kappa_0}}{\left(x + \sqrt{x^2 + \rho_{K_0}} \right)^2} \left(x + \sqrt{x^2 + \rho_{K_0}} \right)^2 + \frac{2\rho_{\kappa_0}^2}{\left(x + \sqrt{x^2 + \rho_{K_0}} \right)^2}$$
$$+ 2\rho_{\kappa_0} \left(1 + \frac{\rho_{\kappa_0}}{\left(x + \sqrt{x^2 + \rho_{K_0}} \right)^2} \right)$$
$$= 4\rho_{\kappa_0} \left(1 + \frac{\rho_{\kappa_0}}{\left(x + \sqrt{x^2 + \rho_{K_0}} \right)^2} \right) .$$

The quantity (2.5.23) is non-positive, since using

$$\left(1 + \frac{x}{\sqrt{x^2 + \rho_{K_0}}} \right) \left(1 + \frac{\rho_{\kappa_0}}{\left(x + \sqrt{x^2 + \rho_{K_0}} \right)^2} \right) =$$
$$= \frac{1}{\sqrt{x^2 + \rho_{K_0}}} \left(x + \sqrt{x^2 + \rho_{K_0}} + \frac{\rho_{\kappa_0}}{x + \sqrt{x^2 + \rho_{K_0}}} \right) ,$$

one can check that the last inequality in (2.5.23) is equivalent to

$$1 + x^2 + \rho_{K_0} \leq \left(x^2 + \rho_{K_0} \right) \frac{4 + \left(x + \sqrt{x^2 + \rho_{K_0}} + \frac{\rho_{\kappa_0}}{x + \sqrt{x^2 + \rho_{K_0}}} \right)^2}{\left(x + \sqrt{x^2 + \rho_{K_0}} + \frac{\rho_{\kappa_0}}{x + \sqrt{x^2 + \rho_{K_0}}} \right)^2}$$

$$= x^2 + \rho_{K_0} + \frac{4(x^2+\rho_{K_0})}{\left(x + \sqrt{x^2+\rho_{K_0}} + \frac{\rho_{\kappa_0}}{x+\sqrt{x^2+\rho_{K_0}}}\right)^2}$$
$$= x^2 + \rho_{K_0} + \frac{4}{\left(1 + \frac{1}{\sqrt{x^2+\rho_{K_0}}}\left(x + \frac{\rho_{\kappa_0}}{x+\sqrt{x^2+\rho_{K_0}}}\right)\right)^2}.$$

Finally, we see that

$$\frac{4}{\left(1 + \frac{1}{\sqrt{x^2+\rho_{K_0}}}\left(x + \frac{\rho_{\kappa_0}}{x+\sqrt{x^2+\rho_{K_0}}}\right)\right)^2} \geq \lim_{x\to\infty} \frac{4}{\left(1 + \frac{1}{\sqrt{x^2+\rho_{K_0}}}\left(x + \frac{\rho_{\kappa_0}}{x+\sqrt{x^2+\rho_{K_0}}}\right)\right)^2} = 1$$

in turn, given by the fact that the function $\frac{1}{\sqrt{x^2+\rho_{K_0}}}\left(x + \frac{\rho_{\kappa_0}}{x+\sqrt{x^2+\rho_{K_0}}}\right)$ is non-negative and has derivative

$$\frac{d}{dx}\left(\frac{x}{\sqrt{x^2+\rho_{K_0}}} + \frac{\rho_{\kappa_0}}{x^2+\rho_{K_0}+x\sqrt{x^2+\rho_{K_0}}}\right) =$$
$$= \frac{\rho_{K_0}}{\left(\sqrt{x^2+\rho_{K_0}}\right)^3} - \frac{\rho_{\kappa_0}\left(2x + \sqrt{x^2+\rho_{K_0}} + \frac{x^2}{\sqrt{x^2+\rho_{K_0}}}\right)}{x^2+\rho_{K_0}+x\sqrt{x^2+\rho_{K_0}}} > 0$$

since $\rho_{K_0} > 0, \rho_{\kappa_0} < 0$.

Now, the monotonicity of h_{κ_0,K_0} given by the computation (2.5.23) yields

$$\widetilde{\mathsf{T}}_{\overline{\mathrm{q}}} - \widetilde{\mathsf{T}}_{\mathrm{p}} =$$
$$= 4m\,\mathrm{artanh}\left(\frac{\mathsf{T}_{\mathrm{p}} + \sqrt{\mathsf{T}_{\mathrm{p}}^2+\rho_{K_0}} - \frac{\rho_{\kappa_0}}{\mathsf{T}_{\mathrm{p}}+\sqrt{\mathsf{T}_{\mathrm{p}}^2+\rho_{K_0}}}}{\sqrt{4 + \left(\mathsf{T}_{\mathrm{p}} + \sqrt{\mathsf{T}_{\mathrm{p}}^2+\rho_{K_0}} + \frac{\rho_{\kappa_0}}{\mathsf{T}_{\mathrm{p}}+\sqrt{\mathsf{T}_{\mathrm{p}}^2+\rho_{K_0}}}\right)^2}}\right) - 4m\,\mathrm{artanh}\left(\frac{\mathsf{T}_{\mathrm{p}}}{\sqrt{1+\mathsf{T}_{\mathrm{p}}^2+\rho_{K_0}}}\right)$$
$$\leq \lim_{\mathsf{T}_{\mathrm{p}}\to 0} 4m\left(\mathrm{artanh}\left(\frac{\mathsf{T}_{\mathrm{p}} + \sqrt{\mathsf{T}_{\mathrm{p}}^2+\rho_{K_0}} - \frac{\rho_{\kappa_0}}{\mathsf{T}_{\mathrm{p}}+\sqrt{\mathsf{T}_{\mathrm{p}}^2+\rho_{K_0}}}}{\sqrt{4 + \left(\mathsf{T}_{\mathrm{p}} + \sqrt{\mathsf{T}_{\mathrm{p}}^2+\rho_{K_0}} + \frac{\rho_{\kappa_0}}{\mathsf{T}_{\mathrm{p}}+\sqrt{\mathsf{T}_{\mathrm{p}}^2+\rho_{K_0}}}\right)^2}}\right) - \mathrm{artanh}\left(\frac{\mathsf{T}_{\mathrm{p}}}{\sqrt{1+\mathsf{T}_{\mathrm{p}}^2+\rho_{K_0}}}\right)\right)$$
$$= 4m\,\mathrm{artanh}\left(\frac{\rho_{K_0} - \rho_{\kappa_0}}{\sqrt{4\rho_{K_0} + (\rho_{K_0}+\rho_{\kappa_0})^2}}\right) = 4m\,\mathrm{artanh}\left(\sqrt{\frac{\rho_{K_0}\left(1 - \frac{\rho_{\kappa_0}}{\rho_{K_0}}\right)^2}{4 + \rho_{K_0}\left(1 + \frac{\rho_{\kappa_0}}{\rho_{K_0}}\right)^2}}\right).$$

Now, defining $C_{\kappa_0,K_0} := 4\,\mathrm{artanh}\left(\sqrt{\frac{\rho_{K_0}\left(1-\frac{\rho_{\kappa_0}}{\rho_{K_0}}\right)^2}{4+\rho_{K_0}\left(1+\frac{\rho_{\kappa_0}}{\rho_{K_0}}\right)^2}}\right)$, we conclude.

□

2.6 Barriers in the asymptotically flat end

As aforementioned, to produce a maximal surface in the class $\mathscr{B}_{\mathrm{b}}$, we first need choose an hypersurface $\mathscr{S}_{\mathrm{b}} \in \mathscr{B}_{\mathrm{b}}$ and construct upper and lower barriers converging to it at infinity. We will show that the freedom of the order $O(\log(|\,\boldsymbol{x}\,|))$ in Definition 2.4.1 enables to fulfill this objective, obtaining the following result, which is also consistent with the outcomes of [BCO90, Lemma 2.3].

Proposition 2.6.1. *For any* $\mathrm{b} \in [0,1)$, *there exists a spacelike hypersurface* $\mathscr{S}_{\mathrm{b}} \in \mathscr{B}_{\mathrm{b}}$ *and a function* $\bar{f}_{\mathrm{b}} \in C^\infty(\mathbb{R}^3 \setminus \overline{B}_1)$ *satisfying*

$$\sup_{\boldsymbol{y} \in \mathbb{R}^3 \setminus \overline{B}_1} \left(\frac{|\bar{f}_{\mathrm{b}}(\boldsymbol{y})|}{\log(|\,\boldsymbol{y}\,|)} + |\,\boldsymbol{y}\,| \sum_{i=1}^{3} |\partial_i \bar{f}_{\mathrm{b}}(\boldsymbol{y})| + |\,\boldsymbol{y}\,|^2 \sum_{i,j=1}^{3} |\partial^2_{ij} \bar{f}_{\mathrm{b}}(\boldsymbol{y})| + |\,\boldsymbol{y}\,|^3 \sum_{i,j,k=1}^{3} |\partial^3_{ijk} \bar{f}_{\mathrm{b}}(\boldsymbol{y})| \right) = C_{\bar{f}_{\mathrm{b}}} \tag{2.6.1}$$

for some constant $0 < C_{\bar{f}_{\mathrm{b}}} < \infty$ *such that, in isotropic coordinates,*

$$\mathscr{S}_{\mathrm{b}} \cap \{|\,\boldsymbol{x}\,| > \bar{K} m\} = \{t = u_{\mathrm{b}}(\boldsymbol{x})\}\,, \qquad u_{\mathrm{b}}(\boldsymbol{x}) \coloneqq \mathrm{b}\, x_1 + m \bar{f}_{\mathrm{b}}\left(\frac{\boldsymbol{x}}{m}\right) \tag{2.6.2}$$

for some constant $\bar{K} \geq 1$ *and such that it admits upper and lower barriers. More precisely, for any* $0 < \varepsilon < 1$, *there exists* $K_{\mathrm{b}} = K_{\mathrm{b}}(\mathrm{b}, \varepsilon, C_{\bar{f}_{\mathrm{b}}}) > \bar{K}$ *such that for any* $K \geq K_{\mathrm{b}}$ *there exist hypersurfaces* $\mathscr{W}^{\pm}_{\mathrm{b},K,\varepsilon}$ *satisfying*

$$\begin{gathered} \mathscr{W}^{\pm}_{\mathrm{b},K,\varepsilon} = \{t = w^{\pm}_{\mathrm{b},K,\varepsilon}(\boldsymbol{x})\} \ \ \textit{spacelike on} \ \ \{|\,\boldsymbol{x}\,| \geq Km\} \\ \mathrm{H}_{\mathfrak{g}^I_m}(\mathscr{W}^{+}_{\mathrm{b},K,\varepsilon}) < 0\,,\ \ 0 < w^{+}_{\mathrm{b},K,\varepsilon} - u_{\mathrm{b}} = O(|\,\boldsymbol{x}\,|^{-\varepsilon})\,,\ \ (w^{+}_{\mathrm{b},K,\varepsilon} - u_{\mathrm{b}})\big|_{\{|\,\boldsymbol{x}\,|=Km\}} \geq K^\varepsilon\, m \sqrt{1-\mathrm{b}^2} \\ \mathrm{H}_{\mathfrak{g}^I_m}(\mathscr{W}^{-}_{\mathrm{b},K,\varepsilon}) > 0\,,\ \ 0 < u_{\mathrm{b}} - w^{-}_{\mathrm{b},K,\varepsilon} = O(|\,\boldsymbol{x}\,|^{-\varepsilon})\,,\ \ (u_{\mathrm{b}} - w^{-}_{\mathrm{b},K,\varepsilon})\big|_{\{|\,\boldsymbol{x}\,|=Km\}} \geq K^\varepsilon\, m \sqrt{1-\mathrm{b}^2} \end{gathered} \tag{2.6.3}$$

with $\left|\mathrm{H}_{\mathfrak{g}^I_m}(w^{\pm}_{\mathrm{b},K,\varepsilon}) \pm \frac{K^{2\varepsilon}\varepsilon(1-\varepsilon)m^{1+\varepsilon}}{|\,\boldsymbol{z}(\boldsymbol{x})|^{2+\varepsilon}}\right| \leq \frac{C_{\mathrm{b}}(1+K^{2\varepsilon})}{|\,\boldsymbol{x}\,|^3}$ *as* $|\,\boldsymbol{x}\,| \to \infty$, *for* C_{b} *is independent of* K.

The proof will be presented in Section 2.6.1, as we first want to discuss the difficulties arising.

We recall that in [Bar84, Section 5] barriers over a time slice in the asymptotically flat end are constructed by considering rotationally symmetric hypersurfaces in Minkowski with mean curvature decaying as $O(r^{-3})$ and estimating their mean curvature when considered in the asymptotically flat end. The estimates are carried assuming $\mathrm{H}^o = O(r^{-3})$, which can and needs to be relaxed for our purposes. We will start with a discussion on the Minkowskian rotationally symmetric surfaces and on the required decay of H^o in order for these surfaces to produce barriers in the asymptotically flat end, so that our choices are heuristically justified.

Rotationally symmetric hypersurfaces in Minkowski. Imposing rotational symmetry $\bar{w} = \bar{w}(r)$, the mean curvature equation in Minkowski reads

$$\mathrm{H}_\eta(\bar{w}) = r^{-2}\,\partial_r\left(\frac{r^2\,\partial_r\bar{w}}{\sqrt{1-(\partial_r\bar{w})^2}}\right) \tag{2.6.4}$$

and aiming to find a solution decaying as $O(r^{-\varepsilon})$ for some $0 < \varepsilon < 1$, one can set $\mathrm{H}_\eta(w) = \pm\frac{\Lambda}{r^{2+\varepsilon}}$ given some constant $\Lambda > 0$, while in [Bar84, Section 5] Bartnik sets it to $\pm\frac{\Lambda}{r^3}$. It is easy to show that the family of solutions is given by

$$\bar{w}^{\pm}_{\Lambda,L,c,\varepsilon}(r) = c + \int_r^\infty \frac{L \pm \Lambda s^{1-\varepsilon}}{\sqrt{s^4 + (L \pm \Lambda s^{1-\varepsilon})^2}}\,\mathrm{d}s \;= O(r^{-\varepsilon})$$

with c, L being integration constants. To simplify the computations, we can consider directly for $\Lambda > 0$ the positive function

$$w^+_\Lambda(r) \coloneqq \Lambda r^{-\varepsilon}, \qquad \partial_r w^+_\Lambda(r) = -\Lambda\varepsilon\, r^{-1-\varepsilon}, \qquad \partial_r^2 w^+_\Lambda(r) = \Lambda(1+\varepsilon)\varepsilon\, r^{-2-\varepsilon} \tag{2.6.5}$$

which is spacelike for sufficiently large radii depending on Λ and ε. By (2.6.4) its mean curvature is given by

$$\begin{aligned}
\mathrm{H}_\eta(w^+_\Lambda) = &= \frac{1}{\sqrt{1-(\partial_r w^+_\Lambda)^2}}\left[\frac{\partial_r^2 w^+_\Lambda}{1-(\partial_r w^+_\Lambda)^2} + \frac{2\,\partial_r w^+_\Lambda}{r}\right] \\
&= \left(1 + O\left(r^{-1-\varepsilon}\right)\right)\left(\Delta_{\delta_{\mathbb{R}^3}} w^+_\Lambda + O\left(r^{-3-2\varepsilon}\right)\right) \\
&= \Delta_{\delta_{\mathbb{R}^3}} w^+_\Lambda + O\left(r^{-3-2\varepsilon}\right) < 0
\end{aligned}$$

where we used that $\Delta_{\delta_{\mathbb{R}^3}} = \partial_r^2 + \frac{2}{r}\partial_r$. The sign is negative for sufficiently big radii, since it holds

$$\Delta_{\delta_{\mathbb{R}^3}} w^+_\Lambda = -\frac{\Lambda\varepsilon(1-\varepsilon)}{r^{2+\varepsilon}} < 0 \tag{2.6.6}$$

The corresponding negative function with positive mean curvature is simply

$$w^-_\Lambda(r) \coloneqq -w^+_\Lambda(r)\,, \qquad \mathrm{H}_\eta(w^-_\Lambda) = -\,\mathrm{H}_\eta(w^+_\Lambda) > 0\,. \tag{2.6.7}$$

The mean curvature of a graph over an asymptotically flat time slice. Expanding and modifying the estimates in [Bar84, Section 5], we recover the mean curvature of the graphs of $w^{\pm}_\Lambda$ as in (2.6.5) and (2.6.7) in an asymptotically flat spacetime (computed with respect to the radial coordinates of the asymptotically flat chart). It will then be evident which conditions we need to require for the mean curvature H^o of the time slices.

Consider an asymptotically flat spacetime as in Definition 2.2.3 with time function

t and with the metric $\mathfrak{g}$ reading in local coordinates (t, y^i) as (2.2.7), so that

$$\alpha_t = 1 + O(|\,\boldsymbol{y}\,|^{-1}), \qquad \beta^i = O(|\,\boldsymbol{y}\,|^{-1}), \qquad {}^t g_{ij} = \delta_{ij} + O(|\,\boldsymbol{y}\,|^{-1})\,. \tag{2.6.8}$$

Given the function (2.6.5) computed as $w := w^+_\Lambda(|\,\boldsymbol{y}\,|)$ and setting $r = |\,\boldsymbol{y}\,|$, we can estimate $\mathrm{H}_{\mathfrak{g}}(w)$ as given in (2.2.19). First, let us recall the quantity $\boldsymbol{v}_t(w)$ defined in (2.2.12), which reads

$$\boldsymbol{v}_t(w) = \frac{1}{\sqrt{1 - |\mathrm{W}|^2_{{}^t g}}}\,, \qquad \mathrm{W} = \frac{\alpha_t\, {}^t\mathrm{D}w}{1 + \langle \beta, {}^t\mathrm{D}w\rangle}.$$

Using that $\delta^{ij}\,\partial_i w\,\partial_j w = (\partial_r w)^2$ we estimate

$$\begin{aligned}\boldsymbol{v}_t(w) =& \frac{1}{\sqrt{1 - \frac{\alpha^2\,(\delta^{ij} + O(|\,\boldsymbol{y}\,|^{-1}))\partial_i w\,\partial_j w}{1 + \partial_r w\, O(|\,\boldsymbol{y}\,|^{-1})}}} = \frac{1}{\sqrt{1 - (\partial_r w)^2\,(1 + O(|\,\boldsymbol{y}\,|^{-1}))(1 + O(|\,\boldsymbol{y}\,|^{-2-\varepsilon}))}}\\ =& \frac{1}{\sqrt{1 - (\partial_r w)^2}} + O(|\,\boldsymbol{y}\,|^{-1}))\,.\end{aligned} \tag{2.6.9}$$

In addition, we have

$$\begin{aligned}\langle \mathrm{W},\, \overline{\nabla}_{\mathscr{T}}\mathscr{T}\rangle =& \left\langle \frac{\alpha\,\mathrm{D}w}{1 + \langle \beta,\,\mathrm{D}w\rangle},\, \frac{1}{\alpha}\overline{\nabla}_{(\partial_t - \beta)}\left(\frac{1}{\alpha}(\partial_t - \beta)\right)\right\rangle\\ =& \left\langle \partial_r w\,(1 + O(|\,\boldsymbol{y}\,|^{-1}))\,\frac{y^i}{|\,\boldsymbol{y}\,|}\,\partial_i,\; \frac{-\left(\partial_t\alpha - \beta^k\partial_k\alpha\right)}{\alpha^2}\,(\partial_t - \beta) + O(|\,\boldsymbol{y}\,|^{-2})\right\rangle\\ =& \,O(|\,\boldsymbol{y}\,|^{-3-\varepsilon})\,,\end{aligned} \tag{2.6.10}$$

where we used that the derivative of $\bar{g}$ and consequently the Christoffel symbols of the metric $\mathfrak{g}$ are $O(|\,\boldsymbol{y}\,|^{-2})$. Similarly, we can compute

$$\begin{aligned}\mathscr{T}(|\mathrm{W}|^2_{{}^t g}) =& \frac{1}{\alpha}\left(\partial_t - \beta^k\partial_k\right)\left(\delta^{ij} + O(|\,\boldsymbol{y}\,|^{-1})\right)\partial_i w\,\partial_j w\\ =& \,O(|\,\boldsymbol{y}\,|^{-1})\,\left(2\partial_r w\,\partial^2_r w + O(|\,\boldsymbol{y}\,|^{-1})\right)\\ =& \,O(|\,\boldsymbol{y}\,|^{-4-\varepsilon})\,.\end{aligned} \tag{2.6.11}$$

Finally, we estimate the following

$$\begin{aligned}\mathrm{div}^o\,(\mathrm{W}\,\boldsymbol{v}_t(w)) =& \,\mathrm{div}^o\left(\frac{\alpha_t\,{}^t\mathrm{D}w}{1 + \langle \beta,\,{}^t\mathrm{D}w\rangle}\left(\frac{1}{\sqrt{1 - (\partial_r w)^2}} + O_1(|\,\boldsymbol{y}\,|^{-1}))\right)\right)\\ =& \frac{1}{\sqrt{\det{}^t g}}\,\partial_i\left(\frac{\sqrt{\det g}\,\alpha_t\,g^{ij}\partial_j w}{1 + \beta_k\,g^{kl}\,\partial_l w}\left(\frac{1}{\sqrt{1 - (\partial_r w)^2}} + O(|\,\boldsymbol{y}\,|^{-1}))\right)\right)\\ =& \,(1 + O_2(|\,\boldsymbol{y}\,|^{-1}))\,\partial_i\Bigg((1 + O(|\,\boldsymbol{y}\,|^{-2-\varepsilon}))\end{aligned}$$

$$
\begin{aligned}
&(\delta^{ij} + O(|\,\boldsymbol{y}\,|^{-1}))\,\partial_j w \left(\frac{1}{\sqrt{1-(\partial_r w)^2}} + O(|\,\boldsymbol{y}\,|^{-1}) \right) \Bigg) \\
&= \partial_i \left(\frac{\delta^{ij}\partial_j w}{\sqrt{1-(\partial_r w)^2}} \right) + \partial_i \left(\delta^{ij}\partial_j w\, O(|\,\boldsymbol{y}\,|^{-1}) \right) + \\
&\qquad \partial_i \left(\frac{\partial_j w}{\sqrt{1-(\partial_r w)^2}}\, O|\,\boldsymbol{y}\,|^{-1}) \right) + \partial_i (O(|\,\boldsymbol{y}\,|^{-1}))\, O(|\,\boldsymbol{y}\,|^{-1}) \\
&= \mathrm{H}_\eta(w) + O(|\,\boldsymbol{y}\,|^{-3-\varepsilon}) + O(|\,\boldsymbol{y}\,|^{-3})\,. \qquad (2.6.12)
\end{aligned}
$$

Combining now the estimates (2.6.9), (2.6.10), (2.6.11) and (2.6.12) in (2.2.19) and plugging in the identity found in (2.6.6), we obtain

$$
\begin{aligned}
\mathrm{H}_{\mathfrak{g}}(w) &= \mathrm{H}_\eta(w) + \frac{\mathrm{H}^o}{\sqrt{1-(\partial_r w)^2}} + O(|\,\boldsymbol{y}\,|^{-3}) \\
&= -\frac{\Lambda\varepsilon(1-\varepsilon)}{|\,\boldsymbol{y}\,|^{2+\varepsilon}} + \mathrm{H}^o\,(1 + O(|\,\boldsymbol{y}\,|^{-1-\varepsilon})) + O(|\,\boldsymbol{y}\,|^{-3})\,. \qquad (2.6.13)
\end{aligned}
$$

From this computation, we immediately notice that in order graph w to be an upper barrier at the spatial infinity, having strictly negative mean curvature, we need to ensure that

$$
\mathrm{H}^o = O(|\,\boldsymbol{y}\,|^{-2-\varepsilon}) \qquad \text{as } |\,\boldsymbol{y}\,| \to \infty\,. \qquad (2.6.14)
$$

This decay condition on the mean curvature of the time slices will rule how to choose a coordinate-dependent boosted slice $\mathscr{S}_{\mathrm{b}} \in \mathscr{B}_{\mathrm{b}}$.

The mean curvature of $\{t = \mathrm{b}\,x_1\}$ in isotropic coordinates. Let us consider the spacelike hypersurface given by the graph of

$$
f_{\mathrm{b}}(\boldsymbol{x}) = \mathrm{b}\,x_1\,, \qquad \text{satisfying} \quad \partial_i f_{\mathrm{b}}(\boldsymbol{x}) = \mathrm{b}\,\delta_{i1}, \quad \partial^2_{ij} f_{\mathrm{b}}(\boldsymbol{x}) = 0 \qquad (2.6.15)
$$

in isotropic coordinates (2.3.10). First, recall that in these coordinates, on $\{|\,\boldsymbol{x}\,| \geq m\}$ there holds

$$
\begin{aligned}
&0 < 1 - N_m = 1 - 1 + \frac{m}{|\,\boldsymbol{x}\,|\left(1 + \frac{m}{2|\,\boldsymbol{x}\,|}\right)} \leq \frac{m}{|\,\boldsymbol{x}\,|}\,, \\
&0 < \varphi_m - 1 = \frac{m}{2|\,\boldsymbol{x}\,|} \\
&|{}^t g_{ij} - \delta_{ij}| \leq |\varphi_m^4 - 1| = \frac{2m}{|\,\boldsymbol{x}\,|}\left(1 + \frac{3m}{4|\,\boldsymbol{x}\,|} + \frac{m^2}{4|\,\boldsymbol{x}\,|^2} + \frac{m^3}{32|\,\boldsymbol{x}\,|^3}\right) \leq \frac{5m}{|\,\boldsymbol{x}\,|}, \\
&|{}^t g^{ij} - \delta^{ij}| \leq |\varphi_m^{-4} - 1| = \frac{|\varphi_m^4 - 1|}{\varphi_m^4} \leq \frac{5m}{|\,\boldsymbol{x}\,|}
\end{aligned}
\qquad (2.6.16)
$$

and

$$\left|\partial_i N_m - \frac{m x_i}{|\,\boldsymbol{x}\,|^3}\right| = \left|\frac{x_i}{|\,\boldsymbol{x}\,|}\partial_r N_m - \frac{m x_i}{|\,\boldsymbol{x}\,|^3}\right| \le \left|\frac{\frac{m}{2|\,\boldsymbol{x}\,|^2}}{1+\frac{m}{2|\,\boldsymbol{x}\,|}} - \frac{\left(1-\frac{m}{2|\,\boldsymbol{x}\,|}\right)\left(-\frac{m}{2|\,\boldsymbol{x}\,|^2}\right)}{\left(1+\frac{m}{2|\,\boldsymbol{x}\,|}\right)^2} - \frac{m}{|\,\boldsymbol{x}\,|^2}\right|$$

$$= \left|\frac{m}{|\,\boldsymbol{x}\,|^2\left(1+\frac{m}{2|\,\boldsymbol{x}\,|}\right)^2} - \frac{m}{|\,\boldsymbol{x}\,|^2}\right| \le \frac{m^2}{|\,\boldsymbol{x}\,|^3}\left(1+\frac{m}{4|\,\boldsymbol{x}\,|}\right) \le \frac{5m^2}{4|\,\boldsymbol{x}\,|^3}\,,$$

$$\partial_i \varphi_m = \frac{x_i}{|\,\boldsymbol{x}\,|}\partial_r \varphi_m = -\frac{m x_i}{2|\,\boldsymbol{x}\,|^3}\,, \tag{2.6.17}$$

and the Christoffel symbols of the metric ${}^t g$ on the time slices read

$$\begin{aligned} {}^t\Gamma^k_{ij} &= \frac{\varphi_m^{-4}}{2}\delta^{kl}\left(\partial_i(\varphi_m^4\delta_{jl}) + \partial_j(\varphi_m^4\delta_{il}) - \partial_l(\varphi_m^4\delta_{ij})\right) \\ &= 2\varphi_m^{-1}\,\delta^{kl}\left(\partial_i\varphi_m\,\delta_{jl} + \partial_j\varphi_m\,\delta_{il} - \partial_l\varphi_m\,\delta_{ij}\right) \\ &= -\frac{m\,\varphi_m^{-1}}{|\,\boldsymbol{x}\,|^3}\left(x_i\,\delta^k_j + x_j\,\delta^k_i - x^k\,\delta_{ij}\right) \end{aligned}$$

and hence can be estimated by

$$\left|{}^t\Gamma^k_{ij} + \frac{m}{|\,\boldsymbol{x}\,|^3}\left(x_i\delta^k_j + x_j\delta^k_i - x^k\delta_{ij}\right)\right| \le \frac{3m}{|\,\boldsymbol{x}\,|^2}\frac{|\varphi_m - 1|}{\varphi_m} \le \frac{3m^2}{2|\,\boldsymbol{x}\,|^3} \tag{2.6.18}$$

Now we can estimate

$$\begin{aligned} \left|\boldsymbol{v}_t(f_\mathrm{b}) - \frac{1}{\sqrt{1-\mathrm{b}^2}}\right| &= \left|\frac{1}{\sqrt{1-\frac{N_m^2(|\,\boldsymbol{x}\,|)}{\varphi_m^4(|\,\boldsymbol{x}\,|)}\,\mathrm{b}^2}} - \frac{1}{\sqrt{1-\mathrm{b}^2}}\right| \\ &= \frac{1}{\sqrt{1-\mathrm{b}^2}}\left(1-\sqrt{1-\mathrm{b}^2\,\frac{1-\frac{N_m^2(|\,\boldsymbol{x}\,|)}{\varphi_m^4(|\,\boldsymbol{x}\,|)}}{1-\frac{N_m^2(|\,\boldsymbol{x}\,|)}{\varphi_m^4(|\,\boldsymbol{x}\,|)}\,\mathrm{b}^2}}\right) \\ &\le \frac{\mathrm{b}^2}{\sqrt{1-\mathrm{b}^2}}\frac{1-\frac{N_m^2(|\,\boldsymbol{x}\,|)}{\varphi_m^4(|\,\boldsymbol{x}\,|)}}{1-\frac{N_m^2(|\,\boldsymbol{x}\,|)}{\varphi_m^4(|\,\boldsymbol{x}\,|)}\,\mathrm{b}^2} \le \frac{\mathrm{b}^2}{(1-\mathrm{b}^2)^{\frac{3}{2}}}\frac{8m}{|\,\boldsymbol{x}\,|} \end{aligned}$$

where we used that

$$\begin{aligned} 0 < 1 - \frac{N_m^2(|\,\boldsymbol{x}\,|)}{\varphi_m^4(|\,\boldsymbol{x}\,|)} &\le 1 - \varphi_m^{-4}\left(1 - \frac{2m}{|\,\boldsymbol{x}\,|\varphi_m(|\,\boldsymbol{x}\,|)} + \frac{m^2}{|\,\boldsymbol{x}\,|^2\varphi_m^2(|\,\boldsymbol{x}\,|)}\right) \\ &\le |1-\varphi_m^{-4}| + \frac{3m}{|\,\boldsymbol{x}\,|} \le \frac{8m}{|\,\boldsymbol{x}\,|}\,. \end{aligned}$$

Then we can estimate the different terms of the formula (2.3.15) as follows

$$\begin{aligned}
&\left|\upsilon_t(f_{\mathrm{b}})\left(1+\upsilon_t^2(f_{\mathrm{b}})\right)\mathrm{d}N_m({}^t\mathrm{D}f_{\mathrm{b}})-\frac{\mathrm{b}m}{\sqrt{1-\mathrm{b}^2}}\left(1+\frac{1}{1-\mathrm{b}^2}\right)\frac{x_1}{|\boldsymbol{x}|^3}\right|\\
&\qquad=\left|\upsilon_t(f_{\mathrm{b}})\left(1+\upsilon_t^2(f_{\mathrm{b}})\right)\varphi_m^{-4}\,\partial_1 N_m\,\mathrm{b}-\frac{\mathrm{b}m}{\sqrt{1-\mathrm{b}^2}}\left(1+\frac{1}{1-\mathrm{b}^2}\right)\frac{x_1}{|\boldsymbol{x}|^3}\right|\\
&\qquad\leq C_{\mathrm{b}}\frac{m^2}{|\boldsymbol{x}|^3}
\end{aligned}\tag{2.6.19}$$

$$\begin{aligned}
&\left|-\upsilon_t(f_{\mathrm{b}})N_m\,{}^tg^{ij}\,{}^t\Gamma_{ij}^k\partial_k f_{\mathrm{b}}+\frac{\mathrm{b}m}{\sqrt{1-\mathrm{b}^2}}\frac{x_1}{|\boldsymbol{x}|^3}\right|\\
&\qquad\leq\frac{\mathrm{b}\,m}{\sqrt{1-\mathrm{b}^2}}\left|\frac{\delta^{ij}}{|\boldsymbol{x}|^3}\left(x_i\delta_j^1+x_j\delta_i^1-x^1\delta_{ij}\right)+\frac{x_1}{|\boldsymbol{x}|^3}\right|+C_{\mathrm{b}}\frac{m^2}{|\boldsymbol{x}|^3}\\
&\qquad=C_{\mathrm{b}}\frac{m^2}{|\boldsymbol{x}|^3}
\end{aligned}\tag{2.6.20}$$

$$\begin{aligned}
&\left|-\upsilon_t^3(f_{\mathrm{b}})N_m^3\,{}^t\mathrm{D}^if_{\mathrm{b}}\,{}^t\mathrm{D}^jf_{\mathrm{b}}\,{}^t\Gamma_{ij}^k\partial_kf_{\mathrm{b}}-\frac{\mathrm{b}^3m}{(1-\mathrm{b}^2)^{\frac{3}{2}}}\frac{x_1}{|\boldsymbol{x}|^3}\right|\\
&\qquad\leq\frac{\mathrm{b}^3m}{(1-\mathrm{b}^2)^{\frac{3}{2}}}\left|\frac{\delta^{i1}\delta^{j1}\delta_{k1}}{|\boldsymbol{x}|^3}\left(x_i\delta_j^1+x_j\delta_i^1-x^1\delta_{ij}\right)-\frac{x_1}{|\boldsymbol{x}|^3}\right|+C_{\mathrm{b}}\frac{m^2}{|\boldsymbol{x}|^3}\\
&\qquad=C_{\mathrm{b}}\frac{m^2}{|\boldsymbol{x}|^3}
\end{aligned}\tag{2.6.21}$$

where C_{b} is a constant depending only on b. Combining (2.6.19), (2.6.20) and (2.6.21) in (2.3.15), we obtain that the mean curvature of $\{t=f_{\mathrm{b}}(\boldsymbol{x})\}$ can be estimated by

$$\left|\mathrm{H}_{\mathfrak{g}_m^I}(f_{\mathrm{b}})-\frac{\mathrm{b}(1+\mathrm{b}^2)m}{(1-\mathrm{b}^2)^{\frac{3}{2}}}\frac{x_1}{|\boldsymbol{x}|^3}\right|\leq C_{\mathrm{b}}\frac{m^2}{|\boldsymbol{x}|^3}\,,\tag{2.6.22}$$

which then reads $\mathrm{H}_{\mathfrak{g}_m^I}(f_{\mathrm{b}})=O(|\boldsymbol{x}|^{-2})$. Now, to be consistent with the discussion of the previous paragraph, we introduce coordinates $(\widetilde{t},\widetilde{\boldsymbol{x}})$ so that $\{t=f_{\mathrm{b}}(\boldsymbol{x})\}=\{\widetilde{t}=0\}$ and we show that the metric is still asymptotically flat in the sense of Definition 2.2.3. Considering the same change of coordinates that corresponds to isometries in Minkowski (2.2.37), the Schwarzschild metric reads

$$\mathfrak{g}_m^I|_{(\widetilde{t},\widetilde{\boldsymbol{x}})}=-\frac{N_m^2-\mathrm{b}^2\,\varphi_m^4}{1-\mathrm{b}^2}\mathrm{d}\widetilde{t}^2+\frac{2\,\mathrm{b}(-N_m^2+\varphi_m^4)}{1-\mathrm{b}^2}\mathrm{d}\widetilde{t}\mathrm{d}\widetilde{x}_1+\varphi_m^4\left(\mathrm{d}\widetilde{x}_2^2+\mathrm{d}\widetilde{x}_3^2\right)+\frac{-\mathrm{b}^2\,N_m^2+\varphi_m^4}{1-\mathrm{b}^2}\mathrm{d}\widetilde{x}_1^2.$$

Now we notice that since the coordinate transformation (2.2.37) are linear, it holds $O_k(|\boldsymbol{x}|^{-l})=O_k(|\widetilde{\boldsymbol{x}}|^{-l})$. These coordinates still capture the asymptotical flatness of Schwarzschild with the same rate of decay, as it is clear from the following computations

$$\frac{N_m^2-\mathrm{b}^2\,\varphi_m^4}{1-\mathrm{b}^2}=\frac{1+O(|\boldsymbol{x}|^{-1})-\mathrm{b}^2\,(1+O(|\boldsymbol{x}|^{-1}))}{1-\mathrm{b}^2}$$

$$=\frac{1-\mathrm{b}^2}{1-\mathrm{b}^2}+O(|\,\boldsymbol{x}\,|^{-1}) = 1+O(|\,\widetilde{\boldsymbol{x}}\,|^{-1})$$

$$\frac{2\,\mathrm{b}}{1-\mathrm{b}^2}(-N_m^2+\varphi_m^4) = \frac{2\,\mathrm{b}}{1-\mathrm{b}^2}(-1+O(|\,\boldsymbol{x}\,|^{-1})+1++O(|\,\boldsymbol{x}\,|^{-1})) = O(|\,\widetilde{\boldsymbol{x}}\,|^{-1})$$

$$\varphi_m^4 = 1+O(|\,\widetilde{\boldsymbol{x}}\,|^{-1})$$

$$\frac{-\,\mathrm{b}^2\,N_m^2+\varphi_m^4}{1-\mathrm{b}^2} = \frac{-\,\mathrm{b}^2\,(1+O(|\,\boldsymbol{x}\,|^{-1})+1+O(|\,\boldsymbol{x}\,|^{-1}))}{1-\mathrm{b}^2} = 1+O(|\,\widetilde{\boldsymbol{x}}\,|^{-1})\,.$$

On the other hand, by (2.6.22), we have that the mean curvature of the time slices $\mathcal{S}_{\widetilde{t}}$ is $\mathrm{H}^o = \mathrm{H}_{\mathfrak{g}^I_m}(f_\mathrm{b}) = O(|\,\widetilde{\boldsymbol{x}}\,|^{-2})$, which does not accomplish the condition (2.6.14) to construct barriers in the aforementioned manner.

Analyzing closely (2.3.15) computed in asymptotically flat coordinates for large radii, we observe that for a sublinear function with decaying derivatives, the dominant term is ${}^t g^{ij}\partial_{ij}(\,\cdot\,) = (\delta^{ij}+O(|\,\boldsymbol{x}\,|^{-1})\partial_{ij}(\,\cdot\,)$, while for a purely linear function like f_b, the mean curvature is dominated by the terms computed in (2.6.19), (2.6.20) and (2.6.21). In fact, considering a solution to the (flat) Poisson equation with right-hand side $-\,\mathrm{H}_{\mathfrak{g}^I_m}(f_\mathrm{b})$ will provide a correction to f_b so that the condition (2.6.14) can be attained.

2.6.1 A hypersurface admitting barriers

The proof of Proposition 2.6.1 will rely on the following result.

Lemma 2.6.2 ([Mey63, Lemma 5], [BCO90, Proposition 2.1])**.** *Let $\phi \in C^{k,\alpha}(\mathbb{R}^3\setminus\overline{B}_1)$ for some $k \geq 0$, $\alpha \in (0,1)$ be a function decaying as $\phi = O_k(|\boldsymbol{y}|^{-2})$ for $|\boldsymbol{y}| \to \infty$. Then there exists a solution $\bar{f} \in C^{k+2,\alpha}(\mathbb{R}^3\setminus\overline{B}_1)$ to the Poisson equation on the external domain*

$$\Delta_{\delta_{\mathbb{R}^3}}\bar{f} = \phi \qquad \text{on } \mathbb{R}^3\setminus\overline{B}_1$$

such that $\bar{f} = O_{k+2}(\log(|\boldsymbol{y}|))$ for $|\boldsymbol{y}| \to \infty$.

Proof of Proposition 2.6.1. Consider a solution $\bar{f}_\mathrm{b} \in C^\infty(\mathbb{R}^3\setminus\overline{B}_m)$ to

$$\left(\Delta_{\delta_{\mathbb{R}^3}}+\frac{\mathrm{b}^2}{1-\mathrm{b}^2}\partial^2_{11}\right)\bar{f}_\mathrm{b}\left(\frac{\boldsymbol{x}}{m}\right) = -\frac{\mathrm{b}(1+\mathrm{b}^2)}{(1-\mathrm{b}^2)}\frac{x_1}{|\,\boldsymbol{x}\,|^3}\,, \tag{2.6.23}$$

where the right-hand side is the dominant part of $-\frac{\sqrt{1-\mathrm{b}^2}}{m}\,\mathrm{H}_{\mathfrak{g}^I_m}(f_\mathrm{b})$. Such a solution exists with $\bar{f}_\mathrm{b}$ satisfying (2.6.1) as it is sufficient to change the coordinates

$$z_1 = \tfrac{x_1}{\sqrt{1-\mathrm{b}^2}},\quad z_a = x_a, \qquad a = 2,3 \tag{2.6.24}$$

and consider for $\boldsymbol{y} = \frac{\boldsymbol{z}}{m}$ the solution to the Poisson equation

$$\Delta_{\delta_{\mathbb{R}^3}} \bar{f}_{\mathrm{b}}(\boldsymbol{y}) = -\frac{\mathrm{b}(1-\mathrm{b}^2)}{\sqrt{1-\mathrm{b}^2}} \frac{y_1}{\left((1-\mathrm{b}^2)y_1^2 + y_2^2 + y_3^2\right)^{\frac{3}{2}}} ,$$

ensured by Lemma 2.6.2.

We now consider on $\{|\,\boldsymbol{x}\,| \geq \bar{K}m\}$ the hypersurface

$$\mathscr{S}_{\mathrm{b}} := \{t = u_{\mathrm{b}}(\boldsymbol{x})\} , \qquad u_{\mathrm{b}}(\boldsymbol{x}) := f_{\mathrm{b}}(\boldsymbol{x}) + m\bar{f}_{\mathrm{b}}\left(\frac{\boldsymbol{x}}{m}\right) \tag{2.6.25}$$

where $f_{\mathrm{b}} = \mathrm{b}\, x_1$ and $\bar{f}_{\mathrm{b}}$ have been defined in isotropic coordinates by (2.6.15) and (2.6.23), respectively. For any fixed $0 < \varepsilon < 1$, we claim that $\mathscr{S}_{\mathrm{b}} \in \mathscr{B}_{\mathrm{b}}$ is the desired hypersurface admitting barriers

$$\mathscr{W}^{\pm}_{\mathrm{b},K,\varepsilon} := \left\{t = w^{\pm}_{\mathrm{b},K,\varepsilon}(\boldsymbol{x})\right\} , \qquad w^{\pm}_{\mathrm{b},K,\varepsilon}(\boldsymbol{x}) := u_{\mathrm{b}}(\boldsymbol{x}) + \widetilde{w}^{\pm}_{K}(\boldsymbol{x}) , \tag{2.6.26}$$

on $\{|\,\boldsymbol{x}\,| \geq Km\}$ for any big enough $K > K_{\mathrm{b}}$ with $K_{\mathrm{b}} = K_{\mathrm{b}}(\mathrm{b}, \varepsilon, C_{\bar{f}_{\mathrm{b}}}) > \bar{K}$ to be determined and where we define

$$\widetilde{w}^{\pm}_{K}(\boldsymbol{x}) = m\, w^{\pm}_{K^{2\varepsilon}}\left(\frac{|\boldsymbol{z}(\boldsymbol{x})|}{m}\right) , \tag{2.6.27}$$

for $w^{\pm}_{\Lambda}$ as in (2.6.5)–(2.6.7) with $\Lambda = K^{2\varepsilon}$ and the coordinates $\boldsymbol{z}$ as in (2.6.24).

It is an immediate consequence of Remark 2.4.2 and the definition of $\bar{f}_{\mathrm{b}}$ in (2.6.23) via Lemma 2.6.2 that $\mathscr{S}_{\mathrm{b}}$ in (2.6.25) is in the class $\mathscr{B}_{\mathrm{b}}$ as in Definition 2.4.1. It is also immediate to see that

$$0 < w^{+}_{\mathrm{b},K,\varepsilon} - u_{\mathrm{b}} = O(|\,\boldsymbol{x}\,|^{-\varepsilon}) , \qquad 0 < u_{\mathrm{b}} - w^{-}_{\mathrm{b},K,\varepsilon} = O(|\,\boldsymbol{x}\,|^{-\varepsilon}) .$$

We can then check that for $\boldsymbol{x}$ such that $|\,\boldsymbol{x}\,| = Km$, we have

$$(w^{+}_{\mathrm{b},K,\varepsilon} - u_{\mathrm{b}})(\boldsymbol{x}) = \widetilde{w}^{+}_{K}(\boldsymbol{x}) = \frac{K^{2\varepsilon}\, m^{1+\varepsilon}}{|\,\boldsymbol{z}(\boldsymbol{x})|^{\varepsilon}} \geq \frac{K^{2\varepsilon}\, m^{1+\varepsilon}\sqrt{1-\mathrm{b}^2}}{|\,\boldsymbol{x}\,|^{\varepsilon}} = K^{\varepsilon}\, m\sqrt{1-\mathrm{b}^2}$$

$$(u_{\mathrm{b}} - w^{-}_{\mathrm{b},K,\varepsilon})(\boldsymbol{x}) = -\widetilde{w}^{-}_{K}(\boldsymbol{x}) = \frac{K^{2\varepsilon}\, m^{1+\varepsilon}}{|\,\boldsymbol{z}(\boldsymbol{x})|^{\varepsilon}} \geq \frac{K^{2\varepsilon}\, m^{1+\varepsilon}\sqrt{1-\mathrm{b}^2}}{|\,\boldsymbol{x}\,|^{\varepsilon}} = K^{\varepsilon}\, m\sqrt{1-\mathrm{b}^2}$$

noticing that $|\boldsymbol{z}(\boldsymbol{x})|^2 = z_1(\boldsymbol{x})^2 + z_2(\boldsymbol{x})^2 + z_3(\boldsymbol{x})^2 = \frac{x_1^2}{1-\mathrm{b}^2} + x_2^2 + x_3^3$ so there holds

$$|\,\boldsymbol{x}\,|^2 \leq |\boldsymbol{z}(\boldsymbol{x})|^2 \leq \frac{1}{1-\mathrm{b}^2}|\,\boldsymbol{x}\,|^2 . \tag{2.6.28}$$

We are left showing that there exists $K_{\mathrm{b}} = K_{\mathrm{b}}(\mathrm{b}, \varepsilon, C_{\bar{f}_{\mathrm{b}}})$ such that for all $K > K_{\mathrm{b}}$ the barriers defined as in (2.6.26)–(2.6.27) on $\{|\,\boldsymbol{x}\,| \geq Km\}$ are spacelike and their mean curvatures have the desired sign. We first compute and estimate by using (2.6.1), (2.6.5) and (2.6.28),

$$\begin{aligned}|\partial_i w^{\pm}_{\mathrm{b},K,\varepsilon}(\boldsymbol{x})| &= \left|\mathrm{b}\ \delta_{i1} + m\,\frac{\partial}{\partial x^i}\bar{f}_{\mathrm{b}}\left(\frac{\boldsymbol{x}}{m}\right) \mp \frac{K^{2\varepsilon}\,\varepsilon\, m^{1+\varepsilon}}{|\,\boldsymbol{z}(\boldsymbol{x})|^{1+\varepsilon}}\right| \\ &\leq \mathrm{b}\ \delta_{i1} + \frac{C_{\bar{f}_{\mathrm{b}}}\, m}{|\,\boldsymbol{x}\,|} + \frac{K^{2\varepsilon}\,\varepsilon\, m^{1+\varepsilon}}{|\,\boldsymbol{x}\,|^{1+\varepsilon}} \\ &\leq \mathrm{b}\ \delta_{i1} + \frac{C_{\bar{f}_{\mathrm{b}}}}{K} + \frac{\varepsilon}{K^{1-\varepsilon}}\end{aligned} \tag{2.6.29}$$

as $\mathrm{b} \in [0,1)$ and it is then clear that there exists a sufficiently large $K_{\mathrm{b}} = K_{\mathrm{b}}(\mathrm{b}, \varepsilon, C_{\bar{f}_{\mathrm{b}}})$ such that for all $K > K_{\mathrm{b}}$

$$\sum_{i=1}^{3}(\partial_i w^{\pm}_{\mathrm{b},K,\varepsilon}(\boldsymbol{x}))^2 < \mathrm{b}^2 + \frac{1-\mathrm{b}^2}{2}, \qquad \boldsymbol{v}_t(w^{\pm}_{\mathrm{b},K,\varepsilon}) < \frac{1}{\sqrt{1-\left(\mathrm{b}^2+\frac{1-\mathrm{b}^2}{2}\right)}} = \sqrt{\frac{2}{1-\mathrm{b}^2}}$$

which ensures that the spacelike condition (2.3.14) is satisfied on $\{|\,\boldsymbol{x}\,| \geq Km\}$. Let us now estimate the mean curvature. We will initially give explicit constants to estimate the asymptotic behaviours of the objects involved, which will be collected at the end in a constant C_{b} depending only on b and independent of m. Using (2.6.29), we estimete

$$\begin{aligned}\left|\boldsymbol{v}_t(w^{\pm}_{\mathrm{b},K,\varepsilon}) - \frac{1}{\sqrt{1-\mathrm{b}^2}}\right| &= \left|\frac{1}{\sqrt{1-\frac{N_m^2(|\,\boldsymbol{x}\,|)}{\varphi_m^4(|\,\boldsymbol{x}\,|)}\sum_{i=1}^3(\partial_i w^{\pm}_{\mathrm{b},K,\varepsilon}(\boldsymbol{x}))}} - \frac{1}{\sqrt{1-\mathrm{b}^2}}\right| \\ &\leq \frac{\boldsymbol{v}_t^3(w^{\pm}_{\mathrm{b},K,\varepsilon})}{2}\left(\frac{N_m^2(|\,\boldsymbol{x}\,|)}{\varphi_m^4(|\,\boldsymbol{x}\,|)}\sum_{i=1}^{3}(\partial_i w^{\pm}_{\mathrm{b},K,\varepsilon}(\boldsymbol{x})) - \mathrm{b}^2\right) \\ &< \frac{C_{\mathrm{b}}(1+K^{\varepsilon})m}{|\,\boldsymbol{x}\,|}\end{aligned}$$

by combining the mean value theorem with the estimates above. Before applying (2.3.15) to $w^{+}_{\mathrm{b},K,\varepsilon}$, we observe that by the definition (2.6.27) of $\widetilde{w}^{\pm}_{K}$, by the change of coordinates (2.6.24) and by the computation (2.6.6), there holds

$$\left(\Delta_{\delta_{\mathbb{R}^3}} + \frac{\mathrm{b}^2}{1-\mathrm{b}^2}\partial^2_{11}\right)\widetilde{w}^{\pm}_{K}(\boldsymbol{x}) = m\sum_{i=1}^{3}\frac{\partial^2}{\partial z_i^2}w^{\pm}_{K}\left(\frac{|\,\boldsymbol{z}(\boldsymbol{x})|}{m}\right) = \mp\frac{K^{2\varepsilon}\varepsilon(1-\varepsilon)m^{1+\varepsilon}}{|\,\boldsymbol{z}(\boldsymbol{x})|^{2+\varepsilon}}\,. \tag{2.6.30}$$

We then estimate

$$\begin{aligned}&\left|\mathrm{H}_{\mathfrak{g}^I_m}(w^{\pm}_{\mathrm{b},K,\varepsilon}) \pm \frac{K^{2\varepsilon}\varepsilon(1-\varepsilon)m^{1+\varepsilon}}{|\,\boldsymbol{z}(\boldsymbol{x})|^{2+\varepsilon}}\right| \\ &\qquad \leq \left|\frac{1}{\sqrt{1-\mathrm{b}^2}}\left(\delta^{ij} + \frac{\mathrm{b}^2}{1-\mathrm{b}^2}\delta^{i1}\delta^{j1}\right)\left(\partial^2_{ij}\left(m\bar{f}_{\mathrm{b}}\left(\frac{\boldsymbol{x}}{m}\right) + \widetilde{w}^{+}_{K}(\boldsymbol{x})\right) - {}^t\Gamma^k_{ij}\,\mathrm{b}\,\delta_{1k}\right)\right.\end{aligned}$$

$$
\begin{aligned}
&+ \frac{1}{\sqrt{1-\mathrm{b}^2}}\left(1+\frac{1}{1-\mathrm{b}^2}\right)\frac{m\,\mathrm{b}\,x_1}{|\,\boldsymbol{x}\,|^3} \pm \frac{K^{2\varepsilon}\varepsilon(1-\varepsilon)m^{1+\varepsilon}}{|\,\boldsymbol{z}(\boldsymbol{x})|^{2+\varepsilon}}\Bigg| + \frac{C_{\mathrm{b}}(1+K^{\varepsilon})m^2}{|\,\boldsymbol{x}\,|^3} \\
&\leq \Bigg| \mathrm{H}_{\mathfrak{g}^I_m}(f_{\mathrm{b}}) + \left(\Delta_{\delta_{\mathbb{R}^3}} + \frac{\mathrm{b}^2}{1-\mathrm{b}^2}\partial^2_{11}\right)\left(m\bar{f}_{\mathrm{b}}\left(\frac{\boldsymbol{x}}{m}\right) + \widetilde{w}^{\pm}_K(\boldsymbol{x})\right) \\
&\quad \pm \frac{K^{2\varepsilon}\varepsilon(1-\varepsilon)m^{1+\varepsilon}}{|\,\boldsymbol{z}(\boldsymbol{x})|^{2+\varepsilon}}\Bigg| + \frac{C_{\mathrm{b}}(1+K^{2\varepsilon})m^2}{|\,\boldsymbol{x}\,|^3} \\
&= \frac{C_{\mathrm{b}}(1+K^{2\varepsilon})m^2}{|\,\boldsymbol{x}\,|^3}
\end{aligned}
\tag{2.6.31}
$$

where we, in particular, have used (2.6.19), (2.6.20), (2.6.21), the specific choice of $\bar{f}_{\mathrm{b}}$ in (2.6.23) and the computation (2.6.30). Again, we highlight that the constant C_{b} is independent of K and m. It is then clear that

$$
\begin{aligned}
\mathrm{H}_{\mathfrak{g}^I_m}(w^{+}_{\mathrm{b},K,\varepsilon})(\boldsymbol{x}) &\leq -\frac{K^{2\varepsilon}\varepsilon(1-\varepsilon)m^{1+\varepsilon}}{|\,\boldsymbol{z}(\boldsymbol{x})|^{2+\varepsilon}} + \frac{C_{\mathrm{b}}(1+K^{2\varepsilon})m^2}{|\,\boldsymbol{x}\,|^3} \\
&\leq -\frac{(1-\mathrm{b}^2)^{2+\varepsilon}K^{2\varepsilon}\varepsilon(1-\varepsilon)m^{1+\varepsilon}}{|\,\boldsymbol{x}\,|^{2+\varepsilon}}\left(1-\frac{C_{\mathrm{b}}(1+K^{-2\varepsilon})m^{1-\varepsilon}}{|\,\boldsymbol{x}\,|^{1-\varepsilon}}\right) < 0
\end{aligned}
$$

which is strictly negative for any $K > K_{\mathrm{b}}(\mathrm{b},\varepsilon,C_{\bar{f}_{\mathrm{b}}})$, provided K_{b} is sufficiently large, since $-2+\varepsilon > -3+2\varepsilon$ for $0<\varepsilon<1$. in fact, we can for example estimate

$$
\frac{C_{\mathrm{b}}(1+K^{-2\varepsilon})m^{1-\varepsilon}}{|\,\boldsymbol{x}\,|^{1-\varepsilon}} \leq \frac{C_{\mathrm{b}}(1+K^{-2\varepsilon})m^{1-\varepsilon}}{K^{1-\varepsilon}} \leq \frac{1}{2}
$$

for K big enough. Similarly, we have

$$
\begin{aligned}
\mathrm{H}_{\mathfrak{g}^I_m}(w^{-}_{\mathrm{b},K,\varepsilon})(\boldsymbol{x}) &\geq \frac{K^{2\varepsilon}\varepsilon(1-\varepsilon)m^{1+\varepsilon}}{|\,\boldsymbol{z}(\boldsymbol{x})|^{2+\varepsilon}} - \frac{C_{\mathrm{b}}(1+K^{2\varepsilon})m^2}{|\,\boldsymbol{x}\,|^3} \\
&\geq \frac{(1-\mathrm{b}^2)^{2+\varepsilon}K^{2\varepsilon}\varepsilon(1-\varepsilon)m^{1+\varepsilon}}{|\,\boldsymbol{x}\,|^{2+\varepsilon}}\left(1-\frac{C_{\mathrm{b}}(1+K^{-2\varepsilon})m^{1-\varepsilon}}{|\,\boldsymbol{x}\,|^{1-\varepsilon}}\right) > 0
\end{aligned}
$$

for any $K > K_{\mathrm{b}}(\mathrm{b},\varepsilon,C_{\bar{f}_{\mathrm{b}}})$ with K_{b} as before. This concludes the proof. □

2.7 Gluing the time functions

The aim of this section is to define a time function on the maximally extended Schwarzschild $\left(\mathfrak{L}^K_m, \mathfrak{g}^K_m\right)$ such that the conditions (2.2.30) on $\{r > \kappa_0 m\} \subset \mathfrak{L}^K_m$ are satisfied for some constant $C_o = C_o(\kappa_0)$ and $\kappa_0 > 0$ and concurrently such that its zero-time slice in the asymptotically flat ends of *region* I and I′ coincide with $\mathcal{S}_{\mathrm{b}}$ as defined in Proposition 2.6.1. This can be obtained by gluing the new time function introduced in Proposition 2.5.2 with the time function having as time slices in *region* I given by $\{t = u_{\mathrm{b}}(\boldsymbol{x}) + c \,:\, c \in \mathbb{R}\}$ for u_{b} as in Proposition 2.6.1.

Gluing $\widetilde{\mathsf{T}}$ to the static time. We start by gluing the new time function $\widetilde{\mathsf{T}}$ defined in (2.5.11) with the static time t. We will show that the corresponding time slices can be glued into spacelike slices. For each $\bar{c} \in \mathbb{R}$, we consider the hypersurface

$$\{t = \bar{c}\} = \left\{ t = 4m \operatorname{arcosh} \sqrt{\frac{1}{1-c^2}} \right\}, \qquad c = \tanh\left(\frac{\bar{c}}{4m}\right) \in (-1, 1)\,.$$

The above identity follows from the property of the hyperbolic functions $1 - \tanh^2 x = \frac{1}{\cosh^2 x}$ yielding $\sqrt{\frac{1}{1-c^2}} = \cosh\left(\frac{\bar{c}}{4m}\right)$. We can then consider the level set of $\widetilde{\mathsf{T}}$ in the regions I and I′, namely on $\{r > 2m\}$ as follows

$$\begin{aligned}
\left\{\widetilde{\mathsf{T}} = \bar{c}\right\} \cap \{r > 2m\} &= \left\{\mathsf{T} = \tanh\left(\frac{\bar{c}}{4m}\right)\sqrt{1+\mathsf{X}^2}\right\} \cap \{r > 2m\} \\
&= \left\{\mathsf{T} = c\,\mathsf{X}\sqrt{1 + \frac{1}{\mathsf{X}^2}}\right\} \cap \{r > 2m\} \\
&= \left\{ t = \frac{\mathsf{X}}{|\mathsf{X}|} 4m \operatorname{arcosh} \sqrt{\frac{1}{1-c^2}\left(1 + c^2 \rho_m^{-1}(r)\right)} \right\}, \qquad (2.7.1)
\end{aligned}$$

for ρ_m defined in (2.3.17). To see that the identities hold, we see that on the hypersurface (2.7.1), due to the change of coordinates (2.3.22), it holds

$$\mathsf{T}^2 - \mathsf{X}^2 = \mathsf{X}^2(c^2 - 1) + c^2 = -\rho_m(r)\cosh^2\left(\frac{t}{4m}\right)(1 - c^2) + c^2$$

which combined with (2.3.17) gives the result. For simplicity, we will consider *region* I where $\frac{\mathsf{X}}{|\mathsf{X}|} = 1$; the computation for *region* I′ are analogous. Consider the function in Schwarzschildean coordinates

$$F(r) := \varsigma_1(r)\left(4m \operatorname{arcosh}\sqrt{\frac{1}{1-c^2}}\right) + (1 - \varsigma_1(r))\left(4m \operatorname{arcosh}\sqrt{\frac{1 + c^2\,\rho_m^{-1}(r)}{1-c^2}}\right)$$

where $\varsigma_1 : (0, +\infty) \to \mathbb{R}$ is a cut-off function

$$\varsigma_1(s) = \begin{cases} 1 & \text{if } s < \kappa_1 m \\ 0 & \text{if } s > 2\kappa_1 m \end{cases}, \qquad |\varsigma_1'(s)| < \frac{2}{\kappa_1 m} \qquad (2.7.2)$$

where $\kappa_1 m > 2m$ is a Schwarzschildean radius to be determined. To check the condition (2.3.11), we then compute

$$|h_m \partial_r F| = 4m\, h_m \left| \varsigma_1'(r)\left(\operatorname{arcosh}\sqrt{\frac{1}{1-c^2}} - \operatorname{arcosh}\sqrt{\frac{1 + c^2\,\rho_m^{-1}(r)}{1-c^2}}\right) + \right. \qquad (2.7.3)$$

$$
\begin{aligned}
&+ (1-\varsigma_1(r)) \frac{1}{\sqrt{\frac{1+c^2\,\rho_m^{-1}(r)}{1-c^2}-1}} \sqrt{\frac{1-c^2}{1+c^2\,\rho_m^{-1}(r)}} \frac{(-c^2)\rho_m^{-2}(r)\rho_m'(r)}{2(1-c^2)} \Bigg| \\
&= 4m\,h_m \Bigg| \varsigma_1'(r) \log\left(\frac{\sqrt{\frac{1}{1-c^2}} + \sqrt{\frac{1}{1-c^2}-1}}{\sqrt{\frac{1+c^2\,\rho_m^{-1}(r)}{1-c^2}} + \sqrt{\frac{1+c^2\,\rho_m^{-1}(r)}{1-c^2}-1}} \right) \\
&\quad - (1-\varsigma_1(r)) \frac{|c|\,\rho_m^{-2}(r)\rho_m'(r)}{2\sqrt{1+\rho_m^{-1}(r)}\sqrt{1+c^2\rho_m^{-1}(r)}} \Bigg| \\
&= 4m\,h_m\,|\varsigma_1'(r)| \log\left(\frac{\sqrt{1+c^2\rho_m^{-1}(r)} + |c|\sqrt{1+\rho_m^{-1}(r)}}{1+|c|} \right) \\
&\quad + (1-\varsigma_1(r)) \frac{|c|\,\rho_m^{-1}(r)}{\sqrt{1+\rho_m^{-1}(r)}\sqrt{1+c^2\rho_m^{-1}(r)}} \\
&\le \frac{16}{\kappa_1}\left(1-\frac{1}{\kappa_1}\right)\sqrt{1+\rho_m^{-1}(\kappa_1 m)} + \rho_m^{-1}(\kappa_1 m) \\
&= \frac{16}{\kappa_1}\left(1-\frac{1}{\kappa_1}\right)\sqrt{1+\frac{e^{-\frac{\kappa_1}{2}}}{\left(\frac{\kappa_1}{2}-1\right)}} + \frac{e^{-\frac{\kappa_1}{2}}}{\left(\frac{\kappa_1}{2}-1\right)}
\end{aligned} \tag{2.7.4}
$$

using that $\frac{d}{dx}\operatorname{arcosh} x = \frac{1}{\sqrt{x^2-1}}$, the identity $\operatorname{arcosh} x = \log(x+\sqrt{x^2-1})$ and that

$$
\rho_m'(r) = \frac{r}{4m^2} e^{\frac{r}{2m}} > 0\,, \qquad \rho_m^{-1}(r)\rho_m'(r) = \frac{1}{2m\,h_m}\,.
$$

The quantity (2.7.3) is strictly less than 1 taking, for example, $\kappa_1 = 16$. We have then constructed a time function which coincides with $\widetilde{\mathsf{T}}$ on $\{r < 16m\}$ and with the static time t on $\{r > 32m\}$.

Gluing the static time to $t - \mathrm{b}\,x_1$. Let us now glue the static time with the boosted time $t - \mathrm{b}\,x_1$, which can be performed exactly as in Minkowski spacetime. For simplicity, we will work in *region* I. For any fixed $\widetilde{\kappa}_2 > \frac{1}{2}$ consider the following Lipschitz compactly supported function

$$
\ell_{\mathrm{b}}(\boldsymbol{x}) = \begin{cases}
0 & \text{if } |\,\boldsymbol{x}\,| \le \widetilde{\kappa}_2 m \\
\frac{\mathrm{b}+1}{2}\left(1-\frac{\widetilde{\kappa}_2 m}{|\,\boldsymbol{x}\,|}\right) x_1 & \text{if } \widetilde{\kappa}_2 m \le |\,\boldsymbol{x}\,| \le \widetilde{\kappa}_2 m \left(1+\frac{2\,\mathrm{b}}{1-\mathrm{b}}\right) \\
\mathrm{b}\,x_1 & \text{if } \widetilde{\kappa}_2 m \left(1+\frac{2\,\mathrm{b}}{1-\mathrm{b}}\right) \le |\,\boldsymbol{x}\,| \le 4\widetilde{\kappa}_2 m \left(1+\frac{2\,\mathrm{b}}{1-\mathrm{b}}\right) \\
\frac{\mathrm{b}+1}{2}\left(\left(1+\frac{4\,\mathrm{b}}{1-\mathrm{b}}\right)\frac{4\widetilde{\kappa}_2 m}{|\,\boldsymbol{x}\,|}-1\right) x_1 & \text{if } 4\widetilde{\kappa}_2 m \left(1+\frac{2\,\mathrm{b}}{1-\mathrm{b}}\right) \le |\,\boldsymbol{x}\,| \le 4\widetilde{\kappa}_2 m \left(1+\frac{4\,\mathrm{b}}{1-\mathrm{b}}\right) \\
0 & \text{if } |\,\boldsymbol{x}\,| \ge 4\widetilde{\kappa}_2 m \left(1+\frac{4\,\mathrm{b}}{1-\mathrm{b}}\right)
\end{cases}
$$

One can check that on $\left\{\widetilde{\kappa}_2 m \le |\,\boldsymbol{x}\,| \le \widetilde{\kappa}_2 m \left(1+\frac{2\,\mathrm{b}}{1-\mathrm{b}}\right)\right\}$, the following holds

$$
\partial_{x_1}\ell_{\mathrm{b}} = \frac{\mathrm{b}+1}{2}\left(1-\frac{\widetilde{\kappa}_2 m}{|\,\boldsymbol{x}\,|}+\widetilde{\kappa}_2 m\frac{x_1^2}{|\,\boldsymbol{x}\,|^3}\right)\,, \qquad \partial_a \ell_{\mathrm{b}} = \frac{\mathrm{b}+1}{2}\widetilde{\kappa}_2 m \frac{x_1 x_a}{|\,\boldsymbol{x}\,|^3} \quad a = 2,3
$$

and consequently

$$\sum_{i=1}^{3}(\partial_i \not{f}_{\mathrm{b}})^2 = \frac{(\mathrm{b}+1)^2}{4}\left[1-\frac{\widetilde{\kappa}_2 m}{|\,\boldsymbol{x}\,|}\left(2-\frac{\widetilde{\kappa}_2 m}{|\,\boldsymbol{x}\,|}\right)\left(1-\frac{x_1^2}{|\,\boldsymbol{x}\,|^2}\right)\right] \leq \frac{(\mathrm{b}+1)^2}{4} < 1$$

as $\mathrm{b} \in [0,1)$. An analogous computation and the same bound can be carried out on $\left\{4\widetilde{\kappa}_2 m\left(1+\frac{2\,\mathrm{b}}{1-\mathrm{b}}\right) \leq |\,\boldsymbol{x}\,| \leq 4\widetilde{\kappa}_2 m\left(1+\frac{4\,\mathrm{b}}{1-\mathrm{b}}\right)\right\}$. Since $\mathrm{b}^2 < \frac{(\mathrm{b}+1)^2}{4}$, we obtain that the Lipschitz constant is given by $\mathrm{Lip}(\not{f}_{\mathrm{b}}) = \frac{1+\mathrm{b}}{2}$. Consider now for sufficiently small $0 < \delta < 1$ a sequence mollifiers $\varphi_\delta \in C_0^\infty((-\delta,\delta))$ and the corresponding mollified functions $\varphi_\delta * \not{f}_{\mathrm{b}} = \not{f}_{\mathrm{b}}^\delta \in C_0^\infty(\mathbb{R}^3)$ with support on $\left\{\widetilde{\kappa}_2 m - 2\delta < |\,\boldsymbol{x}\,| < 4\widetilde{\kappa}_2 m\left(1+\frac{4\,\mathrm{b}}{1-\mathrm{b}}\right)+2\delta\right\}$ and $\mathrm{Lip}(\not{f}_{\mathrm{b}}^\delta) = \frac{1+\mathrm{b}}{2}$. Consider finally a cut-off function $\varsigma_2 : (0,+\infty) \to \mathbb{R}$

$$\varsigma_2(s) = \begin{cases} 1 & \text{if } s < 2\widetilde{\kappa}_2 m\left(1+\frac{2\,\mathrm{b}}{1-\mathrm{b}}\right) \\ 0 & \text{if } s > 3\widetilde{\kappa}_2 m\left(1+\frac{2\,\mathrm{b}}{1-\mathrm{b}}\right) \end{cases}, \qquad |\varsigma_2'(s)| < \frac{2}{\widetilde{\kappa}_2 m\left(1+\frac{2\,\mathrm{b}}{1-\mathrm{b}}\right)} \tag{2.7.5}$$

and define the function

$$\bar{\not{f}}_{\mathrm{b}}(\boldsymbol{x}) \coloneqq \varsigma_2(|\,\boldsymbol{x}\,|)\not{f}_{\mathrm{b}}^\delta + (1-\varsigma_2(|\,\boldsymbol{x}\,|))\,\mathrm{b}\,x_1\,. \tag{2.7.6}$$

One can check that on $\left\{2\widetilde{\kappa}_2 m \leq |\,\boldsymbol{x}\,| \leq 3\widetilde{\kappa}_2 m\left(1+\frac{2\,\mathrm{b}}{1-\mathrm{b}}\right)\right\}$, the following holds

$$\partial_1 \not{f}_{\mathrm{b}} = \frac{x_1}{|\,\boldsymbol{x}\,|}\varsigma_2'\left(\not{f}_{\mathrm{b}}^\delta - \mathrm{b}\,x_1\right) + \varsigma_2\left(\partial_1 \not{f}_{\mathrm{b}}^\delta - \mathrm{b}\right) + \mathrm{b}\,, \qquad \partial_a f_{\mathrm{b}} = \frac{x_a}{|\,\boldsymbol{x}\,|}\varsigma_2'\left(\not{f}_{\mathrm{b}}^\delta - \mathrm{b}\,x_1\right) + \varsigma_2 \partial_a \not{f}_{\mathrm{b}}^\delta$$

for $a = 2,3$. By the standard properties of mollified Lipschitz functions, we have that $\not{f}_{\mathrm{b}}^\delta \rightrightarrows \mathrm{b}\,x_1$ as $\delta \to 0$ on any compact set of $\{\widetilde{\kappa}_2 m - 2\delta < |\,\boldsymbol{x}\,|\}$ and its derivatives converge almost everywhere and in particular on the sets where the derivatives of $\not{f}_{\mathrm{b}}$ are defined. It is then clear that we can choose a fixed $\bar{\delta} > 0$ small enough so that the spacelike condition (2.3.14) is satisfied. We have constructed a time function $t - \bar{\not{f}}_{\mathrm{b}}$ which coincides with the static time t on $\{\widetilde{\kappa}_2 m - 2\bar{\delta} < |\,\boldsymbol{x}\,|\}$ and with $t - \mathrm{b}\,x_1$ on $\left\{|\,\boldsymbol{x}\,| > 3\widetilde{\kappa}_2 m\left(1+\frac{2\,\mathrm{b}}{1-\mathrm{b}}\right)\right\}$.

Gluing $t - \mathrm{b}\,x_1$ to $t - u_{\mathrm{b}}$. Consider the function in isotropic coordinates, for simplicity in *region* I, given by

$$\widetilde{\not{f}}_{\mathrm{b}}(\boldsymbol{x}) = \varsigma_3(|\,\boldsymbol{x}\,|)\,\mathrm{b}\,x_1 + (1-\varsigma_3(|\,\boldsymbol{x}\,|))(\mathrm{b}\,x_1 + \bar{f}_{\mathrm{b}}(\boldsymbol{x})) = \mathrm{b}\,x_1 + (1-\varsigma_3(|\,\boldsymbol{x}\,|))\bar{f}_{\mathrm{b}}(\boldsymbol{x})$$

describing the level sets of the time function obtained gluing $t - \mathrm{b}\,x_1$ to $t - u_{\mathrm{b}}$, where $u_{\mathrm{b}}(\boldsymbol{x}) = \mathrm{b}\,x_1 + \bar{f}_{\mathrm{b}}(\boldsymbol{x})$ for $\bar{f}_{\mathrm{b}} = O(\log(|\,\boldsymbol{x}\,|))$ as in Proposition 2.6.1. The cut-off function $\varsigma_2 : (0,+\infty) \to \mathbb{R}$ is taken so that

$$\varsigma_3(s) = \begin{cases} 1 & \text{if } s < \widetilde{\kappa}_3 m \\ 0 & \text{if } s > 2\widetilde{\kappa}_3 m \end{cases}, \qquad |\varsigma_3'(s)| < \frac{2}{\widetilde{\kappa}_3 m} \tag{2.7.7}$$

where $\widetilde{\kappa}_3 > \frac{1}{2}$ has to be determined so that the condition (2.3.14) is satisfied. We compute

$$\partial_i \widetilde{f}_{\mathrm{b}} = \mathrm{b}\,\delta_{i1} - \varsigma_3'(|\,\boldsymbol{x}\,|)\bar{f}_{\mathrm{b}}(\boldsymbol{x}) + (1 - \varsigma_3(|\,\boldsymbol{x}\,|))\,\partial_i \bar{f}_{\mathrm{b}}(\boldsymbol{x})$$

and considering the constant $C_{\bar{f}_{\mathrm{b}}}$ as in Proposition 2.6.1, we can produce the following rough estimate

$$|\partial_i \widetilde{f}_{\mathrm{b}}| \le |\,\mathrm{b}\,|\delta_{1i} + \frac{4C_{\bar{f}_{\mathrm{b}}}\log(2\widetilde{\kappa}_3)}{\widetilde{\kappa}_3} + \frac{C_{\bar{f}_{\mathrm{b}}}}{\widetilde{\kappa}_3}\,.$$

It is then clear that there exists a sufficiently large $\widetilde{\kappa}_3 = \widetilde{\kappa}_3(C_{\bar{f}_{\mathrm{b}}}, \mathrm{b})$ such that the spacelike condition holds.

The final time function. In the following theorem, we claim the existence of a time function satisfying the crucial conditions (2.2.30) and (2.2.35) and moreover, having time slices in the class $\mathscr{B}_{\mathrm{b}}$ admitting lower and upper barriers for large radii.

Theorem 2.7.1. *For any* $\mathrm{b} \in [0,1)$, *the maximally extended Schwarzschild spacetime* $\left(\mathfrak{L}^K_m, \mathfrak{g}^K_m\right)$ *admits a time function* $t_{\mathrm{b}} \in C^\infty(\mathfrak{L}^K_m)$ *such that the following hold simultaneously.*

(i) The Bartnik's spacetime conditions (2.2.30) *for the norm* (2.2.5) *defined with* t_{b} *are satisfied for some constant* $C_o = C_o(\kappa_0)$ *and* $\kappa_0 > 0$ *on* $\mathfrak{L}^K_m \cap \{r > \kappa_0 m\}$*:*

$$^{t_{\mathrm{b}}}\|\log\alpha_{t_{\mathrm{b}}}\| + {}^{t_{\mathrm{b}}}\|\mathrm{d}\log\alpha_{t_{\mathrm{b}}}\|m + {}^{t_{\mathrm{b}}}\|\,\overline{\nabla}\,\mathscr{T}_{\mathrm{b}}\|m + {}^{t_{\mathrm{b}}}\|\,\overline{\nabla}^2\,\mathscr{T}_{\mathrm{b}}\|m^2 \le C_o(\kappa_0) \qquad (2.7.8)$$

where $\alpha_{t_{\mathrm{b}}}$ *is the lapse function of* t_{b} *as defined in* (2.2.3) *and* $\mathscr{T}_{\mathrm{b}}$ *is the future-pointing unit normal to the time slices* $\mathcal{S}_{t_{\mathrm{b}}}$.

(ii) Each time slice is a Schwarzschildean coordinate boost of parameter $\mathrm{b} \in [0,1)$, *namely* $\mathcal{S}_{t_{\mathrm{b}}} \in \mathscr{B}_{\mathrm{b}}$. *Moreover, the spacelike hypersurface* $\mathcal{S}^o_{t_{\mathrm{b}}} = \{t_{\mathrm{b}} = 0\}$ *is invariant under the isometry* $(\mathsf{X},\mathsf{T}) \mapsto (-\mathsf{X},-\mathsf{T})$ *of the spacetime* $\left(\mathfrak{L}^K_m, \mathfrak{g}^K_m\right)$ *and given any fixed* $0 < \varepsilon < 1$ *admits barriers* $\mathscr{W}^{\pm}_{\mathrm{b},K,\varepsilon}$ *on* $\{|\,\boldsymbol{x}\,| \ge Km\}$ *for any* $K \ge K_{\mathrm{b}}$, *with* $K_{\mathrm{b}} = K_{\mathrm{b}}(\mathrm{b},\varepsilon,C_{\bar{f}_{\mathrm{b}}})$ *as in Proposition 2.6.1.*

(iii) A relaxed version of Bartnik's uniform interior condition (2.2.35) *is satisfied for any* $0 < \kappa_0 < \frac{3}{2}$ *and any* $K_0 > \kappa_6$ *on* $\mathfrak{L}^K_m \cap \{r > \kappa_0 m\}$, *namely there exists a constant* $C^I_{K_0} = C^I_{K_0}(\mathrm{b}, C_{\bar{f}_{\mathrm{b}}}, \kappa_0, K_0, \kappa_1) > 0$ *such that*

$$\begin{aligned}
&\sup_{\mathrm{p}\in\{r=K_0 m\}}\ \sup_{\mathrm{q}\in\{\kappa_0 m<r<K_0 m\}\setminus I^+_{\mathrm{p}}} t_{\mathrm{b}}(\mathrm{q}) - t_{\mathrm{b}}(\mathrm{p}) \le C^I_{K_0} m \qquad \text{if } t(\mathrm{p}) \ge 0\,,\\
&\sup_{\mathrm{p}\in\{r=K_0 m\}}\ \sup_{\mathrm{q}\in\{\kappa_0 m<r<K_0 m\}\setminus I^-_{\mathrm{p}}} t_{\mathrm{b}}(\mathrm{p}) - t_{\mathrm{b}}(\mathrm{q}) \le C^I_{K_0} m \qquad \text{if } t(\mathrm{p}) \le 0\,.
\end{aligned} \qquad (2.7.9)$$

where

$$I_{\mathrm{p}}^{+} := \bigcup_{\overline{\mathrm{p}} \in \{r=K_0 m\} \cap \{t_{\mathrm{b}}=t_{\mathrm{b}}(\mathrm{p})\}} I^{+}(\overline{\mathrm{p}}) \,, \qquad I_{\mathrm{p}}^{-} := \bigcup_{\overline{\mathrm{p}} \in \{r=K_0 m\} \cap \{t_{\mathrm{b}}=t_{\mathrm{b}}(\mathrm{p})\}} I^{-}(\overline{\mathrm{p}}) \,. \tag{2.7.10}$$

Remark 2.7.2. *The* Bartnik's uniform interior condition *is one of the fundamental tools in [Bar84, Theorem 5.3] to get an a priori height estimate. In our a priori height estimate proven in Theorem 2.8.1, we will combine the relaxed version of* Bartnik's uniform interior condition *as in* (2.7.9)*, which relies on the rotational symmetry of the Schwarzschild spacetime, with the control of the oscillation of the height on a large fixed radius as a consequence of* (2.8.9)*.*

Proof of Theorem 2.7.1. Consider the time function

$$t_{\mathrm{b}} := \varsigma_1(r)\widetilde{\mathsf{T}} + \frac{\mathsf{X}}{|\mathsf{X}|}(1 - \varsigma_1(r))\overline{t}_{\mathrm{b}} \tag{2.7.11}$$

with ς_1 as in (2.7.2) and $\kappa_1 = 16$. The function $\overline{t}_{\mathrm{b}}$ is in turn defined by a double gluing, namely

$$\overline{t}_{\mathrm{b}} := \varsigma_3(r)\bar{f}_{\mathrm{b}}(\boldsymbol{x}(r)) + \big(1 - \varsigma_3(r)\big)\big(t - u_{\mathrm{b}}(\boldsymbol{x}(r, \theta_I))\big) \tag{2.7.12}$$

where $\bar{f}_{\mathrm{b}}$ is defined in (2.7.6) via the cut-off function ς_2 given by (2.7.5) and where $\varsigma_3(r) = \varsigma_3(|\,\boldsymbol{x}\,|(r))$ is given by (2.7.7) via the inverse of the change of coordinates (2.3.9) on *region* I, where $\{|\,\boldsymbol{x}\,| > \frac{m}{2}\}$. The constants $\widetilde{\kappa}_2$ and $\widetilde{\kappa}_3$ in ς_2 and ς_3 are chosen so that under the change of coordinates (2.3.9)

$$\kappa_2 = \widetilde{\kappa}_2 \left(1 + \frac{1}{2\widetilde{\kappa}_2}\right)^2 > 32 = 2\kappa_1 \,, \qquad \widetilde{\kappa}_3 > 6\widetilde{\kappa}_2 \left(1 + \frac{2\,\mathrm{b}}{1 - \mathrm{b}}\right) .$$

In view of the discussion above, t_{b} is a time function.

The feature *(i)* follows from the properties of $\widetilde{\mathsf{T}}$ proven in Proposition 2.5.2 on the region $\{r < \kappa_1 m\}$ and of Proposition 2.5.1 on $\{r > \kappa_1 m > 2m\}$.

The property *(ii)* is an immediate consequence of the construction of u_{b} in Proposition 2.6.1. The symmetry of $\mathcal{S}^o_{t_{\mathrm{b}}} = \{t_{\mathrm{b}} = 0\}$ follows again by construction and by the transformation between Schwarzschildean coordinates and Kruskal–Szekeres coordinates (2.3.22). In fact, the level sets $\{t_{\mathrm{b}} = c,\ c \in \mathbb{R}\}$ of t_{b} coincide with the gluing of $\{\widetilde{\mathsf{T}} = c\}$ to $\{t = \pm c\}$, $\{t = \pm c + \mathrm{b}\, r\cos\theta_1\}$ and $\{t = \pm c + u_{\mathrm{b}}(r, \theta)\}$, where the sign $\pm$ in front of the constant c is positive in *region* I and negative in *region* I$'$. It is then evident that while for $c \neq 0$, the time slices are not symmetric under the isometry $(\mathsf{X}, \mathsf{T}) \mapsto (-\mathsf{X}, -\mathsf{T})$, the slice $c = 0$ is in fact invariant.

Finally, Bartnik's interior conditions *(iii)* follow from Proposition 2.5.3 and the fact that $\widetilde{\mathsf{T}}$, t and $t - u_{\mathrm{b}}$ are comparable by fixed constants for points $\mathrm{q} \in \{\kappa_1 m \leq r \leq K_0 m\}$.

In particular, it holds

$$|t(\mathrm{q}) - (t - u_{\mathrm{b}})(\mathrm{q})| \leq \big(\mathrm{b}\, K_0 + C_{\bar{f}_{\mathrm{b}}} \log(K_0)\big)\, m\,.$$

Moreover, using (2.3.22) to get $\mathsf{X}(\mathrm{q}) = \sqrt{\mathsf{T}(\mathrm{q})^2 + \rho_m(r)}$,, we obtain the following comparison

$$\begin{aligned}
|t(\mathrm{q}) - \widetilde{\mathsf{T}}(\mathrm{q})| &= 4m \left| \operatorname{artanh}\left(\frac{\mathsf{T}(\mathrm{q})}{\mathsf{X}(\mathrm{q})}\right) - \operatorname{artanh}\left(\frac{\mathsf{T}(\mathrm{q})}{\sqrt{1+\mathsf{X}(\mathrm{q})^2}}\right)\right| \\
&= 4m \left| \operatorname{artanh}\left(\frac{\mathsf{T}(\mathrm{q})}{\sqrt{\mathsf{T}(\mathrm{q})^2 + \rho_m(r)}}\right) - \operatorname{artanh}\left(\frac{\mathsf{T}(\mathrm{q})}{\sqrt{1+\mathsf{T}(\mathrm{q})^2 + \rho_m(r)}}\right)\right| \\
&\leq 4m \lim_{\mathsf{T}(\mathrm{q})\to+\infty} \left(\operatorname{artanh}\left(\frac{\mathsf{T}(\mathrm{q})}{\sqrt{\mathsf{T}(\mathrm{q})^2 + \rho_m(r)}}\right) - \operatorname{artanh}\left(\frac{\mathsf{T}(\mathrm{q})}{\sqrt{1+\mathsf{T}(\mathrm{q})^2 + \rho_m(r)}}\right)\right) \\
&= 2m \log\left(1 + \frac{1}{\rho_m(r)}\right) \leq 2m \log\left(1 + \frac{1}{\rho_{\kappa_1}}\right)
\end{aligned}$$

for $\rho_{\kappa_1} \coloneqq \rho_m(\kappa_1 m)$. The estimate above follows from the fact that for any fixed $\rho_m(r) > 0$ it holds

$$\begin{aligned}
\frac{d}{dx}\left(\operatorname{artanh}\left(\frac{x}{\sqrt{x^2+\rho_m(r)}}\right) - \operatorname{artanh}\left(\frac{x}{\sqrt{1+x^2+\rho_m(r)}}\right)\right) &= \\
= \frac{1}{\sqrt{x^2+\rho_m(r)}} - \frac{1}{\sqrt{1+x^2+\rho_m(r)}} &> 0
\end{aligned}$$

and from the Taylor expansion for $x \to +\infty$

$$\begin{aligned}
\operatorname{artanh}\left(1 + \frac{\rho_m(r)}{x^2} + O(|x|^{-4})\right) - \operatorname{artanh}\left(1 + \frac{1+\rho_m(r)}{x^2} + O(|x|^{-4})\right) &= \\
= \frac{1}{2}\log\left(\frac{2 + \frac{\rho_m(r)}{x^2} + O(|x|^{-4})}{1 + O(|x|^{-2})} \, \frac{1 + \frac{1}{\rho_m(r)} + O(|x|^{-2})}{2 + \frac{1+\rho_m(r)}{x^2} + O(|x|^{-4})}\right)
\end{aligned}$$

computed via the identity $\operatorname{artanh} y = \frac{1}{2}\log\left(\frac{1+y}{1-y}\right)$. Hence, defining $C_{\kappa_1} \coloneqq 2\log\left(1+\rho_{\kappa_1}^{-1}\right)$, we can take

$$C^I_{K_0} \coloneqq C_{\kappa_1} + C_{\kappa_0,K_0} + \mathrm{b}\, K_0 + C_{\bar{f}_{\mathrm{b}}} \log(K_0)$$

where $C_{\bar{f}_{\mathrm{b}}}$ is defined in (2.6.1) and C_{κ_0,K_0} is given in Proposition 2.5.3.

□

2.8 A priori height estimate

As briefly described in Section 2.4, the main issues to be solved in order to prove Theorem 2.4.4 consisted of constructing a time function satisfying the crucial conditions (2.2.30) and (2.2.35) and moreover, having time slices in the class $\mathscr{B}_{\mathrm{b}}$ admitting lower and upper barriers for large radii. This has been proven in Theorem 2.7.1. In particular, Bartnik's spacetime conditions (2.2.30) allow to employ the pointwise estimate in Theorem 2.2.5-(*ii*) combined with Bartnik's uniform interior condition (2.2.35) and the presence of asymptotic barriers to obtain a uniform a priori height estimate, as we will illustrate in the following Theorem.

Theorem 2.8.1 (A priori height estimate). *Consider a boost parameter* $\mathrm{b} \in [0,1)$ *and the maximally extended Schwarzschild spacetime* $\left(\mathfrak{L}^K_m, \mathfrak{g}^K_m\right)$ *endowed with the time function* $\mathscr{t}_{\mathrm{b}}$ *defined in* (2.7.11) *with properties as in Theorem 2.7.1. Let* $M \hookrightarrow \left(\mathfrak{L}^K_m, \mathfrak{g}^K_m\right)$ *be a compact spacelike maximal surface* $\mathrm{H}_{\mathfrak{g}^K_m}(M) = 0$ *with boundary* $\partial M = \partial M_{\mathrm{I}} \sqcup \partial M_{\mathrm{I}'}$ *contained in the zero-time slice* $\mathscr{S}^o_{\mathscr{t}_{\mathrm{b}}} = \{\mathscr{t}_{\mathrm{b}} = 0\}$ *of* $\mathscr{t}_{\mathrm{b}}$ *and consisting in two connected components* $\partial M_{\mathrm{I}} \subset \mathscr{S}^o_{\mathscr{t}_{\mathrm{b}}} \cap \{\mathsf{X} < 0\}$ *in* region I *and* $\partial M_{\mathrm{I}'} \subset \mathscr{S}^o_{\mathscr{t}_{\mathrm{b}}} \cap \{\mathsf{X} > 0\}$ *in* region I′*. Then there exists a constant* $\bar{C}_{\mathrm{b}} = \bar{C}_{\mathrm{b}}\left(C_o, \mathrm{b}, \varepsilon, K_{\mathrm{b}}, C^I_{K_{\mathrm{b}}}\right)$*, with* C_o *as in Theorem 2.7.1,* K_{b} *as in Proposition 2.6.1 and* $C^I_{K_{\mathrm{b}}}$ *as in Proposition 2.5.3 , such that*

$$\sup_M |u| \le \bar{C}_{\mathrm{b}} m \tag{2.8.1}$$

where u *is the height function of* M *defined as in* (2.2.10) *with respect to* $\mathscr{t}_{\mathrm{b}}$.

Moreover, for any fixed $0 < \varepsilon < 1$*, there exists a constant* $\bar{K} = \bar{K}(\bar{C}_{\mathrm{b}}, \mathrm{b}, \varepsilon) > K_{\mathrm{b}}$ *for* K_{b} *as in Proposition 2.6.1 such that on* $\{|\,\boldsymbol{x}\,| > \bar{K}m\}$ *the hypersurface* M *lies in between the barriers* $\mathscr{W}^{\pm}_{\mathrm{b},\bar{K},\varepsilon}$ *as given in Proposition 2.6.1, namely there holds*

$$\mathscr{t}_{\mathrm{b}}|_{\mathscr{W}^{-}_{\mathrm{b},\bar{K},\varepsilon}} < u < \mathscr{t}_{\mathrm{b}}|_{\mathscr{W}^{+}_{\mathrm{b},\bar{K},\varepsilon}} \tag{2.8.2}$$

on any such fixed radius $|\,\boldsymbol{x}\,|$.

Proof. The proof is similar to [Bar84, Theorem 5.3] with some modifications. We will, in fact, keep working with the spatial isotropic coordinates $\boldsymbol{x} = (x_1, x_2, x_3)$ instead of finding explicit coordinates on the time slices of $\mathscr{t}_{\mathrm{b}}$. Like Bartnik, we will proceed with the estimate of the supremum, and the same can be done for the infimum.

We first observe that once the height estimate (2.8.1) is established, we can easily show the validity of (2.8.2). In fact, considering

$$\bar{K} := \left(\frac{\bar{C}_{\mathrm{b}}}{\sqrt{1-\mathrm{b}^2}}\right)^{\frac{1}{\varepsilon}},$$

the desired estimate is guaranteed by the control ensured by (2.6.3) for the barriers $\mathscr{W}^{\pm}_{\mathrm{b},\bar{K},\varepsilon}$ on $\{|\,\boldsymbol{x}\,| = \bar{K}m\}$ and the maximum principle on the static asymptotically flat end.

To show (2.8.1), let us fix K_{b} as in Proposition 2.6.1. Without loss of generality, we can assume that $K_{\mathrm{b}} > 2\widetilde{\kappa}_3$ for $\widetilde{\kappa}_3$ as in (2.7.7). After fixing $0 < \varepsilon < 1$, consider the height control of the barriers $\mathscr{W}^{\pm}_{\mathrm{b},K_{\mathrm{b}},\varepsilon}$ on $\{|\boldsymbol{x}| = K_{\mathrm{b}}m\}$ provided in (2.6.3) and set the constant

$$C_{\mathrm{b},\varepsilon} := (K_{\mathrm{b}})^{\varepsilon}\sqrt{1-\mathrm{b}}\,. \tag{2.8.3}$$

By the staticity of the asymptotically flat end and the barrier construction, exactly as in [Bar84, Corollary 5.2], it is easy to show that

$$\text{either}\quad \sup_M |u| \leq C_{\mathrm{b},\varepsilon}m \qquad \text{or}\quad \sup_M |u| \leq \sup_{M\cap\{|\boldsymbol{x}|\leq \widetilde{K}_{\mathrm{b}}m\}} |u|$$

for the Schwarzschildean radius $\widetilde{K}_{\mathrm{b}}m$ corresponding to the isotropic radius $K_{\mathrm{b}}m$ via (2.3.9). Since the first option concludes the proof, we will assume that the second alternative holds.

From now onwards, notice that due to Proposition 2.3.2, since $\partial M \subset \{r > \kappa_0 m\}$ for any $0 < \kappa_0 < \frac{3}{2}$, $M \subset \{r > \kappa_0 m\}$ and hence the dependence of any constant on $\kappa_0 > 0$ is not restrictive and can be reduced to a dependence on $\frac{3}{2}$. Also notice that, for the sake of simplicity, we often work in isotropic coordinates in *region* I; the conclusions we get clearly hold on *region* I′ as well simply by translating the results in Schwarzschildian coordinates and by the symmetries of the Schwarzschild spacetime. Consider the new time function on $\{|\boldsymbol{x}| > K_{\mathrm{b}}m\}$ given by

$$\widetilde{\mathscr{t}}_{\mathrm{b}} := \mathscr{t}_{\mathrm{b}} - \widetilde{w}^{+}_{K_{\mathrm{b}}}(\boldsymbol{x}) \tag{2.8.4}$$

for $\widetilde{w}^{+}_{K_{\mathrm{b}}}$ as in (2.6.26)–(2.6.27). Define then $\widetilde{M} = M \cap \{|\boldsymbol{x}| > K_{\mathrm{b}}m\} \cap \{\widetilde{\mathscr{t}}_{\mathrm{b}} \geq 0\}$ to be the set of points of M lying above the barrier. Let $\widetilde{u} := \widetilde{\mathscr{t}}_{\mathrm{b}}|_M$ and let us denote by $\widetilde{\mathscr{v}} = \mathscr{v}_{\widetilde{\mathscr{t}}_{\mathrm{b}}}(u)$ the quantity defined in (2.2.12) with respect to $\widetilde{\mathscr{t}}_{\mathrm{b}}$, by $\widetilde{\mathrm{K}}^{o}$ and by $\widetilde{\mathrm{H}}^{o}$ the second fundamental form and the mean curvature of the time slices of $\widetilde{\mathscr{t}}_{\mathrm{b}}$, respectively. Then the boundary $\partial\widetilde{M} = \{\widetilde{u} = 0\} \sqcup \mathfrak{C}_{K_{\mathrm{b}}}$ where $\mathfrak{C}_{K_{\mathrm{b}}} \subset M \cap \{|\boldsymbol{x}| > K_{\mathrm{b}}m\}$. We now consider the equation (2.2.15) with respect to the time function $\widetilde{\mathscr{t}}_{\mathrm{b}}$, which, due to the geometric nature of the mean curvature, reads

$$0 = \mathrm{div}_M(\alpha_{\widetilde{\mathscr{t}}_{\mathrm{b}}}\,\mathrm{grad}_M\,\widetilde{u}) + \mathrm{div}_M\,\widetilde{\mathscr{T}}_{\mathrm{b}}$$

where $\widetilde{\mathscr{T}}_{\mathrm{b}}$ is the future-pointing unit normal to the time slices of $\widetilde{\mathscr{t}}_{\mathrm{b}}$. Multiplying by $\alpha_{\widetilde{\mathscr{t}}_{\mathrm{b}}}\psi(\widetilde{u})$, where $\psi \in C^1([0,+\infty))$, with $\psi(0) = 0$, $\psi(s), \psi'(s) > 0$ and integrating by parts, we obtain

$$\begin{aligned}
0 &= \int_{\widetilde{M}} \mathrm{div}_M(\alpha^2_{\widetilde{\mathscr{t}}_{\mathrm{b}}}\psi(\widetilde{u})\,\mathrm{grad}_M\,\widetilde{u})\,\mathrm{d}V_M - \int_{\widetilde{M}} \alpha_{\widetilde{\mathscr{t}}_{\mathrm{b}}}\psi(\widetilde{u})\langle \mathrm{grad}_M\,\widetilde{u}, \mathrm{grad}_M\,\alpha_{\widetilde{\mathscr{t}}_{\mathrm{b}}}\rangle\,\mathrm{d}V_M \\
&\quad - \int_{\widetilde{M}} \alpha^2_{\widetilde{\mathscr{t}}_{\mathrm{b}}}\psi'(\widetilde{u})|\,\mathrm{grad}_M\,\widetilde{u}|^2\,\mathrm{d}V_M + \int_{\widetilde{M}} \alpha_{\widetilde{\mathscr{t}}_{\mathrm{b}}}\psi(\widetilde{u})\,\mathrm{div}_M\,\widetilde{\mathscr{T}}_{\mathrm{b}}\,\mathrm{d}V_M \\
&= -\int_{\mathfrak{C}_{K_{\mathrm{b}}}} \alpha^2_{\widetilde{\mathscr{t}}_{\mathrm{b}}}\psi(\widetilde{u})\,\langle \mathrm{grad}_M\,\widetilde{u}, \nu_{\mathfrak{C}_{K_{\mathrm{b}}}}\rangle\,\mathrm{d}\sigma_{\mathfrak{C}_{K_{\mathrm{b}}}} + \int_{\widetilde{M}} \alpha_{\widetilde{\mathscr{t}}_{\mathrm{b}}}\psi(\widetilde{u})\widetilde{\mathscr{v}}\,\widetilde{\mathrm{H}}^{o}\,\mathrm{d}V_M
\end{aligned}$$

$$- \int_{\widetilde{M}} \psi(\widetilde{u})(\widetilde{\upsilon}^2 - 1) \left(\frac{\psi'(\widetilde{u})}{\psi(\widetilde{u})} + \frac{\alpha_{\widetilde{t}_{\mathrm{b}}}}{\widetilde{\upsilon}+1} \widetilde{\mathrm{H}}^o - \alpha_{\widetilde{t}_{\mathrm{b}}} \widetilde{\mathrm{K}}^o_{\widetilde{u}} + \widetilde{\mathscr{T}}_{\mathrm{b}}(\alpha_{\widetilde{t}_{\mathrm{b}}}) \right) \, \mathrm{d}V_M \tag{2.8.5}$$

where $\widetilde{\mathrm{K}}^o_{\widetilde{u}} = \widetilde{\mathrm{K}}^o \left(\frac{\mathrm{grad}_M \widetilde{u}}{|\mathrm{grad}_M \widetilde{u}|}, \frac{\mathrm{grad}_M \widetilde{u}}{|\mathrm{grad}_M \widetilde{u}|} \right)$, the vector field $\nu_{\mathfrak{C}_{K_{\mathrm{b}}}}$ is the unit normal to $\mathfrak{C}_{K_{\mathrm{b}}}$ in $\widetilde{M}$ pointing towards the asymptotic flat end and $\mathrm{d}V_M$, $\mathrm{d}\sigma_{\mathfrak{C}_{K_{\mathrm{b}}}}$ are the volume forms on $\widetilde{M}$ and on the portion of its boundary $\mathfrak{C}_{K_{\mathrm{b}}}$, respectively. In the above identities we used (2.2.14) and (2.2.16) and the identity $1 = \widetilde{\upsilon} - \frac{\widetilde{\upsilon}^2 - 1}{\widetilde{\upsilon}+1}$. Recalling the mean curvature of the barriers $\mathscr{W}^{\pm}_{\mathrm{b},K_{\mathrm{b}},\varepsilon}$ as in Proposition 2.6.1 and specifically by the computation (2.6.31), there holds $\widetilde{\mathrm{H}}^o = O(|\boldsymbol{x}|^{-2-\varepsilon})$. We remark that for all the quantities involved in this proof, the notation $O(|\boldsymbol{x}|^{-\alpha})$ corresponds to a control on $\{|\boldsymbol{x}| \geq K_{\mathrm{b}} m\}$ by constants depending on $\mathrm{b}, \varepsilon, C_{\bar{f}_{\mathrm{b}}}, K_{\mathrm{b}}$ and of m consistently to the scaling.

From now on, all the estimates of this proof will be performed with $m = 1$ and can be suitably rescaled to any m.

We first compute $\alpha_{\widetilde{t}_{\mathrm{b}}}$ and $\widetilde{\mathrm{K}}^o$ to estimate the quantity in the brackets of the last term of (2.8.5). Let us recall the definition of $\widetilde{t}_{\mathrm{b}}$ in (2.7.12) and of t_{b} in (2.7.11), which on $\{|\boldsymbol{x}| > 2\widetilde{\kappa}_3\}$ coincides to $t - u_{\mathrm{b}}(\boldsymbol{x})$ with u_{b} as in Proposition 2.6.1. We then obtain, with respect to isotropic coordinates,

$$\mathrm{d}\widetilde{t}_{\mathrm{b}} = \mathrm{d}t - \mathrm{b}\, \mathrm{d}x_1 - \partial_j \bar{f}_{\mathrm{b}}(\boldsymbol{x})\, \mathrm{d}x_j - \partial_j w^+ (|\boldsymbol{z}(\boldsymbol{x})|)\, \mathrm{d}x_j \,.$$

Since $\partial_j \bar{f}_{\mathrm{b}}(\boldsymbol{x}) = O(|\boldsymbol{x}|^{-1})$ and $\partial_j w^+ (|\boldsymbol{z}(\boldsymbol{x})|) = O(|\boldsymbol{x}|^{-1-\varepsilon})$, we obtain

$$\begin{aligned} \alpha^2_{\widetilde{t}_{\mathrm{b}}} &= -\langle \mathrm{d}\widetilde{t}_{\mathrm{b}}, \mathrm{d}\widetilde{t}_{\mathrm{b}} \rangle^{-1} = \left(N_1^{-2} - \varphi_1^{-4} \left(\mathrm{b}^2 + O(|\boldsymbol{x}|^{-1}) \right) \right)^{-1} \\ &= \frac{1}{1 - \mathrm{b}^2} + O(|\boldsymbol{x}|^{-1}) \,. \end{aligned}$$

Finally, since the tangent vectors are spanned by $X_i = \partial_i + (\mathrm{b} + O(|\boldsymbol{x}|^{-1}))\partial_t$, we see that

$$\widetilde{\mathrm{K}}^o(X_i, X_j) = -\alpha_{\widetilde{t}_{\mathrm{b}}} \mathrm{d}\widetilde{t}_{\mathrm{b}} \left(\overline{\nabla}_{X_i} X_j \right) = O(|\boldsymbol{x}|^{-2}) \,.$$

Hence, we can estimate for some constant $C_1 > 0$ the quantity

$$\left| \frac{\alpha_{\widetilde{t}_{\mathrm{b}}}}{\widetilde{\upsilon}+1} \widetilde{\mathrm{H}}^o - \alpha_{\widetilde{t}_{\mathrm{b}}} \widetilde{\mathrm{K}}^o_{\widetilde{u}} + \widetilde{\mathscr{T}}_{\mathrm{b}}(\alpha_{\widetilde{t}_{\mathrm{b}}}) \right| \leq \frac{C_1}{|\boldsymbol{x}|^2} \,. \tag{2.8.6}$$

To estimate the first term of the right-hand side of (2.8.5), we first assume that

$$\mu \coloneqq \sup_M u - \widetilde{C}^I_{K_{\mathrm{b}}} - C_{\mathrm{b},\varepsilon} > 2K_{\mathrm{b}} > 1 \,, \tag{2.8.7}$$

where $C_{\mathrm{b},\varepsilon}$ as in (2.8.3) and $\widetilde{C}^I_{K_{\mathrm{b}}} \coloneqq C^I_{K_{\mathrm{b}}} + 4(K_{\mathrm{b}} + \mathrm{b}\, K_{\mathrm{b}} + C_{\bar{f}_{\mathrm{b}}} \log(K_{\mathrm{b}}))$, with $C^I_{K_{\mathrm{b}}}$ being the constant given by Proposition 2.5.3. In fact, if this does not hold, we could already conclude. Now, by the properties of t_{b} in Theorem 2.7.1 and by definition of the height function u, we have that for $\mathrm{p} \in \mathfrak{C}_{K_\mathrm{b}} \subset \{|\,\boldsymbol{x}\,| = K_\mathrm{b}\}$ there holds

$$u(\mathrm{p}) - \sup_{\partial M} u \;\geq\; \sup_{M} u - \widetilde{C}^I_{K_\mathrm{b}} \;\geq\; \mu + C_{\mathrm{b},\varepsilon} \;>\; 1 \;>\; 0\,. \tag{2.8.8}$$

The first inequality follows from the fact that $\sup_{\partial M} u = 0$ and from the definition of $\widetilde{C}^I_{K_\mathrm{b}}$ in terms of $C^I_{K_\mathrm{b}}$. In fact, considering the value $\sup_{\mathfrak{C}_{K_\mathrm{b}}} u$, by (2.7.9) in Theorem 2.7.1, we have $\sup_M u - \sup_{\mathfrak{C}_{K_\mathrm{b}}} u \leq C^I_{K_\mathrm{b}}$. The first inequality in (2.8.8) then follows by showing that the oscillation of u on $\mathfrak{C}_{K_\mathrm{b}}$ is bounded by $4(K_\mathrm{b} + \mathrm{b}\,K_\mathrm{b} + C_{\bar{f}_\mathrm{b}} \log(K_\mathrm{b}))$. This can be obtained by assuming without loss of generality that $K_\mathrm{b} \geq 6$, and observing that

$$|{}^t\mathrm{D}u|^2_{\delta_{\mathbb{R}^3}} \leq \varphi_1^4\, N_1^{-2} \leq \tfrac{(2K_\mathrm{b}+1)^6}{(2K_\mathrm{b})^4(2K_\mathrm{b}-1)^2} \leq 2\,. \tag{2.8.9}$$

Now, the estimate (2.8.8) ensures that the hypothesis of Theorem 2.2.5-(ii) are satisfied and this provides for $\mathrm{p} \in \mathfrak{C}_{K_\mathrm{b}}$ the pointwise estimate

$$v_{t_\mathrm{b}}(M)\big|_\mathrm{p} \leq \widetilde{C}\left(C_o, \widetilde{C}^I_{K_\mathrm{b}}\right)\,.$$

We remark that this crucial estimate holds only in virtue of the properties of the time function t_b as in Theorem 2.7.1, ensuring that the time-dependent Bartnik's conditions (2.2.30) are satisfied. Notice that $\widetilde{t}_\mathrm{b} - t_\mathrm{b} = \widetilde{w}^+_{K_\mathrm{b}}(\boldsymbol{x})$ and hence on $\{|\,\boldsymbol{x}\,| = K_\mathrm{b}\}$ the quantity v_{t_b} is comparable with $\widetilde{v}(\mathrm{p})$ by [Bar84, Lemma 3.3] by a fixed constant. This, combined with (2.2.14), yields the estimate

$$\begin{aligned}\left|\int_{\mathfrak{C}_{R_0}} \alpha^2_{\widetilde{t}_\mathrm{b}} \psi(\widetilde{u}) \left\langle \mathrm{grad}_M \widetilde{u}, \nu_{\mathfrak{C}_{R_0}} \right\rangle \mathrm{d}\sigma_{\mathfrak{C}_{R_0}}\right| &\leq \sup_M \psi(\widetilde{u}) \int_{\mathfrak{C}_{R_0}} |\alpha_{\widetilde{t}_\mathrm{b}}| \sqrt{\widetilde{v}^2 - 1}\; \mathrm{d}\sigma_{\mathfrak{C}_{R_0}} \\ &\leq C_2 \sup_M \psi(\widetilde{u})\end{aligned} \tag{2.8.10}$$

for a constant $C_2 = C_2\left(C_o, C^I_{K_\mathrm{b}}, K_\mathrm{b}\right)$.

On the other hand, considering $\widetilde{M}$ as a graph in isotropic coordinates over a domain $\widetilde{\Omega} \subset \{t = 0\}$, the induced metric on it is given by $g_{ij} - N_1^2\, \partial_i u\, \partial_j u$ and hence

$$\mathrm{d}V_M = \sqrt{\det g_{ij}}\; v^{-1}\,.$$

Again, by [Bar84, Lemma 3.3] and the computation (2.6.31), we can then estimate

$$\int_{\widetilde{M}} \alpha_{\widetilde{t}_\mathrm{b}} \psi(\widetilde{u}) \widetilde{v}\, \widetilde{\mathrm{H}}^o\; \mathrm{d}V_M \leq -C_3 \int_{\widetilde{\Omega}} \psi(\widetilde{u})\, |\,\boldsymbol{x}\,|^{-2-\varepsilon}\; \mathrm{d}\,\boldsymbol{x} \tag{2.8.11}$$

for $C_3 = C_3\left(\inf |\alpha_{\widetilde{t}_\mathrm{b}} \sqrt{\det g_{ij}}|, \varepsilon, \mathrm{b}, K_\mathrm{b}\right)$. We can now plug (2.8.6), (2.8.10) and (2.8.11) in (2.8.5) to obtain

$$C_3 \int_{\widetilde{\Omega}} \frac{\psi(\widetilde{u})}{|\,\boldsymbol{x}\,|^{2+\varepsilon}} \,\mathrm{d}\,\boldsymbol{x} \leq C_2 \sup_M \psi(\widetilde{u}) - \int_{\widetilde{M}} \psi(\widetilde{u})(\widetilde{\boldsymbol{v}}^2 - 1) \left(\frac{\psi'(\widetilde{u})}{\psi(\widetilde{u})} - \frac{C_1}{|\,\boldsymbol{x}\,|^2} \right) \mathrm{d}V_M \qquad (2.8.12)$$

Finally, still using (2.8.9), we can estimate for any $\mathrm{q} \in \widetilde{M}$ and $\mathrm{p} \in \{|\,\boldsymbol{x}\,| = K_{\mathrm{b}}\}$ the quantity

$$|t(\mathrm{q}) - t(\mathrm{p})| \leq 2(|\,\boldsymbol{x}(\mathrm{q})| - K_{\mathrm{b}})$$

and consequently, there holds

$$\begin{aligned}
\widetilde{u}(\mathrm{q}) &= t(\mathrm{q}) - u_{\mathrm{b}}(\mathrm{q}) - \widetilde{w}^+_{K_{\mathrm{b}}}(\mathrm{q}) && (2.8.13)\\
&\geq t(\mathrm{p}) - u_{\mathrm{b}}(\mathrm{p}) + u_{\mathrm{b}}(\mathrm{p}) - \widetilde{w}^+_{K_{\mathrm{b}}}(\mathrm{q}) - 2(|\,\boldsymbol{x}(\mathrm{q})| - R_0 m) && (2.8.14)\\
&= u(\mathrm{p}) + u_{\mathrm{b}}(\mathrm{p}) - \widetilde{w}^+_{K_{\mathrm{b}}}(\mathrm{q}) - 2(|\,\boldsymbol{x}(\mathrm{q})| - K_{\mathrm{b}})\\
&\geq \sup_M u - \widetilde{C}^I_{K_{\mathrm{b}}} - C_{K_{\mathrm{b}},\varepsilon} - \mathrm{b}\, K_{\mathrm{b}} - C_{\bar{f}_{\mathrm{b}}} \log(K_{\mathrm{b}}) - 2(|\,\boldsymbol{x}(\mathrm{q})| - K_{\mathrm{b}})\\
&= \mu + (2 - \mathrm{b})K_{\mathrm{b}} - C_{\bar{f}_{\mathrm{b}}} \log(K_{\mathrm{b}}) - 2|\,\boldsymbol{x}(\mathrm{q})|\\
&\geq \mu + 2 - 2|\,\boldsymbol{x}(\mathrm{q})|\,, && (2.8.15)
\end{aligned}$$

where we used that, up to taking K_{b} large enough, $(2-\mathrm{b})K_{\mathrm{b}} - C_{\bar{f}_{\mathrm{b}}} \log(K_{\mathrm{b}}) \geq 2$. The above estimate implies that for any $\mathrm{q} \in \widetilde{M}$ we have

$$|\,\boldsymbol{x}(\mathrm{q})| \geq \frac{1}{2}\,(\mu + 2 - \widetilde{u}(\mathrm{q})) \qquad (2.8.16)$$

Now, we can define a suitable function ψ modifying the one in [Bar84, proof of Theorem 5.3], as follows

$$\log(\psi(s)) = \begin{cases} 4C_1\,(\mu+1)^{-2} \log s & \text{if } 0 < s \leq 1\\ 4C_1\,(\mu+1)^{-1}\,(\mu+2-s)^{-1}\,(s-1) & \text{if } 1 \leq s \leq \mu\\ C_1\,(s + 2 - \mu - 4(\mu+1)^{-1}) & \text{if } \mu \leq s \end{cases} \qquad (2.8.17)$$

with $\psi(0) = 0$, so that

$$\frac{\psi'(s)}{\psi(s)} = \frac{d}{ds}\log(\psi(s)) = \begin{cases} 4C_1\,(\mu+1)^{-2}\,s^{-1} & \text{if } 0 < s \leq 1\\ 4C_1\,(\mu+2-s)^{-2} & \text{if } 1 \leq s \leq \mu\\ C_1 & \text{if } \mu \leq s \end{cases} \qquad (2.8.18)$$

Due to (2.8.16), it holds that $C_1|\,\boldsymbol{x}\,|^{-2} \leq 4C_1\,(\mu + 2 - \widetilde{u})^{-2} = \frac{\psi'(\widetilde{u})}{\psi(\widetilde{u})}$ for $1 \leq \widetilde{u} \leq \mu$ and clearly $C_1|\,\boldsymbol{x}\,|^{-2} \leq C_1 = \frac{\psi'(\widetilde{u})}{\psi(\widetilde{u})}$ for $\widetilde{u} \geq \mu$ since $|\,\boldsymbol{x}\,| \geq K_{\mathrm{b}} > 1$. On the other hand, we have that

$$\widetilde{u} = u - \widetilde{w}^+_{K_\mathrm{b}} \geq \sup_M u - \widetilde{C}^I_{K_\mathrm{b}} - C_{\mathrm{b},\varepsilon} m = \mu > 1$$

and so we conclude that the second term in (2.8.12) can be neglected. Finally, we see that

$$\widetilde{u} \leq \sup_M u + C_{\mathrm{b},\varepsilon} = \mu + \widetilde{C}^I_{K_\mathrm{b}} + 2C_{\mathrm{b},\varepsilon}$$

and consequently, by the definition of ψ, we get

$$\sup_M \psi(\widetilde{u}) \leq \exp\left(2 + \widetilde{C}^I_{K_\mathrm{b}} + 2C_{\mathrm{b},\varepsilon}\right) .$$

Now, since (2.8.13) implies that for $|\,\boldsymbol{x}\,| \leq \frac{\mu}{2}$ it holds $\widetilde{u} \geq \mu + 2 - \mu = 2$, then we conclude that on $\left\{K_\mathrm{b} \leq |\,\boldsymbol{x}\,| \leq \frac{\mu}{2}\right\}$ we have $\psi(\widetilde{u}) \geq 1$ and hence (2.8.12) reads

$$C_3 \int_{K_\mathrm{b} \leq |\,\boldsymbol{x}\,| \leq \frac{\mu}{2}} |\,\boldsymbol{x}\,|^{-2-\varepsilon} \mathrm{d}\,\boldsymbol{x} \leq C_3 \int_{\widetilde{\Omega}} \psi(\widetilde{u}) |\,\boldsymbol{x}\,|^{-2-\varepsilon} \leq C_2 \exp\left(2 + \widetilde{C}^I_{K_\mathrm{b}} + 2C_{\mathrm{b},\varepsilon}\right) .$$

After integrating, we obtain a bound on μ. Rescaling back to any $m > 0$, this allows to conclude that

$$\sup_M u = \left(\mu + \widetilde{C}^I_{K_\mathrm{b}} + C_{\mathrm{b},\varepsilon}\right) m \leq \bar{C}_\mathrm{b} m$$

for a constant $\bar{C}_\mathrm{b} = \bar{C}_\mathrm{b}\left(C_o, C^I_{K_\mathrm{b}}, K_\mathrm{b}, \varepsilon, \mathrm{b}, C_{\mathrm{b},\varepsilon}\right)$. □

2.9 Proof of the main Theorem

Some final preliminary results. We start by addressing the preliminaries for the symmetry statement in the main Theorem 2.4.4, showing that boundary data which are invariant under the isometry $(\mathsf{X}, \mathsf{T}) \mapsto (-\mathsf{X}, -\mathsf{T})$ produce solutions containing the bifurcation sphere. For the sake of the proof, we need the *uniqueness of solution to the compact Dirichlet problem.* This is ensured on globally hyperbolic spacetimes satisfying the so-called *Timelike Convergence Condition* $\mathfrak{Ric}(V, V) \geq 0$ for any timelike vector field V, which is trivially satisfied in our case since $\mathfrak{Ric}_{\mathfrak{g}^K_m} = 0$. This statement has been addressed under different hypotheses in [Cho76; Cho77; BF78; Ger83; MT80], but the formalisation, as we need it, can be found in [Bar88b, Section 1].

Lemma 2.9.1. *Consider a compact maximal hypersurface $M = \{\mathsf{T} = F(\mathsf{X}, \theta_I)\}$, given as a graph of a smooth function F over $\{\mathsf{T} = 0\}$ and satisfying the hypotheses of Theorem 2.8.1. Assume that the two connected components of the boundary are symmetric, namely $\partial M_{\mathrm{I}'}$ coincide with the image of ∂M_I under this isometry. Then $\{\mathsf{X} = \mathsf{T} = 0\} \subset M$, id est M passes through the bifurcation sphere and it is fully contained in the regions* I *and* I$'$.

Proof. Consider a maximal hypersurface $M = \{\mathsf{T} = F(\mathsf{X}, \theta_I)\}$ with symmetric boundary data as in the hypothesis. Applying the isometry above to M produces an hypersurface $\widetilde{M} = \{\mathsf{T} = -F(-\mathsf{X}, \theta_I)\}$, with the same boundary data due to their symmetry. By the uniqueness of the Dirichlet problem –see the discussion above–, we have $M = \widetilde{M}$, namely M is invariant under the isometry. This, in particular, implies that $F(0, \theta_I) = -F(0, \theta_I)$ and this shows that the bifurcation sphere $\{\mathsf{T} = \mathsf{X} = 0\} \subset M$. Again by the spacelike condition, the hypersurface M has to lie between the null hypersurfaces $\{\mathsf{T} = \mathsf{X}\}$ and $\{\mathsf{T} = -\mathsf{X}\}$, namely in the regions I and I′, corresponding to the two copies of exterior Schwarzschild. $\square$

Before proving the main Theorem, we need to construct a suitable family of boundary data on $\mathscr{S}^o_{t_\mathrm{b}}$ for the sequence of compact Dirichlet problems.

Lemma 2.9.2. *The spacelike hypersurface $\mathscr{S}^o_{t_\mathrm{b}} = \{t_\mathrm{b} = 0\}$ can be exhausted by a sequence of bounded connected sets $\{\mathscr{U}_k\}_{k=1}^{\infty}$ with boundaries $\partial\mathscr{U}_k = \Sigma^2_k = (\Sigma^2_k)_\mathrm{I} \sqcup (\Sigma^2_k)_{\mathrm{I}'}$ consisting in two connected components $(\Sigma^2_k)_\mathrm{I}$ in* region I *and $(\Sigma^2_k)_{\mathrm{I}'}$ in* region I′*, such that there exists a uniform constant $C_\Sigma = C_\Sigma(\kappa_3)$ independent of k bounding the spacetime Riemannian norm of their spacetime mean curvature*

$$
{}^{t_\mathrm{b}}\|\vec{\mathscr{H}}_{\mathfrak{g}^K_m}(\Sigma^2_k)\| \leq \frac{C_\Sigma}{m} \tag{2.9.1}
$$

where ${}^{t_\mathrm{b}}\|\cdot\|$ is the time-dependent Riemannian norm defined in (2.2.5) *with respect to the time t_b of Theorem 2.7.1 and $\vec{\mathscr{H}}_{\mathfrak{g}^K_m}(\Sigma^2_k)$ is the spacetime mean curvature of Σ^2_k as defined in* (2.2.11).

Proof. Consider the coordinate spheres $\{r = km\} \cap \{t = 0\}$ in Schwarzschildean coordinates for $k > \kappa_6$ and we take the graph of $u_\mathrm{b}|_{\{r=km\}}$, namely

$$
\Sigma^2_k \coloneqq \{(t, r, \theta_I) \,:\, r = km,\, t = u_\mathrm{b}(r, \theta_I)\} \subset \mathscr{S}^o_{t_\mathrm{b}}
$$

for u_b defined in Proposition 2.6.1. We claim that these boundaries of the exhaustion $\mathscr{S}^o_{t_\mathrm{b}} \cap \{r < km\} = \mathscr{U}_k$ satisfy the desired features. The tangent vectors to Σ^2_k are spanned by $X_I = \partial_I + \partial_I u_\mathrm{b} \partial_t$ and induced metric

$$
(g_{\Sigma^2_k})_{IJ} = (g_k)_{IJ} - h_m\, \partial_I u_\mathrm{b}\, \partial_J u_\mathrm{b}\,, \quad (g_{\Sigma^2_k})^{IJ} = (g_k)^{IJ} + \frac{h_m\, (g_k)^{II} (g_k)^{JJ} \partial_I u_\mathrm{b}\, \partial_J u_\mathrm{b}}{(1 - h_m |\mathrm{d}u_\mathrm{b}|^2_{g_k})}\,,
$$

where $g_k = (km)^2 g_{\mathbb{S}^2}$ is the induced metric on $\{r = km\} \cap \{t = 0\}$. On the other hand, the normal bundle to Σ^2_k is spanned by

$$
\mathscr{T}_\mathrm{b}\big|_{\{r=km\}} = \frac{(\partial_t + h_m (g_k)^{IJ} \partial_J u_\mathrm{b}\, \partial_I)}{\sqrt{h_m}\sqrt{1 - h_m |\mathrm{d}u_\mathrm{b}|^2_{g_k}}}\,, \quad V_r = \frac{\sqrt{h_m}\,(\partial_r + \partial_r u_\mathrm{b} \partial_t)}{\sqrt{1 - h_m^2 (\partial_r u_\mathrm{b})^2}}
$$

which are also elements of an orthonormal adapted frame in the sense of (2.2.5). We then notice that using the form of the Christoffel symbols (2.3.6), (2.3.7) and (2.3.8) we have

$$\overline{\nabla}_{X_I} X_J = \overline{\nabla}_{\partial_I} \partial_J + \partial_{IJ} u_{\mathrm{b}} \, \partial_t = \overline{\Gamma}^r_{IJ} \, \partial_r + \overline{\Gamma}^K_{IJ} \, \partial_K + \partial_{IJ} u_{\mathrm{b}} \, \partial_t \, .$$

Notice that $\partial_I u_{\mathrm{b}}$ and $\partial_{IJ} u_{\mathrm{b}}$ are bounded as $\partial_I = \frac{\partial x_i}{\partial \theta_I} \partial_{x_i}$. Hence, to compute the time-dependent Riemannian norm (2.2.5), we need to check the boundedness independently of k of the following two quantities

$$\begin{aligned} \left| \left\langle \vec{\mathscr{H}}_{\mathfrak{g}^K_m}(\Sigma^2_k), \mathscr{T}_{\mathrm{b}} \right\rangle \right| &= \left| -(g_{\Sigma^2_k})^{IJ} \left\langle \overline{\nabla}_{X_I} X_J, \mathscr{T}_{\mathrm{b}} \right\rangle \right| \\ &\leq \left| (g_{\Sigma^2_k})^{IJ} \right| \sqrt{\frac{h_m}{1 - h_m |\mathrm{d}u_{\mathrm{b}}|^2_{g_k}}} \left(|\partial_{IJ} u_{\mathrm{b}}| + \left| \overline{\Gamma}^K_{IJ} \partial_K u_{\mathrm{b}} \right| \right) \end{aligned}$$

which decays to 0 like $(km)^{-2}$ as $k \to \infty$ and

$$\begin{aligned} \left| \left\langle \vec{\mathscr{H}}_{\mathfrak{g}^K_m}(\Sigma^2_k), V_r \right\rangle \right| &= \left| -(g_{\Sigma^2_k})^{IJ} \left\langle \overline{\nabla}_{X_I} X_J, V_r \right\rangle \right| \\ &\leq \left| (g_{\Sigma^2_k})^{IJ} \right| \sqrt{\frac{h_m}{1 - h_m |\mathrm{d}u_{\mathrm{b}}|^2_{g_k}}} \left((km)(g_{\mathbb{S}^2})_{IJ} + h_m \, |\partial_r u_{\mathrm{b}}| \, |\partial_{IJ} u_{\mathrm{b}}| \right) \end{aligned}$$

which decays to 0 like $(km)^{-1}$ as $k \to \infty$. This concludes the proof. □

Proof of the main Theorem. In the previous section, we established the a priori height estimate in Theorem 2.8.1. This ensures an a priori gradient estimate by Theorem 2.2.5 due to the properties of the time function constructed in Theorem 2.7.1. The existence of such a time function, in turn, relies on Proposition 2.6.1 to ensure the presence of barriers on a suitable choice of time slices for large radii, on the definition of a new time function in Proposition 2.5.2 and on the monotonicity result in Proposition 2.5.3. Having then defined in Lemma 2.9.2 a suitable sequence of codimension-2 boundary data on $\mathscr{S}^o_{t_{\mathrm{b}}}$, we have all the tools to prove the main theorem.

Proof of Theorem 2.4.4. The proof is analogous to [Bar84, proof of Theorem 5.4], but we will expand some of the steps.

Consider the boundaries $\{\Sigma^2_k\}_{k=1}^{\infty}$ of an exhaustion satisfying (2.9.1) as in Lemma 2.9.2 of the spacelike hypersurface $\mathscr{S}^o_{t_{\mathrm{b}}} = \{t_{\mathrm{b}} = 0\}$ given in Theorem 2.7.1. For any fixed k, the existence of a solution to the Dirichlet problem associated to the maximal surface equation with boundary data Σ^2_k, reading

$$\mathrm{H}_{\mathfrak{g}^K_m}(M_k) = 0 \, , \qquad \partial M_k = \Sigma^2_k \tag{2.9.2}$$

is ensured by the presence of the barriers at the crushing singularity as in Lemma 2.3.1–Proposition 2.3.2 and by [Bar84, Theorem 4.3]. We point out that, alternatively, one can argue directly by using the same a priori height estimate and the consequent a priori gradient estimate given by Theorem 2.2.5-(i) that need to be established in the following for constructing the desired hypersurface M_{b} as a (sub)-limit of $\{M_k\}_{k=1}^{\infty}$. As mentioned at the end of Section 2.2.1, the existence of such solutions relies on the *Leray–Schauder fixed point theorem* in the version [Bar84, Theorem 4.1]. Since for this sequence of compact Dirichlet problems, we work on compact regions of the globally hyperbolic spacetime $\big(\mathfrak{L}_m^K, \mathfrak{g}_m^K\big)$, the construction of such solutions is performed by considering graphs in Kruskal–Szekeres coordinates (2.3.16). In particular, all the solutions read $\{\mathsf{T} = F_k(\mathsf{X}, \theta_I)\}$ for smooth functions $F_k : \Omega_k \subset \{\mathsf{T} = 0\} \to \mathbb{R}$.

Now, for any $N \in \mathbb{N}$ let us consider the compact set $\mathscr{U}_N := \{|\mathsf{X}| \leq N,\ \mathsf{T} = 0\} \subset \mathbb{R} \times \mathbb{S}^2$ in Kruskal–Szekeres coordinates. For any sufficiently large k, $\mathscr{U}_N \subset \Omega_k$, so F_k are all defined on $\mathscr{U}_N$. We observe that the result in Theorem 2.8.1 immediately provides a height estimate uniform in k for the sequence $\{M_k\}_{k=1}^{\infty}$, with respect to the time function $\mathscr{t}_{\mathrm{b}}$ of Theorem 2.7.1. We can then apply Theorem 2.2.5-(i) as its hypotheses hold due to the properties of $\mathscr{t}_{\mathrm{b}}$ in Theorem 2.7.1-(i) combined with the fact that all the boundaries Σ_k^2 lie in the time slice $\mathscr{S}_{\mathscr{t}_{\mathrm{b}}}^o = \{\mathscr{t}_{\mathrm{b}} = 0\}$ and satisfy (2.9.1) by Lemma 2.9.2. This provides a uniform control on the geometric quantity $\mathscr{v}_{\mathscr{t}_{\mathrm{b}}}(M_k)$. Now, due to the barriers at the crushing singularity Lemma 2.3.1–Proposition 2.3.2, we can consider the compact region $\big\{|\mathsf{X}| \leq N\big\} \cap \big\{r \geq \frac{3}{2}m\big\}$ and the maximum therein of the quantity $-\langle \mathscr{T}_{\mathrm{b}}, \mathscr{T}\rangle$, where $\mathscr{T}_{\mathrm{b}}$ and $\mathscr{T}$ are the unit normals to the time slices of $\mathscr{t}_{\mathrm{b}}$ and T, respectively. Again, by [Bar84, Lemma 3.3] combined with the uniform bound on $\mathscr{v}_{\mathscr{t}_{\mathrm{b}}}(M_k)$, we find that there exists a constant independent of k

$$C_N = C_N\left(C_o, C_{K_{\mathrm{b}}}^I, K_{\mathrm{b}}, \mathrm{b}, \varepsilon, C_{\bar{f}_{\mathrm{b}}}, \sup_{\{|\mathsf{X}|\leq N\}\cap\left\{r\geq\frac{3}{2}m\right\}} \langle \mathscr{T}_{\mathrm{b}}, \mathscr{T}\rangle\right),$$

with C_o as in Theorem 2.7.1, K_{b} and $C_{\bar{f}_{\mathrm{b}}}$ as in Proposition 2.6.1 and $C_{K_{\mathrm{b}}}^I$ as in Proposition 2.5.3, such that

$$\sup_{\{|\mathsf{X}|\leq N\}\cap\left\{r\geq\frac{3}{2}m\right\}} \mathscr{v}_{\mathsf{T}}(F_k) \leq C_N\,. \tag{2.9.3}$$

Notice that by definition of $\mathscr{v}_{\mathsf{T}}(F_k)$ in (2.2.12), the above estimate is not only ensuring an a priori estimate on $\|F_k\|_{C^1(\mathscr{U}_N)}$ uniform in k, but in particular is giving an a priori estimate on the ellipticity of the operator (2.2.26) and on the spacelike nature of the hypersurfaces, also uniform in k. By Theorem 2.2.7 this, in turn, gives the required bounds, uniform in k, on the coefficients of (2.2.26) and the interior Schauder estimate Theorem 2.2.6 yields

$$\|F_k\|_{C^{2,\alpha}(\mathscr{U}_N')} \leq \bar{C}_N$$

for any $\mathscr{U}_N' \subset\subset \mathscr{U}_N \subset \mathbb{R} \times \mathbb{S}^2$, with $\bar{C}_N = \bar{C}_N(C_N, \mathscr{U}_N', \mathscr{U}_N)$ independent of k. Hence, we can extract a subsequence F_{k_i} converging to a function $F_{\mathrm{b}}^N \in C^{2,\beta}(\mathscr{U}_N)$ for any $0 < \beta <$

$\alpha < 1$, with

$$\|F_{\mathrm{b}}^N - F_{k_i}\|_{C^{2,\beta}(\mathscr{U}_N)} \to 0 \qquad \text{as } i \to \infty\,.$$

Combined with the uniform estimate (2.9.3), which ensures the spacelike condition on F_{b}^N, this convergence implies that $\mathrm{H}_{\mathfrak{g}_m^K}(F_{\mathrm{b}}^N) = 0$. Then, by bootstrapping the equation in the form (2.2.26), one obtains $F_{\mathrm{b}}^N \in C^\infty(\mathscr{U}_N)$. By a standard diagonal argument on the sequence of compact sets $\mathscr{U}_N$ for increasing $N \to \infty$, we have constructed a smooth spacelike entire hypersurface $M_{\mathrm{b}} \hookrightarrow \big(\mathfrak{L}_m^K, \mathfrak{g}_m^K\big)$.

In addition, by construction, the hypersurface M_{b} is asymptotic to $\mathscr{S}^o_{t_{\mathrm{b}}} \in \mathscr{B}_{\mathrm{b}}$ as required in (2.4.2). This follows from the fact that the a priori height estimate in Theorem 2.8.1 ensures that all the solutions M_k satisfy (2.8.2), namely on $\{|\,\boldsymbol{x}\,| \geq \bar{K}\}$ they lie between the barriers $\mathscr{W}^{\pm}_{\mathrm{b},\bar{K},\varepsilon}$ as given in Proposition 2.6.1.

The fact that this entire maximal hypersurface contains the bifurcation sphere $\{\mathsf{T} = \mathsf{X} = 0\} \in M_{\mathrm{b}}$ is an immediate consequence of Lemma 2.9.1 on each of the solutions M_k which have been constructed with symmetric data as a consequence of Theorem 2.7.1-(*ii*) and Lemma 2.9.2. The uniform convergence of their height function to F_{b} transfers the property to M_{b}. Finally, since M_{b} is spacelike, it is contained in the regions delimited by the null hypersurfaces $\{\mathsf{T} = \mathsf{X}\}$ and $\{\mathsf{T} = -\mathsf{X}\}$, that is, in the *regions* I and I$'$. □

2.9.1 Exterior solutions in exterior Schwarzschild

The employed technique to tackle Theorem 2.4.4 can be used to show the existence of exterior solutions in exterior Schwarzschild for a natural class of boundary data, extending the results in Minkowski by [HY21]. We will sketch the proof.

Proposition 2.9.3. *Consider the exterior Schwarzschild spacetime* $\big(\mathfrak{L}_m^S, \mathfrak{g}_m^S\big)$ *given in Schwarzschildian coordinates* (2.3.2) *and a connected codimension-2 surface* Σ *contained in a spacelike hypersurface* $\Sigma \hookrightarrow \mathscr{S}^3 \hookrightarrow \{r > R_o > 2m\}$. *Assume that there exists a smooth function* $\bar{\varphi} : \mathbb{R}^3 \setminus \overline{B}_{R_o} \to \mathbb{R}$ *and a domain* $\Omega \subset \mathbb{R}^3$ *with* $\overline{B}_{R_o} \subset \Omega$ *such that*

$$\mathscr{S}^3 = \{t = \bar{\varphi}(r, \theta_I)\}, \qquad \Sigma = \{t = \bar{\varphi}|_{\partial\Omega}(r, \theta_I)\}\,.$$

Then for any $\mathrm{b} \in (0, 1]$, *there exists an hypersurface* M_{b} *with boundary* Σ *and asymptotic to* $\mathscr{S}_{\mathrm{b}} \in \mathscr{B}_{\mathrm{b}}$ *as in Theorem 2.4.4.*

Proof. The proof is similar to the one of Theorem 2.4.4. We need to construct a time function so that Σ belongs to the zero-time slice. This can be obtained by gluing the time function generated by $\mathscr{S}^3$ via staticity with the time function generated by the hypersurface $\mathscr{S}_{\mathrm{b}} \in \mathscr{B}_{\mathrm{b}}$ given by Proposition 2.6.1. The gluing procedure is similar to the one in Section 2.7. It is then easy to see that the weak version of the *Bartnik's uniform* *interior condition* is satisfied between any two radii in exterior Schwarzschild; combining this to Proposition 2.5.1 and Lemma 2.9.2, we have all the conditions needed to prove the claim as in the proofs of Theorem 2.8.1 and Theorem 2.4.4. □

Appendix

A.1 Bartnik's spacetime conditions on the new time function

The computations of this section are dedicated to proving the bounds (2.5.12) in terms of the time-dependent Riemannian norm (2.2.5) induced by the new time function $\widetilde{\mathsf{T}}$ defined in Proposition 2.5.2. Recall that by the definition of ${}^{\widetilde{\mathsf{T}}}\|\cdot\|$ in (2.2.5), it is sufficient to compute and bound in absolute value the tensors $\log\alpha_{\widetilde{\mathsf{T}}}$, $\overline{\nabla}\widetilde{\mathscr{T}}$ and their covariant derivative on any combination of the adapted orthonormal frame

$$\left\{\widetilde{\mathscr{T}}, \widetilde{V}_{\mathsf{X}}, \widetilde{V}_I\right\}, \quad I = 1, 2$$

as given by (2.5.17), (2.5.18) and (2.5.19).

The notation will be consistent with the one provided for the Kruskal–Szekeres coordinates in Section 2.3.2. In particular, we recall the definition of ξ_m in (2.3.16), which computed on a radius $\kappa_0 m$ reads

$$\xi_m(\kappa_0 m) = \frac{32m^2}{\kappa_0} e^{-\frac{\kappa_0}{2}} = C(\kappa_0)\, m^2\,,$$

with $C(\kappa_0)$ independent of m. On the other hand, we highlight that the Kruskal–Szekeres coordinates T, X are scaling invariant. These two fact allow us to check that all the estimates scale as expected.

Throughout this appendix, when estimating quantities by their value on a radius $\kappa_0 m > 0$, we will denote by $C(\kappa_0)$ any constant depending on κ_0 but not on m.

The desired final estimates to prove (2.5.12) are (A.1.4), (A.1.10), (A.1.22), (A.1.46).

Bound on $\log\alpha_{\widetilde{\mathsf{T}}}$

The lapse computed in (2.5.15) can be written by the identity (2.3.17) as follows

$$\alpha_{\widetilde{\mathsf{T}}}^2 = \frac{2m}{r} \mathrm{e}^{-\frac{r}{2m}} \frac{(1 + \mathsf{X}^2 - \mathsf{T}^2)}{1 + f}$$

$$= \frac{1}{1+f}\,\frac{2m}{r}\mathrm{e}^{-\frac{r}{2m}}\left(1+\left(\frac{r}{2m}-1\right)\mathrm{e}^{\frac{r}{2m}}\right)$$
$$= \frac{1}{1+f}\left(1+\frac{2m}{r}(\mathrm{e}^{-\frac{r}{2m}}-1)\right)$$

Now, we can easily compute the upper bound

$$\alpha_{\tilde{\mathsf{T}}}^2 \leq 1 \tag{A.1.1}$$

where we used that $\frac{1}{1+f} \leq 1$ combined with $\lim_{r\to+\infty}\left(1+\frac{2m}{r}(\mathrm{e}^{-\frac{r}{2m}}-1)\right) = 1$ and

$$\frac{d}{dr}\left(\frac{\xi_m}{16m^2}(1+\mathsf{X}^2-\mathsf{T}^2)\right) = \frac{d}{dr}\left(1+\frac{2m}{r}(\mathrm{e}^{-\frac{r}{2m}}-1)\right) = \frac{2m}{r^2}\left(1-\mathrm{e}^{-\frac{r}{2m}}\right) - \frac{\mathrm{e}^{-\frac{r}{2m}}}{r} > 0\,, \tag{A.1.2}$$

as it is equivalent to $\frac{2m}{r} > \frac{1}{\mathrm{e}^{\frac{r}{2m}}-1}$ or $\mathrm{e}^{\frac{r}{2m}} > \frac{r}{2m}+1$.

On the other hand, a lower bound can be computed by plugging (2.3.17) in (2.5.16) and getting

$$\frac{1}{1+f} = \frac{1}{1+\frac{\mathsf{T}^2}{1+\mathsf{X}^2}\frac{1}{\left(1+\left(\frac{r}{2m}-1\right)\mathrm{e}^{\frac{r}{2m}}\right)}} \geq \frac{1}{1+\frac{1}{\left(1+\left(\frac{r}{2m}-1\right)\mathrm{e}^{\frac{r}{2m}}\right)}}$$

Again, one can check that $\frac{1}{1+\frac{1}{\left(1+\left(\frac{r}{2m}-1\right)\mathrm{e}^{\frac{r}{2m}}\right)}}$ is a monotone increasing function of r and hence for $r > \kappa_0 m$, we have

$$\alpha_{\tilde{\mathsf{T}}}^2 \geq \frac{1}{1+\frac{1}{\left(1+\left(\frac{\kappa_0}{2}-1\right)\mathrm{e}^{\frac{\kappa_0}{2}}\right)}}\left(1+\frac{2}{\kappa_0}(\mathrm{e}^{-\frac{\kappa_0}{2}}-1)\right) > 0\,. \tag{A.1.3}$$

Combining (A.1.3) with (A.1.1), we conclude that on $\{r > \kappa_0 m\}$, there exists a constant $C_o = C_o(\kappa_0)$ such that

$$^{\tilde{\mathsf{T}}}\|\log\alpha_{\tilde{\mathsf{T}}}\| \leq C_o(\kappa_0)\,. \tag{A.1.4}$$

Bound on $\overline{\nabla}\log\alpha_{\tilde{\mathsf{T}}}$

Observe that $2\mathrm{d}\log\alpha_{\tilde{\mathsf{T}}} = 2\alpha_{\tilde{\mathsf{T}}}^{-1}\mathrm{d}\alpha_{\tilde{\mathsf{T}}} = \alpha_{\tilde{\mathsf{T}}}^{-2}\mathrm{d}\alpha_{\tilde{\mathsf{T}}}^2$, so we compute

$$\begin{aligned}
\alpha_{\tilde{\mathsf{T}}}^{-2}\mathrm{d}\alpha_{\tilde{\mathsf{T}}}^2 &= \left(\frac{16m^2(1+f)}{\xi_m\,(1+\mathsf{X}^2-\mathsf{T}^2)}\right)\,\mathrm{d}\left(\frac{\xi_m}{16m^2}\,\frac{(1+\mathsf{X}^2-\mathsf{T}^2)}{1+f}\right)\\
&= \left(\frac{16m^2(1+f)}{\xi_m\,(1+\mathsf{X}^2-\mathsf{T}^2)}\right)\Bigg[\frac{(1+\mathsf{X}^2-\mathsf{T}^2)}{16m^2\,(1+f)}\frac{d\xi_m}{dr}\,\mathrm{d}r + \frac{2\,\xi_m}{16m^2(1+f)}\,(\mathsf{X}\mathrm{d}\mathsf{X}-\mathsf{T}\mathrm{d}\mathsf{T})\\
&\quad - \frac{\xi_m(1+\mathsf{X}^2-\mathsf{T}^2)}{16m^2\,(1+f)^2}\,(\partial_{\mathsf{X}}f\mathrm{d}\mathsf{X}+\partial_{\mathsf{T}}f\mathrm{d}\mathsf{T})\Bigg]\\
&= -\frac{1}{\xi_m}\frac{\xi_m}{2m}\left(1+\frac{2m}{r}\right)\frac{\xi_m}{4m}\,(\mathsf{X}\mathrm{d}\mathsf{X}-\mathsf{T}\mathrm{d}\mathsf{T}) + \frac{2}{1+\mathsf{X}^2-\mathsf{T}^2}\,(\mathsf{X}\mathrm{d}\mathsf{X}-\mathsf{T}\mathrm{d}\mathsf{T})
\end{aligned}$$

$$
\begin{aligned}
&\quad - \frac{1}{(1+f)} \left(\partial_{\mathsf{X}} f \mathrm{d}\mathsf{X} + \partial_{\mathsf{T}} f \mathrm{d}\mathsf{T}\right) \\
&= \left(\frac{2}{1+\mathsf{X}^2-\mathsf{T}^2} - \frac{\xi_m}{8m^2}\left(1+\frac{2m}{r}\right)\right) (\mathsf{X}\mathrm{d}\mathsf{X} - \mathsf{T}\mathrm{d}\mathsf{T}) - \frac{1}{(1+f)} \left(\partial_{\mathsf{X}} f \mathrm{d}\mathsf{X} + \partial_{\mathsf{T}} f \mathrm{d}\mathsf{T}\right)
\end{aligned}
$$

where we used (2.3.24) and (2.3.23). One can then find, recalling the definition of the function f in (2.5.16)

$$
\partial_{\mathsf{T}} f = 2\,\frac{f}{\mathsf{T}}\,\left(1+(1+\mathsf{X}^2)f\right)\ , \qquad \partial_{\mathsf{X}} f = -2\,\frac{f^2}{\mathsf{T}^2}\,\mathsf{X}\left((1+\mathsf{X}^2-\mathsf{T}^2)+(1+\mathsf{X}^2)\right)\ . \tag{A.1.5}
$$

We now have to compute $\mathrm{d}\log \alpha_{\tilde{\mathsf{T}}}$ on each of the elements of the adapted orthonormal frame.

Bound on $\mathrm{d}\log\alpha_{\tilde{\mathsf{T}}}(\widetilde{\mathscr{T}})$**.** For simplicity, we split the calculation in parts, computing

$$
(\mathsf{X}\mathrm{d}\mathsf{X} - \mathsf{T}\mathrm{d}\mathsf{T})\left(\sqrt{1+\mathsf{X}^2}\,\partial_{\mathsf{T}} + \frac{\mathsf{X}\mathsf{T}}{\sqrt{1+\mathsf{X}^2}}\partial_{\mathsf{X}}\right) = \frac{\mathsf{X}^2\mathsf{T}}{\sqrt{1+\mathsf{X}^2}} - \mathsf{T}\sqrt{1+\mathsf{X}^2} = -\frac{\mathsf{T}}{\sqrt{1+\mathsf{X}^2}}
$$

and, recalling the definition of the function f in (2.5.16),

$$
\begin{aligned}
(\partial_{\mathsf{X}} f \mathrm{d}\mathsf{X} + \partial_{\mathsf{T}} f \mathrm{d}\mathsf{T})&\left(\sqrt{1+\mathsf{X}^2}\,\partial_{\mathsf{T}} + \frac{\mathsf{X}\mathsf{T}}{\sqrt{1+\mathsf{X}^2}}\partial_{\mathsf{X}}\right) = \partial_{\mathsf{T}} f\,\sqrt{1+\mathsf{X}^2} + \frac{\partial_{\mathsf{X}} f\,\mathsf{X}\mathsf{T}}{\sqrt{1+\mathsf{X}^2}} \\
&= \frac{2f\,\sqrt{1+\mathsf{X}^2}}{\mathsf{T}}\left(1+(1+\mathsf{X}^2)f - \frac{\mathsf{X}^2}{1+\mathsf{X}^2}f\left((1+\mathsf{X}^2-\mathsf{T}^2)+(1+\mathsf{X}^2)\right)\right) \\
&= \frac{2f\,\sqrt{1+\mathsf{X}^2}}{\mathsf{T}}\left(1+f+f\mathsf{X}^2 - f\mathsf{X}^2 - f\frac{\mathsf{X}^2(1+\mathsf{X}^2-\mathsf{T}^2)}{1+\mathsf{X}^2}\right) \\
&= \frac{2}{1+\mathsf{X}^2-\mathsf{T}^2}\,\frac{\mathsf{T}}{\sqrt{1+\mathsf{X}^2}}\left(1+f-\frac{\mathsf{T}^2\mathsf{X}^2}{(1+\mathsf{X}^2)^2}\right)
\end{aligned}
$$

We can finally compute

$$
\begin{aligned}
\left|\alpha_{\tilde{\mathsf{T}}}^{-2}\mathrm{d}\alpha_{\tilde{\mathsf{T}}}^2(\widetilde{\mathscr{T}})\right| &= \frac{1}{\sqrt{\xi_m(1+f)(1+\mathsf{X}^2-\mathsf{T}^2)}}\,\frac{|\mathsf{T}|}{\sqrt{1+\mathsf{X}^2}}\left|\left(\frac{\xi_m}{8m^2}\left(1+\frac{2m}{r}\right) - \frac{2}{1+\mathsf{X}^2-\mathsf{T}^2}\right)\right. \\
&\qquad \left. - \frac{2}{(1+f)(1+\mathsf{X}^2-\mathsf{T}^2)}\left(1+f-\frac{\mathsf{T}^2\mathsf{X}^2}{(1+\mathsf{X}^2)^2}\right)\right| \\
&\le \frac{1}{\sqrt{\xi(1+\mathsf{X}^2-\mathsf{T}^2)}}\left(\frac{\xi_m\left(1+\frac{2m}{r}\right)}{8m^2} + \frac{5}{1+\mathsf{X}^2-\mathsf{T}^2}\right) \le \frac{C(\kappa_0)}{m}
\end{aligned} \tag{A.1.6}
$$

where we used that $\mathsf{T}^2 < 1+\mathsf{X}^2$, $f \ge 0$, the monotonicity given by (A.1.2) and the fact that

$$
\frac{1}{1+\mathsf{X}^2-\mathsf{T}^2} = \frac{1}{1+\left(\frac{r}{2m}-1\right)e^{\frac{r}{2m}}} \le \frac{1}{1+\left(\frac{\kappa_0}{2}-1\right)e^{\frac{\kappa_0}{2}}} \tag{A.1.7}
$$

again by monotonicity.

Bound on $\mathrm{d}\log\alpha_{\widetilde{\mathsf{T}}}(\widetilde{V}_{\mathsf{X}})$**.** As before, we split the computations, starting with

$$(\mathsf{X}\mathrm{d}\mathsf{X}-\mathsf{T}\mathrm{d}\mathsf{T})\left(\frac{\mathsf{X}\mathsf{T}}{\sqrt{1+\mathsf{X}^2}}\partial_{\mathsf{T}}+\sqrt{1+\mathsf{X}^2}\,\partial_{\mathsf{X}}\right)=-\frac{\mathsf{X}\mathsf{T}^2}{\sqrt{1+\mathsf{X}^2}}+\mathsf{X}\sqrt{1+\mathsf{X}^2}=\mathsf{X}\,\frac{1+\mathsf{X}^2-\mathsf{T}^2}{\sqrt{1+\mathsf{X}^2}}$$

and, using (A.1.5), the following identity

$$\begin{aligned}(\partial_{\mathsf{X}}f\ \mathrm{d}\mathsf{X}+\partial_{\mathsf{T}}f\mathrm{d}\mathsf{T})&\left(\frac{\mathsf{X}\mathsf{T}}{\sqrt{1+\mathsf{X}^2}}\partial_{\mathsf{T}}+\sqrt{1+\mathsf{X}^2}\,\partial_{\mathsf{X}}\right)=\frac{\partial_{\mathsf{T}}f\,\mathsf{X}\mathsf{T}}{\sqrt{1+\mathsf{X}^2}}+\partial_{\mathsf{X}}f\,\sqrt{1+\mathsf{X}^2}\\
&=2f\frac{\mathsf{X}}{\sqrt{1+\mathsf{X}^2}}\left((1+(1+\mathsf{X}^2)f)-\frac{f(1+\mathsf{X}^2)}{\mathsf{T}^2}\left((1+\mathsf{X}^2-\mathsf{T}^2)+(1+\mathsf{X}^2)\right)\right)\\
&=2f\frac{\mathsf{X}}{\sqrt{1+\mathsf{X}^2}}\left(\frac{1+\mathsf{X}^2}{1+\mathsf{X}^2-\mathsf{T}^2}-1-\frac{1+\mathsf{X}^2}{1+\mathsf{X}^2-\mathsf{T}^2}\right)=-\frac{2f\,\mathsf{X}}{\sqrt{1+\mathsf{X}^2}}\,.\end{aligned}$$

Combining the above, we get

$$\begin{aligned}\left|\alpha^{-2}\mathrm{d}\alpha^2(\widetilde{V}_{\mathsf{X}})\right|&=\frac{1}{\sqrt{\xi_m(1+f)(1+\mathsf{X}^2-\mathsf{T}^2)}}\frac{|\mathsf{X}|}{\sqrt{1+\mathsf{X}^2}}\left|\left(\frac{\xi_m\left(1+\frac{2m}{r}\right)}{8m^2}-\frac{2}{1+\mathsf{X}^2-\mathsf{T}^2}\right)\right.\\
&\qquad\left.-\frac{2f}{1+f}\right|\\
&\le\frac{2}{\sqrt{\xi_m(1+\mathsf{X}^2-\mathsf{T}^2)}}\left(\frac{\xi_m\left(1+\frac{2m}{r}\right)}{16m^2}+\frac{1}{1+\mathsf{X}^2-\mathsf{T}^2}+1\right)\le\frac{C(\kappa_0)}{m}\end{aligned}\tag{A.1.8}$$

where the bound follows from the monotonicity of the quantities involved, as computed above.

Bound on $\mathrm{d}\log\alpha_{\widetilde{\mathsf{T}}}(\widetilde{V}_I)$**.** It is easy to see that

$$\left|\alpha_{\widetilde{\mathsf{T}}}^{-2}\mathrm{d}\alpha_{\widetilde{\mathsf{T}}}^2(\widetilde{V}_I)\right|=0\tag{A.1.9}$$

Combining (A.1.6), (A.1.8) and (A.1.9), we conclude

$$^{\widetilde{\mathsf{T}}}\|\,\overline{\nabla}\log\alpha_{\widetilde{\mathsf{T}}}\|\le\frac{C(\kappa_0)}{m}\,.\tag{A.1.10}$$

Bound on $\overline{\nabla}\,\widetilde{\mathscr{T}}$

Bound on $\overline{\nabla}_{\widetilde{V}_{\mathsf{X}}}\widetilde{\mathscr{T}}$**.** First, we have

$$\overline{\nabla}_{\partial_{\mathsf{X}}+\frac{\mathsf{X}\mathsf{T}}{1+\mathsf{X}^2}\partial_{\mathsf{T}}}\left(\frac{\mathsf{X}\mathsf{T}}{1+\mathsf{X}^2}\partial_{\mathsf{X}}+\partial_{\mathsf{T}}\right)=\overline{\nabla}_{\partial_{\mathsf{X}}}\left(\frac{\mathsf{X}\mathsf{T}}{1+\mathsf{X}^2}\partial_{\mathsf{X}}+\partial_{\mathsf{T}}\right)+\frac{\mathsf{X}\mathsf{T}}{1+\mathsf{X}^2}\overline{\nabla}_{\partial_{\mathsf{T}}}\left(\frac{\mathsf{X}\mathsf{T}}{1+\mathsf{X}^2}\partial_{\mathsf{X}}+\partial_{\mathsf{T}}\right)$$

and hence we need the following computations

$$\overline{\nabla}_{\partial_{\mathsf{X}}}\left(\frac{\mathsf{XT}}{1+\mathsf{X}^2}\partial_{\mathsf{X}}+\partial_{\mathsf{T}}\right)=\frac{\mathsf{T}}{1+\mathsf{X}^2}\left(1-\frac{2\mathsf{X}^2}{1+\mathsf{X}^2}\right)\partial_{\mathsf{X}}+\frac{\mathsf{XT}}{1+\mathsf{X}^2}\left(\overline{\Gamma}^{\mathsf{X}}_{\mathsf{XX}}\partial_{\mathsf{X}}+\overline{\Gamma}^{\mathsf{T}}_{\mathsf{XX}}\partial_{\mathsf{T}}+\overline{\Gamma}^{I}_{\mathsf{XX}}\partial_I\right)$$
$$+\overline{\Gamma}^{\mathsf{X}}_{\mathsf{XT}}\partial_{\mathsf{X}}+\overline{\Gamma}^{\mathsf{T}}_{\mathsf{XT}}\partial_{\mathsf{T}}+\overline{\Gamma}^{I}_{\mathsf{XT}}\partial_I \tag{A.1.11}$$

$$\overline{\nabla}_{\partial_{\mathsf{T}}}\left(\frac{\mathsf{XT}}{1+\mathsf{X}^2}\partial_{\mathsf{X}}+\partial_{\mathsf{T}}\right)=\frac{\mathsf{X}}{1+\mathsf{X}^2}\partial_{\mathsf{X}}+\frac{\mathsf{XT}}{1+\mathsf{X}^2}\left(\overline{\Gamma}^{\mathsf{X}}_{\mathsf{TX}}\partial_{\mathsf{X}}+\overline{\Gamma}^{\mathsf{T}}_{\mathsf{TX}}\partial_{\mathsf{T}}+\overline{\Gamma}^{I}_{\mathsf{TX}}\partial_I\right)$$
$$+\overline{\Gamma}^{\mathsf{X}}_{\mathsf{TT}}\partial_{\mathsf{X}}+\overline{\Gamma}^{\mathsf{T}}_{\mathsf{TT}}\partial_{\mathsf{T}}+\overline{\Gamma}^{I}_{\mathsf{TT}}\partial_I \tag{A.1.12}$$

We then obtain

$$\begin{aligned}\left[\overline{\nabla}_{\partial_{\mathsf{X}}+\frac{\mathsf{XT}}{1+\mathsf{X}^2}\partial_{\mathsf{T}}}\left(\frac{\mathsf{XT}}{1+\mathsf{X}^2}\partial_{\mathsf{X}}+\partial_{\mathsf{T}}\right)\right]^{\mathsf{T}}&=\frac{\mathsf{XT}}{1+\mathsf{X}^2}\overline{\Gamma}^{\mathsf{T}}_{\mathsf{XX}}+\overline{\Gamma}^{\mathsf{T}}_{\mathsf{XT}}+\frac{\mathsf{X}^2\mathsf{T}^2}{(1+\mathsf{X}^2)^2}\overline{\Gamma}^{\mathsf{T}}_{\mathsf{TX}}+\frac{\mathsf{XT}}{1+\mathsf{X}^2}\overline{\Gamma}^{\mathsf{T}}_{\mathsf{TT}}\\&=2\frac{\mathsf{XT}}{1+\mathsf{X}^2}\overline{\Gamma}^{\mathsf{T}}_{\mathsf{TT}}+\left(1+\frac{\mathsf{X}^2\mathsf{T}^2}{(1+\mathsf{X}^2)^2}\right)\overline{\Gamma}^{\mathsf{X}}_{\mathsf{XX}}\end{aligned}$$

$$\begin{aligned}\left[\overline{\nabla}_{\partial_{\mathsf{X}}+\frac{\mathsf{XT}}{1+\mathsf{X}^2}\partial_{\mathsf{T}}}\left(\frac{\mathsf{XT}}{1+\mathsf{X}^2}\partial_{\mathsf{X}}+\partial_{\mathsf{T}}\right)\right]^{\mathsf{X}}&=\frac{\mathsf{T}}{1+\mathsf{X}^2}\left(1-\frac{2\mathsf{X}^2}{1+\mathsf{X}^2}\right)+\frac{\mathsf{XT}}{1+\mathsf{X}^2}\overline{\Gamma}^{\mathsf{X}}_{\mathsf{XX}}+\overline{\Gamma}^{\mathsf{X}}_{\mathsf{XT}}\\&\quad+\frac{\mathsf{X}^2\mathsf{T}}{(1+\mathsf{X}^2)^2}+\frac{\mathsf{X}^2\mathsf{T}^2}{(1+\mathsf{X}^2)^2}\overline{\Gamma}^{\mathsf{X}}_{\mathsf{TX}}+\frac{\mathsf{XT}}{1+\mathsf{X}^2}\overline{\Gamma}^{\mathsf{X}}_{\mathsf{TT}}\\&=\frac{\mathsf{T}}{(1+\mathsf{X}^2)^2}+\left(1+\frac{\mathsf{X}^2\mathsf{T}^2}{(1+\mathsf{X}^2)^2}\right)\overline{\Gamma}^{\mathsf{T}}_{\mathsf{TT}}+2\frac{\mathsf{XT}}{1+\mathsf{X}^2}\overline{\Gamma}^{\mathsf{X}}_{\mathsf{XX}}\end{aligned}$$

Combining these results, we have

$$\begin{aligned}&\mathfrak{g}^K_m\left(\overline{\nabla}_{\partial_{\mathsf{X}}+\frac{\mathsf{XT}}{1+\mathsf{X}^2}\partial_{\mathsf{T}}}\left(\frac{\mathsf{XT}}{1+\mathsf{X}^2}\partial_{\mathsf{X}}+\partial_{\mathsf{T}}\right),\,\partial_{\mathsf{X}}+\frac{\mathsf{XT}}{1+\mathsf{X}^2}\partial_{\mathsf{T}}\right)=\\&\qquad=\xi_m\left[\frac{\mathsf{T}}{(1+\mathsf{X}^2)^2}+\left(1+\frac{\mathsf{X}^2\mathsf{T}^2}{(1+\mathsf{X}^2)^2}\right)\overline{\Gamma}^{\mathsf{T}}_{\mathsf{TT}}+2\frac{\mathsf{XT}}{1+\mathsf{X}^2}\overline{\Gamma}^{\mathsf{X}}_{\mathsf{XX}}\right.\\&\qquad\qquad\left.-2\frac{\mathsf{X}^2\mathsf{T}^2}{(1+\mathsf{X}^2)^2}\overline{\Gamma}^{\mathsf{T}}_{\mathsf{TT}}-\frac{\mathsf{XT}}{1+\mathsf{X}^2}\left(1+\frac{\mathsf{X}^2\mathsf{T}^2}{(1+\mathsf{X}^2)^2}\right)\overline{\Gamma}^{\mathsf{X}}_{\mathsf{XX}}\right]\\&\qquad=\xi_m\left[\frac{\mathsf{T}}{(1+\mathsf{X}^2)^2}+\left(1-\frac{\mathsf{X}^2\mathsf{T}^2}{(1+\mathsf{X}^2)^2}\right)\left(\overline{\Gamma}^{\mathsf{T}}_{\mathsf{TT}}+\frac{\mathsf{XT}}{1+\mathsf{X}^2}\overline{\Gamma}^{\mathsf{X}}_{\mathsf{XX}}\right)\right]\\&\qquad=\xi_m\left[\frac{\mathsf{T}}{(1+\mathsf{X}^2)^2}+\left(1-\frac{\mathsf{X}^2\mathsf{T}^2}{(1+\mathsf{X}^2)^2}\right)\left(1-\frac{\mathsf{X}^2}{1+\mathsf{X}^2}\right)\overline{\Gamma}^{\mathsf{T}}_{\mathsf{TT}}\right]\\&\qquad=\xi_m\,\frac{\mathsf{T}}{(1+\mathsf{X}^2)^2}\left[1+\left((1+\mathsf{X}^2-\mathsf{T}^2)+\frac{\mathsf{T}^2}{1+\mathsf{X}^2}\right)\frac{\xi_m}{16m^2}\left(1+\frac{2m}{r}\right)\right]\end{aligned}$$

$$= \xi_m \frac{\mathsf{T}}{(1+\mathsf{X}^2)^2}\left[1 + \frac{\xi_m\,(1+\mathsf{X}^2-\mathsf{T}^2)(1+f)}{16m^2}\left(1+\frac{2m}{r}\right)\right],$$

where we used $\overline{\Gamma}^{\mathsf{X}}_{\mathsf{XX}} = -\frac{\mathsf{X}}{\mathsf{T}}\overline{\Gamma}^{\mathsf{T}}_{\mathsf{TT}}$, as in (2.3.26). Hence, we finally get

$$\begin{aligned}
&\left|\mathfrak{g}^K_m\left(\overline{\nabla}_{\widetilde{V}_{\mathsf{X}}}\widetilde{\mathscr{T}},\widetilde{V}_{\mathsf{X}}\right)\right| = \\
&= \left|\frac{1+\mathsf{X}^2}{\xi_m(1+f)(1+\mathsf{X}^2-\mathsf{T}^2)}\,\mathfrak{g}^K_m\left(\overline{\nabla}_{\partial_{\mathsf{X}}+\frac{\mathsf{XT}}{1+\mathsf{X}^2}\partial_{\mathsf{T}}}\widetilde{\mathscr{T}}\,,\partial_{\mathsf{X}}+\frac{\mathsf{XT}}{1+\mathsf{X}^2}\partial_{\mathsf{T}}\right)\right| \\
&= \left|-\left(\frac{1+\mathsf{X}^2}{\xi_m(1+f)(1+\mathsf{X}^2-\mathsf{T}^2)}\right)^{\frac{3}{2}}\mathfrak{g}^K_m\left(\frac{\mathsf{XT}}{1+\mathsf{X}^2}\partial_{\mathsf{X}}+\partial_{\mathsf{T}}\,,\overline{\nabla}_{\partial_{\mathsf{X}}+\frac{\mathsf{XT}}{1+\mathsf{X}^2}\partial_{\mathsf{T}}}\left(\partial_{\mathsf{X}}+\frac{\mathsf{XT}}{1+\mathsf{X}^2}\partial_{\mathsf{T}}\right)\right)\right| \\
&= \left|\left(\frac{1+\mathsf{X}^2}{\xi_m(1+f)(1+\mathsf{X}^2-\mathsf{T}^2)}\right)^{\frac{3}{2}}\mathfrak{g}^K_m\left(\overline{\nabla}_{\partial_{\mathsf{X}}+\frac{\mathsf{XT}}{1+\mathsf{X}^2}\partial_{\mathsf{T}}}\left(\frac{\mathsf{XT}}{1+\mathsf{X}^2}\partial_{\mathsf{X}}+\partial_{\mathsf{T}}\right)\,,\partial_{\mathsf{X}}+\frac{\mathsf{XT}}{1+\mathsf{X}^2}\partial_{\mathsf{T}}\right)\right| \\
&= \frac{|\mathsf{T}|}{\sqrt{1+\mathsf{X}^2}}\,\frac{\xi_m}{(\xi_m(1+f)(1+\mathsf{X}^2-\mathsf{T}^2))^{\frac{3}{2}}}\left[1+\frac{\xi_m\,(1+\mathsf{X}^2-\mathsf{T}^2)(1+f)}{16m^2}\left(1+\frac{2m}{r}\right)\right] \\
&\le \frac{1}{\sqrt{\xi_m\,(1+\mathsf{X}^2-\mathsf{T}^2)^3}} + \sqrt{\frac{\xi_m}{1+\mathsf{X}^2-\mathsf{T}^2}}\frac{1}{16m^2}\left(1+\frac{2m}{r}\right) \\
&\le \frac{C(\kappa_0)}{m}
\end{aligned} \tag{A.1.13}$$

where we relied on the usual inequalities $\mathsf{T}^2 < 1+\mathsf{X}^2$ and $f \ge 0$.

Now, we easily get the remaining bounds, since

$$2\,\mathfrak{g}^K_m\left(\overline{\nabla}_{\widetilde{V}_{\mathsf{X}}}\widetilde{\mathscr{T}},\widetilde{\mathscr{T}}\right) = \widetilde{V}_{\mathsf{X}}\left(\mathfrak{g}^K_m\left(\widetilde{\mathscr{T}},\,\widetilde{\mathscr{T}}\right)\right) = 0\,. \tag{A.1.14}$$

and since the following computation

$$\left[\overline{\nabla}_{\partial_{\mathsf{X}}+\frac{\mathsf{XT}}{1+\mathsf{X}^2}\partial_{\mathsf{T}}}\left(\frac{\mathsf{XT}}{1+\mathsf{X}^2}\partial_{\mathsf{X}}+\partial_{\mathsf{T}}\right)\right]^J = \frac{\mathsf{XT}}{1+\mathsf{X}^2}\overline{\Gamma}^J_{\mathsf{XX}} + \overline{\Gamma}^J_{\mathsf{XT}} + \frac{\mathsf{X}^2\mathsf{T}^2}{(1+\mathsf{X}^2)^2}\overline{\Gamma}^J_{\mathsf{TX}} + \frac{\mathsf{XT}}{1+\mathsf{X}^2}\overline{\Gamma}^J_{\mathsf{TT}} = 0$$

implies

$$\mathfrak{g}^K_m\left(\overline{\nabla}_{\widetilde{V}_{\mathsf{X}}}\widetilde{\mathscr{T}},\widetilde{V}_I\right) = 0\,. \tag{A.1.15}$$

Bound on $\overline{\nabla}_{\widetilde{V}_I}\widetilde{\mathscr{T}}$. We first compute

$$\begin{aligned}
\overline{\nabla}_{\partial_I}\left(\frac{\mathsf{XT}}{1+\mathsf{X}^2}\partial_{\mathsf{X}}+\partial_{\mathsf{T}}\right) &= \frac{\mathsf{XT}}{1+\mathsf{X}^2}\left(\overline{\Gamma}^{\mathsf{X}}_{I\mathsf{X}}\partial_{\mathsf{X}}+\overline{\Gamma}^{\mathsf{T}}_{I\mathsf{X}}\partial_{\mathsf{T}}+\overline{\Gamma}^{J}_{I\mathsf{X}}\partial_J\right)+\overline{\Gamma}^{\mathsf{X}}_{I\mathsf{T}}\partial_{\mathsf{X}}+\overline{\Gamma}^{\mathsf{T}}_{I\mathsf{T}}\partial_{\mathsf{T}}+\overline{\Gamma}^{J}_{I\mathsf{T}}\partial_J \\
&= \left(\frac{\mathsf{XT}}{1+\mathsf{X}^2}\overline{\Gamma}^J_{I\mathsf{X}}+\overline{\Gamma}^J_{I\mathsf{T}}\right)\partial_J = \left(1-\frac{\mathsf{X}^2}{1+\mathsf{X}^2}\right)\overline{\Gamma}^J_{I\mathsf{T}}\,\partial_J
\end{aligned}$$

$$= \frac{1}{1+\mathsf{X}^2}\overline{\Gamma}^J_{I\mathsf{T}}\,\partial_J\,,$$

where we used that $\overline{\Gamma}^J_{I\mathsf{X}} = -\frac{\mathsf{X}}{\mathsf{T}}\overline{\Gamma}^J_{I\mathsf{T}}$, as in (2.3.26). It thus follows that

$$\begin{aligned}
\mathfrak{g}^K_m\Big(\overline{\nabla}_{\widetilde{V}_I}\widetilde{\mathscr{T}},\widetilde{V}_{\mathsf{X}}\Big) &= \frac{\sqrt{(g_{\mathbb{S}^2})^{II}}}{r}\,\mathfrak{g}^K_m\left(\overline{\nabla}_{\partial_I}\widetilde{\mathscr{T}},\widetilde{V}_{\mathsf{X}}\right)\\
&= -\frac{\sqrt{(g_{\mathbb{S}^2})^{II}}}{r}\sqrt{\frac{1+\mathsf{X}^2}{\xi_m(1+f)(1+\mathsf{X}^2-\mathsf{T}^2)}}\,\mathfrak{g}^K_m\left(\frac{\mathsf{XT}}{1+\mathsf{X}^2}\partial_{\mathsf{X}}+\partial_{\mathsf{T}},\,\overline{\nabla}_{\partial_I}\widetilde{V}_{\mathsf{X}}\right)\\
&= \frac{\sqrt{(g_{\mathbb{S}^2})^{II}}}{r}\sqrt{\frac{1+\mathsf{X}^2}{\xi_m(1+f)(1+\mathsf{X}^2-\mathsf{T}^2)}}\,\mathfrak{g}^K_m\left(\overline{\nabla}_{\partial_I}\left(\frac{\mathsf{XT}}{1+\mathsf{X}^2}\partial_{\mathsf{X}}+\partial_{\mathsf{T}}\right),\,\widetilde{V}_{\mathsf{X}}\right) = 0
\end{aligned} \tag{A.1.16}$$

On the other hand, by similar observations, we obtain

$$\begin{aligned}
&\left|\mathfrak{g}^K_m\Big(\,\overline{\nabla}_{\widetilde{V}_I}\widetilde{\mathscr{T}},\widetilde{V}_K\Big)\right| =\\
&= \frac{\sqrt{(g_{\mathbb{S}^2})^{II}(g_{\mathbb{S}^2})^{KK}}}{r^2}\sqrt{\frac{1+\mathsf{X}^2}{\xi_m(1+f)(1+\mathsf{X}^2-\mathsf{T}^2)}}\,\left|\mathfrak{g}^K_m\left(\overline{\nabla}_{\partial_I}\left(\frac{\mathsf{XT}}{1+\mathsf{X}^2}\partial_{\mathsf{X}}+\partial_{\mathsf{T}}\right),\partial_K\right)\right|\\
&= \sqrt{(g_{\mathbb{S}^2})^{II}(g_{\mathbb{S}^2})^{KK}}\,(g_{\mathbb{S}^2})_{JK}\sqrt{\frac{1+\mathsf{X}^2}{\xi_m(1+f)(1+\mathsf{X}^2-\mathsf{T}^2)}}\frac{1}{1+\mathsf{X}^2}\left|\overline{\Gamma}^J_{I\mathsf{T}}\right|\\
&= \frac{\sqrt{(g_{\mathbb{S}^2})^{II}(g_{\mathbb{S}^2})^{KK}}\,(g_{\mathbb{S}^2})_{JK}\,\delta^J_I}{\sqrt{\xi_m(1+f)(1+\mathsf{X}^2-\mathsf{T}^2)}}\,\frac{|\mathsf{T}|}{\sqrt{1+\mathsf{X}^2}}\frac{\xi_m}{4mr}\\
&\le \frac{1}{4mr}\sqrt{\frac{\xi_m}{1+\mathsf{X}^2-\mathsf{T}^2}}\;\le\;\frac{C(\kappa_0)}{m}\,.
\end{aligned} \tag{A.1.17}$$

Finally, we easily see that

$$2\mathfrak{g}^K_m\left(\overline{\nabla}_{\widetilde{V}_I}\widetilde{\mathscr{T}},\widetilde{\mathscr{T}}\right) = \widetilde{V}_I\left(\mathfrak{g}^K_m\left(\widetilde{\mathscr{T}},\,\widetilde{\mathscr{T}}\right)\right) = 0\,. \tag{A.1.18}$$

Bound on $\overline{\nabla}_{\widetilde{\mathscr{T}}}\widetilde{\mathscr{T}}$. First, we have

$$\overline{\nabla}_{\frac{\mathsf{XT}}{1+\mathsf{X}^2}\partial_{\mathsf{X}}+\partial_{\mathsf{T}}}\left(\frac{\mathsf{XT}}{1+\mathsf{X}^2}\partial_{\mathsf{X}}+\partial_{\mathsf{T}}\right) = \frac{\mathsf{XT}}{1+\mathsf{X}^2}\overline{\nabla}_{\partial_{\mathsf{X}}}\left(\frac{\mathsf{XT}}{1+\mathsf{X}^2}\partial_{\mathsf{X}}+\partial_{\mathsf{T}}\right)+\overline{\nabla}_{\partial_{\mathsf{T}}}\left(\frac{\mathsf{XT}}{1+\mathsf{X}^2}\partial_{\mathsf{X}}+\partial_{\mathsf{T}}\right)$$

Now, directly from (A.1.11) and (A.1.12), we get

$$\begin{aligned}
\left[\overline{\nabla}_{\frac{\mathsf{XT}}{1+\mathsf{X}^2}\partial_{\mathsf{X}}+\partial_{\mathsf{T}}}\left(\frac{\mathsf{XT}}{1+\mathsf{X}^2}\partial_{\mathsf{X}}+\partial_{\mathsf{T}}\right)\right]^{\mathsf{T}} &= \frac{\mathsf{X}^2\mathsf{T}^2}{(1+\mathsf{X}^2)^2}\overline{\Gamma}^{\mathsf{T}}_{\mathsf{XX}}+\frac{\mathsf{XT}}{1+\mathsf{X}^2}\overline{\Gamma}^{\mathsf{T}}_{\mathsf{XT}}+\frac{\mathsf{XT}}{1+\mathsf{X}^2}\overline{\Gamma}^{\mathsf{T}}_{\mathsf{TX}}+\overline{\Gamma}^{\mathsf{T}}_{\mathsf{TT}}\\
&= \left(1+\frac{\mathsf{X}^2\mathsf{T}^2}{(1+\mathsf{X}^2)^2}\right)\overline{\Gamma}^{\mathsf{T}}_{\mathsf{TT}}+2\frac{\mathsf{XT}}{1+\mathsf{X}^2}\overline{\Gamma}^{\mathsf{X}}_{\mathsf{XX}}
\end{aligned}$$

$$\left[\overline{\nabla}_{\frac{\mathsf{XT}}{1+\mathsf{X}^2}\partial_\mathsf{X}+\partial_\mathsf{T}}\left(\frac{\mathsf{XT}}{1+\mathsf{X}^2}\partial_\mathsf{X}+\partial_\mathsf{T}\right)\right]^\mathsf{X} = \frac{\mathsf{XT}^2}{(1+\mathsf{X}^2)^2}\left(1-\frac{2\mathsf{X}^2}{1+\mathsf{X}^2}\right)+\frac{\mathsf{X}^2\mathsf{T}^2}{(1+\mathsf{X}^2)^2}\overline{\Gamma}^\mathsf{X}_\mathsf{XX}$$
$$+\frac{\mathsf{XT}}{1+\mathsf{X}^2}\overline{\Gamma}^\mathsf{X}_\mathsf{XT}+\frac{\mathsf{X}}{1+\mathsf{X}^2}+\frac{\mathsf{XT}}{1+\mathsf{X}^2}\overline{\Gamma}^\mathsf{X}_\mathsf{TX}+\overline{\Gamma}^\mathsf{X}_\mathsf{TT}$$
$$=\frac{\mathsf{X}}{1+\mathsf{X}^2}\left(1+\frac{\mathsf{T}^2}{1+\mathsf{X}^2}\left(\frac{2}{1+\mathsf{X}^2}-1\right)\right)+\frac{2\mathsf{XT}}{1+\mathsf{X}^2}\overline{\Gamma}^\mathsf{T}_\mathsf{TT}$$
$$+\left(1+\frac{\mathsf{X}^2\mathsf{T}^2}{(1+\mathsf{X}^2)^2}\right)\overline{\Gamma}^\mathsf{X}_\mathsf{XX}$$

Combining these results, we have

$$\mathfrak{g}^K_m\left(\overline{\nabla}_{\frac{\mathsf{XT}}{1+\mathsf{X}^2}\partial_\mathsf{X}+\partial_\mathsf{T}}\left(\frac{\mathsf{XT}}{1+\mathsf{X}^2}\partial_\mathsf{X}+\partial_\mathsf{T}\right),\,\partial_\mathsf{X}+\frac{\mathsf{XT}}{1+\mathsf{X}^2}\partial_\mathsf{T}\right)=$$
$$=\xi_m\left[\frac{\mathsf{X}}{1+\mathsf{X}^2}\left(1+\frac{\mathsf{T}^2}{1+\mathsf{X}^2}\left(\frac{2}{1+\mathsf{X}^2}-1\right)\right)+2\frac{\mathsf{XT}}{1+\mathsf{X}^2}\overline{\Gamma}^\mathsf{T}_\mathsf{TT}\right.$$
$$\left.+\left(1+\frac{\mathsf{X}^2\mathsf{T}^2}{(1+\mathsf{X}^2)^2}\right)\overline{\Gamma}^\mathsf{X}_\mathsf{XX}-\frac{\mathsf{XT}}{1+\mathsf{X}^2}\left(1+\frac{\mathsf{X}^2\mathsf{T}^2}{(1+\mathsf{X}^2)^2}\right)\overline{\Gamma}^\mathsf{T}_\mathsf{TT}-2\frac{\mathsf{X}^2\mathsf{T}^2}{(1+\mathsf{X}^2)^2}\overline{\Gamma}^\mathsf{X}_\mathsf{XX}\right]$$
$$=\xi_m\left[\frac{\mathsf{X}}{1+\mathsf{X}^2}\left(1-\frac{\mathsf{T}^2}{1+\mathsf{X}^2}+\frac{2\mathsf{T}^2}{(1+\mathsf{X}^2)^2}\right)+\frac{\mathsf{XT}}{1+\mathsf{X}^2}\left(1-\frac{\mathsf{X}^2\mathsf{T}^2}{(1+\mathsf{X}^2)}\right)\overline{\Gamma}^\mathsf{T}_\mathsf{TT}\right.$$
$$\left.+\left(1-\frac{\mathsf{X}^2\mathsf{T}^2}{(1+\mathsf{X}^2)^2}\right)\overline{\Gamma}^\mathsf{X}_\mathsf{XX}\right]$$
$$=\xi_m\left[\frac{\mathsf{X}}{(1+\mathsf{X}^2)^2}\left((1+\mathsf{X}^2-\mathsf{T}^2)+\frac{2\mathsf{T}^2}{1+\mathsf{X}^2}\right)+\left(1-\frac{\mathsf{X}^2\mathsf{T}^2}{(1+\mathsf{X}^2)^2}\right)\left(1-\frac{\mathsf{T}^2}{1+\mathsf{X}^2}\right)\overline{\Gamma}^\mathsf{X}_\mathsf{XX}\right]$$
$$=\xi_m\,\frac{\mathsf{X}}{(1+\mathsf{X}^2)^2}\left[\left((1+\mathsf{X}^2-\mathsf{T}^2)(1+f)+\frac{\mathsf{T}^2}{1+\mathsf{X}^2}\right)\right.$$
$$\left.-(1+\mathsf{X}^2-\mathsf{T}^2)(1+f)\frac{\xi_m(1+\mathsf{X}^2-\mathsf{T}^2)}{16m^2}\left(1+\frac{2m}{r}\right)\right]$$

where we used that $\overline{\Gamma}^\mathsf{T}_\mathsf{TT}=-\frac{\mathsf{T}}{\mathsf{X}}\overline{\Gamma}^\mathsf{X}_\mathsf{XX}$, as in (2.3.26). Hence, we finally get

$$\left|\mathfrak{g}^K_m\left(\overline{\nabla}_{\widetilde{\mathscr{T}}}\widetilde{\mathscr{T}},\widetilde{V}_\mathsf{X}\right)\right|=$$
$$=\left|\left(\frac{1+\mathsf{X}^2}{\xi_m(1+f)(1+\mathsf{X}^2-\mathsf{T}^2)}\right)^{\frac{3}{2}}\mathfrak{g}^K_m\left(\overline{\nabla}_{\frac{\mathsf{XT}}{1+\mathsf{X}^2}\partial_\mathsf{X}+\partial_\mathsf{T}}\left(\frac{\mathsf{XT}}{1+\mathsf{X}^2}\partial_\mathsf{X}+\partial_\mathsf{T}\right),\partial_\mathsf{X}+\frac{\mathsf{XT}}{1+\mathsf{X}^2}\partial_\mathsf{T}\right)\right|$$
$$=\frac{|\mathsf{X}|}{\sqrt{1+\mathsf{X}^2}}\frac{1}{\sqrt{\xi_m\left((1+f)(1+\mathsf{X}^2-\mathsf{T}^2)\right)^3}}\left|\left((1+\mathsf{X}^2-\mathsf{T}^2)(1+f)+\frac{\mathsf{T}^2}{1+\mathsf{X}^2}\right)\right.$$

$$
\begin{aligned}
&\quad - \frac{\xi_m(1+\mathsf{X}^2-\mathsf{T}^2)^2(1+f)}{16m^2}\left(1+\frac{2m}{r}\right)\Bigg| \\
&\leq \frac{1}{\sqrt{\xi_m(1+\mathsf{X}^2-\mathsf{T}^2)}}\left(1+\frac{2}{\xi_m(1+\mathsf{X}^2-\mathsf{T}^2)}\right)+\frac{\sqrt{\xi_m(1+\mathsf{X}^2-\mathsf{T}^2)}}{16m^2}\left(1+\frac{2m}{r}\right) \\
&\leq \frac{C(\kappa_0)}{m} \qquad\qquad (\mathrm{A}.1.19)
\end{aligned}
$$

where we used that $|\mathsf{X}| < \sqrt{1+\mathsf{X}^2}$ and the usual inequalities $\mathsf{T}^2 < 1+\mathsf{X}^2$ and $f \geq 0$.

Now, we easily get the remaining bounds, since

$$
2\,\mathfrak{g}_m^K\left(\overline{\nabla}_{\widetilde{\mathscr{T}}}\widetilde{\mathscr{T}},\mathscr{T}\right) = \widetilde{\mathscr{T}}\left(\mathfrak{g}_m^K\left(\widetilde{\mathscr{T}},\,\widetilde{\mathscr{T}}\right)\right) = 0\,. \qquad (\mathrm{A}.1.20)
$$

and since the vanishing of the Christoffel symbols $\overline{\Gamma}^J_{\mathsf{XX}}, \overline{\Gamma}^J_{\mathsf{XT}}, \overline{\Gamma}^J_{\mathsf{TT}}$, as in (2.3.26), gives

$$
\left[\overline{\nabla}_{\frac{\mathsf{XT}}{1+\mathsf{X}^2}\partial_\mathsf{X}+\partial_\mathsf{T}}\left(\frac{\mathsf{XT}}{1+\mathsf{X}^2}\partial_\mathsf{X}+\partial_\mathsf{T}\right)\right]^J = 0
$$

which, in turn, implies

$$
\mathfrak{g}_m^K\left(\overline{\nabla}_{\widetilde{\mathscr{T}}}\widetilde{\mathscr{T}},\widetilde{V}_I\right) = 0\,. \qquad (\mathrm{A}.1.21)
$$

Finally, combining (A.1.13), (A.1.14), (A.1.15), (A.1.19), (A.1.20), (A.1.16), (A.1.18) and (A.1.17), we obtain

$$
{}^{\tilde{\mathsf{T}}}\|\,\overline{\nabla}\,\widetilde{\mathscr{T}}\| \leq \frac{C(\kappa_0)}{m}\,. \qquad (\mathrm{A}.1.22)
$$

Bound on $\overline{\nabla}^2\,\widetilde{\mathscr{T}}$

Bound on $\overline{\nabla}^2_{\widetilde{V}_\mathsf{X}\widetilde{V}_\mathsf{X}}\widetilde{\mathscr{T}}$. Recall that

$$
\overline{\nabla}^2_{\widetilde{V}_\mathsf{X}\widetilde{V}_\mathsf{X}}\widetilde{\mathscr{T}} = \overline{\nabla}_{\widetilde{V}_\mathsf{X}}\left(\overline{\nabla}_{\widetilde{V}_\mathsf{X}}\widetilde{\mathscr{T}}\right) - \overline{\nabla}_{\overline{\nabla}_{\widetilde{V}_\mathsf{X}}\widetilde{V}_\mathsf{X}}\widetilde{\mathscr{T}}. \qquad (\mathrm{A}.1.23)
$$

From the computations (A.1.14) and (A.1.15), we have $\overline{\nabla}_{\widetilde{V}_\mathsf{X}}\widetilde{\mathscr{T}} = \mathfrak{g}_m^K\left(\overline{\nabla}_{\widetilde{V}_\mathsf{X}}\widetilde{\mathscr{T}},\widetilde{V}_\mathsf{X}\right)\widetilde{V}_\mathsf{X}$. Hence, we obtain

$$
\overline{\nabla}_{\widetilde{V}_\mathsf{X}}\left(\overline{\nabla}_{\widetilde{V}_\mathsf{X}}\widetilde{\mathscr{T}}\right) = \widetilde{V}_\mathsf{X}\left(\mathfrak{g}_m^K\left(\overline{\nabla}_{\widetilde{V}_\mathsf{X}}\widetilde{\mathscr{T}},\widetilde{V}_\mathsf{X}\right)\right)\widetilde{V}_\mathsf{X} + \mathfrak{g}_m^K\left(\overline{\nabla}_{\widetilde{V}_\mathsf{X}}\widetilde{\mathscr{T}},\widetilde{V}_\mathsf{X}\right)\overline{\nabla}_{\widetilde{V}_\mathsf{X}}\widetilde{V}_\mathsf{X}
$$

On the other hand, it holds

$$
\overline{\nabla}_{\widetilde{V}_\mathsf{X}}\widetilde{V}_\mathsf{X} = -\mathfrak{g}_m^K\left(\overline{\nabla}_{\widetilde{V}_\mathsf{X}}\widetilde{\mathscr{T}},\widetilde{V}_\mathsf{X}\right)\widetilde{\mathscr{T}}
$$

due to the computations

$$
\begin{aligned}
2\mathfrak{g}_m^K\left(\overline{\nabla}_{\widetilde{V}_{\mathsf{X}}}\widetilde{V}_{\mathsf{X}},\,\widetilde{V}_{\mathsf{X}}\right) &= \widetilde{V}_{\mathsf{X}}\left(\mathfrak{g}_m^K\left(\widetilde{V}_{\mathsf{X}},\widetilde{V}_{\mathsf{X}}\right)\right) = 0\,,\\
\mathfrak{g}_m^K\left(\overline{\nabla}_{\widetilde{V}_{\mathsf{X}}}\widetilde{V}_{\mathsf{X}},\,\widetilde{\mathscr{T}}\right) &= \widetilde{V}_{\mathsf{X}}\left(\mathfrak{g}_m^K\left(\widetilde{V}_{\mathsf{X}},\widetilde{\mathscr{T}}\right)\right) - \mathfrak{g}_m^K\left(\widetilde{V}_{\mathsf{X}},\overline{\nabla}_{\widetilde{V}_{\mathsf{X}}}\widetilde{\mathscr{T}}\right) = -\mathfrak{g}_m^K\left(\overline{\nabla}_{\widetilde{V}_{\mathsf{X}}}\widetilde{\mathscr{T}},\widetilde{V}_{\mathsf{X}}\right)\,,
\end{aligned}
$$

which can be found in (A.1.13) and the fact that

$$
\mathfrak{g}_m^K\left(\overline{\nabla}_{\widetilde{V}_{\mathsf{X}}}\widetilde{V}_{\mathsf{X}},\,\widetilde{V}_I\right) = 0\,,
$$

due to the vanishing of the Christoffel symbols $\overline{\Gamma}^J_{\mathsf{XX}}, \overline{\Gamma}^J_{\mathsf{XT}}, \overline{\Gamma}^J_{\mathsf{TT}}$, as in (2.3.26). This gives

$$
-\overline{\nabla}_{\overline{\nabla}_{\widetilde{V}_{\mathsf{X}}}\widetilde{V}_{\mathsf{X}}}\widetilde{\mathscr{T}} = \mathfrak{g}_m^K\left(\overline{\nabla}_{\widetilde{V}_{\mathsf{X}}}\widetilde{\mathscr{T}},\widetilde{V}_{\mathsf{X}}\right)\,\overline{\nabla}_{\widetilde{\mathscr{T}}}\widetilde{\mathscr{T}}\,.
$$

Combining the above, (A.1.23) reads

$$
\begin{aligned}
\overline{\nabla}^2_{\widetilde{V}_{\mathsf{X}}\widetilde{V}_{\mathsf{X}}}\widetilde{\mathscr{T}} = {}& \widetilde{V}_{\mathsf{X}}\left(\mathfrak{g}_m^K\left(\overline{\nabla}_{\widetilde{V}_{\mathsf{X}}}\widetilde{\mathscr{T}},\widetilde{V}_{\mathsf{X}}\right)\right)\widetilde{V}_{\mathsf{X}} - \mathfrak{g}_m^K\left(\overline{\nabla}_{\widetilde{V}_{\mathsf{X}}}\widetilde{\mathscr{T}},\widetilde{V}_{\mathsf{X}}\right)\mathfrak{g}_m^K\left(\overline{\nabla}_{\widetilde{V}_{\mathsf{X}}}\widetilde{\mathscr{T}},\widetilde{V}_{\mathsf{X}}\right)\widetilde{\mathscr{T}}\\
&+ \mathfrak{g}_m^K\left(\overline{\nabla}_{\widetilde{V}_{\mathsf{X}}}\widetilde{\mathscr{T}},\widetilde{V}_{\mathsf{X}}\right)\,\overline{\nabla}_{\widetilde{\mathscr{T}}}\widetilde{\mathscr{T}}
\end{aligned}
\tag{A.1.24}
$$

Now, computing the second and third term on the right-hand side of (A.1.24) against any vector of the adapted frame gives a quantity that can be estimated by $\frac{C(\kappa_0)}{m^2}$, just by applying (A.1.22). We are then left with estimating $\widetilde{V}_{\mathsf{X}}\left(\mathfrak{g}_m^K\left(\overline{\nabla}_{\widetilde{V}_{\mathsf{X}}}\widetilde{\mathscr{T}},\widetilde{V}_{\mathsf{X}}\right)\right)$. Now, using the explicit computations (A.1.13), we find

$$
\begin{aligned}
&\partial_{\mathsf{X}}\left(\mathfrak{g}_m^K\left(\overline{\nabla}_{\widetilde{V}_{\mathsf{X}}}\widetilde{\mathscr{T}},\widetilde{V}_{\mathsf{X}}\right)\right) =\\
&= \partial_{\mathsf{X}}\left[\frac{\mathsf{T}}{\sqrt{1+\mathsf{X}^2}}\,\frac{1}{\sqrt{\xi_m\left((1+f)(1+\mathsf{X}^2-\mathsf{T}^2)\right)^3}}\left(1+\frac{\xi_m\,(1+\mathsf{X}^2-\mathsf{T}^2)(1+f)\left(1+\frac{2m}{r}\right)}{16m^2}\right)\right]\\
&= -\frac{\mathsf{TX}}{(1+\mathsf{X}^2)^{\frac{3}{2}}}\frac{1}{\sqrt{\xi_m\left((1+f)(1+\mathsf{X}^2-\mathsf{T}^2)\right)^3}}\left(1+\frac{\xi_m\,(1+\mathsf{X}^2-\mathsf{T}^2)(1+f)\left(1+\frac{2m}{r}\right)}{16m^2}\right)\\
&\quad+\frac{\mathsf{T}}{\sqrt{1+\mathsf{X}^2}}\Bigg[\left(-\frac{\frac{d\xi_m}{dr}}{2\sqrt{\left(\xi_m(1+f)(1+\mathsf{X}^2-\mathsf{T}^2)\right)^3}}-\frac{\frac{d\xi_m}{dr}\left(1+\frac{2m}{r}\right)-\frac{2m}{r^2}\xi_m}{32m^2\sqrt{\xi_m(1+f)(1+\mathsf{X}^2-\mathsf{T}^2)}}\right)\frac{\partial r}{\partial\mathsf{X}}\\
&\quad-\frac{3\left(\partial_{\mathsf{X}}f(1+\mathsf{X}^2-\mathsf{T}^2)+2\mathsf{X}(1+f)\right)}{2\sqrt{\xi_m\left((1+f)(1+\mathsf{X}^2-\mathsf{T}^2)\right)^5}}-\frac{\sqrt{\xi_m}\left(1+\frac{2m}{r}\right)\left(\partial_{\mathsf{X}}f(1+\mathsf{X}^2-\mathsf{T}^2)+2\mathsf{X}(1+f)\right)}{32m^2\sqrt{\left((1+f)(1+\mathsf{X}^2-\mathsf{T}^2)\right)^3}}\Bigg]
\end{aligned}
$$

and

$$
\partial_{\mathsf{T}}\left(\mathfrak{g}_m^K\left(\overline{\nabla}_{\widetilde{V}_{\mathsf{X}}}\widetilde{\mathscr{T}},\widetilde{V}_{\mathsf{X}}\right)\right) =
$$

$$= \partial_{\mathsf{T}} \left[\frac{\mathsf{T}}{\sqrt{1+\mathsf{X}^2}} \frac{1}{\sqrt{\xi_m\left((1+f)(1+\mathsf{X}^2-\mathsf{T}^2)\right)^3}} \left(1 + \frac{\xi_m\,(1+\mathsf{X}^2-\mathsf{T}^2)(1+f)\left(1+\frac{2m}{r}\right)}{16m^2} \right) \right]$$

$$= \frac{1}{\sqrt{1+\mathsf{X}^2}} \frac{1}{\sqrt{\xi_m\left((1+f)(1+\mathsf{X}^2-\mathsf{T}^2)\right)^3}} \left(1 + \frac{\xi_m\,(1+\mathsf{X}^2-\mathsf{T}^2)(1+f)\left(1+\frac{2m}{r}\right)}{16m^2} \right)$$

$$+ \frac{\mathsf{T}}{\sqrt{1+\mathsf{X}^2}} \left[\left(-\frac{\frac{d\xi_m}{dr}}{2\sqrt{(\xi_m(1+f)(1+\mathsf{X}^2-\mathsf{T}^2))^3}} - \frac{\frac{d\xi_m}{dr}\left(1+\frac{2m}{r}\right) - \frac{2m}{r^2}\xi_m}{32m^2\sqrt{\xi_m(1+f)(1+\mathsf{X}^2-\mathsf{T}^2)}} \right) \frac{\partial r}{\partial \mathsf{T}} \right.$$

$$\left. - \frac{3\left(\partial_{\mathsf{T}} f(1+\mathsf{X}^2-\mathsf{T}^2) - 2\mathsf{T}(1+f)\right)}{2\sqrt{\xi_m\left((1+f)(1+\mathsf{X}^2-\mathsf{T}^2)\right)^5}} - \frac{\sqrt{\xi_m}\left(1+\frac{2m}{r}\right)\left(\partial_{\mathsf{T}} f(1+\mathsf{X}^2-\mathsf{T}^2) - 2\mathsf{T}(1+f)\right)}{32m^2\sqrt{\left((1+f)(1+\mathsf{X}^2-\mathsf{T}^2)\right)^3}} \right].$$

Combining the two above and plugging in the derivative of ξ_m given by (2.3.24) and the derivatives of the Schwarzschild radial coordinate r with respect to the Kruskal–Szekeres coordinates T, X as in (2.3.25), we obtain

$$\partial_{\mathsf{X}} \left(\mathfrak{g}_m^K \left(\overline{\nabla}_{\widetilde{V}_{\mathsf{X}}} \widetilde{\mathscr{T}}, \widetilde{V}_{\mathsf{X}} \right) \right) + \frac{\mathsf{TX}}{1+\mathsf{X}^2} \partial_{\mathsf{T}} \left(\mathfrak{g}_m^K \left(\overline{\nabla}_{\widetilde{V}_{\mathsf{X}}} \widetilde{\mathscr{T}}, \widetilde{V}_{\mathsf{X}} \right) \right) =$$

$$= \frac{\frac{\mathsf{XT}}{(1+\mathsf{X}^2)^{\frac{3}{2}}} - \frac{\mathsf{XT}}{(1+\mathsf{X}^2)^{\frac{3}{2}}}}{\sqrt{\xi_m\left((1+f)(1+\mathsf{X}^2-\mathsf{T}^2)\right)^3}} \left(1 + \frac{\xi_m\,(1+\mathsf{X}^2-\mathsf{T}^2)(1+f)\left(1+\frac{2m}{r}\right)}{16m^2} \right)$$

$$+ \frac{\mathsf{TX}}{\sqrt{1+\mathsf{X}^2}} \left[\frac{\left(1-\frac{\mathsf{T}^2}{1+\mathsf{X}^2}\right)(\xi_m)^{\frac{3}{2}}}{256m^4} \left(\frac{16m^2\left(1+\frac{2m}{r}\right)}{\xi_m\sqrt{\left((1+f)(1+\mathsf{X}^2-\mathsf{T}^2)\right)^3}} + \frac{\left(1+\frac{2m}{r}\right)^2 + \frac{4m^2}{r^2}}{\sqrt{(1+f)(1+\mathsf{X}^2-\mathsf{T}^2)}} \right) \right.$$

$$+ \left(-\left(\frac{\mathsf{T}}{1+\mathsf{X}^2}\partial_{\mathsf{T}} f + \frac{\partial_{\mathsf{X}} f}{\mathsf{X}} \right)(1+\mathsf{X}^2-\mathsf{T}^2) + 2(1+f)\left(\frac{\mathsf{T}^2}{1+\mathsf{X}^2} - 1 \right) \right)$$

$$\left. \left(\frac{3}{2\sqrt{\xi_m\left((1+f)(1+\mathsf{X}^2-\mathsf{T}^2)\right)^5}} + \frac{\sqrt{\xi_m}}{32m^2\sqrt{\left((1+f)(1+\mathsf{X}^2-\mathsf{T}^2)\right)^3}} \right) \right]$$

$$= \frac{\mathsf{TX}}{\sqrt{1+\mathsf{X}^2}} \left[\frac{(1+\mathsf{X}^2-\mathsf{T}^2)(\xi_m)^{\frac{3}{2}}}{(1+\mathsf{X}^2)256m^4} \left(\frac{16m^2\left(1+\frac{2m}{r}\right)}{\xi_m\sqrt{\left((1+f)(1+\mathsf{X}^2-\mathsf{T}^2)\right)^3}} + \frac{\left(1+\frac{2m}{r}\right)^2 + \frac{4m^2}{r^2}}{\sqrt{(1+f)(1+\mathsf{X}^2-\mathsf{T}^2)}} \right) \right.$$

$$- \left(2f\frac{(1+\mathsf{X}^2-\mathsf{T}^2)}{1+\mathsf{X}^2}\left(1 + f(1+\mathsf{X}^2) - 1 - \frac{1+\mathsf{X}^2}{1+\mathsf{X}^2-\mathsf{T}^2} \right) + 2\frac{(1+f)(1+\mathsf{X}^2-\mathsf{T}^2)}{1+\mathsf{X}^2} \right)$$

$$\left. \left(\frac{3}{2\sqrt{\xi_m\left((1+f)(1+\mathsf{X}^2-\mathsf{T}^2)\right)^5}} + \frac{\sqrt{\xi_m}}{32m^2\sqrt{\left((1+f)(1+\mathsf{X}^2-\mathsf{T}^2)\right)^3}} \right) \right]$$

$$= \frac{\mathsf{T}}{\sqrt{1+\mathsf{X}^2}} \frac{\mathsf{X}(1+\mathsf{X}^2-\mathsf{T}^2)}{(1+\mathsf{X}^2)16m^2} \left[\frac{\left(1+\frac{2m}{r}\right)}{\sqrt{\xi_m\left((1+f)(1+\mathsf{X}^2-\mathsf{T}^2)\right)^3}} + \frac{(\xi_m)^{\frac{3}{2}}\left(\left(1+\frac{2m}{r}\right)^2 + \frac{4m^2}{r^2}\right)}{16m^2\sqrt{(1+f)(1+\mathsf{X}^2-\mathsf{T}^2)}} \right.$$

$$\left. - 2\left(-f+1+f\right)\left(\frac{3}{2\sqrt{\xi_m\left((1+f)(1+\mathsf{X}^2-\mathsf{T}^2)\right)^5}} + \frac{\sqrt{\xi_m}}{32m^2\sqrt{\left((1+f)(1+\mathsf{X}^2-\mathsf{T}^2)\right)^3}} \right) \right]$$

$$= \frac{\mathsf{T}}{\sqrt{1+\mathsf{X}^2}} \frac{\mathsf{X}}{1+\mathsf{X}^2} \left[\frac{\sqrt{\xi_m}}{8mr\sqrt{(1+f)^3(1+\mathsf{X}^2-\mathsf{T}^2)}} + \frac{(\xi_m)^{\frac{3}{2}}\sqrt{(1+\mathsf{X}^2-\mathsf{T}^2)}\left(1+\frac{4m}{r}+\frac{8m^2}{r^2}\right)}{256m^4\sqrt{1+f}} \right.$$

$$- \frac{3}{\sqrt{\xi_m(1+f)^5(1+\mathsf{X}^2-\mathsf{T}^2)^3}}\Bigg],$$

where we used the identities (A.1.5). It follows that

$$\begin{aligned}
&\left|\widetilde{V}_{\mathsf{X}}\left(\not{g}\left(\overline{\nabla}_{\widetilde{V}_{\mathsf{X}}}\widetilde{\mathscr{T}},\widetilde{V}_{\mathsf{X}}\right)\right)\right| \\
&= \sqrt{\frac{1+\mathsf{X}^2}{\xi_m(1+f)(1+\mathsf{X}^2-\mathsf{T}^2)}}\frac{|\mathsf{T}|}{\sqrt{1+\mathsf{X}^2}}\frac{|\mathsf{X}|}{1+\mathsf{X}^2}\Bigg[\frac{\sqrt{\xi_m}}{8mr\sqrt{(1+f)^3(1+\mathsf{X}^2-\mathsf{T}^2)}} \\
&\quad + \frac{\sqrt{\xi_m(1+\mathsf{X}^2-\mathsf{T}^2)}\left(1+\frac{4m}{r}+\frac{8m^2}{r^2}\right)}{256m^4\sqrt{1+f}} - \frac{3}{\sqrt{\xi_m(1+f)^5(1+\mathsf{X}^2-\mathsf{T}^2)^3}}\Bigg] \\
&\le \frac{1}{\sqrt{\xi_m(1+\mathsf{X}^2-\mathsf{T}^2)}}\Bigg[\frac{\sqrt{\xi_m}}{8mr\sqrt{(1+f)^3(1+\mathsf{X}^2-\mathsf{T}^2)}} \\
&\quad + \frac{(\xi_m)^{\frac{3}{2}}\sqrt{(1+\mathsf{X}^2-\mathsf{T}^2)}\left(1+\frac{4m}{r}+\frac{8m^2}{r^2}\right)}{256m^4\sqrt{1+f}} - \frac{3}{\sqrt{\xi_m(1+f)^5(1+\mathsf{X}^2-\mathsf{T}^2)^3}}\Bigg] \\
&\le \frac{1}{8mr\left(1+\left(\frac{r}{2m}-1\right)\mathrm{e}^{\frac{r}{2m}}\right)} + \frac{e^{-\frac{r}{2m}}}{8mr}\left(1+\frac{4m}{r}+\frac{8m^2}{r^2}\right) + \frac{3}{\xi_m\left(1+\left(\frac{r}{2m}-1\right)\mathrm{e}^{\frac{r}{2m}}\right)^2} \\
&\le \frac{C(\kappa_0)}{m^2}\,. \qquad \text{(A.1.25)}
\end{aligned}$$

Combining this last estimate (A.1.25) with the observations for (A.1.24), we obtain

$$\left|\mathfrak{g}_m^K\left(\overline{\nabla}^2_{\widetilde{V}_{\mathsf{X}}\widetilde{V}_{\mathsf{X}}}\widetilde{\mathscr{T}},\widetilde{V}_{\mathsf{X}}\right)\right| + \left|\mathfrak{g}_m^K\left(\overline{\nabla}^2_{\widetilde{V}_{\mathsf{X}}\widetilde{V}_{\mathsf{X}}}\widetilde{\mathscr{T}},\widetilde{\mathscr{T}}\right)\right| + \left|\mathfrak{g}_m^K\left(\overline{\nabla}^2_{\widetilde{V}_{\mathsf{X}}\widetilde{V}_{\mathsf{X}}}\widetilde{\mathscr{T}},\widetilde{V}_I\right)\right| \le \frac{C(\kappa_0)}{m^2}\,. \qquad \text{(A.1.26)}$$

Bound on $\overline{\nabla}^2_{\widetilde{V}_{\mathsf{X}}\widetilde{V}_I}\widetilde{\mathscr{T}}$. Recall that

$$\overline{\nabla}^2_{\widetilde{V}_{\mathsf{X}}\widetilde{V}_I}\widetilde{\mathscr{T}} = \overline{\nabla}_{\widetilde{V}_I}\left(\overline{\nabla}_{\widetilde{V}_{\mathsf{X}}}\widetilde{\mathscr{T}}\right) - \overline{\nabla}_{\overline{\nabla}_{\widetilde{V}_I}\widetilde{V}_{\mathsf{X}}}\widetilde{\mathscr{T}}.$$

Again, from the computations (A.1.14) and (A.1.15), $\overline{\nabla}_{\widetilde{V}_{\mathsf{X}}}\widetilde{\mathscr{T}} = \mathfrak{g}_m^K\left(\overline{\nabla}_{\widetilde{V}_{\mathsf{X}}}\widetilde{\mathscr{T}},\widetilde{V}_{\mathsf{X}}\right)\widetilde{V}_{\mathsf{X}}$. This implies

$$\begin{aligned}
\overline{\nabla}_{\widetilde{V}_I}\left(\overline{\nabla}_{\widetilde{V}_{\mathsf{X}}}\widetilde{\mathscr{T}}\right) &= \widetilde{V}_I\left(\mathfrak{g}_m^K\left(\overline{\nabla}_{\widetilde{V}_{\mathsf{X}}}\widetilde{\mathscr{T}},\widetilde{V}_{\mathsf{X}}\right)\right)\widetilde{V}_{\mathsf{X}} + \mathfrak{g}_m^K\left(\overline{\nabla}_{\widetilde{V}_{\mathsf{X}}}\widetilde{\mathscr{T}},\widetilde{V}_{\mathsf{X}}\right)\overline{\nabla}_{\widetilde{V}_I}\widetilde{V}_{\mathsf{X}} \\
&= \mathfrak{g}_m^K\left(\overline{\nabla}_{\widetilde{V}_{\mathsf{X}}}\widetilde{\mathscr{T}},\widetilde{V}_{\mathsf{X}}\right)\overline{\nabla}_{\widetilde{V}_I}\widetilde{V}_{\mathsf{X}}
\end{aligned}$$

Now, we compute the projections of $\overline{\nabla}_{\widetilde{V}_I}\widetilde{V}_{\mathsf{X}}$, as follows

$$2\mathfrak{g}_m^K\left(\overline{\nabla}_{\widetilde{V}_I}\widetilde{V}_{\mathsf{X}},\widetilde{V}_{\mathsf{X}}\right) = \widetilde{V}_I\left(\mathfrak{g}_m^K\left(\widetilde{V}_{\mathsf{X}},\widetilde{V}_{\mathsf{X}}\right)\right) = 0$$

$$
\begin{aligned}
&\left|\mathfrak{g}_m^K\left(\overline{\nabla}_{\widetilde{V}_I}\widetilde{V}_{\mathsf{X}},\widetilde{V}_J\right)\right| = \\
&\quad= \left|\frac{\sqrt{(g_{\mathbb{S}^2})^{II}}\sqrt{(g_{\mathbb{S}^2})^{JJ}}}{r^2}\sqrt{\frac{1+\mathsf{X}^2}{\xi_m(1+f)(1+\mathsf{X}^2-\mathsf{T}^2)}}\,\mathfrak{g}_m^K\left(\overline{\nabla}_{\partial_I}\left(\partial_{\mathsf{X}}+\frac{\mathsf{XT}}{1+\mathsf{X}^2}\partial_{\mathsf{T}}\right),\partial_J\right)\right| \\
&\quad= \left|\sqrt{(g_{\mathbb{S}^2})^{II}}\sqrt{(g_{\mathbb{S}^2})^{JJ}}\sqrt{\frac{1+\mathsf{X}^2}{\xi_m(1+f)(1+\mathsf{X}^2-\mathsf{T}^2)}}\left(\overline{\Gamma}^I_{I\mathsf{X}}+\frac{\mathsf{XT}}{1+\mathsf{X}^2}\overline{\Gamma}^I_{I\mathsf{T}}\right)(g_{\mathbb{S}^2})_{IJ}\right| \\
&\quad= \left|\delta_{IJ}\sqrt{\frac{1+\mathsf{X}^2}{\xi_m(1+f)(1+\mathsf{X}^2-\mathsf{T}^2)}}\frac{\xi_m}{4m\,r}\mathsf{X}\left(1-\frac{\mathsf{T}^2}{1+\mathsf{X}^2}\right)\right| \\
&\quad= \left|\frac{\delta_{IJ}}{4mr}\sqrt{\frac{\xi_m(1+\mathsf{X}^2-\mathsf{T}^2)}{(1+f)}}\frac{\mathsf{X}}{\sqrt{1+\mathsf{X}^2}}\right| \le \frac{C(\kappa_0)}{m}
\end{aligned}
\tag{A.1.27}
$$

where we used $\overline{\Gamma}^I_{I\mathsf{T}} = -\frac{\mathsf{T}}{\mathsf{X}}\overline{\Gamma}^I_{I\mathsf{X}}$, as in (2.3.26). Moreover, we have

$$
\mathfrak{g}_m^K\left(\overline{\nabla}_{\widetilde{V}_I}\widetilde{V}_{\mathsf{X}},\widetilde{\mathscr{T}}\right) = -\mathfrak{g}_m^K\left(\widetilde{V}_{\mathsf{X}},\overline{\nabla}_{\widetilde{V}_I}\widetilde{\mathscr{T}}\right)\,,
$$

which we have bounded in (A.1.16). Since all the terms involving the first derivatives have already been estimated in (A.1.22), using the above estimate (A.1.27), we conclude that

$$
\left|\mathfrak{g}_m^K\left(\overline{\nabla}^2_{\widetilde{V}_{\mathsf{X}}\widetilde{V}_I}\widetilde{\mathscr{T}},\widetilde{V}_{\mathsf{X}}\right)\right| + \left|\mathfrak{g}_m^K\left(\overline{\nabla}^2_{\widetilde{V}_{\mathsf{X}}\widetilde{V}_I}\widetilde{\mathscr{T}},\widetilde{\mathscr{T}}\right)\right| + \left|\mathfrak{g}_m^K\left(\overline{\nabla}^2_{\widetilde{V}_{\mathsf{X}}\widetilde{V}_I}\widetilde{\mathscr{T}},\widetilde{V}_J\right)\right| \le \frac{C(\kappa_0)}{m^2}\,. \tag{A.1.28}
$$

Bound on $\overline{\nabla}^2_{\widetilde{V}_{\mathsf{X}}\widetilde{\mathscr{T}}}\widetilde{\mathscr{T}}$. Recall that

$$
\overline{\nabla}^2_{\widetilde{V}_{\mathsf{X}}\widetilde{\mathscr{T}}}\widetilde{\mathscr{T}} = \overline{\nabla}_{\widetilde{V}_{\mathsf{X}}}\left(\overline{\nabla}_{\widetilde{\mathscr{T}}}\widetilde{\mathscr{T}}\right) - \overline{\nabla}_{\overline{\nabla}_{\widetilde{V}_{\mathsf{X}}}\widetilde{\mathscr{T}}}\widetilde{\mathscr{T}}.
$$

Again, from the computations (A.1.14)–(A.1.15), $\overline{\nabla}_{\widetilde{V}_{\mathsf{X}}}\widetilde{\mathscr{T}} = \mathfrak{g}_m^K\left(\overline{\nabla}_{\widetilde{V}_{\mathsf{X}}}\widetilde{\mathscr{T}},\widetilde{V}_{\mathsf{X}}\right)\widetilde{V}_{\mathsf{X}}$ and from (A.1.20)–(A.1.21), $\overline{\nabla}_{\widetilde{\mathscr{T}}}\widetilde{\mathscr{T}} = \mathfrak{g}_m^K\left(\overline{\nabla}_{\widetilde{\mathscr{T}}}\widetilde{\mathscr{T}},\widetilde{V}_{\mathsf{X}}\right)\widetilde{V}_{\mathsf{X}}$. These imply

$$
\overline{\nabla}_{\widetilde{V}_{\mathsf{X}}}\left(\overline{\nabla}_{\widetilde{\mathscr{T}}}\widetilde{\mathscr{T}}\right) = \widetilde{V}_{\mathsf{X}}\left(\mathfrak{g}_m^K\left(\overline{\nabla}_{\widetilde{\mathscr{T}}}\widetilde{\mathscr{T}},\widetilde{V}_{\mathsf{X}}\right)\right)\widetilde{V}_{\mathsf{X}} + \mathfrak{g}_m^K\left(\overline{\nabla}_{\widetilde{\mathscr{T}}}\widetilde{\mathscr{T}},\widetilde{V}_{\mathsf{X}}\right)\mathfrak{g}_m^K\left(\overline{\nabla}_{\widetilde{V}_{\mathsf{X}}}\widetilde{V}_{\mathsf{X}},\widetilde{\mathscr{T}}\right)\widetilde{\mathscr{T}} \tag{A.1.29}
$$

and

$$
-\overline{\nabla}_{\overline{\nabla}_{\widetilde{V}_{\mathsf{X}}}\widetilde{\mathscr{T}}}\widetilde{\mathscr{T}} = -\mathfrak{g}_m^K\left(\overline{\nabla}_{\widetilde{\mathscr{T}}}\widetilde{\mathscr{T}},\widetilde{V}_{\mathsf{X}}\right)^2\widetilde{V}_{\mathsf{X}}\,. \tag{A.1.30}
$$

Given the estimate (A.1.22), we are then left with estimating $\widetilde{V}_{\mathsf{X}}\left(\mathfrak{g}_m^K\left(\overline{\nabla}_{\widetilde{\mathscr{T}}}\widetilde{\mathscr{T}},\widetilde{V}_{\mathsf{X}}\right)\right)\widetilde{V}_{\mathsf{X}}$. Using the explicit computation (A.1.19), we work out the following

$$
\partial_{\mathsf{X}}\left(\mathfrak{g}_m^K\left(\overline{\nabla}_{\widetilde{\mathscr{T}}}\widetilde{\mathscr{T}},\widetilde{V}_{\mathsf{X}}\right)\right) =
$$

$$
\begin{aligned}
&= \frac{1}{(1+\mathsf{X}^2)^{\frac{3}{2}}}\left(\frac{1-\frac{\xi_m}{16m^2}\left(1+\frac{2m}{r}\right)}{\sqrt{\xi_m(1+f)(1+\mathsf{X}^2-\mathsf{T}^2)}}+\frac{\mathsf{T}^2}{1+\mathsf{X}^2}\frac{1}{\sqrt{\xi_m\left((1+f)(1+\mathsf{X}^2-\mathsf{T}^2)\right)^3}}\right)\\
&+\frac{\mathsf{X}}{\sqrt{1+\mathsf{X}^2}}\frac{1}{\sqrt{\xi_m}}\left\{\left[-\frac{1}{2\xi_m}\frac{d\xi_m}{dr}\left(\frac{1-\frac{\xi_m}{16m^2}\left(1+\frac{2m}{r}\right)}{\sqrt{(1+f)(1+\mathsf{X}^2-\mathsf{T}^2)}}+\frac{\frac{\mathsf{T}^2}{1+\mathsf{X}^2}}{\sqrt{((1+f)(1+\mathsf{X}^2-\mathsf{T}^2))^3}}\right)\right.\right.\\
&\left.-\frac{\frac{d\xi_m}{dr}\left(1+\frac{2m}{r}\right)-\frac{2m}{r^2}\xi_m}{16m^2\sqrt{(1+f)(1+\mathsf{X}^2-\mathsf{T}^2)}}\right]\frac{\partial r}{\partial\mathsf{X}}\\
&-\frac{\left(1-\frac{\xi_m}{16m^2}\left(1+\frac{2m}{r}\right)\right)\left(\partial_{\mathsf{X}}f(1+\mathsf{X}^2-\mathsf{T}^2)+2\mathsf{X}(1+f)\right)+\frac{2\mathsf{X}\mathsf{T}^2}{(1+\mathsf{X}^2)^2}}{2\sqrt{((1+f)(1+\mathsf{X}^2-\mathsf{T}^2))^3}}\\
&\left.-\frac{\mathsf{T}^2}{1+\mathsf{X}^2}\frac{3\left(\partial_{\mathsf{X}}f(1+\mathsf{X}^2-\mathsf{T}^2)+2\mathsf{X}(1+f)\right)}{2\sqrt{((1+f)(1+\mathsf{X}^2-\mathsf{T}^2))^5}}\right\}
\end{aligned}
\tag{A.1.31}
$$

and

$$
\begin{aligned}
&\partial_{\mathsf{T}}\left(\mathfrak{g}^K_m\left(\overline{\nabla}_{\widetilde{\mathscr{T}}}\widetilde{\mathscr{T}},\widetilde{V}_{\mathsf{X}}\right)\right)=\\
&=\frac{\mathsf{X}}{\sqrt{1+\mathsf{X}^2}}\frac{1}{\sqrt{\xi_m}}\left\{\left[-\frac{1}{2\xi_m}\frac{d\xi_m}{dr}\left(\frac{1-\frac{\xi_m}{16m^2}\left(1+\frac{2m}{r}\right)}{\sqrt{(1+f)(1+\mathsf{X}^2-\mathsf{T}^2)}}+\frac{\frac{\mathsf{T}^2}{1+\mathsf{X}^2}}{\sqrt{((1+f)(1+\mathsf{X}^2-\mathsf{T}^2))^3}}\right)\right.\right.\\
&\left.-\frac{\frac{d\xi_m}{dr}\left(1+\frac{2m}{r}\right)-\frac{2m}{r^2}\xi_m}{16m^2\sqrt{(1+f)(1+\mathsf{X}^2-\mathsf{T}^2)}}\right]\frac{\partial r}{\partial\mathsf{T}}\\
&-\frac{\left(1-\frac{\xi_m}{16m^2}\left(1+\frac{2m}{r}\right)\right)\left(\partial_{\mathsf{T}}f(1+\mathsf{X}^2-\mathsf{T}^2)-2\mathsf{T}(1+f)\right)-\frac{2\mathsf{T}}{1+\mathsf{X}^2}}{2\sqrt{((1+f)(1+\mathsf{X}^2-\mathsf{T}^2))^3}}\\
&\left.-\frac{\mathsf{T}^2}{1+\mathsf{X}^2}\frac{3\left(\partial_{\mathsf{T}}f(1+\mathsf{X}^2-\mathsf{T}^2)-2\mathsf{T}(1+f)\right)}{2\sqrt{((1+f)(1+\mathsf{X}^2-\mathsf{T}^2))^5}}\right\}
\end{aligned}
\tag{A.1.32}
$$

Combining the two above and plugging in the derivative of ξ_m given by (2.3.24) and the derivatives of the Schwarzschild radial coordinate r with respect to the Kruskal–Szekeres coordinates T,X as in (2.3.25), we obtain

$$
\begin{aligned}
&\partial_{\mathsf{X}}\left(\mathfrak{g}^K_m\left(\overline{\nabla}_{\widetilde{\mathscr{T}}}\widetilde{\mathscr{T}},\widetilde{V}_{\mathsf{X}}\right)\right)+\frac{\mathsf{T}\mathsf{X}}{1+\mathsf{X}^2}\partial_{\mathsf{T}}\left(\mathfrak{g}^K_m\left(\overline{\nabla}_{\widetilde{\mathscr{T}}}\widetilde{\mathscr{T}},\widetilde{V}_{\mathsf{X}}\right)\right)=\\
&= \frac{1}{(1+\mathsf{X}^2)^{\frac{3}{2}}}\left(\frac{1-\frac{\xi_m}{16m^2}\left(1+\frac{2m}{r}\right)}{\sqrt{\xi_m(1+f)(1+\mathsf{X}^2-\mathsf{T}^2)}}+\frac{\mathsf{T}^2}{1+\mathsf{X}^2}\frac{1}{\sqrt{\xi_m\left((1+f)(1+\mathsf{X}^2-\mathsf{T}^2)\right)^3}}\right)\\
&+\frac{\mathsf{X}}{\sqrt{1+\mathsf{X}^2}}\frac{1}{\sqrt{\xi_m}}\left\{\left[-\frac{1}{2\xi_m}\frac{d\xi_m}{dr}\left(\frac{1-\frac{\xi_m}{16m^2}\left(1+\frac{2m}{r}\right)}{\sqrt{(1+f)(1+\mathsf{X}^2-\mathsf{T}^2)}}+\frac{\frac{\mathsf{T}^2}{1+\mathsf{X}^2}}{\sqrt{((1+f)(1+\mathsf{X}^2-\mathsf{T}^2))^3}}\right)\right.\right.\\
&\left.-\frac{\frac{d\xi_m}{dr}\left(1+\frac{2m}{r}\right)-\frac{2m}{r^2}\xi_m}{16m^2\sqrt{(1+f)(1+\mathsf{X}^2-\mathsf{T}^2)}}\right]\left(\frac{\partial r}{\partial\mathsf{X}}+\frac{\mathsf{T}\mathsf{X}}{1+\mathsf{X}^2}\frac{\partial r}{\partial\mathsf{T}}\right)
\end{aligned}
$$

$$
\begin{aligned}
&- \frac{\left(1-\frac{\xi_m}{16m^2}\left(1+\frac{2m}{r}\right)\right)\left(\left(\partial_{\mathsf{X}} f + \frac{\mathsf{T}\mathsf{X}}{1+\mathsf{X}^2}\partial_{\mathsf{T}} f\right)(1+\mathsf{X}^2-\mathsf{T}^2) + 2\left(\mathsf{X} - \frac{\mathsf{T}^2\mathsf{X}}{1+\mathsf{X}^2}\right)(1+f)\right)}{2\sqrt{((1+f)(1+\mathsf{X}^2-\mathsf{T}^2))^3}} \\
&- \frac{\mathsf{T}^2}{1+\mathsf{X}^2}\frac{3\left(\left(\partial_{\mathsf{X}} f + \frac{\mathsf{T}\mathsf{X}}{1+\mathsf{X}^2}\partial_{\mathsf{T}} f\right)(1+\mathsf{X}^2-\mathsf{T}^2) + 2\left(\mathsf{X} - \frac{\mathsf{T}^2\mathsf{X}}{1+\mathsf{X}^2}\right)(1+f)\right)}{2\sqrt{((1+f)(1+\mathsf{X}^2-\mathsf{T}^2))^5}} \Bigg\} \\
= {} & \frac{1}{(1+\mathsf{X}^2)^{\frac{3}{2}}}\left(\frac{1-\frac{\xi_m}{16m^2}\left(1+\frac{2m}{r}\right)}{\sqrt{\xi_m(1+f)(1+\mathsf{X}^2-\mathsf{T}^2)}} + \frac{\mathsf{T}^2}{1+\mathsf{X}^2}\frac{1}{\sqrt{\xi_m\left((1+f)(1+\mathsf{X}^2-\mathsf{T}^2)\right)^3}}\right) \\
&+ \frac{\mathsf{X}}{\sqrt{1+\mathsf{X}^2}}\frac{1}{\sqrt{\xi_m}}\Bigg\{\Bigg[-\frac{1}{2\xi_m}\frac{d\xi_m}{dr}\left(\frac{1-\frac{\xi_m}{16m^2}\left(1+\frac{2m}{r}\right)}{\sqrt{(1+f)(1+\mathsf{X}^2-\mathsf{T}^2)}} + \frac{\frac{\mathsf{T}^2}{1+\mathsf{X}^2}}{\sqrt{((1+f)(1+\mathsf{X}^2-\mathsf{T}^2))^3}}\right) \\
&- \frac{\frac{d\xi_m}{dr}\left(1+\frac{2m}{r}\right) - \frac{2m}{r^2}\xi_m}{16m^2\sqrt{(1+f)(1+\mathsf{X}^2-\mathsf{T}^2)}}\Bigg]\frac{\mathsf{X}}{1+\mathsf{X}^2}\frac{\xi_m(1+\mathsf{X}^2-\mathsf{T}^2)}{4m} \\
&- \frac{\left(1-\frac{\xi_m}{16m^2}\left(1+\frac{2m}{r}\right)\right)\frac{\mathsf{X}}{1+\mathsf{X}^2}(1+\mathsf{X}^2-\mathsf{T}^2)}{\sqrt{((1+f)(1+\mathsf{X}^2-\mathsf{T}^2))^3}} - \frac{\mathsf{T}^2}{1+\mathsf{X}^2}\frac{3\frac{\mathsf{X}}{1+\mathsf{X}^2}(1+\mathsf{X}^2-\mathsf{T}^2)}{\sqrt{((1+f)(1+\mathsf{X}^2-\mathsf{T}^2))^5}}\Bigg\} \\
= {} & \frac{1}{(1+\mathsf{X}^2)^{\frac{3}{2}}}\left(\frac{1-\frac{\xi_m}{16m^2}\left(1+\frac{2m}{r}\right)}{\sqrt{\xi_m(1+f)(1+\mathsf{X}^2-\mathsf{T}^2)}} + \frac{\mathsf{T}^2}{1+\mathsf{X}^2}\frac{1}{\sqrt{\xi_m\left((1+f)(1+\mathsf{X}^2-\mathsf{T}^2)\right)^3}}\right) \\
&+ \frac{\mathsf{X}^2}{(1+\mathsf{X}^2)^{\frac{3}{2}}}\frac{(1+\mathsf{X}^2-\mathsf{T}^2)}{\sqrt{\xi_m}}\Bigg\{\Bigg[\frac{\left(1+\frac{2m}{r}\right)}{4m}\Bigg(\frac{1-\frac{\xi_m}{16m^2}\left(1+\frac{2m}{r}\right)}{\sqrt{(1+f)(1+\mathsf{X}^2-\mathsf{T}^2)}} \\
&\quad + \frac{\frac{\mathsf{T}^2}{1+\mathsf{X}^2}}{\sqrt{((1+f)(1+\mathsf{X}^2-\mathsf{T}^2))^3}}\Bigg) + \frac{\frac{\xi_m}{2m}\left(1+\frac{2m}{r}\right)^2 + \frac{2m}{r^2}\xi_m}{16m^2\sqrt{(1+f)(1+\mathsf{X}^2-\mathsf{T}^2)}}\Bigg]\frac{\xi_m}{4m} \\
&\quad - \frac{\left(1-\frac{\xi_m}{16m^2}\left(1+\frac{2m}{r}\right)\right)}{\sqrt{((1+f)(1+\mathsf{X}^2-\mathsf{T}^2))^3}} - \frac{3\frac{\mathsf{T}^2}{1+\mathsf{X}^2}}{\sqrt{((1+f)(1+\mathsf{X}^2-\mathsf{T}^2))^5}}\Bigg\}
\end{aligned}
$$

where we used, by employing (2.3.25), the following identity

$$
\frac{\partial r}{\partial \mathsf{X}} + \frac{\mathsf{T}\mathsf{X}}{1+\mathsf{X}^2}\frac{\partial r}{\partial \mathsf{T}} = \frac{\xi_m}{4m}\mathsf{X}\left(1-\frac{\mathsf{T}^2}{1+\mathsf{X}^2}\right) = \frac{\mathsf{X}}{1+\mathsf{X}^2}\frac{\xi_m(1+\mathsf{X}^2-\mathsf{T}^2)}{4m}\,,
$$

and, due to (A.1.5), the computation

$$
\begin{aligned}
\partial_{\mathsf{X}} f + \frac{\mathsf{T}\mathsf{X}}{1+\mathsf{X}^2}\partial_{\mathsf{T}} f &= -2f\frac{\mathsf{X}}{\mathsf{T}}\left(\frac{f}{\mathsf{T}}(1+\mathsf{X}^2-\mathsf{T}^2) + \frac{f}{\mathsf{T}}(1+\mathsf{X}^2) - \frac{\mathsf{T}}{1+\mathsf{X}^2} - \mathsf{T}f\right) \\
&= -2f\frac{\mathsf{X}}{\mathsf{T}}\left(\frac{\mathsf{T}}{1+\mathsf{X}^2} + \frac{\mathsf{T}}{1+\mathsf{X}^2-\mathsf{T}^2} - \frac{\mathsf{T}}{1+\mathsf{X}^2} - \frac{\mathsf{T}^3}{(1+\mathsf{X}^2-\mathsf{T}^2)(1+\mathsf{X}^2)}\right) \\
&= -2\frac{\mathsf{T}^2\mathsf{X}}{(1+\mathsf{X}^2-\mathsf{T}^2)(1+\mathsf{X}^2)}\frac{1+\mathsf{X}^2-\mathsf{T}^2}{(1+\mathsf{X}^2-\mathsf{T}^2)(1+\mathsf{X}^2)} \\
&= -2\frac{\mathsf{T}^2\mathsf{X}}{(1+\mathsf{X}^2-\mathsf{T}^2)(1+\mathsf{X}^2)^2} = -2f\frac{\mathsf{X}}{1+\mathsf{X}^2}\,.
\end{aligned}
$$

We finally obtain

$$
\begin{aligned}
&\left|\widetilde{V}_{\mathsf{X}}\left(\mathfrak{g}^K_m\left(\overline{\nabla}_{\widetilde{\mathscr{T}}}\widetilde{\mathscr{T}},\widetilde{V}_{\mathsf{X}}\right)\right)\right| = \\
&= \sqrt{\frac{1+\mathsf{X}^2}{\xi_m(1+f)(1+\mathsf{X}^2-\mathsf{T}^2)}}\left|\partial_{\mathsf{X}}\left(\mathfrak{g}^K_m\left(\overline{\nabla}_{\widetilde{\mathscr{T}}}\widetilde{\mathscr{T}},\widetilde{V}_{\mathsf{X}}\right)\right)+\frac{\mathsf{TX}}{1+\mathsf{X}^2}\partial_{\mathsf{T}}\left(\mathfrak{g}^K_m\left(\overline{\nabla}_{\widetilde{\mathscr{T}}}\widetilde{\mathscr{T}},\widetilde{V}_{\mathsf{X}}\right)\right)\right| \\
&= \frac{1}{\sqrt{\xi_m(1+f)(1+\mathsf{X}^2-\mathsf{T}^2)}}\left|\frac{1}{1+\mathsf{X}^2}\left(\frac{1-\frac{\xi_m}{16m^2}\left(1+\frac{2m}{r}\right)}{\sqrt{\xi_m(1+f)(1+\mathsf{X}^2-\mathsf{T}^2)}}\right.\right. \\
&\qquad\left.+\frac{\mathsf{T}^2}{1+\mathsf{X}^2}\frac{1}{\sqrt{\xi_m\left((1+f)(1+\mathsf{X}^2-\mathsf{T}^2)\right)^3}}\right) \\
&\quad+\frac{\mathsf{X}^2}{1+\mathsf{X}^2}\frac{(1+\mathsf{X}^2-\mathsf{T}^2)}{\sqrt{\xi_m}}\left\{\left[\frac{\left(1+\frac{2m}{r}\right)}{4m}\left(\frac{1-\frac{\xi_m}{16m^2}\left(1+\frac{2m}{r}\right)}{\sqrt{(1+f)(1+\mathsf{X}^2-\mathsf{T}^2)}}\right.\right.\right. \\
&\qquad\left.+\frac{\frac{\mathsf{T}^2}{1+\mathsf{X}^2}}{\sqrt{\left((1+f)(1+\mathsf{X}^2-\mathsf{T}^2)\right)^3}}\right)+\frac{\frac{\xi_m}{2m}\left(1+\frac{2m}{r}\right)^2+\frac{2m}{r^2}\xi_m}{16m^2\sqrt{(1+f)(1+\mathsf{X}^2-\mathsf{T}^2)}}\left.\vphantom{\frac{1}{1}}\right]\frac{\xi_m}{4m} \\
&\quad\left.\left.-\frac{\left(1-\frac{\xi_m}{16m^2}\left(1+\frac{2m}{r}\right)\right)}{\sqrt{\left((1+f)(1+\mathsf{X}^2-\mathsf{T}^2)\right)^3}}-\frac{3\frac{\mathsf{T}^2}{1+\mathsf{X}^2}}{\sqrt{\left((1+f)(1+\mathsf{X}^2-\mathsf{T}^2)\right)^5}}\right\}\right| \\
&\le \frac{2}{\xi_m(1+\mathsf{X}^2-\mathsf{T}^2)}\left(1+\frac{\xi_m}{16m^2}\left(1+\frac{2m}{r}\right)+\frac{2}{1+\mathsf{X}^2-\mathsf{T}^2}\right) \\
&\quad+\frac{1}{16m^2}\left[\left(1+\frac{2m}{r}\right)\left(1+\frac{\xi_m}{16m^2}\left(1+\frac{2m}{r}\right)\right)+\frac{1}{1+\mathsf{X}^2-\mathsf{T}^2}\right. \\
&\qquad\left.+\frac{\xi_m}{8m^2}\left(1+\frac{4m}{r}+\frac{8m^2}{r^2}\right)\right] \\
&\le \frac{C(\kappa_0)}{m^2}
\end{aligned}
\tag{A.1.33}
$$

Combining this last estimate (A.1.33) with the observations on (A.1.29)–(A.1.30), we obtain

$$
\left|\mathfrak{g}^K_m\left(\overline{\nabla}^2_{\widetilde{V}_{\mathsf{X}}\widetilde{\mathscr{T}}}\widetilde{\mathscr{T}},\widetilde{V}_{\mathsf{X}}\right)\right|+\left|\mathfrak{g}^K_m\left(\overline{\nabla}^2_{\widetilde{V}_{\mathsf{X}}\widetilde{\mathscr{T}}}\widetilde{\mathscr{T}},\widetilde{\mathscr{T}}\right)\right|+\left|\mathfrak{g}^K_m\left(\overline{\nabla}^2_{\widetilde{V}_{\mathsf{X}}\widetilde{\mathscr{T}}}\widetilde{\mathscr{T}},\widetilde{V}_I\right)\right|\le\frac{C(\kappa_0)}{m^2}. \tag{A.1.34}
$$

Bound on $\overline{\nabla}^2_{\widetilde{V}_I\widetilde{V}_J}\widetilde{\mathscr{T}}$. Recall that

$$
\overline{\nabla}^2_{\widetilde{V}_I\widetilde{V}_J}\widetilde{\mathscr{T}}=\overline{\nabla}_{\widetilde{V}_I}\left(\overline{\nabla}_{\widetilde{V}_J}\widetilde{\mathscr{T}}\right)-\overline{\nabla}_{\overline{\nabla}_{\widetilde{V}_I}\widetilde{V}_J}\widetilde{\mathscr{T}}.
$$

From the computations (A.1.16)–(A.1.18), $\overline{\nabla}_{\widetilde{V}_J}\widetilde{\mathscr{T}}=\mathfrak{g}^K_m\left(\overline{\nabla}_{\widetilde{V}_J}\widetilde{\mathscr{T}},\widetilde{V}_J\right)\widetilde{V}_J$. This implies

$$
\overline{\nabla}_{\widetilde{V}_I}\left(\overline{\nabla}_{\widetilde{V}_J}\widetilde{\mathscr{T}}\right)=\widetilde{V}_I\left(\mathfrak{g}^K_m\left(\overline{\nabla}_{\widetilde{V}_J}\widetilde{\mathscr{T}},\widetilde{V}_J\right)\right)\widetilde{V}_J+\mathfrak{g}^K_m\left(\overline{\nabla}_{\widetilde{V}_J}\widetilde{\mathscr{T}},\widetilde{V}_J\right)\overline{\nabla}_{\widetilde{V}_I}\widetilde{V}_J
$$

$$= \mathfrak{g}_m^K\left(\overline{\nabla}_{\widetilde{V}_J}\widetilde{\mathscr{T}}, \widetilde{V}_J\right)\overline{\nabla}_{\widetilde{V}_I}\widetilde{V}_J\,,$$

as one can check by (A.1.17) that the quantity $\mathfrak{g}_m^K\left(\overline{\nabla}_{\widetilde{V}_J}\widetilde{\mathscr{T}}, \widetilde{V}_J\right)$ does not depend on the spherical coordinates. Let us now simply write

$$\begin{aligned}\overline{\nabla}_{\widetilde{V}_I}\widetilde{V}_J &= \mathfrak{g}_m^K\left(\overline{\nabla}_{\widetilde{V}_I}\widetilde{V}_J, \widetilde{V}_{\mathsf{X}}\right)\widetilde{V}_{\mathsf{X}} + \mathfrak{g}_m^K\left(\overline{\nabla}_{\widetilde{V}_I}\widetilde{V}_J, \widetilde{V}_K\right)\widetilde{V}_K + \mathfrak{g}_m^K\left(\overline{\nabla}_{\widetilde{V}_I}\widetilde{V}_J, \widetilde{\mathscr{T}}\right)\widetilde{\mathscr{T}}\\ &= -\mathfrak{g}_m^K\left(\overline{\nabla}_{\widetilde{V}_I}\widetilde{V}_{\mathsf{X}}, \widetilde{V}_J\right)\widetilde{V}_{\mathsf{X}} + \mathfrak{g}_m^K\left(\overline{\nabla}_{\widetilde{V}_I}\widetilde{V}_J, \widetilde{V}_K\right)\widetilde{V}_K - \mathfrak{g}_m^K\left(\overline{\nabla}_{\widetilde{V}_I}\widetilde{\mathscr{T}}, \widetilde{V}_J\right)\widetilde{\mathscr{T}}\,.\end{aligned}$$

Using $\overline{\nabla}_{\widetilde{\mathscr{T}}}\widetilde{\mathscr{T}} = \mathfrak{g}_m^K\left(\overline{\nabla}_{\widetilde{\mathscr{T}}}\widetilde{\mathscr{T}}, \widetilde{V}_{\mathsf{X}}\right)\widetilde{V}_{\mathsf{X}}$ and $\overline{\nabla}_{\widetilde{V}_{\mathsf{X}}}\widetilde{\mathscr{T}} = \mathfrak{g}_m^K\left(\overline{\nabla}_{\widetilde{V}_{\mathsf{X}}}\widetilde{\mathscr{T}}, \widetilde{V}_{\mathsf{X}}\right)\widetilde{V}_{\mathsf{X}}$, we then have

$$\begin{aligned}-\overline{\nabla}_{\overline{\nabla}_{\widetilde{V}_I}\widetilde{V}_J}\widetilde{\mathscr{T}} &= \mathfrak{g}_m^K\left(\overline{\nabla}_{\widetilde{V}_I}\widetilde{V}_{\mathsf{X}}, \widetilde{V}_J\right)\overline{\nabla}_{\widetilde{V}_{\mathsf{X}}}\widetilde{\mathscr{T}} - \mathfrak{g}_m^K\left(\overline{\nabla}_{\widetilde{V}_I}\widetilde{V}_J, \widetilde{V}_K\right)\overline{\nabla}_{\widetilde{V}_K}\widetilde{\mathscr{T}} + \mathfrak{g}_m^K\left(\overline{\nabla}_{\widetilde{V}_I}\widetilde{\mathscr{T}}, \widetilde{V}_J\right)\overline{\nabla}_{\widetilde{\mathscr{T}}}\widetilde{\mathscr{T}}\\ &= \mathfrak{g}_m^K\left(\overline{\nabla}_{\widetilde{V}_I}\widetilde{V}_{\mathsf{X}}, \widetilde{V}_J\right)\mathfrak{g}_m^K\left(\overline{\nabla}_{\widetilde{\mathscr{T}}}\widetilde{\mathscr{T}}, \widetilde{V}_{\mathsf{X}}\right)\widetilde{V}_{\mathsf{X}} - \mathfrak{g}_m^K\left(\overline{\nabla}_{\widetilde{V}_I}\widetilde{V}_J, \widetilde{V}_K\right)\mathfrak{g}_m^K\left(\overline{\nabla}_{\widetilde{V}_K}\widetilde{\mathscr{T}}, \widetilde{V}_K\right)\widetilde{V}_K\\ &\quad + \mathfrak{g}_m^K\left(\overline{\nabla}_{\widetilde{V}_I}\widetilde{\mathscr{T}}, \widetilde{V}_J\right)\mathfrak{g}_m^K\left(\overline{\nabla}_{\widetilde{V}_{\mathsf{X}}}\widetilde{\mathscr{T}}, \widetilde{V}_{\mathsf{X}}\right)\widetilde{V}_{\mathsf{X}}\end{aligned}$$

Hence, we get

$$\begin{aligned}&\mathfrak{g}_m^K\left(\overline{\nabla}^2_{\widetilde{V}_I\widetilde{V}_J}\widetilde{\mathscr{T}}, \widetilde{V}_{\mathsf{X}}\right) =\\ &\quad = \mathfrak{g}_m^K\left(\overline{\nabla}_{\widetilde{V}_J}\widetilde{\mathscr{T}}, \widetilde{V}_J\right)\mathfrak{g}_m^K\left(\overline{\nabla}_{\widetilde{V}_I}\widetilde{V}_J, \widetilde{V}_{\mathsf{X}}\right) + \mathfrak{g}_m^K\left(\overline{\nabla}_{\widetilde{V}_I}\widetilde{V}_{\mathsf{X}}, \widetilde{V}_J\right)\mathfrak{g}_m^K\left(\overline{\nabla}_{\widetilde{\mathscr{T}}}\widetilde{\mathscr{T}}, \widetilde{V}_{\mathsf{X}}\right)\\ &\qquad\quad + \mathfrak{g}_m^K\left(\overline{\nabla}_{\widetilde{V}_I}\widetilde{\mathscr{T}}, \widetilde{V}_J\right)\mathfrak{g}_m^K\left(\overline{\nabla}_{\widetilde{V}_{\mathsf{X}}}\widetilde{\mathscr{T}}, \widetilde{V}_{\mathsf{X}}\right)\\ &\quad = -\mathfrak{g}_m^K\left(\overline{\nabla}_{\widetilde{V}_J}\widetilde{\mathscr{T}}, \widetilde{V}_J\right)\mathfrak{g}_m^K\left(\overline{\nabla}_{\widetilde{V}_I}\widetilde{V}_{\mathsf{X}}, \widetilde{V}_J\right) + \mathfrak{g}_m^K\left(\overline{\nabla}_{\widetilde{V}_I}\widetilde{V}_{\mathsf{X}}, \widetilde{V}_J\right)\mathfrak{g}_m^K\left(\overline{\nabla}_{\widetilde{\mathscr{T}}}\widetilde{\mathscr{T}}, \widetilde{V}_{\mathsf{X}}\right)\\ &\qquad\quad + \mathfrak{g}_m^K\left(\overline{\nabla}_{\widetilde{V}_I}\widetilde{\mathscr{T}}, \widetilde{V}_J\right)\mathfrak{g}_m^K\left(\overline{\nabla}_{\widetilde{V}_{\mathsf{X}}}\widetilde{\mathscr{T}}, \widetilde{V}_{\mathsf{X}}\right)\,,\end{aligned} \tag{A.1.35}$$

where all the terms involved have already been computed and estimated by $\frac{C(\kappa_0)}{m^2}$ due to (A.1.22). In addition, it holds

$$\mathfrak{g}_m^K\left(\overline{\nabla}^2_{\widetilde{V}_I\widetilde{V}_J}\widetilde{\mathscr{T}}, \widetilde{\mathscr{T}}\right) = \mathfrak{g}_m^K\left(\overline{\nabla}_{\widetilde{V}_J}\widetilde{\mathscr{T}}, \widetilde{V}_J\right)\mathfrak{g}_m^K\left(\overline{\nabla}_{\widetilde{V}_I}\widetilde{V}_J, \widetilde{\mathscr{T}}\right) = -\mathfrak{g}_m^K\left(\overline{\nabla}_{\widetilde{V}_J}\widetilde{\mathscr{T}}, \widetilde{V}_J\right)^2 \tag{A.1.36}$$

which can also be estimated by $\frac{C(\kappa_0)}{m^2}$ due to (A.1.22). To conclude, we compute

$$\begin{aligned}\mathfrak{g}_m^K\left(\overline{\nabla}^2_{\widetilde{V}_I\widetilde{V}_J}\widetilde{\mathscr{T}}, \widetilde{V}_L\right) &= \mathfrak{g}_m^K\left(\overline{\nabla}_{\widetilde{V}_J}\widetilde{\mathscr{T}}, \widetilde{V}_J\right)\mathfrak{g}_m^K\left(\overline{\nabla}_{\widetilde{V}_I}\widetilde{V}_J, \widetilde{V}_L\right)\\ &\quad - \mathfrak{g}_m^K\left(\overline{\nabla}_{\widetilde{V}_I}\widetilde{V}_J, \widetilde{V}_K\right)\mathfrak{g}_m^K\left(\overline{\nabla}_{\widetilde{V}_K}\widetilde{\mathscr{T}}, \widetilde{V}_K\right)\delta_{KL} = 0\,.\end{aligned} \tag{A.1.37}$$

Combining the last estimates (A.1.35)–(A.1.36)–(A.1.37) with (A.1.22), we obtain

$$\left|\mathfrak{g}_m^K\left(\overline{\nabla}^2_{\widetilde{V}_I\widetilde{V}_J}\widetilde{\mathscr{T}}, \widetilde{V}_{\mathsf{X}}\right)\right| + \left|\mathfrak{g}_m^K\left(\overline{\nabla}^2_{\widetilde{V}_I\widetilde{V}_J}\widetilde{\mathscr{T}}, \widetilde{\mathscr{T}}\right)\right| + \left|\mathfrak{g}_m^K\left(\overline{\nabla}^2_{\widetilde{V}_I\widetilde{V}_J}\widetilde{\mathscr{T}}, \widetilde{V}_L\right)\right| \leq \frac{C(\kappa_0)}{m^2}\,. \tag{A.1.38}$$

Bound on $\overline{\nabla}^2_{\widetilde{V}_I\widetilde{\mathscr{T}}}\widetilde{\mathscr{T}}$. Recall that

$$\overline{\nabla}^2_{\widetilde{V}_I\widetilde{\mathscr{T}}}\widetilde{\mathscr{T}} = \overline{\nabla}_{\widetilde{V}_I}\left(\overline{\nabla}_{\widetilde{\mathscr{T}}}\widetilde{\mathscr{T}}\right) - \overline{\nabla}_{\overline{\nabla}_{\widetilde{V}_I}\widetilde{\mathscr{T}}}\widetilde{\mathscr{T}}.$$

From the explicit computations (A.1.20)–(A.1.21), it follows $\overline{\nabla}_{\widetilde{\mathscr{T}}}\widetilde{\mathscr{T}} = \mathfrak{g}^K_m\left(\overline{\nabla}_{\widetilde{\mathscr{T}}}\widetilde{\mathscr{T}}, \widetilde{V}_{\mathsf{X}}\right)\widetilde{V}_{\mathsf{X}}$, while from (A.1.16)–(A.1.18) there holds $\overline{\nabla}_{\widetilde{V}_I}\widetilde{\mathscr{T}} = \mathfrak{g}^K_m\left(\overline{\nabla}_{\widetilde{V}_I}\widetilde{\mathscr{T}}, \widetilde{V}_I\right)\widetilde{V}_I$. These imply

$$\begin{aligned}
\overline{\nabla}_{\widetilde{V}_I}\left(\overline{\nabla}_{\widetilde{\mathscr{T}}}\widetilde{\mathscr{T}}\right) &= \widetilde{V}_I\left(\mathfrak{g}^K_m\left(\overline{\nabla}_{\widetilde{\mathscr{T}}}\widetilde{\mathscr{T}}, \widetilde{V}_{\mathsf{X}}\right)\right)\widetilde{V}_{\mathsf{X}} + \mathfrak{g}^K_m\left(\overline{\nabla}_{\widetilde{\mathscr{T}}}\widetilde{\mathscr{T}}, \widetilde{V}_{\mathsf{X}}\right)\overline{\nabla}_{\widetilde{V}_I}\widetilde{V}_{\mathsf{X}} = \mathfrak{g}^K_m\left(\overline{\nabla}_{\widetilde{\mathscr{T}}}\widetilde{\mathscr{T}}, \widetilde{V}_{\mathsf{X}}\right)\overline{\nabla}_{\widetilde{V}_I}\widetilde{V}_{\mathsf{X}} \\
&= -\mathfrak{g}^K_m\left(\overline{\nabla}_{\widetilde{\mathscr{T}}}\widetilde{\mathscr{T}}, \widetilde{V}_{\mathsf{X}}\right)\mathfrak{g}^K_m\left(\overline{\nabla}_{\widetilde{V}_I}\widetilde{\mathscr{T}}, \widetilde{V}_{\mathsf{X}}\right)\widetilde{\mathscr{T}} + \mathfrak{g}^K_m\left(\overline{\nabla}_{\widetilde{\mathscr{T}}}\widetilde{\mathscr{T}}, \widetilde{V}_{\mathsf{X}}\right)\mathfrak{g}^K_m\left(\overline{\nabla}_{\widetilde{V}_I}\widetilde{V}_{\mathsf{X}}, \widetilde{V}_K\right)\widetilde{V}_K \\
&= \mathfrak{g}^K_m\left(\overline{\nabla}_{\widetilde{\mathscr{T}}}\widetilde{\mathscr{T}}, \widetilde{V}_{\mathsf{X}}\right)\mathfrak{g}^K_m\left(\overline{\nabla}_{\widetilde{V}_I}\widetilde{V}_{\mathsf{X}}, \widetilde{V}_K\right)\widetilde{V}_K
\end{aligned} \tag{A.1.39}$$

and

$$-\overline{\nabla}_{\overline{\nabla}_{\widetilde{V}_I}\widetilde{\mathscr{T}}}\widetilde{\mathscr{T}} = -\mathfrak{g}^K_m\left(\overline{\nabla}_{\widetilde{V}_I}\widetilde{\mathscr{T}}, \widetilde{V}_I\right)\overline{\nabla}_{\widetilde{V}_I}\widetilde{\mathscr{T}} = -\mathfrak{g}^K_m\left(\overline{\nabla}_{\widetilde{V}_I}\widetilde{\mathscr{T}}, \widetilde{V}_I\right)^2\widetilde{V}_I \tag{A.1.40}$$

Combining the computations (A.1.39)–(A.1.40) with (A.1.22), we immediately obtain

$$\left|\mathfrak{g}^K_m\left(\overline{\nabla}^2_{\widetilde{V}_I\widetilde{\mathscr{T}}}\widetilde{\mathscr{T}}, \widetilde{V}_{\mathsf{X}}\right)\right| + \left|\mathfrak{g}^K_m\left(\overline{\nabla}^2_{\widetilde{V}_I\widetilde{\mathscr{T}}}\widetilde{\mathscr{T}}, \widetilde{\mathscr{T}}\right)\right| + \left|\mathfrak{g}^K_m\left(\overline{\nabla}^2_{\widetilde{V}_I\widetilde{\mathscr{T}}}\widetilde{\mathscr{T}}, \widetilde{V}_L\right)\right| \leq \frac{C(\kappa_0)}{m^2}. \tag{A.1.41}$$

Bound on $\overline{\nabla}^2_{\widetilde{\mathscr{T}}\widetilde{\mathscr{T}}}\widetilde{\mathscr{T}}$. Recall that

$$\overline{\nabla}^2_{\widetilde{\mathscr{T}}\widetilde{\mathscr{T}}}\widetilde{\mathscr{T}} = \overline{\nabla}_{\widetilde{\mathscr{T}}}\left(\overline{\nabla}_{\widetilde{\mathscr{T}}}\widetilde{\mathscr{T}}\right) - \overline{\nabla}_{\overline{\nabla}_{\widetilde{\mathscr{T}}}\widetilde{\mathscr{T}}}\widetilde{\mathscr{T}}.$$

From the computations (A.1.20)–(A.1.21), it follows $\overline{\nabla}_{\widetilde{\mathscr{T}}}\widetilde{\mathscr{T}} = \mathfrak{g}^K_m\left(\overline{\nabla}_{\widetilde{\mathscr{T}}}\widetilde{\mathscr{T}}, \widetilde{V}_{\mathsf{X}}\right)\widetilde{V}_{\mathsf{X}}$ and from (A.1.14)–(A.1.15), it holds $\overline{\nabla}_{\widetilde{V}_{\mathsf{X}}}\widetilde{\mathscr{T}} = \mathfrak{g}^K_m\left(\overline{\nabla}_{\widetilde{V}_{\mathsf{X}}}\widetilde{\mathscr{T}}, \widetilde{V}_{\mathsf{X}}\right)\widetilde{V}_{\mathsf{X}}$. We then get

$$-\overline{\nabla}_{\overline{\nabla}_{\widetilde{\mathscr{T}}}\widetilde{\mathscr{T}}}\widetilde{\mathscr{T}} = \mathfrak{g}^K_m\left(\overline{\nabla}_{\widetilde{\mathscr{T}}}\widetilde{\mathscr{T}}, \widetilde{V}_{\mathsf{X}}\right)\overline{\nabla}_{\widetilde{V}_{\mathsf{X}}}\widetilde{\mathscr{T}} = \mathfrak{g}^K_m\left(\overline{\nabla}_{\widetilde{\mathscr{T}}}\widetilde{\mathscr{T}}, \widetilde{V}_{\mathsf{X}}\right)\mathfrak{g}^K_m\left(\overline{\nabla}_{\widetilde{V}_{\mathsf{X}}}\widetilde{\mathscr{T}}, \widetilde{V}_{\mathsf{X}}\right)\widetilde{V}_{\mathsf{X}} \tag{A.1.42}$$

and

$$\begin{aligned}
\overline{\nabla}_{\widetilde{\mathscr{T}}}\left(\overline{\nabla}_{\widetilde{\mathscr{T}}}\widetilde{\mathscr{T}}\right) &= \widetilde{\mathscr{T}}\left(\mathfrak{g}^K_m\left(\overline{\nabla}_{\widetilde{\mathscr{T}}}\widetilde{\mathscr{T}}, \widetilde{V}_{\mathsf{X}}\right)\right)\widetilde{V}_{\mathsf{X}} + \mathfrak{g}^K_m\left(\overline{\nabla}_{\widetilde{\mathscr{T}}}\widetilde{\mathscr{T}}, \widetilde{V}_{\mathsf{X}}\right)\overline{\nabla}_{\widetilde{\mathscr{T}}}\widetilde{V}_{\mathsf{X}} \\
&= \widetilde{\mathscr{T}}\left(\mathfrak{g}^K_m\left(\overline{\nabla}_{\widetilde{\mathscr{T}}}\widetilde{\mathscr{T}}, \widetilde{V}_{\mathsf{X}}\right)\right)\widetilde{V}_{\mathsf{X}} - \mathfrak{g}^K_m\left(\overline{\nabla}_{\widetilde{\mathscr{T}}}\widetilde{\mathscr{T}}, \widetilde{V}_{\mathsf{X}}\right)^2\widetilde{\mathscr{T}},
\end{aligned} \tag{A.1.43}$$

where we computed $\overline{\nabla}_{\widetilde{\mathscr{T}}}\widetilde{V}_{\mathsf{X}} = -\mathfrak{g}^K_m\left(\overline{\nabla}_{\widetilde{\mathscr{T}}}\widetilde{\mathscr{T}}, \widetilde{V}_{\mathsf{X}}\right)\widetilde{\mathscr{T}}$ since the following identity hold

$$\begin{aligned}
2\mathfrak{g}^K_m\left(\overline{\nabla}_{\widetilde{\mathscr{T}}}\widetilde{V}_{\mathsf{X}}, \widetilde{V}_{\mathsf{X}}\right) &= \widetilde{\mathscr{T}}\left(\mathfrak{g}^K_m\left(\widetilde{V}_{\mathsf{X}}, \widetilde{V}_{\mathsf{X}}\right)\right) = 0 \\
\mathfrak{g}^K_m\left(\overline{\nabla}_{\widetilde{\mathscr{T}}}\widetilde{V}_{\mathsf{X}}, \widetilde{\mathscr{T}}\right) &= \widetilde{\mathscr{T}}\left(\mathfrak{g}^K_m\left(\widetilde{V}_{\mathsf{X}}, \widetilde{\mathscr{T}}\right)\right) - \mathfrak{g}^K_m\left(\overline{\nabla}_{\widetilde{\mathscr{T}}}\widetilde{\mathscr{T}}, \widetilde{V}_{\mathsf{X}}\right) = -\mathfrak{g}^K_m\left(\overline{\nabla}_{\widetilde{\mathscr{T}}}\widetilde{\mathscr{T}}, \widetilde{V}_{\mathsf{X}}\right)
\end{aligned}$$

$$\mathfrak{g}_m^K\left(\overline{\nabla}_{\widetilde{\mathscr{T}}}\widetilde{V}_{\mathsf{X}},\widetilde{V}_I\right)=-\mathfrak{g}_m^K\left(\widetilde{V}_{\mathsf{X}},\overline{\nabla}_{\widetilde{\mathscr{T}}}\widetilde{V}_I\right)=0\,.$$

Notice that the last identity holds due to the vanishing of $\overline{\Gamma}^{\mathsf{X}}_{I\mathsf{X}},\overline{\Gamma}^{\mathsf{T}}_{I\mathsf{X}},\overline{\Gamma}^{\mathsf{X}}_{I\mathsf{T}},\overline{\Gamma}^{\mathsf{T}}_{I\mathsf{T}}$, as in (2.3.26). It is clear from (A.1.42)–(A.1.43) that, due to (A.1.22), we are just left with estimating $\widetilde{\mathscr{T}}\left(\mathfrak{g}_m^K\left(\overline{\nabla}_{\widetilde{\mathscr{T}}}\widetilde{\mathscr{T}},\widetilde{V}_{\mathsf{X}}\right)\right)$. Now, by the computations (A.1.31)–(A.1.32), we get

$$\begin{aligned}
&\frac{\mathsf{TX}}{1+\mathsf{X}^2}\partial_{\mathsf{X}}\left(\mathfrak{g}_m^K\left(\overline{\nabla}_{\widetilde{\mathscr{T}}}\widetilde{\mathscr{T}},\widetilde{V}_{\mathsf{X}}\right)\right)+\partial_{\mathsf{T}}\left(\mathfrak{g}_m^K\left(\overline{\nabla}_{\widetilde{\mathscr{T}}}\widetilde{\mathscr{T}},\widetilde{V}_{\mathsf{X}}\right)\right)=\\
&=\frac{\mathsf{TX}}{(1+\mathsf{X}^2)^{\frac{5}{2}}}\left(\frac{1-\frac{\xi_m}{16m^2}\left(1+\frac{2m}{r}\right)}{\sqrt{\xi_m(1+f)(1+\mathsf{X}^2-\mathsf{T}^2)}}+\frac{\mathsf{T}^2}{1+\mathsf{X}^2}\frac{1}{\sqrt{\xi_m\left((1+f)(1+\mathsf{X}^2-\mathsf{T}^2)\right)^3}}\right)\\
&\quad+\frac{\mathsf{X}}{\sqrt{1+\mathsf{X}^2}}\frac{1}{\sqrt{\xi_m}}\left\{\left[-\frac{1}{2\xi_m}\frac{d\xi_m}{dr}\left(\frac{1-\frac{\xi_m}{16m^2}\left(1+\frac{2m}{r}\right)}{\sqrt{(1+f)(1+\mathsf{X}^2-\mathsf{T}^2)}}+\frac{\frac{\mathsf{T}^2}{1+\mathsf{X}^2}}{\sqrt{\left((1+f)(1+\mathsf{X}^2-\mathsf{T}^2)\right)^3}}\right)\right.\right.\\
&\quad\left.-\frac{\frac{d\xi_m}{dr}\left(1+\frac{2m}{r}\right)-\frac{2m}{r^2}\xi_m}{16m^2\sqrt{(1+f)(1+\mathsf{X}^2-\mathsf{T}^2)}}\right]\left(\frac{\mathsf{TX}}{1+\mathsf{X}^2}\frac{\partial r}{\partial\mathsf{X}}+\frac{\partial r}{\partial\mathsf{T}}\right)\\
&\quad-\frac{1}{2\sqrt{\left((1+f)(1+\mathsf{X}^2-\mathsf{T}^2)\right)^3}}\left[\left(1-\frac{\xi_m}{16m^2}\left(1+\frac{2m}{r}\right)\right)\right.\\
&\quad\left(\left(\frac{\mathsf{TX}}{1+\mathsf{X}^2}\partial_{\mathsf{X}}f+\partial_{\mathsf{T}}f\right)(1+\mathsf{X}^2-\mathsf{T}^2)+2\left(\frac{\mathsf{TX}^2}{1+\mathsf{X}^2}-\mathsf{T}\right)(1+f)\right)\\
&\quad\left.+\frac{2\mathsf{X}^2\mathsf{T}^3}{(1+\mathsf{X}^2)^3}-\frac{2\mathsf{T}}{1+\mathsf{X}^2}\right]\\
&\quad\left.-\frac{\mathsf{T}^2}{1+\mathsf{X}^2}\frac{3\left(\left(\frac{\mathsf{TX}}{1+\mathsf{X}^2}\partial_{\mathsf{X}}f+\partial_{\mathsf{T}}f\right)(1+\mathsf{X}^2-\mathsf{T}^2)+2\left(\frac{\mathsf{TX}^2}{1+\mathsf{X}^2}-\mathsf{T}\right)(1+f)\right)}{2\sqrt{\left((1+f)(1+\mathsf{X}^2-\mathsf{T}^2)\right)^5}}\right\}\\
&=\frac{\mathsf{TX}}{(1+\mathsf{X}^2)^{\frac{5}{2}}}\left(\frac{1-\frac{\xi_m}{16m^2}\left(1+\frac{2m}{r}\right)}{\sqrt{\xi_m(1+f)(1+\mathsf{X}^2-\mathsf{T}^2)}}+\frac{\mathsf{T}^2}{1+\mathsf{X}^2}\frac{1}{\sqrt{\xi_m\left((1+f)(1+\mathsf{X}^2-\mathsf{T}^2)\right)^3}}\right)\\
&\quad+\frac{\mathsf{X}}{\sqrt{1+\mathsf{X}^2}}\frac{1}{\sqrt{\xi_m}}\left\{\left[\frac{\left(1+\frac{2m}{r}\right)}{4m}\left(\frac{1-\frac{\xi_m}{16m^2}\left(1+\frac{2m}{r}\right)}{\sqrt{(1+f)(1+\mathsf{X}^2-\mathsf{T}^2)}}+\frac{\frac{\mathsf{T}^2}{1+\mathsf{X}^2}}{\sqrt{\left((1+f)(1+\mathsf{X}^2-\mathsf{T}^2)\right)^3}}\right)\right.\right.\\
&\quad\left.+\frac{\frac{\xi_m}{2m}\left(1+\frac{2m}{r}\right)^2-\frac{2m}{r^2}\xi_m}{16m^2\sqrt{(1+f)(1+\mathsf{X}^2-\mathsf{T}^2)}}\right]\left(-\frac{\mathsf{T}}{1+\mathsf{X}^2}\frac{\xi_m}{4m}\right)\\
&\quad-\frac{1}{2\sqrt{\left((1+f)(1+\mathsf{X}^2-\mathsf{T}^2)\right)^3}}\left[\left(1-\frac{\xi_m}{16m^2}\left(1+\frac{2m}{r}\right)\right)\right.\\
&\quad\left(\frac{2\mathsf{T}}{1+\mathsf{X}^2}\left(1+f-\frac{\mathsf{T}^2\mathsf{X}^2}{(1+\mathsf{X}^2)^2}\right)-\frac{2\mathsf{T}}{1+\mathsf{X}^2}(1+f)\right)\\
&\quad\left.-\frac{2\mathsf{T}}{(1+\mathsf{X}^2)^2}\left((1+\mathsf{X}^2-\mathsf{T}^2)+\frac{\mathsf{T}^2}{1+\mathsf{X}^2}\right)\right]
\end{aligned}$$

$$
\begin{aligned}
&- \frac{\mathsf{T}^2}{1+\mathsf{X}^2} \frac{3\left(\frac{2\mathsf{T}}{1+\mathsf{X}^2}\left(1+f-\frac{\mathsf{T}^2\mathsf{X}^2}{(1+\mathsf{X}^2)^2}\right)-\frac{2\mathsf{T}}{1+\mathsf{X}^2}(1+f)\right)}{2\sqrt{((1+f)(1+\mathsf{X}^2-\mathsf{T}^2))^5}}\Bigg\}\\
&= \frac{\mathsf{TX}}{(1+\mathsf{X}^2)^{\frac{5}{2}}}\left(\frac{1-\frac{\xi_m}{16m^2}\left(1+\frac{2m}{r}\right)}{\sqrt{\xi_m(1+f)(1+\mathsf{X}^2-\mathsf{T}^2)}}+\frac{\mathsf{T}^2}{1+\mathsf{X}^2}\frac{1}{\sqrt{\xi_m\left((1+f)(1+\mathsf{X}^2-\mathsf{T}^2)\right)^3}}\right)\\
&- \frac{\mathsf{TX}}{(1+\mathsf{X}^2)^{\frac{3}{2}}}\frac{\sqrt{\xi_m}}{16m^2}\Bigg\{\Bigg[\left(1+\frac{2m}{r}\right)\left(\frac{1-\frac{\xi_m}{4m}\left(1+\frac{2m}{r}\right)}{\sqrt{(1+f)(1+\mathsf{X}^2-\mathsf{T}^2)}}\right.\\
&\left.+\frac{\frac{\mathsf{T}^2}{1+\mathsf{X}^2}}{\sqrt{((1+f)(1+\mathsf{X}^2-\mathsf{T}^2))^3}}\right)+\frac{\frac{\xi_m}{2m}\left(1+\frac{2m}{r}\right)^2-\frac{2m}{r^2}\xi_m}{16m^2\sqrt{(1+f)(1+\mathsf{X}^2-\mathsf{T}^2)}}\Bigg]\\
&+\frac{\mathsf{T}}{1+\mathsf{X}^2}\frac{\left(1-\frac{\xi_m}{16m^2}\left(1+\frac{2m}{r}\right)\right)\frac{\mathsf{T}^2\mathsf{X}^2}{(1+\mathsf{X}^2)^2}+\frac{(1+\mathsf{X}^2-\mathsf{T}^2)}{1+\mathsf{X}^2}+\frac{\mathsf{T}^2}{(1+\mathsf{X}^2)^2}}{\sqrt{((1+f)(1+\mathsf{X}^2-\mathsf{T}^2))^3}}\\
&+\frac{\mathsf{T}}{1+\mathsf{X}^2}\frac{\frac{3\mathsf{T}^4\mathsf{X}^2}{(1+\mathsf{X}^2)^3}}{\sqrt{((1+f)(1+\mathsf{X}^2-\mathsf{T}^2))^5}}\Bigg\}
\end{aligned}
$$

where we used, by employing (2.3.25) and (2.3.24), the following identity

$$
\frac{\mathsf{TX}}{1+\mathsf{X}^2}\frac{\partial r}{\partial \mathsf{X}}+\frac{\partial r}{\partial \mathsf{T}}=\frac{\xi_m}{4m}\mathsf{T}\left(\frac{\mathsf{X}^2}{1+\mathsf{X}^2}-1\right)=-\frac{\mathsf{T}}{1+\mathsf{X}^2}\frac{\xi_m}{4m},
$$

and, due to (A.1.5), the computation

$$
\begin{aligned}
\frac{\mathsf{TX}}{1+\mathsf{X}^2}\partial_{\mathsf{X}}f+\partial_{\mathsf{T}}f &= 2\frac{f}{\mathsf{T}}\left(-f\frac{\mathsf{X}^2}{1+\mathsf{X}^2}\left((1+\mathsf{X}^2-\mathsf{T}^2)+(1+\mathsf{X}^2)\right)+1+f(1+\mathsf{X}^2)\right)\\
&= 2\frac{f}{\mathsf{T}}\left(f(1+\mathsf{X}^2)\left(1-\frac{\mathsf{X}^2}{1+\mathsf{X}^2}\right)+1-\frac{\mathsf{T}^2\mathsf{X}^2}{(1+\mathsf{X}^2)^2}\right)\\
&= 2\frac{f}{\mathsf{T}}\left(1+f-\frac{\mathsf{T}^2\mathsf{X}^2}{(1+\mathsf{X}^2)^2}\right)
\end{aligned}
$$

We finally have

$$
\begin{aligned}
&\left|\widetilde{\mathscr{T}}\left(\mathfrak{g}_m^K\left(\overline{\nabla}_{\widetilde{\mathscr{T}}}\widetilde{\mathscr{T}},\widetilde{V}_{\mathsf{X}}\right)\right)\right| =\\
&= \sqrt{\frac{1+\mathsf{X}^2}{\xi_m(1+f)(1+\mathsf{X}^2-\mathsf{T}^2)}}\left|\frac{\mathsf{TX}}{1+\mathsf{X}^2}\partial_{\mathsf{X}}\left(\mathfrak{g}_m^K\left(\overline{\nabla}_{\widetilde{\mathscr{T}}}\widetilde{\mathscr{T}},\widetilde{V}_{\mathsf{X}}\right)\right)+\partial_{\mathsf{T}}\left(\mathfrak{g}_m^K\left(\overline{\nabla}_{\widetilde{\mathscr{T}}}\widetilde{\mathscr{T}},\widetilde{V}_{\mathsf{X}}\right)\right)\right|\\
&= \frac{1}{\sqrt{\xi_m(1+f)(1+\mathsf{X}^2-\mathsf{T}^2)}}\left|\frac{\mathsf{TX}}{(1+\mathsf{X}^2)^{\frac{3}{2}}}\left(\frac{1-\frac{\xi_m}{16m^2}\left(1+\frac{2m}{r}\right)}{\sqrt{\xi_m(1+f)(1+\mathsf{X}^2-\mathsf{T}^2)}}\right.\right.\\
&\quad\left.+\frac{\frac{\mathsf{T}^2}{1+\mathsf{X}^2}}{\sqrt{\xi_m\left((1+f)(1+\mathsf{X}^2-\mathsf{T}^2)\right)^3}}\right)
\end{aligned}
$$

$$
\begin{aligned}
&- \frac{\mathsf{TX}}{1+\mathsf{X}^2}\frac{\sqrt{\xi_m}}{4m}\Bigg\{\Bigg[\frac{1}{4m}\left(1+\frac{2m}{r}\right)\left(\frac{1-\frac{\xi_m}{16m^2}\left(1+\frac{2m}{r}\right)}{\sqrt{(1+f)(1+\mathsf{X}^2-\mathsf{T}^2)}}\right.\\
&\quad \left.+\frac{\frac{\mathsf{T}^2}{1+\mathsf{X}^2}}{\sqrt{((1+f)(1+\mathsf{X}^2-\mathsf{T}^2))^3}}\right)+\frac{\frac{\xi_m}{2m}\left(1+\frac{2m}{r}\right)^2-\frac{2m}{r^2}\xi_m}{16m^2\sqrt{(1+f)(1+\mathsf{X}^2-\mathsf{T}^2)}}\Bigg]\\
&+\frac{\mathsf{T}}{\sqrt{1+\mathsf{X}^2}}\frac{\left(1-\frac{\xi_m}{16m^2}\left(1+\frac{2m}{r}\right)\right)\frac{\mathsf{T}^2\mathsf{X}^2}{(1+\mathsf{X}^2)^2}+\frac{(1+\mathsf{X}^2-\mathsf{T}^2)}{1+\mathsf{X}^2}+\frac{\mathsf{T}^2}{(1+\mathsf{X}^2)^2}}{\sqrt{((1+f)(1+\mathsf{X}^2-\mathsf{T}^2))^3}}\\
&+\frac{\mathsf{T}}{\sqrt{1+\mathsf{X}^2}}\frac{\frac{3\mathsf{T}^4\mathsf{X}^2}{(1+\mathsf{X}^2)^3}}{\sqrt{((1+f)(1+\mathsf{X}^2-\mathsf{T}^2))^5}}\Bigg\}\Bigg|\\
\le{}& \frac{1}{\xi_m(1+\mathsf{X}^2-\mathsf{T}^2)}\left(1+\frac{\xi_m}{16m^2}\left(1+\frac{2m}{r}\right)+\frac{1}{\sqrt{1+\mathsf{X}^2-\mathsf{T}^2}}\right)\\
&+\frac{1}{16m^2(1+\mathsf{X}^2-\mathsf{T}^2)}\Bigg[\left(1+\frac{2m}{r}\right)\left(1+\frac{\xi_m}{16m^2}\left(1+\frac{2m}{r}\right)+\frac{1}{1+\mathsf{X}^2-\mathsf{T}^2}\right)\\
&\quad+\frac{\xi_m\left(1+\frac{4m}{r}+\frac{8m^2}{r^2}\right)}{8m^2}+\frac{\left(1-\frac{\xi_m}{16m^2}\left(1+\frac{2m}{r}\right)\right)+(1+\mathsf{X}^2-\mathsf{T}^2)+1}{1+\mathsf{X}^2-\mathsf{T}^2}\\
&\quad+\frac{3}{\sqrt{((1+f)(1+\mathsf{X}^2-\mathsf{T}^2))^3}}\Bigg]\\
\le{}& \frac{C(\kappa_0)}{m^2}
\end{aligned}
\tag{A.1.44}
$$

Combining the computation (A.1.44) with (A.1.42)–(A.1.43) and the bound (A.1.22), we get

$$
\left|\mathfrak{g}_m^K\left(\overline{\nabla}^2_{\widetilde{\mathscr{T}}\widetilde{\mathscr{T}}}\widetilde{\mathscr{T}},\widetilde{V}_{\mathsf{X}}\right)\right|+\left|\mathfrak{g}_m^K\left(\overline{\nabla}^2_{\widetilde{\mathscr{T}}\widetilde{\mathscr{T}}}\widetilde{\mathscr{T}},\widetilde{\mathscr{T}}\right)\right|+\left|\mathfrak{g}_m^K\left(\overline{\nabla}^2_{\widetilde{\mathscr{T}}\widetilde{\mathscr{T}}}\widetilde{\mathscr{T}},\widetilde{V}_I\right)\right|\le\frac{C(\kappa_0)}{m^2}\,. \tag{A.1.45}
$$

Finally, the estimates (A.1.26), (A.1.28), (A.1.34), (A.1.38), (A.1.41) and (A.1.45) imply

$$
{}^{\widetilde{\mathsf{T}}}\|\,\overline{\nabla}^2\,\widetilde{\mathscr{T}}\|\le\frac{C(\kappa_0)}{m^2}\,. \tag{A.1.46}
$$

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Additional articles

This chapter comprises three articles co-authored with collaborators, reproduced in their original form as published papers or preprints on ArXiv. As is usual in mathematics, the authors of the manuscripts are listed alphabetically. The papers are, in order of appearance:

1. Stefano Borghini, Carla Cederbaum and Albachiara Cogo, *Black Hole and Equipotential Photon Surface Uniqueness in 4-dimensional Asymptotically Flat Electrostatic Electro-Vacuum Spacetimes*, published in: Annales Henri Poincaré (December 2024).

2. Carla Cederbaum, Albachiara Cogo, Benedito Leandro and João Paulo dos Santos, *Uniqueness of static vacuum asymptotically flat black holes and equipotential photon surfaces in (n+1) dimensions à la Robinson*, arXiv:math/2403.14422, 2024, under revision after submission to a journal.

3. Albachiara Cogo, Stephen Lynch and Olivia Vičánek Martínez, *Rotational symmetry of ancient solutions to fully nonlinear curvature flows*, arXiv:math/2310.08301, 2023, under revision after submission to a journal.

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Black Hole and Equipotential Photon Surface Uniqueness in Four-Dimensional Asymptotically Flat Electrostatic Electro-Vacuum Spacetimes

Stefano Borghini, Carla Cederbaum and Albachiara Cogo

Abstract. We study four-dimensional asymptotically flat electrostatic electro-vacuum spacetimes with a connected black hole, photon sphere, or equipotential photon surface inner boundary. Our analysis, inspired by the potential theory approach by Agostiniani–Mazzieri, allows to give self-contained proofs of known uniqueness theorems of the sub-extremal, extremal, and super-extremal Reissner–Nordström spacetimes. We also obtain new results for connected photon spheres and equipotential photon surfaces in the extremal case. Finally, we provide, up to a restriction on the range of their radii, the uniqueness result for connected (both non-degenerate and degenerate) equipotential photon surfaces in the super-extremal case, not yet treated in the literature.

1. Introduction

The celebrated static vacuum black hole uniqueness theorem first proved by Israel [42] states that the only asymptotically flat static vacuum four-dimensional spacetime containing a black hole is the Schwarzschild spacetime (of positive mass). It has since been established in various degrees of generality—fewer technical assumptions, weaker asymptotic assumptions, allowing for multiple black holes, and for higher dimensions—and using many different strategies of proof. More information and a (then) complete list of references can be found in the reviews [30,40,61]. More recently, new proofs have been given in [6,19,23], see also Simon's proof described in [59, Appendix A].

The static vacuum black hole uniqueness theorem has also been extended to other matter models, most notably to electro-vacuum, first by Israel [43]

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and then by others in various degrees of generality and using many different strategies of proof. In the electro-vacuum case, one needs to distinguish between non-degenerate and degenerate black holes. In the non-degenerate case or if there is a single degenerate black hole, one recovers a sub-extremal resp. extremal Reissner–Nordström spacetime. In the degenerate case with multiple black holes, one recovers a Majumdar–Papapetrou spacetime. We refer the interested reader to [30,40,61], also for results for other matter models, and to [44,50,53] for more recent results in electro-vacuum. Related results have been established in the presence of a cosmological constant, see [9,14,31,51,73].

More recently, the static (electro-)vacuum black hole uniqueness theorems have been adapted to other situations, replacing black holes by photon spheres and, more generally, by equipotential photon surfaces. Here, *photon surfaces* are totally null geodesic timelike hypersurfaces in any spacetime; in a static spacetime, a photon surface is *equipotential* if the static lapse function is (spatially) constant along each of its canonical time-slices, and it is a *photon sphere* if the static lapse function is constant along it. They are *non-degenerate* if $dN \neq 0$ along them. In vacuum, uniqueness of photon spheres and non-degenerate equipotential photon surfaces was first shown by Cederbaum [21] and by Cederbaum and Galloway [19,26]. It has since been generalized in the same directions as the black hole uniqueness results, including to electro-vacuum, to multiple components, and to higher dimensions by Cederbaum–Galloway, Jahns, Raulot, Yazadjiev–Lazov, see [22–25,28,34,44,59,75]. For related works, including results for other matter models, see [10,33,37,47,54,58,60,62,64,68–72,74,76–78].

In this paper, we give a new proof of the static electro-vacuum black hole uniqueness theorem as well as the corresponding theorems for photon spheres and equipotential photon surfaces. We allow for both non-degenerate and degenerate black holes and for both non-degenerate and degenerate sub-extremal, extremal, and super-extremal photon surfaces (under a technical condition in the last case) but our results are restricted to a single black hole/photon sphere/photon surface and to 4 spacetime dimensions. We require very weak asymptotic decay assumptions. Our proof is based on a cylindrical ansatz inspired by the work by Agostiniani–Mazzieri [6], see Sect. 4, and may be of independent interest.

Comparison to previous uniqueness results in electro-vacuum: For non-degenerate black holes, we do not make the technical assumption that the static lapse function regularly foliates the spacetime made in Israel's non-degenerate black hole uniqueness theorem [43]. Thus, we fully recover the uniqueness results by Müller zum Hagen–Robinson–Seifert [79] and by Simon [65], allowing for weaker asymptotic decay. On the other hand, we do not fully recover Masood-ul-Alam's and Ruback's uniqueness results [55,63] for a priori multiple non-degenerate black holes as our strategy of proof a priori assumes that there is only a single black hole. Yet, in the single black hole case, we recover their results and weaken their asymptotic decay assumptions. We do not include magnetic charge, so we do not recover the result by Heusler [38], but see [65] showing a reduction from the full to the purely electric case.

In the case of degenerate black holes, we do not fully recover either of the uniqueness results by Chruściel [29] who obtains uniqueness for multiple degenerate and non-degenerate black holes with charges of the same sign, nor those of Heusler [41] who considers multiple degenerate black holes, nor those of Chruściel–Tod [32] who consider the most general scenario of combinations of multiple degenerate and non-degenerate black holes, again because we only allow for a single black hole. Yet, in the single black hole case, we recover their results and weaken their asymptotic decay assumptions.

Similarly, for photon spheres, we only recover the multiple non-degenerate black hole and photon sphere results by Cederbaum–Galloway [24] in the single photon sphere case, but allow for weaker asymptotic decay; moreover, we do not assume sub-extremality of the photon sphere. We also recover the results, remove Israel's technical condition, and weaken the asymptotic decay assumptions of the uniqueness result by Yazadjiev–Lazov [75], and other than them do not exclude extremal photon spheres.

Finally, for equipotential photon surfaces, we again only recover the results by Cederbaum–Jahns–Vičánek Martínez [28] in the single equipotential photon surface case, but allow for weaker asymptotic decay; moreover, we do not assume sub-extremality nor non-degeneracy. See Remark 2.1 for a discussion of the asymptotic decay assumptions, Sect. 8.2 for a comparison between the different definitions of sub-extremality, extremality, and super-extremality used in the above works, and Sect. 4.3 for a discussion on the special aspects of the degenerate case.

Structure of the article In Sect. 2, we will introduce the setup and the relevant definitions for the static electro-vacuum uniqueness theorems. Our results are stated in Sect. 3. In Sect. 4, we explain the strategy of our proof. Section 5 is dedicated to setting up the cylindrical ansatz formulation. Section 6 contains the proof of our black hole uniqueness theorems. Section 7 is dedicated to proving our photon sphere and equipotential photon surface uniqueness results. We end the paper by a discussion in Sect. 8.

2. Setup and Preliminaries

In this paper, we study tuples (M, g_0, N, Ψ), where (M, g_0) is a three-dimensional Riemannian manifold and $N\colon M \to \mathbb{R}^+$, $\Psi\colon M \to \mathbb{R}$ are smooth functions satisfying the *electrostatic electro-vacuum equations*

$$\begin{cases} N \,\mathrm{Ric} &= \mathrm{D}^2 N - \frac{2}{N}\, d\Psi \otimes d\Psi + \frac{1}{N}|\mathrm{D}\Psi|^2 g_0, \\ \Delta N &= \frac{1}{N}\,|\mathrm{D}\Psi|^2, \\ \Delta\Psi &= \frac{1}{N}\,\langle \mathrm{D}N \,|\, \mathrm{D}\Psi\rangle \end{cases} \tag{2.1}$$

on M, where D, D^2, Δ, Ric, and $|\cdot|$ denote the covariant derivative, covariant Hessian, Laplacian, Ricci curvature, and tensor norm with respect to g_0, respectively. We will call tuples (M, g_0, N, Ψ) as above *electrostatic electro-vacuum systems*, with M the *spatial manifold*, g_0 the *spatial metric*, N the *lapse function* or *static potential*, and Ψ the *electric potential* of the system.

From a physical point of view, a solution (M, g_0, N, Ψ) of (2.1) gives rise to a (standard) static spacetime $(\mathfrak{L}, \mathfrak{g}) = (\mathbb{R} \times M, -N^2 dt^2 + g_0)$ that satisfies the *source-free Einstein–Maxwell equations*

$$\mathfrak{Ric}_{\alpha\beta} - \frac{1}{2}\mathfrak{R}\,\mathfrak{g}_{\alpha\beta} = 2\left(F_{\alpha\gamma}F_{\beta}{}^{\gamma} - \frac{1}{4}\mathfrak{g}_{\alpha\beta}F^{\mu\nu}F_{\mu\nu}\right), \qquad (\mathfrak{div}F)_\alpha = 0$$

on $\mathbb{R} \times M$ with respect to the Maxwell tensor[1] $F = d\Psi \wedge dt$. Here, $\mathfrak{Ric}$, $\mathfrak{R}$, and $\mathfrak{div}$ denote the Ricci curvature, scalar curvature, and divergence with respect to the metric $\mathfrak{g}$.

Throughout this work, we assume that the spatial manifold M has a non-empty *connected* boundary ∂M. The connectedness of ∂M is important for our strategy of proof, see Sect. 4.1. We also assume that N and Ψ smoothly extend to ∂M as constants: $N|_{\partial M} =: N_0 \geq 0$ and $\Psi|_{\partial M} =: \Psi_0 \in \mathbb{R}$.

The first and most classical boundary condition we will consider is $N_0 = 0$. We call ∂M a *degenerate (static) horizon* if $|dN| \to 0$ when approaching ∂M; otherwise, we call ∂M a *non-degenerate (static) horizon*. In the non-degenerate case, we assume furthermore that g_0 smoothly extends to ∂M; moreover, it is well known that ∂M is totally geodesic in (M, g_0). From a physical point of view, a non-degenerate static horizon ∂M arises as a cross section of a non-degenerate ∂_t-Killing horizon in some electrostatic electro-vacuum spacetime $(\overline{\mathfrak{L}}, \overline{\mathfrak{g}})$ containing the standard static spacetime $(\mathfrak{L}, \mathfrak{g})$ with *surface gravity* $|dN|_{\overline{\mathfrak{g}}} > 0$. In the degenerate case, we assume that g_0 can be written as

$$g_0 = \frac{1}{Az^2} dz \otimes dz - \frac{1}{Az}(dz \otimes \omega + \omega \otimes dz) + b + \frac{1}{A}\omega \otimes \omega \tag{2.2}$$

on a one-sided tubular neighborhood $(0, \varepsilon) \times \partial M$ near $\partial M = \{z = 0\} \approx \mathbb{S}^2$ in M, where $z \in (0, \varepsilon)$. Here, $A = A(z, \cdot) > 0$, $\omega = \omega|_{(z,\cdot)}$, and $b = b|_{(z,\cdot)}$ are a smooth family of functions, one-forms, and Riemannian metrics on ∂M, respectively, smoothly depending on $z \in [0, \varepsilon)$. As usual, we are extending ω and b to $[0, \varepsilon) \times \partial M$ by setting $\omega(\partial_z) := 0$, $b(\partial_z, \cdot) := 0$. Moreover, we assume that there exists a constant $C > 0$ such that

$$\begin{aligned} &A(0, \cdot) = C^2, \quad \omega_I|_{(z,\cdot)} = O(z), \quad b_{IJ}|_{(0,\cdot)} = \frac{1}{C^2}(g_{\mathbb{S}^2})_{IJ}, \\ &\Psi(0, \cdot) = \Psi_0, \quad \partial_z \Psi(0, \cdot) = C, \quad N(z, \cdot) = Cz + O(z^2) \end{aligned} \tag{2.3}$$

on ∂M as $z \to 0^+$, where capital indices $I, J = 1, 2$ refer to local coordinates on ∂M and $g_{\mathbb{S}^2}$ denotes the canonical metric on $\mathbb{S}^2$. Notice that due to the smoothness of N up to ∂M and it being constant on ∂M, from (2.3) it follows that on ∂M as $z \to 0^+$ we have

$$\partial_z N(z, \cdot) = C + O(z), \quad \partial_I N(z, \cdot) = o(z). \tag{2.4}$$

[1] In higher dimensions, the assumed structure of the Maxwell tensor encapsulates the additional assumption that the magnetic field vanishes; however, in the 4 spacetime dimensions we are working in, one can assume this structure without loss of generality by the so-called electromagnetic duality, see, e.g., [44,50].

From a physical point of view, ∂M arises as the $v \to -\infty$ limit in Isenberg–Moncrief Gaussian null coordinates[2] (see [57]) on a degenerate ∂_t-Killing horizon $\partial\mathfrak{L}$ admitting a compact cross section, in some electrostatic electro-vacuum spacetime $(\overline{\mathfrak{L}}, \overline{\mathfrak{g}})$ containing the standard static spacetime $(\mathfrak{L}, \mathfrak{g})$. The notion of degeneracy of the horizon consists in the vanishing *surface gravity* $|dN|_{\overline{\mathfrak{g}}} = 0$. Notice that given this space-time setup, the form of the metric (2.2) and the asymptotic behaviors (2.3) are natural assumptions as they can be explicitly derived (see [32]).

The second boundary condition we will consider is $N_0 > 0$. In this case, we will assume that ∂M arises as a time-slice of a "photon sphere" or a time-slice of an "equipotential photon surface". Let us begin by giving the relevant definitions. Let (M, g_0, N, Ψ) be an electro-vacuum electrostatic system and let $(\mathfrak{L}, \mathfrak{g}) = (\mathbb{R} \times M, -N^2 dt^2 + g_0)$ be the corresponding spacetime. A hypersurface $P \hookrightarrow \mathfrak{L}$ is called a *photon surface* if every null geodesic initially tangent to P remains tangent to P as long as it exists or in other words if P is *null totally geodesic*. Following [28] (see also [20,21,24,26,33,44,75]), a photon surface $P \hookrightarrow \mathfrak{L}$ is called a *photon sphere* if $P = \mathbb{R} \times \Sigma$ for some smooth hypersurface $\Sigma \hookrightarrow M$ and if the lapse function N, the electric potential Ψ, and the length of its derivative $|d\Psi|_{\mathfrak{g}}$ are constant[3] along P. More generally, a photon surface $P \hookrightarrow \mathfrak{L}$ is called *equipotential* if N, Ψ and $|d\Psi|_{\mathfrak{g}}$ are constant[3] on each *(canonical) time-slice* $\Sigma(t_0) := P \cap \{t = t_0\}$. An equipotential photon surface P (and in particular a photon sphere) is called *non-degenerate* if $dN \neq 0$ on P, otherwise it is called *degenerate*. Photon spheres are always non-degenerate, see Sect. 4.3.

In light of these definitions, if ∂M is a time-slice of an equipotential photon surface or of a photon sphere, N, Ψ, and g_0 smoothly extend to the boundary ∂M, with $N|_{\partial M} =: N_0 > 0$, $\Psi|_{\partial M} =: \Psi_0 \in \mathbb{R}$ being constant. It is also known from [28, Proposition 5.5] that, if ∂M is a time-slice of a non-degenerate equipotential photon surface, ∂M is totally umbilic in M, has constant mean curvature H_ν and constant normal derivative $\nu(N)$ with respect to any unit normal ν to ∂M in M, and constant scalar curvature $\mathrm{R}^{\partial M}$

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[2] We consider Isenberg–Moncrief Gaussian null coordinates (v, z, x^1, x^2) on a neighborhood $[0, \varepsilon) \times \partial\mathfrak{L}$ of the Killing horizon $\partial\mathfrak{L}$: v is a parameter value along the integral curve of the Killing vector field (KVF), which is the null generator of $\partial\mathfrak{L}$; $\{x^I\}_{I=1}^2$ are local coordinates on the compact cross section; z is the projection onto the first factor so that $\{z = 0\} = \partial\mathfrak{L}$. For more details see, e.g., [49]. The spacetime metric reads

$$\mathfrak{g} = -z^2 A \, dv \otimes dv + 2dv \otimes dz + z \, dv \otimes \omega + b_{IJ} \, dx^I \otimes dx^J,$$

where $A = A(z, x^I) > 0$, $\omega = \omega(z, x^I)$, $b = b(z, x^I)$, are a smooth function, a one-form and a non-degenerate metric, respectively. They are independent of v, as $\partial\mathfrak{L}$ is a Killing horizon and ∂_v corresponds to the KVF in the whole neighborhood $[0, \varepsilon) \times \partial\mathfrak{L}$. In particular, in a static spacetime the KVF is $\partial_v = \partial_t$ in the interior $\mathfrak{L}$ (the choice of the sign is standard, see, e.g., [46]) and thus is normal to the time-slice M.

[3] In fact, the defining assumption that $|d\Psi|_{\mathfrak{g}}$ must be constant on each time-slice of an equipotential photon surface or fully constant on a photon sphere is automatically satisfied in the connected setting we are studying. This can be seen by realizing that it is neither used in the proof of Proposition 4.1 nor in the proof of [28, Proposition 5.5] by [28, Remark 5.11].

satisfying the *photon surface identity*

$$\mathrm{R}^{\partial M} = \frac{2\nu(\Psi)^2}{N^2} + \frac{2\,\mathrm{H}_\nu\,\nu(N)}{N} + \frac{\mathrm{H}_\nu^2}{2} \tag{2.5}$$

on ∂M. Note that (2.5) is invariant under the choice of unit normal ν and that $\nu(\Psi)^2 = |d\Psi|^2_{\mathfrak{g}} = |\mathrm{D}\Psi|^2$ and $\nu(N)^2 = |dN|^2_{\mathfrak{g}} = |\mathrm{D}N|^2$ on ∂M.

As in [28, Remark 5.7], in order to make our technique effective, provided that $\mathrm{H}_\nu \neq 0$, it will be convenient to introduce the quantity

$$c := \frac{2\nu(N)}{N\,\mathrm{H}_\nu}, \tag{2.6}$$

as this allows us to rewrite (2.5) as the two identities

$$\mathrm{R}^{\Sigma} = \frac{2\,|\mathrm{D}\Psi|^2}{N^2} + \left(c + \frac{1}{2}\right)\mathrm{H}_\nu^2, \tag{2.7}$$

$$2\nu(N) = cN\,\mathrm{H}_\nu\,. \tag{2.8}$$

It is shown in [24, Proposition 2.4] and [28, Corollary 5.6] that $c = 1$ holds on time-slices of photon spheres. Furthermore, [28, Lemma 5.9] tells us that if $c = 1$ holds on one time-slice $\Sigma(t_0)$ of a non-degenerate equipotential photon surface then $\mathbb{R} \times \Sigma(t_0)$ is a photon sphere. Concerning the condition that $\mathrm{H}_\nu \neq 0$, it follows from the proof of [28, Theorem 5.22] applied to the case of a connected non-degenerate equipotential photon surface boundary ∂M that $\mathrm{H}_\nu > 0$ holds on ∂M when choosing the unit normal ν pointing toward infinity (see below; think of pointing into M). It follows that c is well defined and that it satisfies $c > 0$ if and only if $\nu(N) > 0$ (for which, in words, we say that the photon surface is *outward directed*). In Sect. 7, we independently re-prove this result.

Notation 1. We note that ∂M is necessarily compact under all the above boundary conditions. Conventions vary on whether $\partial M \subset M$ for manifolds with boundary. Here, whenever confusion may arise, we will be explicitly referring to $\overline{M} := M \cup \partial M$ or to $M \setminus \partial M$.

In our analysis, we will focus on electro-vacuum electrostatic systems (M, g_0, N, Ψ) which are *asymptotically flat*: we assume that there is a compact[4] set $K \subset M$ with $K \supset \partial M$ such that $M \setminus K$ is diffeomorphic to $\mathbb{R}^3 \setminus B$ for some closed ball B and such that, with respect to the coordinates (x^i) induced by this chart, one has

$$\begin{aligned} (g_0)_{ij} &= (\delta_{ij} + \eta_{ij})\,dx^i \otimes dx^j, \qquad && \eta_{ij} = o_1(1),\\ N &= 1 - \frac{\mu}{|x|} + o_2(|x|^{-1}), && \Psi = o(1) \end{aligned} \tag{2.9}$$

[4]In the degenerate horizon case, we abuse conventions and mean "compact" to refer to the manifold topology on $M \cup \partial M$, that is, we consider $[0, \frac{\varepsilon}{2}] \times \partial M$ to be compact, see also Remark 1 and Remark 2.2. This is just to simplify language and has no consequences for our proof.

as $|x| \to \infty$, where $\mu \in \mathbb{R}$ is a constant called the *mass (parameter)*. When $\mu = 0$, we ask in addition that

$$\Psi = \frac{\kappa}{|x|} + o_2(|x|^{-1}) \tag{2.10}$$

as $|x| \to \infty$ for some constant $\kappa \in \mathbb{R}$ called the *charge (parameter)*. This extra condition will be important to get enough information on the asymptotic behavior of the level sets of the lapse function N in Sect. 7.2.

Here and throughout the paper, for a given smooth function $f : U \to \mathbb{R}$ on some open domain $U \subset \mathbb{R}^3$, the notation $f = o_l(|x|^\alpha)$ for some $l \in \mathbb{N}$, $\alpha \in \mathbb{R}$ means that

$$\sum_{|J| \leq l} |x|^{\alpha+|J|} |\partial^J f| = o(1) \tag{2.11}$$

as $|x| \to \infty$. The meaning of the notation $f = O_l(|x|^\alpha)$ is analogous, substituting $o(1)$ by $O(1)$ in Eq. 2.11.

Remark 2.1 (Asymptotic decay). In the standard definition of asymptotic flatness, one usually requires stronger asymptotic conditions, namely that $\eta_{ij} = O_2(|x|^{-\frac{1}{2}-\varepsilon})$ for some $\varepsilon > 0$ and that the scalar curvature R of g_0 is integrable. Under these additional assumptions, it can be seen by a standard computation that μ coincides with the ADM-mass. However, for our purposes, the asymptotic assumptions in (2.9) (combined with (2.10) when $\mu = 0$) will be enough. Our decay assumptions are very weak when compared with the other electrostatic electro-vacuum uniqueness results discussed in Sect. 1; these assume that the metric g_0 is asymptotically flat with $\varepsilon = \frac{1}{2}$ ([75]) or asymptotic to the Reissner–Nordström metric of mass μ and charge q ([24,28,44]) and that Ψ is asymptotic to the Reissner–Nordström potential $\frac{q}{|x|}$, see (2.12). The same is true for the static vacuum uniqueness results discussed in Sect. 1 with the notable exceptions of [6,22,23] who make the same assumption on the decay of N as we make. Moreover, in [6] (and consequently in its application in [22]), it is assumed that $\eta_{ij} = o_2(|x|^{-\frac{1}{2}})$. On the other hand, [23] makes the same asymptotic assumptions we make. Note however that it is conceivable that, when combined with (2.1), our asymptotic decay can be boot-strapped to stronger decay assumptions as, e.g., in [45].

The following remark will be useful for our strategy of proof, see Sect. 4.

Remark 2.2 (Geodesic and metric completeness). For later convenience, we note that asymptotically flat electrostatic systems, with or without boundary, are necessarily metrically complete and geodesically complete (up to the boundary) with at most finitely many boundary components which are necessarily all closed, see, for example, [27, Appendix] (where stronger asymptotic flatness assumptions are made but not used in the proof). This continues to apply in the degenerate horizon case as cylindrical ends are also geodesically complete up to the boundary by similar reasoning. Note however that while ∂M has a pre-compact tubular neighborhood, the topology on this neighborhood does not coincide with the topology induced by the Riemannian distance function d_{g_0} induced by g_0 in the degenerate horizon case as this tubular neighborhood/cylindrical end is necessarily unbounded with respect to d_{g_0}.

The most important asymptotically flat, electrostatic electro-vacuum systems in view of this paper are the *Reissner–Nordström systems* given by

$$M = (r_0, \infty) \times \mathbb{S}^2, \quad g_0 = \frac{dr \otimes dr}{N^2} + r^2 g_{\mathbb{S}^2}, \quad N = \sqrt{1 - \frac{2m}{r} + \frac{q^2}{r^2}}, \quad \Psi = \frac{q}{r} \tag{2.12}$$

for fixed parameters $m, q \in \mathbb{R}$ and $r_0 \geq 0$ chosen such that N is well defined for all $r > r_0$. Here, m and q are called the *mass* and the *charge* of the Reissner–Nordström system. When $q = 0$, the Reissner–Nordström system of mass m becomes the more widely known Schwarzschild system of mass m, while $m = q = 0$ corresponds to the Minkowski system (also known as Euclidean space). The behavior of the Reissner–Nordström system and the maximal interval (r_0, ∞) on which it is well defined depend on the relation between m and q. In particular, one distinguishes three cases (not containing the Minkowski case):

- The *sub-extremal case*, when $m > |q|$: the lapse function N vanishes for two positive radii whose maximum is $r_{m,q} := m + \sqrt{m^2 - q^2}$. Hence the maximal interval of definition of our system is $(r_{m,q}, \infty)$. It is well known that $\partial M := \{r = r_{m,q}\}$ is a non-degenerate static horizon (e.g., by a change to Kruskal–Szekeres coordinates). Thus, we suggestively consider the then asymptotically flat system on $[r_{m,q}, \infty)$.
- The *extremal case*, when $m = |q| \neq 0$: the lapse function N vanishes for the single value $r = r_{m,q} = m$, so the maximal interval of definition of our system is $(r_{m,q} = m, \infty)$. It is well known and can be seen by switching to the coordinate $z := r - r_{m,q}$ that ∂M constitutes a degenerate static horizon. Thus, we will suggestively consider the then asymptotically flat system on $[m, \infty)$. Remark 2.2 and Footnote 4 apply.
- The *super-extremal case*, when $m < |q|$: the system is well defined on $(0, \infty)$. It is well known that there is no horizon boundary. To make the system asymptotically flat and geodesically complete, we need to restrict to $[r_0, \infty)$ for $r_0 > 0$.

It is well known (see, e.g., [28, Corollary 3.2]) that the Reissner–Nordström spacetime of mass m and charge q – that is, the standard static spacetime arising from the corresponding Reissner–Nordström system –

- has a unique photon sphere at $r_* = \frac{3m + \sqrt{9m^2 - 8q^2}}{2}$ when $m \geq |q|$, $m \neq 0$,
- has precisely two photon spheres at $r_{*,\pm} = \frac{3m \pm \sqrt{9m^2 - 8q^2}}{2}$ when $|q| > m > \frac{2\sqrt{2}}{3}|q|$,
- has a unique photon sphere at $r_* = \frac{3m}{2}$ when $m = \frac{2\sqrt{2}}{3}|q|$,
- and no photon spheres otherwise.

It is shown in [28, Theorems 3.7, 3.9, 3.10] that, for each radius $r_1 > r_0$, r_0 being the radius corresponding to the maximal existence interval in (2.12), all Reissner–Nordström spacetimes have spherically symmetric equipotential photon surfaces going through $\{r = r_1\}$. Moreover, [28, Corollary 2.25] asserts that they contain no other equipotential photon surfaces (except in the Minkowski spacetime). Finally, we recall that $dN \neq 0$ holds in all non-Minkowski Reissner–Nordström spacetimes except on $\{r = \frac{q^2}{m}\}$ when $0 < m < |q|$. Thus spherically symmetric/equipotential photon surfaces in Reissner–Nordström spacetimes are always non-degenerate unless they cross $\{r = \frac{q^2}{m}\}$ when $0 < m < |q|$. Hence we do not get around also treating degenerate equipotential photon surfaces in this paper if we want to include all equipotential photon surfaces (or at least all those staying in $\{r \geq m\}$, see Remark 3.4, noting that $\frac{q^2}{m} > m$ when $q > m > 0$).

3. Results

For our first result, we will restrict our attention to the horizon case $N_0 = 0$. Hence, our first result is a version of the well-known *electrostatic electro-vacuum black hole uniqueness theorem* for a single horizon, see Sect. 1 for a detailed comparison with other proofs.

Theorem 3.1. *Let (M, g_0, N, Ψ) be an asymptotically flat electrostatic system with connected boundary ∂M of mass μ solving (2.1). Suppose that ∂M is a static horizon so that in particular $N|_{\partial M} = N_0 = 0$. Then (M, g_0, N, Ψ) is isometric to a sub-extremal or extremal Reissner–Nordström system of mass μ and some charge q, with interval of definition $(r_{\mu,q}, \infty)$. In particular, if the horizon ∂M is non-degenerate, this Reissner–Nordström system is sub-extremal, $\mu > |q|$, while it is extremal, $\mu = |q| \neq 0$, if the horizon ∂M is degenerate.*

Our second result shows *electrostatic equipotential photon surface uniqueness* in electro-vacuum. Exactly as in the first result, it shows that an asymptotically flat electrostatic electro-vacuum spacetime that admits a connected equipotential photon surface as its inner boundary must be isometric to a Reissner–Nordström spacetime of the same mass. As discussed in more detail in Sect. 1, our result recovers previously known results but also extends beyond what was previously known.

Theorem 3.2. *Let (M, g_0, N, Ψ) be an asymptotically flat electrostatic system with connected boundary ∂M of mass μ (and of charge κ if $\mu = 0$) solving (2.1). Suppose that ∂M is a time-slice of an equipotential photon surface, hence in particular satisfies that $N|_{\partial M} = N_0 > 0$, $\Psi|_{\partial M} = \Psi_0$ are constants. Unless $N_0 = 1$ and $\Psi_0 = 0$ both hold and assuming that $N_0^2 \geq |1 - \Psi_0^2|$ holds if $N_0^2 > (1 - |\Psi_0|)^2$, the system (M, g_0, N, Ψ) can be isometrically embedded into a Reissner–Nordström system of mass μ and some charge q, with $q = \kappa$ when $\mu = 0$. In particular, this Reissner–Nordström system is*

- sub-extremal *if $N_0^2 < (1 - |\Psi_0|)^2$,*
- extremal *if $N_0^2 = (1 - |\Psi_0|)^2$, and*

- super-extremal *if* $N_0^2 > (1-|\Psi_0|)^2$.

Remark 3.3. Notice that when $\Psi_0 = 0$, then $\Psi \equiv 0$ in view of Proposition 4.1 and hence the problem reduces to the analysis of the vacuum static system (M, g_0, N) provided in [22] which gives the isometric embedding of (M, g_0, N) into a Schwarzschild system with positive mass $\mu > 0$ if $N_0 < 1$ and into a Schwarzschild system with negative mass $\mu < 0$ if $N_0 > 1$. When $N_0 = 1$, it follows from the Hopf lemma[5] that $N \equiv 1$ on M, $\mu = 0$, and (M, g_0) is Ricci-flat and hence flat. However, to the best knowledge of the authors, it is not known whether this implies that (M, g_0) can be isometrically embedded into Euclidean space under the very weak decay assumption made on η_{ij} in (2.9), see also [22]. This is why we exclude that both $N_0 = 1$ and $\Psi_0 = 0$ hold. Thus, Theorem 3.2 can be interpreted as an extension of [22] to the electro-vacuum case. For a more extended discussion, see Sect. 4.1.

Remark 3.4. Assuming that $N_0^2 \geq |1-\Psi_0^2|$ holds if $N_0^2 > (1-|\Psi_0|)^2$ is a technical condition needed for our method of proof. In the Reissner–Nordström model case, this corresponds to excluding equipotential photon surfaces with $r < m$ in super-extremal Reissner–Nordström spacetimes of positive mass $m > 0$. See the end of Sect. 2 for a discussion on why this is a restriction and Sect. 8.1 for a discussion of why this condition arises in our approach.

Remark 3.5. As can be seen from the proof of Theorem 3.2, instead of assuming that ∂M arises as a time-slice of an equipotential photon surface, it suffices to assume that $N|_{\partial M} = N_0$, $\Psi|_{\partial M} = \Psi_0$ are constant, that ∂M is umbilic and has constant mean curvature H_ν and scalar curvature $\mathrm{R}^{\partial M}$, and constant normal derivatives $\nu(\Psi)$ and $\nu(N)$ such that the photon surface identity (2.5) holds on ∂M. Moreover, in case dN does not vanish entirely on ∂M, it follows from the proof of [28, Proposition 5.5] that assuming $N|_{\partial M} = N_0$, $\Psi|_{\partial M} = \Psi_0$, umbilicity of ∂M, constancy of $\nu(\Psi)$ and of either $\nu(N)$ or H_ν imply all the other assumptions, i.e., constancy of H_ν, respectively, $\nu(N)$, constancy of $\mathrm{R}^{\partial M}$, and (2.5) (see also Sect. 4.3).

4. Strategy of Proof and Comments

The proofs of our results are based on the cylindrical ansatz strategy introduced by Agostiniani and Mazzieri in [5]. Since then, the same strategy has found several applications to potential theory [3,7,35], manifolds with non-negative Ricci curvature [2,11], and static spacetimes [6,16–18,22]. Other related ideas and methods have been developed in [1,4,8,13,15]. Here we review and comment on the method in some detail.

To set up the cylindrical ansatz, the first step is to write down the radial coordinate r of the model system (the Reissner–Nordström system (2.12) in our case) as a function of the lapse function N. This is of course possible only as long as N is a monotonic function of the radius r. For sub-extremal

[5] Similarly modified as the maximum principle in the proof of Proposition 4.1.

and extremal Reissner–Nordström systems, this is indeed the case: It can be checked by a direct computation that N is a strictly increasing function of the radius r as long as $m \geq |q|$ unless $m = q = 0$. Taking the inverse of the relation between r and N, we can then write $r = \rho(N)$ for a suitable *pseudo-radial* function ρ.

The proofs of Theorem 3.1 and of the sub-extremal, extremal, and $m \leq 0$-super-extremal cases of Theorem 3.2 loosely follow the original strategy of [7]. The equipotential photon surface case is handled similarly to the application of the results of [7] in [22]: First, one exploits the Bochner formula to produce a vector field depending on ρ whose divergence is non-negative and equal to zero if and only if the system is isometric to the Reissner–Nordström system, see Sect. 5.3. Then, to conclude the proof, one proves that the equality is necessarily saturated using the properties of a horizon or of a time-slice of an equipotential photon surface, respectively. This is the objective of Sect. 6 in the black hole case, and of Sect. 7.1 and 7.2 in the (sub-)extremal and in the $m \leq 0$-super-extremal equipotential photon surface cases, respectively. We emphasize however that there are a couple of technical but important points that need to be addressed in these proofs:

1. There are two natural choices of pseudo-radial functions: we will make suitable choices in each case, giving the asymptotic behavior and justifying their well-definedness and smoothness.
2. To work with the weaker notion of asymptotic flatness, see (2.9) and (2.10), we need a more careful analysis of the asymptotics in the cylindrical ansatz, see Sect. 5.4.
3. For the degenerate case of Theorem 3.1, we need a careful analysis near the degenerate horizon to conclude rigidity. To proceed similarly to the non-degenerate case, we first use the additional asymptotic assumption (6.4). Notice that this would be a consequence of a more accurate decay control in (2.3) of the coefficients of the metric (2.2). However, this additional assumption is not needed in view of a different proof provided in Sect. 7.7 and relying on the monotonicity analysis of Sect. 7.5.

On the other hand, in the $m > 0$-super-extremal case, the lapse function of the Reissner–Nordström system is *not* a monotonic function of the radial coordinate anymore: N is strictly decreasing when $r < \frac{q^2}{m}$, has a global minimum at $r = \frac{q^2}{m}$, and is strictly increasing when $r > \frac{q^2}{m}$. Therefore, it is only possible to write the radial coordinate r as a function of N if we restrict our attention to $r \leq \frac{q^2}{m}$ or to $r \geq \frac{q^2}{m}$. In order to perform the conformal change prescribed by the cylindrical ansatz on a general manifold in the $m > 0$-super-extremal case, we hence need a more involved procedure: The first step will be to show that, given an $m > 0$-super-extremal system (M, g, N, Ψ) of (2.1), a suitably chosen "critical set" $\mathfrak{S}$ mimicking the set of global minima either coincides with ∂M or divides the manifold into two pieces, a compact one and a non-compact one. The first case is treated in Sect. 7.3 and still roughly follows the same strategy as the other cases. In the second case, we perform different conformal changes in the two regions, i.e., use different pseudo-radial functions, and analyze them separately, see Sect. 7.6. As a result of our analysis, in both regions, we produce quantities that are monotonic along the level sets of the lapse function N.

Remarkably, these monotonic quantities coincide on $\mathfrak{S}$ and the monotonicities combine (see Proposition 7.2, and in particular formulas (7.21) and (7.22)). This allows us to compare the value of our monotonic quantities at infinity and on the time-slice of the photon surface, obtaining valuable insights into the geometry of the time-slice of the photon surface and allowing us to conclude the rigidity of the super-extremal Reissner–Nordström system, at least for time-slices of photon surfaces of radius $r \geq m$, see also Remark 3.4.

The strategy outlined above resembles the one explored recently by Borghini–Mazzieri in [18] for the study of vacuum black hole uniqueness with positive cosmological constant. In that paper, the authors divide the manifold along the set of the global maxima and then apply the cylindrical ansatz strategy separately in every region. There are however some interesting and crucial differences. The first one is that, while in the present paper we prove that the set of local minima $\mathfrak{S}$ separates the manifold in exactly two regions, in the setting of [18] it is not known how many regions are produced when cutting along the set of the global maxima: there are models where cutting along the set of maxima produces two regions (Schwarzschild–de Sitter and Nariai spacetimes), but it is also possible for the cut to produce just a single region (de Sitter spacetime) and there is currently no way to rule out the case of more than two regions. Furthermore, while in [18] the authors also manage to exploit the cylindrical ansatz strategy to produce monotonic quantities (see [18, Corollary 3.5]), these quantities do not combine across the set of maxima. The authors thus had to resort to other methods in order to obtain rigidity statements.

4.1. Simplifications in Our Framework

In this subsection, we will discuss some preliminary observations, leading to a helpful rewriting of the rigidity/uniqueness problem. First, we observe that it is not restrictive to limit our attention to the case where the boundary value Ψ_0 of Ψ is positive. In fact, notice that if (M, g_0, N, Ψ) is a solution to (2.1), then $(M, g_0, N, -\Psi)$ is a solution as well. As a consequence, up to changing the sign of Ψ, we can assume that Ψ_0 is non-negative. As already pointed out in Remark 3.3, in the case $\Psi_0 = 0$, the rigidity/uniqueness problem reduces to the vacuum case which has already been studied in depth in several papers, see Sect. 1. In particular, when $N_0 \neq 1$, the results by Cederbaum–Cogo–Fehrenbach [22] and by Cederbaum–Cogo–Leandro–Paulo dos Santos [23] prove the vacuum case of our Theorem 3.2, under the same decay assumptions. The aim of this work is that of employing and expanding on the cylindrical ansatz technique presented in the vacuum case by Agostiniani and Mazzieri [6] for a connected (and necessarily non-degenerate, see, e.g., [46]) horizon and by Cederbaum–Cogo–Fehrenbach [22] for a connected time-slice of a non-degenerate equipotential photon surface to deal with the case where Ψ is non-trivial.

In fact, our asymptotic analysis in Sect. 4.4 and 5.4 can be used to extend the vacuum uniqueness results by [6] and hence also those by [22] to the weaker asymptotic assumptions (2.9). To see this requires to discuss the case when $\mu = 0$ in which we make the additional asymptotic assumption (2.10) not present in vacuum: As vacuum black holes are necessarily non-degenerate (see, e.g., [46]), we find $\nu(N) \neq 0$ on ∂M. Hence, $\mu = 0$ is immediately ruled out for vacuum horizons by using the divergence theorem applied to the vacuum equation $\Delta N = 0$ and exploiting the asymptotic assumption on N to obtain

$$\int_{\partial M} \nu(N)\,\mathrm{d}\sigma \;=\; \lim_{R\to+\infty} \int_{\{|x|=R\}} \nu(N)\,\mathrm{d}\sigma \;=\; 4\pi\mu.$$

The same argument applies to non-degenerate time-slices of equipotential photon surfaces. For degenerate such time-slices, one instead argues that $\nu(N) = 0$ gives $N = 1$ on M by the Hopf lemma[5] applied to $\Delta N = 0$ and thus $N_0 = 1$ which is ruled out in Theorem 3.2.

To set up the cylindrical ansatz, we first employ the strategy presented in [39, Corollary 9.6] and also exploited in [75] to show how our assumptions of asymptotic flatness and connected boundary help to simplify the problem significantly by relating the electric potential explicitly with the lapse function. As discussed, we will only deal with the case $\Psi_0 \geq 0$, recalling that one can treat the opposite sign by a global sign change of Ψ.

Proposition 4.1 (Reduction). *Let (M, g_0, N, Ψ) be an asymptotically flat electrostatic electro-vacuum system such that M has a connected boundary ∂M to which N and Ψ smoothly extend to constants $N|_{\partial M} = N_0 \geq 0$, $\Psi|_{\partial M} = \Psi_0 \geq 0$. Then, if $\Psi_0 > 0$,*

$$N^2 = \Psi^2 + 1 + \left(\frac{N_0^2}{\Psi_0} - \Psi_0 - \frac{1}{\Psi_0}\right)\Psi \tag{4.1}$$

holds on $\overline{M}$ and $0 < \Psi < \Psi_0$ holds on M. If, on the other hand, $\Psi_0 = 0$, then $\Psi \equiv 0$ on $\overline{M}$.

Proof. Using (2.1), a simple computation shows that the quantity

$$X_{ab} := \frac{a}{2}\left(\Psi^2 + 1 - N^2\right) + b\Psi$$

for any $a, b \in \mathbb{R}$ satisfies the elliptic PDE

$$\Delta X_{ab} = \frac{1}{N}\langle \mathrm{D}X_{ab} \,|\, \mathrm{D}N\rangle_{g_0} \tag{4.2}$$

on M. From the asymptotic flatness assumption (2.9), we know that $N \to 1$ and $\Psi \to 0$ at infinity, from which it follows that $X_{ab} \to 0$ at infinity for any choice of a, b. Assume first that $\Psi_0 > 0$. Choosing $a = 2$, $b = \frac{N_0^2}{\Psi_0} - \Psi_0 - \frac{1}{\Psi_0}$, X_{ab} specializes to

$$X := X_{a=2\ b=\frac{N_0^2}{\Psi_0}-\Psi_0-\frac{1}{\Psi_0}} = \Psi^2 + 1 - N^2 + \left(\frac{N_0^2}{\Psi_0} - \Psi_0 - \frac{1}{\Psi_0}\right)\Psi$$

which satisfies $X = 0$ on ∂M. If instead $\Psi_0 = 0$, we choose $a = 0$, $b = 1$ and obtain

$$Y := X_{a=0\ b=1} = \Psi$$

which also satisfies $Y = 0$ on ∂M. We will now apply the maximum and minimum principle to show that $X \equiv 0$ and $Y \equiv 0$ on M. We have to be a bit careful because the manifold M is not compact and in case $N_0 = 0$ the coefficient $\frac{1}{N}$ in (4.2) blows up on ∂M, and, in addition, g_0 does not extend smoothly to ∂M in the degenerate horizon case. However, this is easy to work around: Suppose that $\{X > 0\} \neq \emptyset$. Since $X = 0$ at the boundary and $X \to 0$ at infinity, then X must have a positive maximum at a point $p_0 \in M \setminus \partial M$. Now let $\Omega \subset M \setminus \partial M$ be a neighborhood of p_0 with smooth boundary, large enough to contain some $q \in \Omega$ with $X(q) < X(p_0)$ (if $X = X(p_0)$ on any such neighborhood then X is necessarily constant, a contradiction). We can hence apply the strong maximum principle to $X|_\Omega$ where in particular $N \neq 0$, obtaining a contradiction. The cases $\{X < 0\} \neq \emptyset$ and $\{Y > 0\} \neq \emptyset$, $\{Y < 0\} \neq \emptyset$ can be treated analogously. Similarly, for $\Psi_0 > 0$, using Y, one finds $0 < \Psi < \Psi_0$ on $M \setminus \partial M$. □

From now on, under the assumptions of Proposition 4.1, we restrict to the case when $\Psi_0 > 0$. In view of (4.1) and of the nature of N and Ψ in the Reissner–Nordström systems (2.12) with positive charge (corresponding to $\Psi_0 > 0$), we aim at finding parameters $m \in \mathbb{R}$, $q > 0$ satisfying

$$\frac{N_0^2}{\Psi_0} - \Psi_0 - \frac{1}{\Psi_0} = -\frac{2m}{q}. \tag{4.3}$$

Note that (4.3) fixes the ratio between m and q, while the actual values of one of m and q still needs to be defined (with the exception of $\frac{m}{q} = 0$ when $m = 0$ and $q > 0$ is free). From Proposition 4.1, it then follows that

$$N^2 = \Psi^2 - \frac{2m}{q}\Psi + 1 \tag{4.4}$$

holds on M. In particular, the two potentials N, Ψ are functions one of the other and we can equivalently rewrite (2.1) in terms of just N by

$$\begin{cases} N\,\mathrm{Ric} = \mathrm{D}^2 N - \frac{2N}{N^2+k}\, dN \otimes dN + \frac{N}{N^2+k}\, |\mathrm{D}N|^2\, g_0, \\ \Delta N \quad = \frac{N}{N^2+k}\, |\mathrm{D}N|^2 \end{cases} \tag{4.5}$$

on M, where

$$k := \frac{m^2}{q^2} - 1. \tag{4.6}$$

Notice that, by (4.4), we have

$$N^2 + k = \left(\Psi - \frac{m}{q}\right)^2 \geq 0 \tag{4.7}$$

on $\overline{M}$. When $k \leq 0$, it is possible for the quantity $N^2 + k$ to reach the value zero, in which case the equations (4.5) are ill-posed. Nevertheless, from (4.4), (4.7), we obtain the identity

$$N^2 \, |\mathrm{D}N|^2 = (N^2 + k) \, |\mathrm{D}\Psi|^2 \tag{4.8}$$

which can be used to make sense of the equations in (4.5) even when $N^2 + k$ vanishes.

4.2. Sub-Extremality, Extremality, and Super-Extremality

Before moving on, let us note that via (4.3), the ratio of m and $q > 0$ is fully determined by the boundary values N_0 and Ψ_0. Consequently, transferring the definitions of sub-extremal, extremal, and super-extremal Reissner–Nordström systems in terms of their mass and charge parameters m and q directly to general asymptotically flat electrostatic electro-vacuum systems (M, g_0, N, Ψ) with constant lapse function N_0 and constant electric potential $\Psi_0 > 0$ on the connected boundary ∂M, we say that the system (M, g_0, N, Ψ) and/or its boundary $(\partial M, N_0, \Psi_0)$ is *sub-extremal* if $m > q$, *extremal* if $m = q \neq 0$, and *super-extremal* if $m < q$. Note that the comparison between m and q does not depend on how we will later choose m and q but just on their ratio fixed by (4.3). A direct computation then shows that sub-extremality corresponds to $N_0^2 < (1 - \Psi_0)^2$, extremality corresponds to $N_0^2 = (1 - \Psi_0)^2$, and super-extremality corresponds to $N_0^2 > (1 - \Psi_0)^2$. Recalling that the case $\Psi_0 < 0$ can be addressed by a sign change on Ψ, the technical condition in Theorem 3.2 hence only applies in the super-extremal case, see also Remark 3.4. In particular, if ∂M is a horizon, then (M, g_0, N, Ψ) is sub-extremal if and only if $\Psi_0 \neq 1$, extremal if $\Psi_0 = 1$, and never super-extremal as to be expected.

Moreover, if ∂M is a horizon, as $\Psi_0 + \frac{1}{\Psi_0} \geq 2$, it follows that $m \geq q > 0$ and thus $k \geq 0$. This is expected, since we want m and q to represent mass and charge of a Reissner–Nordström system, and hence a horizon should only be present in the sub-extremal and extremal cases.[6] Also, tracing the first equation in (2.1) and combining it with the second one, we obtain

$$2|\mathrm{D}\Psi|^2 = N^2 \, \mathrm{R} \tag{4.9}$$

on $M \setminus \partial M$ which combines with (4.8) to give

$$2|\mathrm{D}N|^2 = (N^2 + k) \, \mathrm{R} \tag{4.10}$$

on $M \setminus \partial M$ because $N^2 + k > 0$ on $M \setminus \partial M$ by the strong maximum principle[7] applied to the second equation in (2.1) and because $k \geq 0$. If the horizon ∂M is non-degenerate so that $|\mathrm{D}N| > 0$ on ∂M, (4.10) implies $k > 0$ and $\mathrm{R}\,|_{\partial M} > 0$ by (4.9). In particular, any asymptotically flat electrostatic electro-vacuum system (M, g_0, N, Ψ) with non-degenerate horizon boundary is sub-extremal. If the horizon ∂M is degenerate, taking the derivative with respect to z of (4.4) and using the asymptotic expansions (2.3) and (2.4) as $z \to 0^+$, we get $\Psi_0 = \frac{m}{q}$. Plugging this information back into (4.4), we conclude that $k = 0$. Thus,

[6]Recall that we have excluded $q = 0$ by assumption and $q < 0$ by alluding to a sign change for Ψ.

any asymptotically flat electrostatic electro-vacuum system (M, g_0, N, Ψ) with degenerate horizon boundary has $m = q \neq 0$ and is extremal.

On the other hand, if ∂M has $N_0 > 0$ and $\Psi_0 > 0$, the Hopf lemma[5] applied to the last equation in (2.1) and the assumptions $\Psi_0 > 0$ and $\Psi \to 0$ at infinity give that $|\mathrm{D}\Psi||_{\partial M} > 0$. Thus, (4.8) tells us that this time-slice is non-degenerate, i.e., $dN \neq 0$ on ∂M, unless $N_0^2 + k = 0$ or equivalently $\Psi_0 = \frac{m}{q} > 0$ by (4.7). Moreover, the expansion (2.4) and $C > 0$ directly imply that N regularly foliates a neighborhood of ∂M.

4.3. Consequences for Degenerate Equipotential Photon Surfaces

Let us now briefly investigate the properties of degenerate equipotential photon surfaces, exploiting the above simplification. We continue to work under the assumptions of Proposition 4.1, with $\Psi_0 > 0$, and assume that $N_0 > 0$ so that we investigate an equipotential photon surface inner boundary. Let us first note that photon spheres are always non-degenerate in this context because they have non-vanishing mean curvature H_ν by [24, Lemma 2.6] (see also [36, Theorem 3.1]) and $c = 1$ or in other words they satisfy $N\,\mathrm{H}_\nu = 2\nu(N)$ (see [28, Proposition 5.3] noting that the non-degeneracy condition is in fact not used in the proof of the photon sphere case).

Now observe that unless $N_0^2 + k = 0$ holds on ∂M, we can appeal to the Hopf lemma[5] applied to the second equation in (4.5) to see that $\nu(N) \neq 0$: In case $N^2 + k > 0$ on M, this follows from the asymptotic assumption $N \to 1$ at infinity (as $N_0 = 1$ is ruled out by the maximum principle[7] applied to the same equation, noting that $N \equiv 1$ contradicts $\Psi_0 > 0$ and $\Psi \to 0$ at infinity by (4.4)). On the other hand, if $\{N^2 + k = 0\} \cap (M \setminus \partial M) \neq \emptyset$, the same argument can be given using some regular value N_1 of N with $N_0 > N_1 > \sqrt{-k}$ (which exists by Sard's lemma). Hence time-slices of equipotential photon surfaces are necessarily *non-degenerate* (meaning $dN \neq 0$ on the time-slice) unless they satisfy $N_0 = \sqrt{-k}$ in which case they are necessarily *degenerate* by (4.8), and indeed satisfy $dN \equiv 0$ on the time-slice.

Next, note that the properties of time-slices of non-degenerate equipotential photon surfaces listed on pp. 5, in particular (2.5), also hold for non-degenerate time-slices of degenerate equipotential photon surfaces as these necessarily have an open neighborhood of non-degenerate time-slices by continuity and because the proof of these properties (see [28, Proposition 5.5]) is completely local. Furthermore, by the Morse–Sard theorem [66] applied to the function N which we know to be real analytic on M from [67], we know that the critical values of N are isolated so that a degenerate slice of an equipotential photon surface must have a neighborhood in which it is the only degenerate slice. In the last argument, we implicitly used that photon spheres are non-degenerate. Consequently, the properties listed on pp. 5 also apply to time-slices of degenerate equipotential photon surfaces.

[7]Adapted to this context as in the proof of Proposition 4.1

4.4. Asymptotic Consequences of the Simplification

We continue to work under the assumptions of Proposition 4.1, with $\Psi_0 > 0$. The goal of this section is to establish asymptotic decay on relevant quantities derived from N and g_0 and to bootstrap the assumed decay on Ψ. Recalling our asymptotic assumptions from (2.9), the Christoffel symbols Γ^k_{ij} of g_0 with respect to the asymptotic coordinates (x^i) satisfy $\Gamma^k_{ij} = o(|x|^{-1})$ and the area element $\mathrm{d}\sigma$ of g_0 on coordinate spheres satisfies

$$\mathrm{d}\sigma = \left(|x|^2 + o\left(|x|^2\right)\right) \mathrm{d}\sigma_{g_{\mathbb{S}^2}} \tag{4.11}$$

as $|x| \to \infty$. We also compute the following asymptotic expansions for the derivatives of the lapse function N

$$\frac{\partial N}{\partial x^i} = \frac{\mu x_i}{|x|^3} + o_1\left(|x|^{-2}\right), \qquad |\mathrm{D}N| = \frac{|\mu|}{|x|^2} + o_1(|x|^{-2}) \tag{4.12}$$

as $|x| \to \infty$. Next, let ν denote the unit normal to a coordinate sphere pointing toward infinity. Considering $|x|$ as the radial coordinate in the corresponding spherical polars, $\frac{\partial}{\partial |x|}$ denotes the radial coordinate vector field. By standard computations, it follows that

$$\nu = \frac{\partial}{\partial |x|} + \alpha^i \partial_{x^i}, \qquad \alpha^i = o(1) \tag{4.13}$$

for $i = 1, 2, 3$ as $|x| \to \infty$. Combining (2.9) with (4.4), we find

$$\Psi\left(\Psi - \frac{2m}{q}\right) = -\frac{2\mu}{|x|} + o_2(|x|^{-1}) \tag{4.14}$$

as $|x| \to \infty$. If $m \neq 0$, $\Psi = o(1)$ then leads to $\Psi = \frac{q\mu}{m|x|} + o(|x|^{-1})$ and thus, plugging this back into (4.14), we find

$$\Psi = \frac{q\mu}{m|x|} + o_2(|x|^{-1}) \tag{4.15}$$

as $|x| \to \infty$. Next, still assuming $m \neq 0$, by the last equation in (2.1), (4.11), (4.15), (4.13), (2.9), and the divergence theorem, we find that

$$\int_\Sigma \frac{\nu(\Psi)}{N}\, \mathrm{d}\sigma = \lim_{R\to\infty} \int_{\mathbb{S}^2_R(0)} \frac{\nu(\Psi)}{N}\, \mathrm{d}\sigma = -\frac{4\pi q\mu}{m} \tag{4.16}$$

for any closed, orientable surface Σ homologous to ∂M, where ν now also denotes the g_0-unit normal to Σ pointing toward the asymptotic end. On the other hand, applying the Hopf lemma[5] to the last equation in (2.1) and recalling $\Psi_0 > 0$ and $\Psi \to 0$ at infinity, we find that $\nu(\Psi) = -|\mathrm{D}\Psi| < 0$ if Σ is sufficiently close to ∂M (say $\Sigma = \partial M$ in the non-degenerate horizon and in the equipotential photon surface case and $\Sigma = \{z = \varepsilon\}$ for sufficiently small $\varepsilon > 0$ in the degenerate horizon case). Hence, by $q > 0$, we have asserted that $\frac{\mu}{m} > 0$ when $m \neq 0$. Alternatively, one can argue via (4.8) and the Hopf lemma[5] to get information about the decay of $\nu(\Psi)$ without first computing the decay of Ψ, as will be done below.

Let us now consider the case $m = 0$ and suppose toward a contradiction that $\mu \neq 0$. Then, (4.8) and $\nu(N)^2 = |\mathrm{D}N|^2$ combined with (2.9), (4.6) and (4.12) giving $k = -1$ lead to

$$\nu(\Psi)^2 = -\frac{\mu}{2|x|^3} + o(|x|^{-3}) \tag{4.17}$$

as $|x| \to \infty$ and thus in particular $\mu < 0$. Arguing again by the Hopf lemma[5], one finds that $\nu(\Psi) < 0$ asymptotically. Applying the divergence theorem as in (4.16), this leads to

$$\int_\Sigma \frac{\nu(\Psi)}{N}\,\mathrm{d}\sigma = \int_{\mathbb{S}^2_R(0)} \frac{\nu(\Psi)}{N}\,\mathrm{d}\sigma = -4\pi\sqrt{-\frac{\mu R}{2}} + o(\sqrt{R}) \tag{4.18}$$

as $R \to \infty$ for any closed, orientable surface Σ homologous to ∂M. Fixing Σ as above, this leads to the desired contradiction. We have hence proved the following lemma.

Lemma 4.2 (Signs of m and μ). *Under the assumptions of Proposition 4.1 with* $\Psi_0 > 0$, *either* $\mu = m = 0$ *or* $\mu m > 0$.

Using the additional decay assumption (2.10) we are making in case $\mu = 0$ combined with (4.13), we find the improved decay

$$\nu(\Psi) = -\frac{\kappa}{|x|^2} + o(|x|^{-2}) \tag{4.19}$$

as $|x| \to \infty$. Hence the above divergence theorem argument gives

$$\int_\Sigma \frac{\nu(\Psi)}{N}\,\mathrm{d}\sigma = -4\pi\kappa + o(1) \tag{4.20}$$

as $r \to +\infty$ for any closed, orientable surface Σ homologous to ∂M. Again, by the Hopf lemma[5], this proves the following lemma.

Lemma 4.3 (Sign of κ). *Under the assumptions of Proposition 4.1 with* $\Psi_0 > 0$, *assuming* $\mu = m = 0$ *and* (2.10), *we have* $\kappa > 0$.

5. The Cylindrical Ansatz

In this section, we will set up the cylindrical ansatz we will use in all the different parts of the proof of Theorems 3.1 and 3.2. We will continue to work under the assumptions of Proposition 4.1, with $\Psi_0 > 0$. The analysis of this section is performed for any pair of parameters $m \in \mathbb{R}$ and $q > 0$ whose ratio satisfies relation (4.3). In Sect. 5.5, we will show how to fix the values of m, q appropriately.

5.1. Setting Up the Cylindrical Ansatz

In the spirit of [18] and inspired by the explicit formula for N as a function of the radial coordinate for the Reissner–Nordström system (2.12), we define the *pseudo-radial function* $\rho\colon \overline{M} \to \mathbb{R}$ as a function satisfying one of the two equivalent relations

$$N^2 = 1 - \frac{2m}{\rho} + \frac{q^2}{\rho^2}, \tag{5.1}$$

$$N^2 + k = \left(\frac{m}{q} - \frac{q}{\rho}\right)^2. \tag{5.2}$$

One can solve equation (5.1) explicitly to find two choices for ρ, namely

$$\rho_+ = \frac{q^2}{m + q\sqrt{N^2+k}}, \tag{5.3}$$

$$\rho_- = \frac{q^2}{m - q\sqrt{N^2+k}}. \tag{5.4}$$

Before analyzing the properties of these pseudo-radial functions, let us explain when we will use which one. The choice that we will use for most of this work is ρ_-. In fact, it will be the one we will consider when dealing with black holes (see Sect. 6.2 for the non-degenerate and Corollary 6.4 for the degenerate case) and with both the sub-extremal and extremal cases for equipotential photon surfaces (see Sect. 7.1). The reason for this is that we expect the pseudo-radial function to mimic the behavior of the radial coordinate r in Reissner–Nordström systems, so in particular we want the pseudo-radial function to diverge at infinity, see Lemmas 5.1 and 5.2.

On the other hand, when dealing with the super-extremal case for equipotential photon surfaces, there will be two important occasions in which the choice ρ_+ will actually be the natural one. The first case is when $m \leq 0$: In this case, the behavior at infinity of ρ_- and ρ_+ is reversed, and ρ_+ is this time the one diverging as expected. Another, more subtle, case is when the *critical set* $\mathfrak{S} = \{p \in \overline{M} \,|\, N(p)^2 + k = 0\}$ is non-empty: In this case, we will show that $\mathfrak{S}$ divides the manifold into two pieces, an asymptotically flat one (where the correct choice for the pseudo-radial function is again ρ_-) and a compact one (where the right choice is now ρ_+). These special cases will be discussed in Sect. 7.2 and Sect. 7.4, respectively. In the super-extremal case with $\mathfrak{S} = \emptyset$ and $m > 0$, we still use ρ_-, see Sect. 7.3.

Let us now summarize the properties of the two pseudo-radial functions $\rho_\pm$ for later convenience. As already visible in the reduced equations (4.5), the critical set $\mathfrak{S}$ plays a special role also for the smoothness of the pseudo-radial functions. Note that $\mathfrak{S}$ can a priori contain ∂M: This happens for a degenerate horizon boundary where $N_0 = 0$ and $k = 0$ (see Sect. 4.2) and for a degenerate time-slice of an equipotential photon surface for which $\Psi_0 = \frac{m}{q}$. For a non-degenerate horizon, $\mathfrak{S} \cap \partial M = \emptyset$ because $k > 0$ as discussed in Sect. 4.2. Also, for a non-degenerate equipotential photon surface boundary ∂M, $\mathfrak{S} \cap \partial M = \emptyset$ holds by (4.8).

Lemma 5.1 (Properties of ρ_+). *The pseudo-radial function ρ_+ is well defined and continuous on $\overline{M}$ and smooth away from $\mathfrak{S}$. Moreover, $\rho_+ > 0$ on $\overline{M}$. If $m > 0$, one has $\rho_+ \to \frac{q^2}{2m}$ in the asymptotically flat end of M. If $m \leq 0$, one has $\rho_+ \to \infty$ in the asymptotically flat end of M. Last but not least, $\rho_+(1+N) > m$ on $\overline{M}$ if and only if $m \leq 0$ or $q > m > 0$.*

Proof. To see that ρ_+ is well defined and continuous on $\overline{M}$, we observe that by (4.7) and $q > 0$, a zero of its denominator can only occur for $m \leq 0$ and hence when $\Psi = \frac{2m}{q} \leq 0$, in contradiction to Proposition 4.1. By its definition in (5.3), ρ_+ is smooth away from $\mathfrak{S}$. The asymptotic claims are immediate when recalling that $N \to 1$ in the asymptotically flat end, taking into account the definition of k in (4.6). Positivity of ρ_+ is immediate from the asymptotic behavior upon noticing that $\rho_+ \neq 0$ and recalling (4.6) and $q > 0$. Finally, $\rho_+(1+N) > m$ obviously holds by positivity of ρ_+ provided that $m \leq 0$. Recalling $q > 0$ and assuming $m > 0$ it now suffices to show that $\rho_+(1+N) > m$ is equivalent to $q^2 > m^2$. To see this, we first compute

$$\rho_+(1+N) > m \iff \frac{1+N}{\frac{m}{q} + \sqrt{N^2+k}} > \frac{m}{q} \iff 1 - \frac{m^2}{q^2} > -N + \frac{m}{q}\sqrt{N^2 + \frac{m^2}{q^2} - 1}.$$

On the other hand, using $N^2 + \frac{m^2}{q^2} \geq -k + \frac{m^2}{q^2} = 1$, we find

$$-N + \frac{m}{q}\sqrt{N^2 + \frac{m^2}{q^2} - 1} \geq 0 \iff \left(N^2 + \frac{m^2}{q^2}\right)\left(1 - \frac{m^2}{q^2}\right) \leq 0 \iff 1 \leq \frac{m^2}{q^2}.$$

The second chain of equivalences shows that assuming $q^2 > m^2$ readily implies that

$$-N + \frac{m}{q}\sqrt{N^2 + \frac{m^2}{q^2} - 1} < 0 < 1 - \frac{m^2}{q^2}.$$

Combining this with the first chain of equivalences shows that $\rho_+(1+N) > m$. Conversely, assuming toward a contradiction that $\rho_+(1+N) > m$ and $q^2 \leq m^2$ both hold, the first chain of equivalences shows that

$$0 \geq 1 - \frac{m^2}{q^2} > -N + \frac{m}{q}\sqrt{N^2 + \frac{m^2}{q^2} - 1},$$

contradicting the second chain of equivalences. □

Lemma 5.2 (Properties of ρ_-). *The pseudo-radial function ρ_- is well defined and continuous on $\overline{M} \setminus \{N = 1\}$ and well defined and continuous on $\{N = 1\}$ if and only if $m < 0$. It is smooth in its domain of definition away from $\mathfrak{S}$. Moreover, if $m > 0$, $\rho_- > 0$ on $\overline{M} \cap \{N < 1\}$ while $\rho_- < 0$ on $M \cap \{N > 1\}$. On the other hand, $\rho_- < 0$ on $\overline{M}$ if $m \leq 0$. If $m > 0$, one has $\rho_- \to \infty$ in the asymptotically flat end of M. If $m \leq 0$, one has $\rho_- \to \frac{q^2}{2m}$ in the asymptotically flat end of M. Last but not least, $\rho_-(1+N) > m$ on $\overline{M}$ if $m \leq 0$, $\rho_-(1+N) < m$ on $\overline{M} \cap \{N > 1\}$ if $m > 0$, and $\rho_-(1+N) > m$ on $\overline{M} \cap \{N < 1\}$ if $m > 0$.*

Proof. To see that ρ_- is well defined and continuous away from $\{N=1\}$ and well defined and continuous on $\{N=1\}$ if and only if $m<0$, note that by (4.7) and $q>0$, a zero of its denominator will occur if and only if $m \geq 0$ and $\Psi = \frac{2m}{q}$ by Proposition 4.1. The second condition is equivalent to $N=1$ by (4.4). The smoothness claim readily follows from (5.4). A simple computation using (4.4) gives that $\rho_- > 0$ only holds if $m>0$ and $N<1$ but $\rho_- < 0$ holds if $m \leq 0$ or if $m>0$ and $N>1$. The asymptotic claims follows as in the proof of Lemma 5.1. Last but not least, $\rho_-(1+N) > m$ obviously holds on M by positivity of ρ_- when $m \leq 0$. Similarly, $\rho_-(1+N) < m$ obviously holds on $\overline{M} \cap \{N>1\}$ by negativity of ρ_- when $m>0$. Finally, if $q \geq m > 0$, one estimates

$$\rho_-(1+N) - m \geq \frac{m^2}{m - q\sqrt{N^2+k}} - m = \frac{q\sqrt{N^2+k}}{m - q\sqrt{N^2+k}} = \frac{\sqrt{N^2+k}}{q}\rho_- > 0$$

on $\overline{M} \cap \{N<1\}$ as $N^2+k>0$ where $N<1$ by (4.6), and using $N \geq 0$. On the other hand, if $m>q>0$, one estimates

$$\rho_-(1+N) - m \geq \frac{q^2 - m^2 + mq\sqrt{k}}{m - q\sqrt{N^2+k}}$$

on $\overline{M} \cap \{N<1\}$, again using $N \geq 0$ and the fact that $k>0$ when $m>q$ by (4.6). As $m>q$, $q^2-m^2+mq\sqrt{k}>0$ is equivalent to $0<m^2-q^2<mq\sqrt{k}$ and thus, by squaring, equivalent to $q^2>0$ which is clearly true. This implies the last claim. ☐

In particular, Lemmas 5.1 and 5.2 establish that we can use the pseudo-radial functions $\rho_\pm$ as explained in the above discussion, once we establish that $N \neq 1$ on $M \backslash \partial M$ when using ρ_- so that ρ_- is well defined. Moreover, Lemma 5.1 and 5.2 assert that once the pseudo-radial function $\rho_\pm$ has been chosen as explained above (and is well defined), we can then introduce the *pseudo-affine functions* $\varphi_\pm \colon \overline{M} \to \mathbb{R}$, defined by

$$\varphi_\pm := \log\left[\rho_\pm(1+N) - m\right] \tag{5.5}$$

and the *cylindrical ansatz metric*

$$g_\pm := \frac{g_0}{\rho_\pm^2} \tag{5.6}$$

on M and on ∂M except in the degenerate black horizon case. These are defined such that if $\rho_\pm$ is chosen to be the radial coordinate r in the Reissner–Nordström system of mass m and charge $q>0$ as in the above explanation, one can check that $g_\pm$ is a round cylindrical metric of radius 1 and $\varphi_\pm$ is an affine function such that $\nabla\varphi_\pm$ is the splitting direction for $g_\pm$, where ∇ denotes the covariant derivative with respect to $g_\pm$. The cylindrical metric is smooth where it is defined and the pseudo-affine functions $\varphi_\pm$ are smooth away from $\mathfrak{S}$ where well defined.

Quite surprisingly, almost all formulas that we obtain through local computations (without using any global insights) give the same expressions independently of whether we work with ρ_+ or ρ_-, as long as $\varphi_\pm$ is well defined and

$\rho_\pm > 0$ (see Lemmas 5.1 and 5.2). This prompts us to simplify the notation as follows.

Notation 2. In order to ease the notation, when there is no risk of confusion, we will avoid to write explicitly the subscript $\pm$: we will simply write ρ, without explicitly saying which among the pseudo-radial functions (5.3) and (5.4) we are choosing. Analogously, we will denote by φ, g the pseudo-affine function and the cylindrical ansatz metric with respect to the chosen pseudo-radial function ρ (whenever φ is well defined and $\rho > 0$, see Lemmas 5.1 and 5.2.Moreover, it will be understood that we can only use ρ_- where $N \neq 1$ (unless $m < 0$) and that we stay away from the critical set $\mathfrak{S} = \{N^2 + k = 0\}$ and from ∂M unless explicitly saying otherwise.

Now let ρ be a pseudo-radial function, defined as in (5.3) or (5.4), and let φ, g be defined by (5.5) and (5.6), respectively. We will investigate how the equations in (4.5) transform under the conformal change (5.6), or in other words to the new variables (g, φ). The computations for a conformal change of this particular form have been discussed with some care in [18, Section 3]. Starting from [18, Formula (3.3)], with some calculations we obtain the following relation between $\nabla^2\varphi$ and $\mathrm{D}^2 N$, where ∇ and D denote the Levi–Civita connections of g and g_0, respectively, obtaining

$$\nabla^2\varphi = \frac{\rho^2}{m\rho - q^2}\,\mathrm{D}^2 N + \frac{N\rho^4(3m\rho - 4q^2)}{(m\rho - q^2)^3}\,dN \otimes dN - \frac{N\rho^4}{(m\rho - q^2)^2}\,|\mathrm{D}N|^2\,g_0. \tag{5.7}$$

Note that $m\rho - q \neq 0$ where $N^2 + k \neq 0$. Taking the trace of (5.7) and recalling the second equation in (4.5), we find out that φ is g-harmonic,

$$\Delta_g \varphi = 0. \tag{5.8}$$

From (5.5), one computes

$$d\varphi = \mp \frac{\rho_\pm}{q\sqrt{N^2 + k}}\,dN \tag{5.9}$$

from which one obtains a very useful relation between $|\nabla\varphi|_g$ and $|\mathrm{D}N|$, namely

$$|\nabla\varphi|_g^2 = \frac{\rho^4\,|\mathrm{D}N|^2}{q^2(N^2 + k)} = \frac{\rho^6\,|\mathrm{D}N|^2}{(m\rho - q^2)^2}. \tag{5.10}$$

Another important relation is the one between the Hessians of ρ and φ, namely

$$\nabla^2\rho = \rho N\,\nabla^2\varphi + (\rho - m)\,d\varphi \otimes d\varphi. \tag{5.11}$$

In particular, tracing the second equation and recalling $\Delta_g\varphi = 0$, we get

$$\Delta_g \rho = (\rho - m)\,|\nabla\varphi|_g^2 \tag{5.12}$$

Finally, recall from [12, Theorem 1.159] the formula for the conformal transformation of the Ricci tensor

$$\mathrm{Ric} = \mathrm{Ric}_g - \frac{1}{\rho}\,\nabla^2\rho + \frac{2}{\rho^2}\,d\rho \otimes d\rho - \frac{1}{\rho}\,\Delta_g\rho\,g. \tag{5.13}$$

Using the first equation in (4.5) to write down Ric_{g_0} as a function of $\mathrm{D}^2 N$ and using the above formulæ (5.11) and (5.7), after some calculations we get

$$\mathrm{Ric}_g = \frac{1}{N}\left(1 - \frac{m}{\rho}\right)\nabla^2\varphi - d\varphi \otimes d\varphi + |\nabla\varphi|_g^2\, g. \tag{5.14}$$

The reformulation (5.14), (5.8) of (4.5) in terms of ρ, g, and φ will play a crucial role in the proofs of Theorems 3.1 and 3.2. Our strategy will be to exploit the geometry of the level sets, studied in Sects. 5.2 and 5.3, and the properties of black holes and equipotential photon surfaces to prove that the quantity $|\nabla^2\varphi|_g$ vanishes identically. This will lead us to conclude that the system is isometric to a (half) round cylinder (see for instance the final part of the proof of [6, Proposition 4.2]), which in turn will imply that g_0 is the spatial Reissner–Nordström metric.

5.2. Geometry of the Level Sets

In this section, we will carefully study the geometry of the level sets of N, appealing to Notation 2. To this end, let ρ be a pseudo-radial function, defined as in (5.3) or (5.4), and let φ, g be defined by (5.5) and (5.6), respectively. In this subsection we study the extrinsic curvature of a level set $\Sigma = \{N = s\}$. We start by discussing the case when s is a regular value of the function N and hence also of the function φ by (5.10). Let $\mathrm{n} := \frac{\mathrm{D}N}{|\mathrm{D}N|}$ and $\mathrm{n}_g := \frac{\nabla\varphi}{|\nabla\varphi|_g}$ be our choices of g_0-unit normal and of g-unit normal to Σ, respectively, and let H and H_g be the mean curvatures of Σ with respect to g_0 and g, respectively.

Since Σ is also a level set of φ and since φ is g-harmonic by (5.8), arguing as, for example, in [18, Section 3.2], we deduce that

$$\mathrm{H}_g\, |\nabla\varphi|_g = -\nabla^2\varphi(\mathrm{n}_g, \mathrm{n}_g), \tag{5.15}$$

$$\mathrm{H}\, |\mathrm{D}N| = \Delta N - \mathrm{D}^2 N(\mathrm{n}, \mathrm{n}). \tag{5.16}$$

Next, writing $\nabla^2\varphi$ in terms of $\mathrm{D}^2 N$ using (5.11) and then using the first equation in (4.5) to write $\mathrm{D}^2 N$ in terms of Ric, we compute

$$\mathrm{H}_g\, |\nabla\varphi|_g = -\frac{N\rho^4}{m\rho - q^2}\,\mathrm{Ric}(\mathrm{n},\mathrm{n}) - \frac{2N\rho^6}{(m\rho - q^2)^2}\,|\mathrm{D}N|^2. \tag{5.17}$$

Recalling the first equation in (4.5) and using (5.16), we can rewrite (5.17) as

$$\mathrm{H}_g\, |\nabla\varphi|_g = \frac{\rho^4}{m\rho - q^2}\left[\mathrm{H}\,|\mathrm{D}N| - \frac{2N\rho^2}{m\rho - q^2}\,|\mathrm{D}N|^2\right]. \tag{5.18}$$

This is a useful formula to compare the mean curvatures with respect to the two metrics on a given level set. An alternative useful modification of (5.17) is obtained by means of the Gauss–Codazzi equation

$$2\,\mathrm{Ric}(\mathrm{n},\mathrm{n}) = \mathrm{R} - \mathrm{R}^\Sigma - |\,\mathrm{h}\,|^2 + \mathrm{H}^2 \tag{5.19}$$

where R^Σ denotes the scalar curvature of Σ with respect to g_0 and h denotes its second fundamental form with respect to g_0. Recalling

$$\mathrm{R} = \frac{2q^2\rho^2}{(m\rho - q^2)^2}\,|\mathrm{D}N|^2 \tag{5.20}$$

(this can be obtained by tracing the first equation in (4.5) and then applying some algebraic manipulations), we find

$$\mathrm{H}_g\,|\nabla\varphi|_g = \frac{N\rho^4}{2(m\rho - q^2)}\left(\mathrm{R}^\Sigma + |\mathring{\mathrm{h}}|^2 - \frac{\mathrm{H}^2}{2}\right) - \frac{(2m\rho - q^2)N\rho^6}{(m\rho - q^2)^3}\,|\mathrm{D}N|^2, \tag{5.21}$$

where $\mathring{\mathrm{h}} = \mathrm{h} - \frac{\mathrm{H}}{2} g_0$ denotes the traceless part of the second fundamental form h of Σ with respect to g_0.

The normals n and n_g are related by $\mathrm{n}_g = \mp\rho_\pm\,\mathrm{n}$ as one sees from a direct computation using (5.10) and (5.9). From the usual law of conformal transformation of mean curvature, we find $\mathrm{H}_g = \mp\left(\rho_\pm\,\mathrm{H} - 2\,\mathrm{n}(\rho_\pm)\right)$.

Let us end this subsection by discussing the non-regular level sets of N. Let $\mathrm{Crit}(N)$ be the set of the critical points of N and suppose that $\{N = s\} \cap \mathrm{Crit}(N) \neq \emptyset$. We recall from [67] that N is a real analytic function on M (with respect to g_0-harmonic coordinates). In particular, the set of critical values of N is discrete. Furthermore, from the results in [52] (see also [48, Theorem 6.3.3]) on the structure of the critical set of a real analytic function, it follows that there exists a smooth surface $S \subseteq \mathrm{Crit}(N)$ with Hausdorff measure $\mathcal{H}^2(\mathrm{Crit}(N) \setminus S) = 0$. As a consequence, the unit normal, the second fundamental form, and the mean curvature are well-defined $\mathcal{H}^2$-almost everywhere on the critical level set $\{N = s\}$ as well, with respect to g_0. Proceeding now exactly as at the end of [18, Section 3.2], one computes $\mathrm{H}_g = \mp\rho_\pm\,\mathrm{H}$ at the points of S. It follows then that (5.21) holds at the points of S, which in particular implies that (5.21) holds $\mathcal{H}^2$-almost everywhere on the critical level set $\{N = s\}$ as well.

5.3. Bochner Formula

In this section, we will study some useful consequences of the Bochner formula in the cylindrical picture, appealing again to Notation 2. Again, let ρ be a pseudo-radial function, defined as in (5.3) or (5.4), and let φ, g be defined by (5.5) and (5.6), respectively. The Bochner formula for the g-harmonic function φ then reads

$$\begin{aligned}\Delta_g|\nabla\varphi|_g^2 &= 2\,|\nabla^2\varphi|_g^2 + 2\,\mathrm{Ric}_g(\nabla\varphi,\nabla\varphi)\\ &= 2\,|\nabla^2\varphi|_g^2 + \frac{1}{N}\left(1 - \frac{m}{\rho}\right)\langle\nabla|\nabla\varphi|_g^2\,|\,\nabla\varphi\rangle_g,\end{aligned} \tag{5.22}$$

where we have used (5.14) in the second equality. More generally, for any $\beta \geq 2$, one has the following formula

$$\begin{aligned}\Delta_g|\nabla\varphi|_g^\beta &= \beta\,|\nabla\varphi|_g^{\beta-2}\,|\nabla^2\varphi|_g^2 + \frac{1}{N}\left(1 - \frac{m}{\rho}\right)\langle\nabla|\nabla\varphi|_g^\beta\,|\,\nabla\varphi\rangle_g\\ &\quad + \beta(\beta-2)|\nabla\varphi|_g^{\beta-2}\,|\nabla|\nabla\varphi|_g|_g^2,\end{aligned} \tag{5.23}$$

which in particular implies

$$\Delta_g|\nabla\varphi|_g^\beta - \frac{1}{N}\left(1 - \frac{m}{\rho}\right)\langle\nabla|\nabla\varphi|_g^\beta\,|\,\nabla\varphi\rangle_g \geq 0. \tag{5.24}$$

This elliptic inequality for $|\nabla\varphi|_g^\beta$ will play an important role later. An alternative useful way to rewrite (5.24) is the following

$$\mathrm{div}_g\left(\frac{1}{\rho N}\nabla|\nabla\varphi|_g^\beta\right) = \frac{1}{\rho N}\left[\Delta_g|\nabla\varphi|_g^\beta - \frac{1}{N}\left(1-\frac{m}{\rho}\right)\langle\nabla|\nabla\varphi|_g^\beta\,|\,\nabla\varphi\rangle_g\right] \geq 0. \tag{5.25}$$

In other words, we have found a family of vector fields

$$X_\beta := \frac{1}{\rho N}\nabla|\nabla\varphi|_g^\beta \tag{5.26}$$

with non-negative g-divergence as desired, for each $\beta \geq 2$.

5.4. Asymptotic Analysis of the Conformal Metric

The purpose of this section is to discuss the asymptotic behavior of our solution. More precisely, we show that the gradient and Hessian of φ have bounded norm at infinity and that the metric g is *asymptotically cylindrical*, namely the level sets of φ converge to round spheres of fixed radius with respect to the metric g. The arguments are similar to the ones presented in [6, Lemma 3.1], but since we are using a weaker notion of asymptotic flatness, we prefer to present all the details.

First of all, in slight contrast with Sect. 4.4, here it is convenient to work with the level sets of the lapse N instead of working with coordinate spheres. Let ν be the g_0-unit normal to the level sets of N pointing toward $+\infty$ and let $\mathrm{n} = \frac{\mathrm{D}N}{|\mathrm{D}N|}$. Clearly we have $\mathrm{n} = \pm\nu$, depending on the direction where N is growing. From (4.12), we deduce

$$\nu = \left(\frac{x^i}{|x|} + \sigma^i\right)\frac{\partial}{\partial x^i} = \frac{\partial}{\partial|x|} + \sigma^i\frac{\partial}{\partial x^i}, \qquad \sigma^i = o_1(1) \tag{5.27}$$

as $|x| \to \infty$. From (4.12) if $m \neq 0$ and from $N^2 = 1+\Psi^2$ and (2.10) if $m = 0$, we deduce that we can always find a neighborhood U of infinity, contained in the asymptotic end $M \setminus K$, such that $|\mathrm{D}N| \neq 0$ in U. If $\mathrm{n} = \nu$ (resp., $\mathrm{n} = -\nu$), we deduce that we can find t_0 close enough to 1 such that a connected component of $\{t_0 \leq N < 1\}$ (resp., $\{1 < N \leq t_0\}$) is contained in U and contains the asymptotic end. It follows from Morse Theory [56] that $\{t_0 \leq N < 1\} \cap U$ (resp., $\{1 < N \leq t_0\} \cap U$) is diffeomorphic to $\Sigma \times [t_0, 1)$ (resp., $\Sigma \times (1, t_0]$), with the level set $\{N = t\}$ corresponding to $\Sigma \times \{t\}$ for all $t \geq t_0$.

On the other hand, asymptotic flatness implies that the topology of the asymptotic end $M \setminus K$ must be that of $\mathbb{R}^3 \setminus B$ for some closed ball B. Since $\mathbb{R}^3 \setminus B$ retracts to $\mathbb{S}^2$, it follows that Σ does as well. In particular, the fundamental group of Σ is the same as the one of $\mathbb{S}^2$. From the classification of surfaces, it follows that Σ is diffeomorphic to $\mathbb{S}^2$. In other words, $\{t_0 \leq N < 1\} \cap U$ (resp., $\{1 < N \leq t_0\} \cap U$) is diffeomorphic to $\mathbb{S}^2 \times [t_0, 1)$ (resp., $\mathbb{S}^2 \times (1, t_0]$). By means of this diffeomorphism, the function N is just the projection onto the second factor and can be used as a coordinate. It is well known (see, e.g., [42]) that, choosing coordinates (θ^1, θ^2) on the first factor $\mathbb{S}^2$, the metric g_0

can be rewritten as

$$g_0 = \frac{dN \otimes dN}{|\mathrm{D}N|^2} + \bar{g}_{ab}(N, \theta^1, \theta^2)\, d\theta^a \otimes d\theta^b, \tag{5.28}$$

with $a, b = 1, 2$. Notice that, for a fixed t, the coefficients of the metric induced on the necessarily regular level set $\{N = t\}$ are given by $\bar{g}_{ab}(t, \theta^1, \theta^2)$.

Let us now study the asymptotic behavior of the metric $\bar{g}$ as $t \to 1$. Since the asymptotic end of M is diffeomorphic to $\mathbb{R}^3 \setminus B \supset \{N \geq t_0\}$, we can also consider standard spherical coordinates $(|x|, \vartheta^1, \vartheta^2)$. Recalling the asymptotic behavior (2.9) of g_0, we can write

$$\begin{aligned} g_0 &= (1+\omega)\, d|x| \otimes d|x| + |x|\zeta_a\, (d|x| \otimes d\vartheta^a + d\vartheta^a \otimes d|x|) \\ &\quad + |x|^2 \left[(g_{\mathbb{S}^2})_{ab} + \xi_{ab}\right] d\vartheta^a \otimes d\vartheta^b, \\ \omega, \xi_{ab}, \zeta_a &= o(1) \end{aligned} \tag{5.29}$$

as $|x| \to \infty$. Exploiting the fact that $\frac{\partial}{\partial \theta^a}$ is g_0-orthogonal to $\mathrm{D}N$, we can rewrite (5.27) in spherical coordinates, obtaining

$$\nu = (1+\lambda)\frac{\partial}{\partial |x|} + \frac{\lambda^a}{|x|}\frac{\partial}{\partial \vartheta^a}, \qquad \lambda, \lambda^a = o(1) \tag{5.30}$$

as $|x| \to \infty$. Since the normal ν is converging to $\frac{\partial}{\partial |x|}$ at infinity, we expect the level sets of N to become round as we approach infinity. For the sake of completeness, we now show how to prove this in detail. For $a = 1, 2$ and using (5.29), (5.30), we compute

$$0 = g_0\left(\frac{\partial}{\partial \theta^a}, \mathrm{n}\right) = \left[(1+\omega)(1+\lambda) + \zeta_c\lambda^c\right]\frac{\partial |x|}{\partial \theta^a} + |x|\left[\lambda^b\left[(g_{\mathbb{S}^2})_{bc} + \xi_{bc}\right] + (1+\lambda)\zeta_c\right]\frac{\partial \vartheta^c}{\partial \theta^a}$$

for $a = 1, 2$ as $|x| \to \infty$. It follows then that $\frac{\partial |x|}{\partial \theta^a} = \tau_b \frac{\partial \vartheta^b}{\partial \theta^a}$, where $\tau_b = o(|x|)$ as $|x| \to \infty$, $b = 1, 2$. With this insight, recalling (5.29) we are finally able to compute the asymptotic behavior of $\bar{g}_{ab}$, obtaining

$$\begin{aligned} \bar{g}_{ab} &= g_0\left(\frac{\partial}{\partial \theta^a}, \frac{\partial}{\partial \theta^b}\right) \\ &= (1+\omega)\frac{\partial |x|}{\partial \theta^a}\frac{\partial |x|}{\partial \theta^b} + |x|\zeta_c\left(\frac{\partial |x|}{\partial \theta^a}\frac{\partial \vartheta^c}{\partial \theta^b} + \frac{\partial \vartheta^c}{\partial \theta^a}\frac{\partial |x|}{\partial \theta^b}\right) + |x|^2\left[(g_{\mathbb{S}^2})_{cd} + \xi_{cd}\right]\frac{\partial \vartheta^c}{\partial \theta^a}\frac{\partial \vartheta^d}{\partial \theta^b} \\ &= |x|^2\left[(g_{\mathbb{S}^2})_{cd} + \tau_{cd}\right] d\vartheta^c \otimes d\vartheta^d\left(\frac{\partial}{\partial \theta^a}, \frac{\partial}{\partial \theta^b}\right) \end{aligned} \tag{5.31}$$

for $a, b = 1, 2$ as $|x| \to \infty$, where $\tau_{cd} = o(1)$ for $c, d = 1, 2$. In other words, the metric $\bar{g}$ induced on a level set $\{N = t\}$ can be written as $\bar{g} = |x|^2\left[(g_{\mathbb{S}^2})_{cd} + \tau_{cd}\right]$ $d\tilde{\vartheta}^c \otimes d\tilde{\vartheta}^d$ where $\tilde{\vartheta}^a$ is the restriction of ϑ^a to $\{N = t\}$. We have thus shown that the metric $\bar{g}$ induced on the level sets of N converges to $g_{\mathbb{S}^2}$ at infinity, as expected.

In order to conclude our study of the asymptotic behavior of the conformal quantities ρ, φ and g, it is convenient to distinguish between the case $m \neq 0$ and $m = 0$.

5.4.1. The Case $m \neq 0$. If $m \neq 0$ then $\mu \neq 0$ as well and m, μ have the same sign thanks to Lemma 4.2. Recall that we choose ρ either as (5.3) or (5.4) in order to have $\rho \to +\infty$ as we approach the infinity, according to Lemma 5.1 and 5.2. With this choice of ρ, recalling also (5.5), we easily see that

$$\rho = \frac{m|x|}{\mu} + o_2(|x|), \tag{5.32}$$

$$\varphi = \log\left(\frac{2m|x|}{\mu}\right) + o_2(1) \tag{5.33}$$

as $|x| \to \infty$. Substituting (5.32), (5.33) into (5.10) and (5.7), we deduce

$$|\nabla\varphi|_g = \frac{m}{\mu} + o_1(1), \tag{5.34}$$

$$\nabla^2_{ij}\varphi = o(|x|^{-2}) \tag{5.35}$$

as $|x| \to \infty$. Furthermore, by (5.6) and (5.32), we get

$$|\nabla^2\varphi|_g = o(1) \tag{5.36}$$

as $|x| \to \infty$. Let us now move our attention to the cylindrical ansatz metric $g = \frac{g_0}{\rho^2}$. This time we use φ as a coordinate on our end instead of N. Dividing formula (5.28) by ρ^2 and using (5.9) for ρ_+ and (5.10) for ρ_-, we obtain

$$g = \frac{d\varphi \otimes d\varphi}{|\nabla\varphi|_g^2} + \frac{\bar{g}_{ab}}{\rho^2}\, d\theta^a \otimes d\theta^b. \tag{5.37}$$

From (5.32) and (5.31), we deduce that the metric induced by g on the level sets of φ is

$$\frac{\bar{g}_{ab}}{\rho^2} = \frac{\mu^2}{m^2}\, g_{\mathbb{S}^2}\left(\frac{\partial}{\partial\theta^a}, \frac{\partial}{\partial\theta^b}\right) + o(1) \tag{5.38}$$

as $|x| \to \infty$. In other words, the metric induced by g on the level sets of N (and thus of φ) converges to $\frac{\mu^2}{m^2} g_{\mathbb{S}^2}$. Denoting by $\mathrm{d}\sigma_g$ the area element induced by g, it follows that

$$\lim_{s\to\infty} \int_{\{\varphi = s\}} \mathrm{d}\sigma_g = \frac{\mu^2}{m^2}\, |\mathbb{S}^2|. \tag{5.39}$$

5.4.2. The Case $m = 0$. Let us now discuss the case when $m = 0$ and hence $\mu = 0$ by Lemma 4.2. From Lemma 4.3, we also know that $\kappa > 0$. Using $N^2 = 1 + \Psi^2$ from (4.4) and (4.19) with $\kappa > 0$ so that $\nu(\Psi) < 0$ on the leaves of N, we find $\nu(N) = -|\mathrm{D}N|$ on the leaves of N and thus $\mathrm{n} = -\nu$ asymptotically. Defining the pseudo-radial function ρ by (5.3) and φ by (5.5), we immediately find $\rho = \frac{q}{\Psi}$ and hence $\varphi = \log(\frac{q(N+1)}{\Psi})$. Exploiting (2.10) with $\kappa > 0$ as well $q > 0$, we thus find from (5.4), (5.5) that

$$\rho = \frac{q|x|}{\kappa} + o_2(|x|), \tag{5.40}$$

$$\varphi = \log\left(\frac{2q|x|}{\kappa}\right) + o_2(1) \tag{5.41}$$

as $|x| \to \infty$. From (5.10) and (5.7), we then compute the following expansions

$$|\nabla\varphi|_g = \frac{q}{\kappa} + o_1(1), \tag{5.42}$$

$$\nabla^2_{ij}\varphi = o(|x|^{-3}) \tag{5.43}$$

as $|x| \to \infty$. Furthermore, by (5.6) and (5.40), we get

$$|\nabla^2\varphi|_g = o(|x|^{-1}) \tag{5.44}$$

as $|x| \to \infty$. Arguing as in Sect. 5.4.1, we obtain

$$\lim_{s\to\infty} \int_{\{\varphi=s\}} \mathrm{d}\sigma_g = \frac{\kappa^2}{q^2}\,|\mathbb{S}^2|. \tag{5.45}$$

5.5. Choosing the Parameters m and q

We recall that the quotient $\frac{m}{q}$ has already been fixed in (4.3) depending on the values N_0 and Ψ_0 of the lapse function N and the electric potential Ψ on ∂M. However, we still have a degree of freedom in the choice of the two parameters. It will be convenient to remove this extra degree of freedom and fix the values of m and q by imposing that $|\nabla\varphi|_g = 1$ shall hold on[8] ∂M. This makes g a unit radius cylinder in the equality case in view of (5.37). Recall that we know that $|\mathrm{D}N| = |dN|$ is constant on[8] ∂M. Thus, in the non-degenerate (black hole or photon surface) case where $|\mathrm{D}N| > 0$ on ∂M and thus $N_0^2 + k > 0$ as discussed in Sect. 4.2, recalling (5.10) and (4.6), this amounts to requiring

$$q = \frac{q^2}{\rho_0^2}\,\frac{\sqrt{N_0^2+k}}{|\mathrm{D}N||_{\partial M}}, \tag{5.46}$$

where the right hand side is well defined thanks to (4.7) and is a function of $\frac{m}{q}$ (whose value has already been fixed) and of N_0 and $|\mathrm{D}N|^2|_{\partial M} > 0$ which are given constants on ∂M. It follows that the parameters m and q are now uniquely defined in the non-degenerate case.

It remains to discuss the choice of m and q in the degenerate (black hole or photon surface) case, i.e., when $|dN| = 0$ on[8] ∂M. From Sect. 4.2, we know that $N_0^2 + k = 0$; hence $\partial M \subseteq \mathfrak{S}$ and $\Psi_0 = \frac{m}{q}$. We claim that indeed $\mathfrak{S} = \partial M$. To see this, let us exploit (4.7) to find $\mathfrak{S} = \{\Psi = \frac{m}{q}\}$. We now use that the last equation in (2.1) is invariant under addition of constants to Ψ so that it also applies to $\overline{\Psi} := \Psi - \frac{m}{q}$. Clearly, $\overline{\Psi}$ satisfies $\overline{\Psi}|_{\partial M} = 0$, while the asymptotic decay assumption on Ψ gives $\overline{\Psi} \to -\frac{m}{q} < 0$ at infinity. Applying the strong maximum principle[7] to the last equation in (2.1) for $\overline{\Psi}$ thus gives $\overline{\Psi} < 0$ or in other words $\Psi < \frac{m}{q}$ in $M \setminus \partial M$ and hence $\mathfrak{S} = \partial M$ as claimed. From this, we deduce that (5.10) holds on $M \setminus \partial M$. In combination with (4.8), this gives

$$|\nabla\varphi|_g^2 = \frac{\rho^4\,|\mathrm{D}\Psi|^2}{q^2 N^2} \tag{5.47}$$

[8]Suppressing a limit $z \to 0^+$ in the degenerate horizon case.

on $M \setminus \partial M$. Next, from the definition of ρ and from $N_0^2 + k = 0$, we know that

$$\rho_0 = \frac{q^2}{m}. \tag{5.48}$$

Let us first focus on the photon surface case $N_0 > 0$. In this case, (5.47) continuously extends to ∂M which corresponds to choosing

$$q = \frac{m^2}{q^2} \frac{N_0}{|\mathrm{D}\Psi||_{\partial M}} \tag{5.49}$$

which is clearly a function of $\frac{m}{q}$ (whose value has already been fixed) and of N_0 and $|\mathrm{D}\Psi||_{\partial M}$ which are given positive constants on ∂M, see Sect. 4.2 with $|\nabla\varphi|_g \to 1$ uniformly as one approaches ∂M. In the degenerate horizon case $N_0 = 0$, it follows from (5.48) and $k = 0$ (this has been shown at the end of Sect. 4.2) that $\rho_0 = m = q$. In addition, the decay assumptions (2.3) imply that the coefficients of the inverse of the metric read

$$(g_0)^{zz} = z^2 A + o(z^2), \qquad (g_0)^{zI} = O(z^2), \qquad (g_0)^{IJ} = b^{IJ} + o(1) \tag{5.50}$$

and consequently, using again the expansions (2.3) combined with (2.4), we get

$$\frac{|\mathrm{D}N|^2}{N^2} = \frac{(g_0)^{zz}(\partial_z N)^2 + o(z^2)}{z^2 C^2 + o(z^2)} = \frac{z^2 C^4 + o(z^2)}{z^2 C^2 + o(z^2)} \to C^2 \tag{5.51}$$

as $z \to 0^+$. Taking the limit of (5.10) as $z \to 0^+$ in combination with the desired normalization $|\nabla\varphi|_g \to 1$ as $z \to 0^+$ leads to the choice

$$q = \frac{1}{C}. \tag{5.52}$$

It follows that the parameters m and q are now uniquely fixed also in the degenerate case, with $|\nabla\varphi|_g \to 1$ uniformly as $z \to 0^+$ in the degenerate horizon case.

6. Black Hole Uniqueness in Electro-Vacuum

This section is dedicated to the proof of Theorem 3.1, that is, we want to characterize sub-extremal and extremal Reissner–Nordström systems as the only asymptotically flat, electrostatic electro-vacuum spacetimes with a connected horizon. To this end, we will exploit the cylindrical ansatz formalism discussed in Sect. 5 and will continue to assume $\Psi_0 > 0$. The parameters m and q shall be chosen as discussed in Sect. 5.5. Recall from Sect. 4.2 that $m > q > 0$ and $\mathfrak{S} = \emptyset$ when ∂M is non-degenerate, while $m = q \neq 0$ and $\mathfrak{S} = \partial M$ when ∂M is degenerate. This tells us that the equations in (4.5) are well defined everywhere on $M \setminus \partial M$. Moreover, the maximum principle[7] applied to the second equation in (2.1), exploiting $N_0 = 0$ and the asymptotic assumption $N \to 1$ at infinity, ensures that $0 < N < 1$ in $M \setminus \partial M$ and in particular $N \neq 1$ in M. Hence, we can apply the conformal change discussed in Sect. 5 on $M \setminus \partial M$ with the pseudo-radial function ρ_- from (5.4), see Lemma 5.2. We then choose the pseudo-affine function φ_- and cylindrical ansatz metric g_- accordingly via

(5.5), (5.6) and note that they are well defined and smooth on $M \setminus \partial M$ by Lemma 5.2. In the non-degenerate horizon case, all these smoothness claims extend to ∂M.

6.1. First Consequences of the Bochner Formula

In this section, we will discuss a couple of consequences of the Bochner formula from Sect. 5.3 in the black hole case. Similar consequences of the Bochner formula will be discussed for equipotential photon surfaces in Sect. 7. In particular, we will exploit inequalities (5.24) and (5.25) to obtain some crucial information on the function φ. For our first result, we start from the vector fields X_β with non-positive divergence found in (5.26) and we prove an integral inequality on the mean curvature H_g of the level sets of φ.

Proposition 6.1. *Let (M, g_0, N, Ψ) be an asymptotically flat electrostatic electro-vacuum system of mass μ such that M has a connected horizon boundary ∂M. Let $\rho = \rho_-$, $\varphi = \varphi_-$, $g = g_-$ be defined as in (5.4), (5.5), and (5.6), with respect to $\mathfrak{m}, q$ satisfying (4.3). Then, for any $\beta \geq 2$ and any $s < \infty$ such that $\{\varphi = s\} \subset M \setminus \partial M$, one has*

$$\int_{\{\varphi=s\}} \frac{\mathrm{H}_g\, |\nabla\varphi|_g^\beta}{N}\, \mathrm{d}\sigma_g \geq 0, \tag{6.1}$$

where H_g is the mean curvature of the level set $\{\varphi = s\}$ with respect to the metric g. Furthermore, if equality holds in (6.1) for some s and β, then $(\{\varphi \geq s\}, g_0)$ isometrically embeds into the $\frac{\mu^2}{\mathfrak{m}^2}$-multiple of the Reissner–Nordström metric of mass $\mathfrak{m}$ and charge q, with N and Ψ corresponding to the respective Reissner–Nordström quantities.

In addition, if (5.46), respectively, (5.52) holds in the non-degenerate, respectively, degenerate horizon case and if equality holds in (6.1) in the limit $s \to \varphi_0$ where $\partial M = \{\varphi = \varphi_0\}$ then $(M\setminus\partial M, g_0, N, \Psi)$ is isometric to the Reissner–Nordström system of mass $\mathfrak{m} = \mu$ and charge q. This isometry extends to the boundary ∂M in the non-degenerate horizon case.

Remark 6.2. It is worth mentioning that one can refine the above proposition and show that the same result remains true for any $\beta \geq \frac{1}{2}$, at the cost of making the proof more complicated (a more careful analysis of the critical points is needed, in the spirit of [7, Section 3.2]). However, the above proposition is enough for our purposes.

Remark 6.3. The above proposition can be applied to obtain a monotonicity formula for the functions $F_\beta(s) = \int_{\{\varphi=s\}} |\nabla\varphi|_g^{\beta+1}\, \mathrm{d}\sigma_g$. This is the main tool employed in [6] and [22] to prove the uniqueness in vacuum for black holes and non-degenerate equipotential photon surfaces, respectively. We could repeat the same arguments in our framework as well, with small modifications, but we prefer to show a faster strategy that requires far less technicalities. Nevertheless, the method involving the monotonicity of the functions F_β will still be of use later on, see Sect. 7.5 and Proposition 7.3.

Proof. From (5.33) or Lemma 5.2, we know that $\{\varphi \geq s\}$ contains the asymptotic end (or a suitable piece thereof for large s). On the other hand, $\{\varphi \geq s\}$ does not contain ∂M, as N (and thus φ) achieves its minimum value on ∂M. Furthermore, $\{\varphi \geq s\}$ is connected: In fact, if $\{\varphi \geq s\}$ had a second connected component, this would have to be compact with boundary contained in $\{\varphi = s\}$. Since φ is g-harmonic, by the maximum and minimum principle[7] we would get that φ is constant on this component and thus everywhere by analyticity. Under our hypotheses, the functions ρ, φ and the metric g are well defined and smooth on M. By continuity, the level sets $\{\varphi = S\}$ are non-empty for any $S > s$.

Let $s < S < \infty$ be two regular values for φ, and let us integrate the inequality (5.25) over $\{s < \varphi < S\}$. As an application of the divergence theorem, we get

$$-\int_{\{\varphi=s\}} \frac{1}{\rho N} \langle \nabla |\nabla\varphi|_g^\beta \,|\, \mathrm{n}_g \rangle \, \mathrm{d}\sigma_g + \int_{\{\varphi=S\}} \frac{1}{\rho N} \langle \nabla |\nabla\varphi|_g^\beta \,|\, \mathrm{n}_g \rangle \, \mathrm{d}\sigma_g \geq 0. \tag{6.2}$$

Since $\langle \nabla |\nabla\varphi|_g^\beta \,|\, \mathrm{n}_g \rangle = \beta |\nabla\varphi|_g^{\beta-1} \nabla^2\varphi(\mathrm{n}_g, \mathrm{n}_g)$, recalling (5.15), we immediately obtain

$$\int_{\{\varphi=s\}} \frac{1}{\rho N} \, \mathrm{H}_g \, |\nabla\varphi|_g^\beta \, \mathrm{d}\sigma_g \geq \int_{\{\varphi=S\}} \frac{1}{\rho N} \, \mathrm{H}_g \, |\nabla\varphi|_g^\beta \, \mathrm{d}\sigma_g. \tag{6.3}$$

In case s or S are critical values for φ, the same inequality still holds. This can be proven exactly as in [6], by observing that the vector field X_β from (5.26) is bounded with bounded divergence for any $\beta \geq 2$, so that the divergence theorem still applies.

In order to conclude the proof of (6.1), it suffices to show that the right-hand side of (6.3) goes to zero as $S \to \infty$. To this end, we use (5.15) to deduce that $|\mathrm{H}_g| |\nabla\varphi|_g = |\nabla^2\varphi(\mathrm{n}_g, \mathrm{n}_g)| \leq |\nabla^2\varphi|_g$. Furthermore, from (5.34), (5.36), and (5.39) we know that $|\nabla\varphi|_g$, $|\nabla^2\varphi|_g$ and $\int_{\{\varphi=S\}} \mathrm{d}\sigma_g$ are bounded near infinity. Since also $N \to 1$, $\rho \to \infty$ at infinity, it follows immediately that the right-hand side of (6.3) goes to zero as $S \to \infty$ as desired.

Let us now prove the rigidity statement. Retracing our proof, we deduce that in the equality case, the right hand side of (5.25) is equal to zero on $\{\varphi \geq s\}$ or in other words, equality holds in (5.24) on $\{\varphi \geq s\}$. From (5.23) it then follows that $|\nabla^2\varphi|_g \equiv 0$ and $\nabla|\nabla\varphi|_g \equiv 0$ on $\{\varphi \geq s\}$, so that the asymptotic assertion (5.34) gives $|\nabla\varphi|_g \equiv \frac{m}{\mu}$. It is now straightforward to conclude by applying [5, Theorem 4.1] that $(\{\varphi \geq s\}, g)$ is isometric to one half of a cylinder over $(\{\varphi = s\}, g|_{\{\varphi=s\}})$. Here, we implicitly used Remark 2.2 and Lemma 4.2. On the other hand, the asymptotic assertion (5.38) informs us that this half cylinder must indeed be round and of radius $\frac{\mu}{m}$. Returning back to the original metric and using (5.6), (5.5), (5.1), (4.4), and the asymptotic decay assumption on Ψ from (2.9), we deduce that $g_0 = \frac{\mu^2}{m^2}\left(\frac{1}{N^2} d\rho^2 + \rho^2 g_{\mathbb{S}^2}\right)$ on $\{\varphi \geq s\}$, with $N = \sqrt{1 - \frac{2m}{\rho} + \frac{q^2}{\rho^2}}$ and $\Psi = \frac{q}{\rho}$ (up to the embedding). To see this, we also exploit that $d\varphi = \pm\frac{m}{\mu} dt$ for the cylindrical height coordinate t from the proof of [5, Theorem 4.1] in combination with $|\nabla\varphi|_g \equiv \frac{m}{\mu}$.

It remains to address the rigidity case in the limit $s \to \varphi_0$. If the left hand side of (6.3) tends to 0 as $s \to \varphi_0$, we can deduce as above that $\nabla^2\varphi = 0$ and $|\nabla\varphi|_g \equiv \frac{m}{\mu}$ on $M \setminus \partial M$. Using that (5.46), respectively (5.52), ensure $|\nabla\varphi|_g \to 1$ as $s \to \varphi_0$, this shows $m = \mu$. We can then conclude as before, obtaining the desired isometry up to (and including ∂M in the non-degenerate case). Note that, in the degenerate case, we apply [5, Theorem 4.1] on $\{\varphi \geq -K\}$ for an arbitrary very large K, recalling $\varphi_0 = -\infty$ in this case to get isometry to a cylinder of radius 1 over the base $(\{\varphi = K\}, g|_{\{\varphi=K\}})$ and then take $K \to \varphi_0 = \infty$. □

This proposition is sufficient to conclude rigidity in the degenerate black hole case under the additional asymptotic assumption that

$$\frac{\mathrm{H}}{N} \to 2C \tag{6.4}$$

uniformly as $N \to 0^+$, where H denotes the mean curvature of the N-level sets with respect to g_0 and to the g_0-unit normal pointing to the asymptotically flat end. As previously remarked, this assumption can be dropped, see Sect. 7.7.

Corollary 6.4 (Degenerate black hole uniqueness). *Theorem 3.1 holds for systems with connected degenerate horizon inner boundary, under the assumption (6.4).*

Proof. Recall that in the degenerate case we have $k = 0$ and hence $m = q$. From (5.51) and (4.10) we immediately deduce

$$\mathrm{R} \to 2C^2 \tag{6.5}$$

uniformly as $z \to 0^+$, where R denotes the scalar curvature of g_0. The decay of the derivatives of N in (2.4) informs us that this uniform convergence also holds as $N \to 0^+$ and hence as $s \to \varphi_0 = -\infty$. Aiming to apply the rigidity statement of Proposition 6.1 in the limit $\varphi \to \varphi_0 = -\infty$, we use (5.18), $q = m = \frac{1}{C}$, and $\rho = \frac{m}{1-N}$, $m\rho - q^2 = \frac{m^2 N}{1-N}$ to compute

$$\frac{\mathrm{H}_g\,|\nabla\varphi|_g}{N} = \frac{m^2}{(1-N)^3}\left[\frac{\mathrm{H}}{N}\frac{|\mathrm{D}N|}{N} - \frac{2}{1-N}\frac{|\mathrm{D}N|^2}{N^2}\right]$$

on any regular level set $\{\varphi = s\}$ for $s > \varphi_0 = -\infty$. From (4.10) and $k = 0$, we learn that

$$\frac{\mathrm{H}_g\,|\nabla\varphi|_g}{N} = \frac{m^2}{(1-N)^3}\left[\frac{\mathrm{H}}{N}\sqrt{\frac{\mathrm{R}}{2}} - \frac{\mathrm{R}}{1-N}\right]. \tag{6.6}$$

We would now like to take the limit $s \to \varphi_0 = -\infty$ of this equation exploiting that we have the existence of a regular foliation by N-level sets near ∂M and already know from the assumption (6.4) that $\frac{\mathrm{H}}{N} \to 2C$ uniformly as $N \to 0^+$ and hence as $s \to \varphi_0 = -\infty$. Now, using again $m = \frac{1}{C}$ and (6.5), the right hand side of (6.6) uniformly converges to 0 as $s \to \varphi_0 = -\infty$. Furthermore, we know that $|\nabla\varphi|_g \to 1$ uniformly as $z \to 0^+$ from Sect. 4.2 which allows us to conclude that

$$\frac{\mathrm{H}_g\,|\nabla\varphi|_g^\beta}{N} \to 0$$

uniformly as $s \to \varphi_0 = -\infty$ for all $\beta \geq 2$. In order to trigger the rigidity statement in Proposition 6.1, it is then enough to show that

$$\int_{\{\varphi=s\}} \mathrm{d}\sigma_g = |\{\varphi = s\}|_g$$

has a finite limit as $s \to \varphi_0 = -\infty$. We will conclude the proof showing that in fact the g-area $|\{\varphi = s\}|_g$ or equivalently $|\{N = t\}|_g$ converges to $|\mathbb{S}^2|$. Let (z, x^I) for $I = 1, 2$ be the Isenberg-Moncrief Gaussian null coordinates near the degenerate horizon. Since by (2.4), $\partial_z N$ converges to $C > 0$, for t sufficiently close to 0 we can write $\{N = t\} = \{z = \phi(x^I)\}$ for some smooth function ϕ. From the expansion of N in (2.3) we have $\phi = \frac{t}{C} + \eta(\phi, x^I)$ with $\eta = O(z^2)$ and from (2.4) we deduce $\partial_z\eta = O(z)$ and $\partial_I\eta = o(z)$. As a consequence

$$\partial_I\phi = \partial_z\eta\, \partial_I\phi + \partial_I\eta = O(z)\, \partial_I\phi + o(z),$$

which implies $\partial_I\phi = o(z)$. Recalling the form of g_0 in (2.2) and the asymptotic behavior (2.3), we conclude that the induced metric on $\{N = t\} = \{z = \phi(x^1, x^2)\}$ is $g_0|_{\{N=t\}} = \frac{1}{C^2} g_{\mathbb{S}^2} + o(1) = \rho_0^2\, g_{\mathbb{S}^2} + o(1)$ as $z \to 0^+$. Since $g_0 = \rho^2\, g$, this proves the claim. $\square$

For the non-degenerate black hole case, we will need the following second application of the Bochner formula consisting of an application of a minimum principle argument in the spirit of [18, Proposition 3.3].

Proposition 6.5. *Let (M, g_0, N, Ψ) be an asymptotically flat electrostatic electro-vacuum system with mass parameter μ and with connected boundary ∂M. Let $\rho = \rho_-$, $\varphi = \varphi_-$, $g = g_-$ be defined as in (5.3), (5.5), (5.6), respectively, with respect to the parameters $m > q > 0$ determined by (4.3) and (5.46). Then,*

$$|\nabla\varphi|_g \leq 1 \tag{6.7}$$

holds on $\overline{M}$ and we have

$$m \leq \mu. \tag{6.8}$$

Furthermore, if equality holds in (6.7) at some point of $M \setminus \partial M$ then (M, g_0, N, Ψ) is isometric to the sub-extremal Reissner–Nordström system (2.12) of mass $\mu = m$ and charge q, including ∂M.

Proof. Consider the function $w := \frac{1}{\rho}(1 - |\nabla\varphi|_g^2)$. From (5.1), (5.5), and (4.6), we compute

$$\nabla w = -Nw\nabla\varphi - \frac{1}{\rho}\nabla|\nabla\varphi|_g^2$$

on M, from which, using (5.22), we obtain

$$\Delta_g w - \left(1 - \frac{3m}{\rho} + \frac{2q^2}{\rho^2}\right)\langle\nabla w|\nabla\varphi\rangle_g = -\frac{2}{\rho}|\nabla^2\varphi|_g^2 \leq 0$$

on M. Therefore, the minimum principle[7] applies, meaning that w takes its minimum value on[8] the boundary ∂M or at infinity. On the boundary, we have $|\nabla\varphi|_g = 1$ by our choice of m and q, hence $w = 0$ on ∂M. By (5.34) and (5.32), the quantity $|\nabla\varphi|_g$ is bounded at infinity, whereas $\frac{1}{\rho} \to 0$ and thus $w \to 0$ at infinity. It follows that we must have $w \geq 0$, or equivalently $|\nabla\varphi|_g \leq 1$, on $\overline{M}$. Appealing again to (5.34), this shows (6.8).

Furthermore, if $|\nabla\varphi|_g = 1$ (equivalently, $w = 0$) at some interior point, then the strong minimum principle[7] tells us that indeed $|\nabla\varphi|_g \equiv 1$ on $\overline{M}$ and thus in particular $\mu = m$ by (5.34). Substituting this into (5.22), we also get $|\nabla^2\varphi|_g \equiv 0$ in M. Then the same arguments given at the end of the proof of Proposition 6.1 demonstrate the remaining claims. $\square$

6.2. Proof of Theorem 3.1 for Non-Degenerate Horizons

In this section, we will give a proof of Theorem 3.1 for non-degenerate horizons. Recall that we already know that $m \leq \mu$ from Proposition 6.5. We will now proceed to prove the opposite inequality which will allow us to conclude rigidity in the non-degenerate horizon case. To show $m \geq \mu$, we start by recalling formula (5.21) for the mean curvature of a level set of φ, replacing $|\mathrm{D}N|$ by $|\nabla\varphi|_g$ via (5.10), and obtain

$$\frac{|\nabla\varphi|_g\,\mathrm{H}_g}{N} = \frac{\rho^4}{2(m\rho - q^2)}\left(\mathrm{R}^\Sigma + |\mathring{\mathrm{h}}|^2 - \frac{\mathrm{H}^2}{2}\right) - \frac{2m\rho - q^2}{m\rho - q^2}\,|\nabla\varphi|_g^2 \tag{6.9}$$

on any level set of φ in M. We now want to take the limit of this formula as $N \to 0$, that is, as we approach the horizon ∂M. Since m and q have been chosen so that $|\nabla\varphi|_g = 1$ on ∂M, recalling that non-degenerate horizons are totally geodesic with respect to g_0 (see Sect. 2), it is easy to compute from (6.9) that

$$\lim_{N\to 0}\frac{|\nabla\varphi|_g^\beta\,\mathrm{H}_g}{N} = \frac{\rho_0}{q\sqrt{k}}\left(\frac{\rho_0^2}{2}\,\mathrm{R}^\partial\,M - 1\right) \tag{6.10}$$

for every $\beta \geq 2$. Applying Proposition 6.1 and taking the limit as we approach ∂M, we find

$$\int_{\partial M}\left(\frac{\rho_0^2}{2}\,\mathrm{R}^{\partial M} - 1\right)\mathrm{d}\sigma \geq 0, \tag{6.11}$$

exploiting that $k > 0$, $q > 0$, and $\rho_0 > 0$. From the Gauss–Bonnet formula, we first deduce that ∂M is topologically spherical and then consequently obtain the area bound

$$|\partial M| \leq 4\pi\rho_0^2 = |\mathbb{S}^2|\rho_0^2, \tag{6.12}$$

where $|\partial M|$ denotes the area of ∂M with respect to g_0 and $|\mathbb{S}^2|$ denotes the area of $\mathbb{S}^2$ with respect to $g_{\mathbb{S}^2}$. Let us now exploit the g-harmonicity of φ. Integrating $\Delta_g\varphi = 0$ between two level sets of φ, we find out immediately that the function $F(s) := \int_{\{\varphi=s\}} |\nabla\varphi|_g\,\mathrm{d}\sigma_g$ is constant which in particular implies

$$\int_{\partial M} |\nabla\varphi|_g\,\mathrm{d}\sigma_g = \lim_{s\to\infty}\int_{\{\varphi=s\}} |\nabla\varphi|_g\,\mathrm{d}\sigma_g. \tag{6.13}$$

Recalling that $|\nabla\varphi|_g = 1$ and $\rho = \rho_0$ on ∂M, combining this identity with (6.12), we obtain

$$|\mathbb{S}^2| \geq \lim_{s\to\infty}\int_{\{\varphi=s\}} |\nabla\varphi|_g\,\mathrm{d}\sigma_g. \tag{6.14}$$

On the other hand, from the asymptotic behaviors (5.34) and (5.39), it follows that

$$\lim_{s\to\infty}\int_{\{\varphi=s\}}|\nabla\varphi|_g\,\mathrm{d}\sigma_g=\frac{\mu}{m}\,|\mathbb{S}^2|. \tag{6.15}$$

Substituting this into (6.14), we find $m\geq\mu$ as desired.

Combining this with (6.8), we conclude that $m=\mu$. It follows that equality must hold in all inequalities in this subsection. In particular, (6.11) must be an identity and, through the limit statement in Proposition 6.1, this triggers the rigidity statement of Proposition 6.5, allowing us to conclude that (M,g_0,N,Ψ) is isometric to the sub-extremal Reissner–Nordström system (2.12) of mass $\mu=m$ and charge q. This proves the non-degenerate horizon case of Theorem 3.1. $\square$

7. Equipot. Photon Surface Uniqueness in Electro-Vacuum

This section is dedicated to the proof of Theorem 3.2, split into several different cases, closely related to if and where $dN=0$ in the different Reissner–Nordström spacetimes discussed at the end of Sect. 2. Let m and $q>0$ be chosen as in Sect. 5.5. We will first discuss the case where the critical set $\mathfrak{S}=\{N^2+k=0\}$ is empty and $N_0\leq 1$ in Sect. 7.1. This part covers all previously known equipotential photon surface uniqueness results, see Sect. 1. In particular, our analysis will characterize all—necessarily non-degenerate (see Sect. 4.3)— time-slices of equipotential photon surfaces in the sub-extremal ($m>q>0$) and the extremal case ($m=q>0$) as well as all super-extremal ones with $q>m>$0 and $\rho_0>\frac{q^2}{m}$, where ρ_0 denotes the value of the pseudo-radial function ρ on ∂M. Its proof resembles the one given for non-degenerate black holes in Sect. 6.2.

Using slight modifications of the same strategy, we will then treat the cases $\mathfrak{S}=\emptyset$ and $N_0>1$ in Sect. 7.2 and $N_0=\sqrt{-k}$ (and thus $\mathfrak{S}\neq\emptyset$) in Sect. 7.3. This will characterize all time-slices of equipotential photon surfaces in the super-extremal case with $m\leq 0<q$ and with $q>m>0$ and $\rho_0=\frac{q^2}{m}$, respectively. As discussed in Sect. 4.3, the first ones are necessarily (locally) non-degenerate while the second one is necessarily degenerate.

Finally, after a careful analysis of the critical set $\mathfrak{S}$ and its complement when $\mathfrak{S}\subset(M\setminus\partial M)$ in Sect. 7.4, we will treat the case $\mathfrak{S}\neq\emptyset$ and $N_0>\sqrt{-k}$ in Sect. 7.6. This proof deviates the most from the black hole case and uses an approach via monotonicity formulas derived in Sect. 7.5. This will characterize all — again necessarily (locally) non-degenerate (see Sect. 4.3)—time-slices of equipotential photon surfaces in the super-extremal case with $q>m>0$ and $\rho_0<\frac{q^2}{m}$.

Note that $\mathfrak{S}\neq\emptyset$ implies $k<0$ by positivity of N in the equipotential photon surface case, thus the condition $N_0\geq\sqrt{-k}$ is well defined; moreover, by (4.6) and (4.3), $N_0<\sqrt{-k}$ is not possible. Hence, our analysis covers all cases of Theorem 3.2.

As announced in Sect. 2, we will denote by ν the unit normal to the boundary ∂M pointing toward the asymptotically flat end. Notice that ν may coincide with the normal $\mathrm{n} = \frac{\mathrm{D}N}{|\mathrm{D}N|}$ introduced in Sect. 5.2 or may point in the opposite direction.

7.1. Proof of Theorem 3.2 for $\mathfrak{S} = \emptyset$ and $N_0 \leq 1$

Recalling $\Psi_0 > 0$ and using $N_0 \leq 1$, this immediately asserts that $m > 0$ and hence $\mu > 0$ by Lemma 4.2. Next, the assumption $\mathfrak{S} = \emptyset$ ensures that problem (4.5) is well posed in M. In particular, the second equation in (4.5) gives that N satisfies the maximum and minimum principle[7] in M. Since $N \to 1$ at infinity, it follows immediately that $N_0 \neq 1$, otherwise we would have $N \equiv 1$ in M so that, since Ψ is a function of N and $\Psi \to 0$ at infinity, this would imply that Ψ is trivial, which is against our hypothesis $\Psi_0 > 0$. We conclude that $N_0 < 1$ and, by the maximum and minimum principle[7] and the Hopf lemma[5], we have $N_0 < N < 1$ in M and $\nu(N) > 0$ on ∂M or equivalently $\nu = \mathrm{n}$ on ∂M. Moreover, $N \neq 1$ on M so that ρ_- is well defined and smooth on M by Lemma 5.2, using again our assumption $\mathfrak{S} = \emptyset$. This allows us to choose the pseudo-radial function ρ_- from (5.4) as in the black hole case treated in Sect. 6, with $\varphi = \varphi_-$ and $g = g_-$ as in (5.5), (5.6), respectively. Finally, $\mathfrak{S} = \emptyset$ implies that ∂M is non-degenerate by the discussion in Sect. 4.3 so that the properties of time-slices of non-degenerate equipotential photon surfaces listed on pp. 5 apply.

It is now easy to see that the arguments given in Sect. 6.1 work in this framework as well. Proceeding exactly in the same way as in the proof of Proposition 6.1 and exploiting that m and q satisfy (5.46) establishes (6.1) for $\partial M = \{\varphi = \varphi_0\}$ and any $\beta \geq 2$, with equality if and only if (M, g_0, N, Ψ) is isometric to a suitable piece of the Reissner–Nordström system of mass $m = \mu$ and charge q, with N and Ψ corresponding to the respective Reissner–Nordström quantities, including the boundary ∂M. Similarly, one can adapt the proof of Proposition 6.5 to find that $|\nabla\varphi|_g \leq 1$ in M and $m \leq \mu$.

To prove the opposite inequality $m \geq \mu$ and trigger equality in (6.1), we proceed slightly differently than in Sect. 6.2. Since we know that the mean curvature H of a time-slice of a non-degenerate equipotential photon surface is constant from pp. 5, we deduce from (5.10) and (5.18) that H_g is also constant on ∂M. Thus, (6.1) gives $\mathrm{H}_g \geq 0$ on ∂M. This inequality gives us a couple of important consequences. First of all, from (5.46) and the relation (5.18) by $\nu = \mathrm{n}$ between $\mathrm{H} = \mathrm{H}_\nu$ and H_g, we observe that

$$m\rho_0 - q^2 = \frac{q\sqrt{N_0^2 + k}\,(m + q\sqrt{N_0^2 + k})}{1 - N_0^2} > 0, \tag{7.1}$$

where ρ_0 is the value of ρ on ∂M. From this, we deduce that

$$\mathrm{H}_\nu \geq \frac{2N_0\rho_0^2}{m\rho_0 - q^2}\,|\mathrm{D}N| = \frac{2N_0}{\rho_0} \tag{7.2}$$

so that in particular $\mathrm{H}_\nu > 0$ (compare the discussion after (2.8), recalling $\nu = \mathrm{n}$ in this case). On the other hand, recalling umbilicity of ∂M from pp. 5, combining $\mathrm{H}_g \geq 0$ with (5.4), (5.21), (5.46), and (7.2), we get

$$\mathrm{R}^{\partial M} \geq \frac{\mathrm{H}_\nu^2}{2} + \frac{2\rho_0^2(2m\rho_0 - q^2)}{(m\rho_0 - q^2)^2}\,|\mathrm{D}N|^2 \geq \frac{2N_0^2}{\rho_0^2} + \frac{2(2m\rho_0 - q^2)}{\rho_0^4} = \frac{2}{\rho_0^2}. \tag{7.3}$$

Integrating this over ∂M and using the Gauss–Bonnet formula, we obtain

$$4\pi\chi(\partial M) = \int_{\partial M} \mathrm{R}^{\partial M}\,\mathrm{d}\sigma \geq \frac{2|\partial M|}{\rho_0^2}. \tag{7.4}$$

It follows that the Euler characteristic of ∂M is positive, meaning that ∂M is diffeomorphic to a sphere. Hence $\chi(\partial M) = 2$ and $|\partial M| \leq 4\pi\rho_0^2$. In other words, we have recovered the area bound (6.12) and we can now conclude the proof exactly as in Sect. 6.2; namely we combine (6.13) with (6.15) to get $|\partial M| = \frac{\mu}{m}\,|\mathbb{S}^2|$, from which it follows that $\mu \leq m$ as desired. Also, this triggers rigidity in (7.3) and thus allows us to conclude as in the non-degenerate black hole case treated in Sect. 6.2. Moreover, we learned from (7.1) that $\rho_0 > \frac{q^2}{m}$ which imposes the anticipated restriction on ρ_0 when $q > m > 0$ and no restriction on ρ_0 when $m \geq q > 0$ as $\rho_0 > r_{m,q} > \frac{q^2}{m}$ in that case.

7.2. Proof of Theorem 3.2 for $\mathfrak{S} = \emptyset$ and $N_0 > 1$

Again, we start by observing that, since the critical set $\mathfrak{S}$ is empty, the second equation in (4.5) is a well-posed elliptic equation for N on the whole manifold M. Since $N \to 1$ at infinity, we conclude by the maximum and minimum principle[7] and the Hopf lemma[5] that $1 < N < N_0$ holds on M and $\nu(N) < 0$ or in other words $\nu = -\mathrm{n} = \mathrm{n}_g$ holds on ∂M. Next, as usual, let $m \in \mathbb{R}$, $q > 0$ be chosen as in Sect. 5.5. Since $N > 1$ by the above and $\Psi > 0$ in M by Proposition 4.1, we deduce from(4.4) that $\Psi > \frac{2m}{q}$ in M. Since $\Psi \to 0$ at infinity, we conclude that $m \leq 0$ as expected. Since $m \leq 0$, Lemma 5.1 and 5.2 suggest to use the pseudo-radial function ρ_+ from (5.4), with $\varphi = \varphi_+$ and $g = g_+$ from (5.5) and (5.6), respectively. Moreover, $\mathfrak{S} = \emptyset$ implies that ∂M is non-degenerate by the discussion in Sect. 4.3 so that the properties of time-slices of non-degenerate equipotential photon surfaces listed on pp. 5 apply.

We plan to apply the strategy from Sect. 6.1 and then conclude in a way similar to Sect. 7.1. We will first discuss the argument for $m < 0$ and then treat the case $m = 0$ in Sect. 7.2.2; note that the latter case needs to be handled separately because of its different asymptotic behavior.

7.2.1. The Sub-Case $m < 0$. Recalling that $\mu < 0$ follows from $m < 0$ by Lemma 4.2, it is easy to see that the arguments given in Sect. 6.1 work in this framework as well. In particular, (6.1) holds on ∂M for all $\beta \geq 2$ with the identical rigidity assertion as we have chosen m and q as in (5.46). Again, an adaptation of the proof of Proposition 6.5 allows us to find that $|\nabla\varphi|_g \leq 1$ in $\overline{M}$. Exploiting (5.34), this gives $m \geq \mu$ because $\mu < 0$.

prove the opposite inequality $m \le \mu$, we proceed as in Sect. 7.1. Since we know that the mean curvature H of a time-slice of a non-degenerate equipotential photon surface is constant from pp. 5, we deduce from (5.10) and (5.18) that H_g is also constant on ∂M. Thus, (6.1) gives $\mathrm{H}_g \ge 0$ on ∂M also here. Since clearly $m\rho_0 - q^2 < 0$ in this case, we deduce from (5.18) that $\mathrm{H} < 0$ on ∂M. As $\mathrm{n} = -\nu$, this shows that the mean curvature $\mathrm{H}_\nu = -\mathrm{H}$ with respect to the normal ν pointing toward the asymptotically flat end is again positive (compare the discussion after (2.8)). More precisely, using

$$1 = |\nabla\varphi|_g = -\frac{\rho_0^3}{m\rho_0 - q^2}\,|\mathrm{D}N| \tag{7.5}$$

on ∂M, (5.18) gives

$$\mathrm{H}_\nu \ge -\frac{2N_0\rho_0^2}{m\rho_0 - q^2}|\mathrm{D}N| = \frac{2N_0}{\rho_0}, \tag{7.6}$$

whereas from (5.21) and (7.5), we get

$$\mathrm{R}^{\partial M} \le \frac{\mathrm{H}_\nu^2}{2} + \frac{2(2m\rho_0 - q^2)}{\rho_0^4}, \tag{7.7}$$

using again the properties of time-slices of non-degenerate equipotential photon surfaces discussed on pp. 5. This time, combining the scalar curvature estimate (7.7) with the Gauss–Bonnet formula is inconclusive. Instead, we will exploit the photon surface constraint (2.5) in the form of (2.7) and (2.8), recalling that we have already established $\mathrm{H}_\nu > 0$. Rewriting these conditions using (5.46), (7.5), $\nu(N) = -|\mathrm{D}N|$, and (4.8), we find

$$\mathrm{R}^{\partial M} = \frac{2q^2}{\rho_0^4} + \left(c + \frac{1}{2}\right)\mathrm{H}_\nu^2, \tag{7.8}$$

$$c\,\mathrm{H}_\nu = \frac{2(m\rho_0 - q^2)}{N_0\rho_0^3}. \tag{7.9}$$

From (7.6) and (7.9), we then deduce

$$\frac{m\rho_0 - q^2}{N_0^2\rho_0^2} \le c < 0. \tag{7.10}$$

Computing H_ν from (7.9) and substituting it into (7.8), we get by (5.1) that

$$\begin{aligned}\frac{c^2N_0^2\rho_0^6}{2}\,\mathrm{R}^{\partial M} &= c^2N_0^2\rho_0^2q^2 + (2c+1)(m\rho_0 - q^2)^2\\ &= q^2\left[\frac{N_0^2\rho_0^2c - (m\rho_0 - q^2)}{N_0\rho_0}\right]^2 + \rho_0(m\rho_0 - q^2)^2\left[\frac{2mc}{m\rho_0 - q^2} + \frac{\rho_0 - 2m}{N_0^2\rho_0^2}\right].\end{aligned}$$

Recalling that m and $m\rho_0 - q^2$ are both negative, we can then apply (7.10) in the term in the last square bracket above to deduce

$$\mathrm{R}^{\partial M} \ge \frac{2(m\rho_0 - q^2)^2}{N_0^4\rho_0^6c^2} \ge \frac{2}{\rho_0^2}, \tag{7.11}$$

where in the second inequality we have again used (7.10). We now proceed as in Sect. 7.1, namely integrating this bound on the scalar curvature and using the Gauss–Bonnet formula to obtain (7.4) and hence deduce the area bound (6.12). As in Sect. 6.2, we conclude via g-harmonicity of φ that $\frac{\mu}{m} \leq 1$ which leads to $m \leq \mu$ as $\mu < 0$ which, using $m \geq \mu$ from above, triggers equality in (7.11). This allows us to conclude as in Sect. 6.2 for $m < 0$. $\square$

7.2.2. The Sub-Case $m = 0$. By utilizing the asymptotic information provided in Sect. 5.4.2 combined with (5.15), it is straightforward to see that (6.1) holds on ∂M for all $\beta \geq 2$ with equality if and only if (M, g_0, N, Ψ) is isometric to a suitable piece of the Reissner–Nordström system (2.12) with mass $m = \mu = 0$ and charge $\kappa = q$, appealing to the choice of q made in (5.46). Proceeding as in Sect. 7.1, we can now apply (the proof of) Proposition 6.5 to find that $|\nabla\varphi|_g \leq 1$ in $\overline{M}$ and thus $q \leq \kappa$ by (5.42), see Lemma 4.3. Arguing as in Sect. 7.2.1, we again find the scalar curvature bound (7.11) on ∂M and thus the area bound (6.12). As in Sect. 6.2, using (5.45), we conclude via g-harmonicity of φ that $\kappa \leq q$ which, using $\kappa \geq q$ from above, triggers equality in (7.11). This allows us to conclude as in Sect. 6.2 for $m = 0$. $\square$

7.3. Proof of Theorem 3.2 for $N_0 = \sqrt{-k}$

From $N_0 = \sqrt{-k}$ we find $\partial M \subseteq \mathfrak{S}$ by definition of $\mathfrak{S}$ as well as $\Psi_0 = \frac{m}{q}$ by (4.7) so that $0 < m < q$ follows from our assumption $\Psi_0 > 0$. We choose m and q as specified in (5.49). Next, from (4.8), we find that ∂M is degenerate (as a time-slice of an equipotential photon surface). By Sect. 4.3, we know that the properties of time-slices of equipotential photon surface listed on pp. 5 hold also in this case.

We recall from Sect. 5.5 that $\mathfrak{S} = \partial M$ so that the equations in (4.5) are well defined everywhere in the interior of M as in the degenerate horizon case. Moreover, by the maximum principle[7] applied to the second equation in (2.1), exploiting $N_0 = \sqrt{-k} < 1$ and the asymptotic assumption $N \to 1$ at infinity, ensures that $\sqrt{-k} < N < 1$ in $M \setminus \partial M$ and in particular $N \neq 1$ in M. Hence we can use the pseudo-radial function ρ_- from (5.4) also in this case, as well as $\varphi = \varphi_-$ and $g = g_-$ from (5.5), (5.6), respectively, and note that they are well defined and smooth on $M \setminus \partial M$ by Lemma 5.2. Consequently, using $m > 0$ and thus $\mu > 0$ by Lemma 4.2, the asymptotic analysis from Sect. 5.4 also applies here.

Now recall that we have shown in Sect. 5.5 that $|\mathrm{D}\Psi| > 0$ on ∂M. As $N > \sqrt{-k}$ by the above, this tells us that level sets $\{N = N_0 + \varepsilon\}$ are regular by (4.8) for suitably small $\varepsilon > 0$. Also, they are connected and converge to ∂M as $\varepsilon \to 0^+$. Hence we can apply (the proof of) Proposition 6.1 to obtain (6.1) on $\{N = N_0 + \varepsilon\}$ for all $\beta \geq 2$ and all suitably small $\varepsilon > 0$, with the usual equality assertion, but not necessarily having $m = \mu$ as we cannot yet appeal to (5.49). Moreover, using (5.49), the proof of Proposition 6.5 can be adjusted[7] to give $|\nabla\varphi|_g \leq 1$ on M and $m \leq \mu$.

To prove the opposite inequality $m \geq \mu$, we proceed similarly as in Sect. 7.2.1. First, we extend (6.1) to ∂M: To see that the left hand side in (6.1) converges as $\varepsilon \to 0^+$, we apply (5.18) and (4.8) to get

$$\mathrm{H}_g\,|\nabla\varphi|_g = \frac{\rho^3}{q}\left[\frac{\mathrm{H}\,|\mathrm{D}\Psi|}{N} - \frac{2\rho\,|\mathrm{D}\Psi|^2}{qN}\right] \tag{7.12}$$

on $\{N = N_0 + \varepsilon\}$ for $\varepsilon > 0$ sufficiently small. Using (5.38) and (5.49), the right hand side of (7.12) converges to a well-defined constant as $\varepsilon \to 0^+$ as ρ is continuous up to ∂M. Noting (4.7), we find that $\rho = \frac{q}{\Psi}$ is in fact smooth up to ∂M, hence this convergence is uniform by the mean value theorem. Hence, applying (6.1) in the limit $\varepsilon \to 0^+$, we obtain

$$\mathrm{H} \geq \frac{2m\sqrt{-k}}{q^2} > 0 \tag{7.13}$$

on ∂M. In particular, we have $\mathrm{H} = \mathrm{H}_\nu > 0$ as $N > \sqrt{-k} = N_0$ and hence $\nu = \mathrm{n}$ (compare the discussion after (2.8)). Thus, from (2.8) and $\nu(N) = 0$ we find $c = 0$ and thus

$$\mathrm{R}^{\partial M} \geq \frac{2m^2}{q^4} = \frac{2}{\rho_0^2} \tag{7.14}$$

by (2.7), (5.49), and (5.38). Arguing as in Sect. 7.2.1 and using smoothness of ρ up to ∂M, we again find the area bound (6.12). Using g-harmonicity of φ as before, this leads to $m \geq \mu$ and thus in combination with $m \leq \mu$ triggers rigidity in (7.14). By smoothness of ρ up to ∂M, this isometry extends to ∂M. $\square$

7.4. The Structure of $\mathfrak{S}$

In order to conclude the proof of Theorem 3.2, it remains to consider the case where $\mathfrak{S} \neq \emptyset$ and $N_0 > \sqrt{-k}$, so that $\partial M \cap \mathfrak{S} = \emptyset$. As discussed in Sect. 7.3, it follows from $\mathfrak{S} \neq \emptyset$ that $q > m > 0$. Moreover, it follows from (4.8) and the definition of the critical set $\mathfrak{S}$ that the gradient of N must vanish on $\mathfrak{S}$ and that $\sqrt{-k}$ is in fact necessarily the minimum value attained by N in M. As a consequence, we have to be more careful when applying the maximum and minimum principle to the second equation in (4.5). In order to prove Theorem 3.2 in this setting, we first need to gain a better understanding of the structure of the set $\mathfrak{S} = \{N = \sqrt{-k}\}$. This is the purpose of this subsection.

First of all, we recall from [67] that N is a real analytic function on M. In particular, since $\mathfrak{S}$ is clearly compact, the Łojasiewicz Structure Theorem on the level sets of a real analytic function (established in [52], see also [48, Theorem 6.3.3]) guarantees us that $\mathfrak{S}$ admits a decomposition

$$\mathfrak{S} = \mathfrak{S}_0 \sqcup \mathfrak{S}_1 \sqcup \mathfrak{S}_2, \tag{7.15}$$

where $\mathfrak{S}_0$ is a finite collection of points, $\mathfrak{S}_1$ is a finite collection of real analytic curves and $\mathfrak{S}_2$ is a finite collection of real analytic surfaces. A point $p \in \mathfrak{S}$ belongs to $\mathfrak{S}_i$, $i = 0, 1, 2$, if i is the largest integer such that there exists a neighborhood Ω of p in M and a real analytic diffeomorphism $\phi : \Omega \to \mathbb{R}^3$ such that $\phi(\Omega \cap \mathfrak{S}_i) = L \cap \phi(\Omega)$ for some i-dimensional linear space L.

This structure of $\mathfrak{S}$ is crucial for the following result, which tells us that the set $\mathfrak{S}$ disconnects the manifold.

Lemma 7.1. *Let (M, g_0, N, Ψ) be an asymptotically flat electrostatic electro-vacuum system with connected boundary ∂M. Let $N = N_0$ and $\Psi = \Psi_0$ hold on ∂M for constants $N_0 > \sqrt{-k}$, $\Psi_0 > 0$ and assume that $\mathfrak{S} = \{N^2 + k = 0\} \neq \emptyset$.*

Then $\mathfrak{S}$ disconnects the manifold M into two pieces, one bounded inner piece *M_i containing ∂M and one* outer piece *M_o containing the asymptotically flat end. Furthermore, every connected component of $\mathfrak{S}$ intersects $\Gamma := \overline{M_i} \cap \overline{M_o} \subset \mathfrak{S}$ and, at any point $p \in \mathfrak{S}$, $|\mathrm{D}\Psi|(p) \neq 0$ if and only if $p \in \mathfrak{S}_2 \cap \Gamma$.*

Proof. First, recall that $q > m > 0$ follows from $\mathfrak{S} \neq \emptyset$. Then, for any choice of pseudo-radial function $\rho_\pm$, we compute

$$\varphi|_{\mathfrak{S}} = \log\left(\frac{q^2(1+\sqrt{-k})}{m} - m\right) =: \varphi_{\mathfrak{S}}. \tag{7.16}$$

Next, we note that $\varphi_{\mathfrak{S}}$ is in fact a locally minimal value of φ near $\mathfrak{S}$: if $\rho = \rho_-$, using $m > 0$ and $N > \sqrt{-k}$ away from $\mathfrak{S}$ readily gives $\varphi_- > \varphi_{\mathfrak{S}}$ on $M \setminus (\mathfrak{S} \cup \{N = 1\})$ (recalling Lemma 5.2). As $N = \sqrt{-k} \neq 1$ on $\mathfrak{S}$ by $m > 0$, $N \neq 1$ holds near $\mathfrak{S}$. If $\rho = \rho_+$, one finds from $m > 0$ that $\varphi_+ > \varphi_{\mathfrak{S}}$ in $\{\sqrt{-k} + \delta > N > \sqrt{-k}\}$ for suitably small $\delta > 0$. By compactness of $\mathfrak{S}$, there hence exists a bounded neighborhood Ω of $\mathfrak{S}$ with smooth boundary such that $\varphi_\pm > \varphi_{\mathfrak{S}}$ or in other words $\varphi > \varphi_{\mathfrak{S}}$ on $\overline{\Omega} \setminus \mathfrak{S}$.

Now suppose toward a contradiction that $\mathfrak{S} \neq \emptyset$ and $M \setminus \mathfrak{S}$ is connected. From (4.7), it then follows that $\Psi - \frac{m}{q}$ is either positive or negative on $M \setminus \mathfrak{S}$ (as it can only vanish on $\mathfrak{S}$) and thus its value $\frac{m}{q}$ on $\mathfrak{S}$ is either a maximum or a minimum value of Ψ in M. As a consequence, we have $|\mathrm{D}\Psi| = 0$ on $\mathfrak{S}$. Next, by (5.10) and (4.8), we get $|\nabla\varphi|_g^2 \to 0$ as we approach $\mathfrak{S}$, regardless of which choice of pseudo-radial function $\rho_\pm$ we use. We can now use the g-harmonicity of φ to get a contradiction as follows, again without specifying which choice of pseudo-radial function we make: consider a regular value s of φ satisfying $\varphi_{\mathfrak{S}} < s < \min_{\partial\Omega}(\varphi)$. Then, the set $\{\varphi \leq s\} \cap \overline{\Omega}$ is compact with smooth boundary $\{\varphi = s\} \cap \overline{\Omega}$. Given any small neighborhood $U \subset \Omega$ of $\mathfrak{S}$ with smooth boundary ∂U, integrating the identity $\Delta_g \varphi = 0$ in $(\{\varphi \leq s\} \setminus U) \cap \Omega$ and using the divergence theorem, we obtain

$$\int_{\partial U} \langle \nabla\varphi \,|\, \mathrm{n}_g \rangle_g \, \mathrm{d}\sigma_g = \int_{\{\varphi = s\} \cap \Omega} |\nabla\varphi|_g \, \mathrm{d}\sigma_g. \tag{7.17}$$

Since we have shown that $|\nabla\varphi|_g$ goes to zero as we approach $\mathfrak{S}$, the left hand side of the above identity can be made as small as we want by choosing a small enough open set U (and appealing to the Cauchy–Schwarz inequality). It follows that the right hand side must be equal to zero. Since this is true for any regular value s of φ close enough to φ_0 and as regular values of φ are dense by Sard's lemma, it follows that $|\nabla\varphi|_g$ vanishes identically in $\Omega \setminus \mathfrak{S}$, contradicting that $\varphi > \varphi_0$ in $\Omega \setminus \mathfrak{S}$, $\varphi = \varphi_{\mathfrak{S}}$ on $\mathfrak{S}$, and continuity of φ on Ω. This gives the desired contradiction, so it follows that $\mathfrak{S}$ must disconnect M into more than one connected component.

We now proceed to showing that there are exactly two connected components M_i and M_o of $M \setminus \mathfrak{S}$ as in the claim of Lemma 7.1. As $N \to 1$ at infinity and $\sqrt{-k} < 1$, it is clear that the asymptotically flat end of M is contained in one connected component M_o which is then non-compact. Next, as ∂M is connected and $N_0 > \sqrt{-k}$, ∂M is also entirely contained in one connected component of $M \setminus \mathfrak{S}$; this could a priori also be the connected component M_o. Recall that $\mathfrak{S}$ separates M into at least two components, hence there exists a connected component M_i of $M \setminus \mathfrak{S}$ such that $M_i \neq M_o$. If we had $M_i \cap \partial M = \emptyset$, then M_i would be bounded with $\partial M_i \subseteq \mathfrak{S}$ which leads to a contradiction via the maximum principle applied to the second equation in (4.5) as $N > \sqrt{-k}$ in M_i and $N = \sqrt{-k}$ on $\mathfrak{S}$. Thus M_i must contain ∂M, and the same argument by contradiction shows that M_i, M_o must be the only connected components of $M \setminus \mathfrak{S}$.

We will now establish that every connected component of $\mathfrak{S}$ must intersect $\Gamma = \overline{M_i} \cap \overline{M_o} \subset \mathfrak{S}$. In fact, if there were some other connected component C of $\mathfrak{S}$ inside $\overline{M_i}$ or $\overline{M_o}$ disconnected from Γ, i.e., satisfying $C \cap \Gamma = \emptyset$, we would get a contradiction by applying the arguments explained at the beginning of the proof. Namely, we show that $|\mathrm{D}\Psi| = 0$ on C as $\Psi - \frac{m}{q}$ has a sign on M_i and M_o and vanishes on C and hence $|\nabla \varphi|_g = 0$ on C; we then apply the divergence theorem to the integral of $\Delta_g \varphi$ in a small neighborhood U of C to prove that $|\nabla \varphi|_g$ vanishes identically in U, contradicting that $\varphi_{\mathfrak{S}}$ is a locally minimal value near $\mathfrak{S}$.

Finally, let $p \in \Gamma$. If $p \in \mathfrak{S}_2 \cap \Gamma$, then p belongs to the smooth portion of the boundary of $\overline{M_i}$. The boundary of $\overline{M_i}$ is the union of ∂M, where $\Psi = \Psi_0 \neq \frac{m}{q}$ by (4.7), and of Γ, where $\Psi = \frac{m}{q}$. Then the Hopf lemma, applied in M_i to the third equation in (2.1) (which is strongly elliptic as $N \geq N_0 > \sqrt{-k}$ in $\overline{M_i}$), tells us that $|\mathrm{D}\Psi|(p) \neq 0$. It remains to show that $|\mathrm{D}\Psi|(p) = 0$ if $p \notin \mathfrak{S}_2 \cap \Gamma$. Assuming this were not the case, we would get that $\Delta N|_p \neq 0$ from the second equation in (2.1) which gives the desired contradiction by [14, Theorem 3.3]. □

Thanks to Lemma 7.1, applying the Hopf lemma[5] to the second equation in (4.5) on M_i, we immediately deduce that $\nu(N) < 0$ and hence $\nu = -\mathrm{n}$ on ∂M.

7.5. A Monotonicity Argument

We have seen in the previous subsection that if $\mathfrak{S} \neq \emptyset$ and $N_0 > \sqrt{-k}$, $\mathfrak{S}$ disconnects the manifold M into two pieces, an inner one M_i containing ∂M and an outer one M_o containing the asymptotically flat end. Since $N \neq \sqrt{-k}$ away from $\mathfrak{S}$, the second equation in (4.5) is well posed in both M_i and M_o. It follows that the maximum and minimum principle[7] apply to N in $M_o \setminus U$ and $M_i \setminus U$, where U is any small neighborhood of $\mathfrak{S}$. Since the neighborhood U can be taken to be arbitrarily small and N is continuous on M, recalling that $N \to 1$ at infinity, we deduce that

$$\begin{aligned} \sqrt{-k} < N < N_0 \ \text{inside}\, M_i, \\ \sqrt{-k} < N < 1 \quad \text{inside}\, M_o. \end{aligned} \tag{7.18}$$

Let us now perform our conformal change in both M_i and M_o. In M_o, we use the pseudo-radial function ρ_- defined by (5.4) which is well defined there as $N \neq 1$ by (7.18), whereas in M_i, we use the pseudo-radial function ρ_+ defined by (5.4). By definition, $\rho_+ = \rho_- = \frac{q^2}{m}$ holds on $\mathfrak{S}$; hence, the combined function

$$\rho := \begin{cases} \rho_+ & \text{in } \overline{M_i}, \\ \rho_- & \text{in } \overline{M_o}, \end{cases} \tag{7.19}$$

is well defined and continuous in M and smooth in $M \setminus \mathfrak{S}$ by Lemma 5.1 and 5.2. The same naturally applies to the combined function φ and metric g. However, a priori there is no reason to expect higher regularity. In particular, it is hard to exploit the maximum principle argument described in Proposition 6.5 in the present situation. Instead, we will rely on an alternative argument, more in line with [6], finding functions that are monotonic on the level sets of φ in both the inner region M_i and the outer region M_o, and then showing that the monotonicity formulas in the two regions combine appropriately.

First of all, let us observe that Proposition 6.1 is in force in this setting as well.

Proposition 7.2. *In the setting of Lemma 7.1, let ρ be defined by (5.4) in M_o and by (5.3) in M_i, with respect to parameters m, q satisfying (4.3) and with asymptotic mass parameter μ. Finally, let φ, g be defined by (5.5) and (5.6), respectively. Then, for any $\beta \geq 2$ and any $s < \infty$ such that $\{\varphi = s\} \subset M$, we have*

$$\int_{\{\varphi=s\}} \mathrm{H}_g \, |\nabla \varphi|_g^\beta \, \mathrm{d}\sigma_g \geq 0. \tag{7.20}$$

Furthermore, if equality holds in (7.20) for some s and β, the same rigidity statement as in Proposition 6.1 applies.

Proof. Giving the same arguments used at the beginning of the proof of Proposition 6.1, we see that $\{\varphi \geq s\}$ is connected and contains the infinity. If $\{\varphi \geq s\}$ does not contain $\mathfrak{S}$ then $\{\varphi \geq s\} \subseteq M_o$. We can then proceed as in Proposition 6.1 to obtain the desired inequality immediately, recalling that we use $\rho = \rho_-$ in M_o. This concludes the proof for those values s for which $\{\varphi \geq s\} \subseteq M_o$. In particular,

$$\lim_{s \to \varphi_{\mathfrak{S}}^+} \int_{\{\varphi=s\}} \mathrm{H}_g \, |\nabla \varphi|_g^\beta \, \mathrm{d}\sigma_g \geq 0,$$

where $\varphi_{\mathfrak{S}}$ is given by (7.16), provided this limit exists. We know from Lemma 7.1 that $|\mathrm{D}\Psi| = 0$ on $\mathfrak{S} \setminus (\mathfrak{S}_2 \cap \Gamma)$ and $|\mathrm{D}\Psi| \neq 0$ on $\mathfrak{S}_2 \cap \Gamma$. By (5.10) and (4.8), we conclude that $|\nabla \varphi|_g = 0$ on $\mathfrak{S} \setminus (\mathfrak{S}_2 \cap \Gamma)$ and $|\nabla \varphi|_g \neq 0$ on $\mathfrak{S}_2 \cap \Gamma$. As $\mathfrak{S}_2 \cap \Gamma$ is dense in Γ by definition of $\mathfrak{S}_2$, we deduce that

$$\lim_{s \to \varphi_{\mathfrak{S}}^+} \int_{\{\varphi=s\}} \mathrm{H}_g \, |\nabla \varphi|_g^\beta \, \mathrm{d}\sigma_g = \int_\Gamma \mathrm{H}_g \, |\nabla \varphi|_g^\beta \, \mathrm{d}\sigma_g \geq 0 \tag{7.21}$$

converges as H_g is defined almost everywhere on a critical level set (see the end of Sect. 5.2) and because g is continuous across $\mathfrak{S}$.

It remains to consider the case where $\{\varphi = s\} \subset M_i$, in which case $\{\varphi \geq s\} \supset \mathfrak{S}$. In M_i, we can still proceed as in the proof of Proposition 6.1, integrating (5.25) and using the divergence theorem, obtaining (6.3) for any $s < S < \varphi_{\mathfrak{S}}$. Taking the limit as $\{\varphi = S\}$ approaches the level set Γ, this time we obtain

$$\int_{\{\varphi=s\}} \frac{1}{\rho N} \, \mathrm{H}_g \, |\nabla\varphi|_g^\beta \, \mathrm{d}\sigma_g \geq \int_\Gamma \frac{1}{\rho N} \, \mathrm{H}_g \, |\nabla\varphi|_g^\beta \, \mathrm{d}\sigma_g. \tag{7.22}$$

Now, recall that inequalities (7.21) and (7.22) are computed with respect to different choices of the pseudo-radial function ρ. Since the combined ρ, g, φ are continuous along and across $\mathfrak{S}$, in order to combine the two inequalities we only need to check that the integrand $\mathrm{H}_g \, |\nabla\varphi|_g^\beta$ has the same value on Γ from both sides. By (5.10), (4.8), and the fact that $\rho_\pm = \frac{q^2}{m}$ when $N_0 = \sqrt{-k}$, we find that $|\nabla\varphi|_g$ coincides on Γ from both sides. Moreover, arguing as around (7.12), we find that $\mathrm{H}_g \, |\nabla\varphi|_g$ and thus H_g coincide on Γ from both sides, keeping in mind that H is computed with respect to $\nu = \mathrm{n}$. Hence, the estimates in (7.21) and (7.22) can be combined, leading to (7.20). The rigidity statement is proven as usual, first in M_o and subsequently extending to M_i if necessary. $\square$

Let us now discuss a first consequence of Proposition 7.2. Consider the function

$$F_\beta(s) := \int_{\{\varphi=s\}} |\nabla\varphi|_g^{\beta+1} \, \mathrm{d}\sigma_g \tag{7.23}$$

for all s in the range of φ. As before, integrating $\Delta_g \varphi = 0$ between two level sets of φ, we find out immediately that F_0 is constant, which in particular implies

$$\int_{\partial M} |\nabla\varphi|_g \, \mathrm{d}\sigma_g = \lim_{s\to\infty} \int_{\{\varphi=s\}} |\nabla\varphi|_g \, \mathrm{d}\sigma_g. \tag{7.24}$$

In the same spirit, from Proposition 7.2, we obtain the following:

Proposition 7.3. *In the setting of Lemma 7.1, let ρ be defined by (5.4) in M_o and by (5.3) in M_i, with respect to parameters m, q satisfying (4.3) and asymptotic mass parameter μ. Finally, let φ, g be defined by (5.5) and (5.6), respectively. Then, for any $\beta \geq 2$, the function F_β is locally absolutely continuous, with weak derivative*

$$F_\beta'(s) = -\beta \int_{\{\varphi=s\}} \mathrm{H}_g \, |\nabla\varphi|_g^\beta \, \mathrm{d}\sigma_g \leq 0 \tag{7.25}$$

for all s in the range of φ. Furthermore, if equality holds in (7.25) for some s and β, the same rigidity statement as in Proposition 6.1 applies.

Remark 7.4. With some more work, following the steps in [6, Theorem 3.2], one can prove more, namely that F_β is actually continuous and differentiable. However, Proposition 7.3 is enough for our purposes.

Proof. The proof follows the strategy from [11, Proposition 3.5]. We start by looking at $|\nabla\varphi|_g$. We know that $|\nabla\varphi|_g$ is bounded at infinity because of the asymptotic flatness assumption. Furthermore, $|\nabla\varphi|_g$ is a smooth function away from $\mathfrak{S}$ and it is bounded on $\mathfrak{S}$ thanks to (5.10) and (4.8). It follows easily from these observations that $|\nabla\varphi|_g$ is uniformly bounded, namely, there exists $C > 0$ such that $|\nabla\varphi|_g \leq C$ on M. As a consequence, for any regular value s of φ, we have

$$F_\beta(s) = \int_{\{\varphi=s\}} |\nabla\varphi|_g^{\beta+1}\,\mathrm{d}\sigma_g \leq C^\beta \int_{\{\varphi=s\}} |\nabla\varphi|_g\,\mathrm{d}\sigma_g = C^\beta F_0(s). \tag{7.26}$$

Since F_0 is constant, it follows that, for every $\beta \geq 0$, the function $F_\beta(s)$ is essentially uniformly bounded, namely there exists a constant K such that $F_\beta(s) \leq K$ for almost every value s (in fact we have proven that this holds for any regular value s, and these are dense because of Sard's lemma). This enables us to employ the coarea formula, a tool that will be crucial in the proof, also thanks to [11, Remark 2.2], using the fact that $\nabla|\nabla\varphi|_g^\beta$ vanishes at the critical points of φ for any $\beta \geq 2$.

Next, let $a, b \in \mathbb{R}$ be constants with $a < b < \varphi_\mathfrak{S}$ or $\varphi_\mathfrak{S} < a < b$, where $\varphi_\mathfrak{S}$ again denotes the value of φ on $\mathfrak{S}$. Let also $\eta \in \mathcal{C}_c^\infty(a,b)$. By means of the coarea formula, we then compute

$$\begin{aligned}
\int_a^b \eta'(s) F_\beta(s)\,\mathrm{d}s &= \int_a^b \eta'(s) \int_{\{\varphi=s\}} |\nabla\varphi|_g^{\beta+1}\,\mathrm{d}\sigma_g\,\mathrm{d}s \\
&= \int_{\{a<\varphi<b\}} \langle \nabla(\eta(\varphi)) \,|\, \nabla\varphi\rangle_g\, |\nabla\varphi|_g^\beta\,\mathrm{d}V_g,
\end{aligned}$$

where $\mathrm{d}V_g$ denotes the volume element on M induced by g. Integrating by parts, recalling $\Delta_g\varphi = 0$, and applying again the coarea formula, we get

$$\begin{aligned}
\int_a^b \eta'(s) F_\beta(s)\,\mathrm{d}s &= -\int_{\{a<\varphi<b\}} \eta(\varphi) \big\langle \nabla|\nabla\varphi|_g^\beta \,\big|\, \nabla\varphi \big\rangle_g\,\mathrm{d}V_g \\
&= -\int_a^b \eta(s) \int_{\{\varphi=s\}} \left\langle \nabla|\nabla\varphi|_g^\beta \,\middle|\, \frac{\nabla\varphi}{|\nabla\varphi|_g} \right\rangle_g \mathrm{d}\sigma_g\,\mathrm{d}s.
\end{aligned}$$

On the other hand, if $a < \varphi_\mathfrak{S} < b$, one obtains the same formula by working separately in $(a, \varphi_\mathfrak{S})$ and $(\varphi_\mathfrak{S}, b)$ and then summing the two integrals (recall that $\mathfrak{S} = \{\varphi = \varphi_\mathfrak{S}\}$ has zero volume). It follows that F_β belongs to $W^{1,1}_{\mathrm{loc}}$, with weak derivative satisfying

$$F'_\beta(s) = \int_{\{\varphi=s\}} \left\langle \nabla|\nabla\varphi|_g^\beta \,\middle|\, \frac{\nabla\varphi}{|\nabla\varphi|_g} \right\rangle_g \mathrm{d}\sigma_g. \tag{7.27}$$

Recalling (5.15), the claim follows. □

As a consequence of the previous result, since F_β is locally absolutely continuous and its weak derivative is non-positive, it follows that F_β is monotonically decreasing. In particular, we have

$$\int_{\partial M} |\nabla\varphi|_g^{\beta+1}\,\mathrm{d}\sigma_g \geq \lim_{s\to\infty}\int_{\{\varphi=s\}} |\nabla\varphi|_g^{\beta+1}\,\mathrm{d}\sigma_g. \tag{7.28}$$

7.6. Proof of Theorem 3.2 with $\mathfrak{S}\neq\emptyset$ and $N_0>\sqrt{-k}$

We are now in the position to exploit the monotonicity of the functions F_β defined in (7.23), exploiting (7.5) and (5.46). Recalling (7.24), (7.28), we deduce

$$\frac{|\partial M|}{\rho_0^2} = \lim_{s\to\infty}\int_{\{\varphi=s\}} |\nabla\varphi|_g\,\mathrm{d}\sigma_g, \tag{7.29}$$

$$\frac{|\partial M|}{\rho_0^2} \geq \lim_{s\to\infty}\int_{\{\varphi=s\}} |\nabla\varphi|_g^{\beta+1}\,\mathrm{d}\sigma_g \tag{7.30}$$

for $\beta\geq 2$. Now we exploit asymptotic flatness. Clearly the expansions presented in Sect. 5.4 work in the current framework as $m>0$ and hence $\mu>0$ by Lemma 4.2. In particular, from (5.34) and (5.39), it follows that

$$\lim_{s\to\infty}\int_{\{\varphi=s\}} |\nabla\varphi|_g\,\mathrm{d}\sigma_g = \frac{\mu}{m}\,|\mathbb{S}^2|.$$

$$\lim_{s\to\infty}\int_{\{\varphi=s\}} |\nabla\varphi|_g^{\beta+1}\,\mathrm{d}\sigma_g = \left(\frac{m}{\mu}\right)^{\beta-1}|\mathbb{S}^2|$$

for $\beta\geq 2$. Substituting this into (7.29) and (7.30), we obtain

$$\frac{1}{\rho_0^2}|\partial M| = \frac{\mu}{m}\,|\mathbb{S}^2|, \tag{7.31}$$

$$\frac{1}{\rho_0^2}|\partial M| \geq \left(\frac{m}{\mu}\right)^{\beta-1}|\mathbb{S}^2| \tag{7.32}$$

for $\beta\geq 2$. Combining (7.31) and (7.32) with $m>0$ and $\mu>0$, we then immediately deduce $m\leq\mu$. As usual, we now proceed to showing that the opposite inequality is also in place, thus proving $m=\mu$, from which the result follows.

Since H_g is constant on ∂M, it follows from Proposition 7.2 that $\mathrm{H}_g\geq 0$ on ∂M. Next,

$$\rho_0 = \frac{q^2}{m+q\sqrt{N_0^2+k}} < \frac{q^2}{m}, \tag{7.33}$$

holds as expected, hence in particular $m\rho_0-q^2<0$. Recalling that $\mathrm{H}=-\mathrm{H}_\nu$ as $\nu=-\mathrm{n}$ was established at the end of Sect. 7.4, it follows as in Sect. 7.2.1 that (7.6), (7.7), (7.8), and (7.9) hold on ∂M. In particular, $\mathrm{H}_\nu>0$ (compare the discussion after (2.8)). This time however, the parameter m is positive and this changes the argument. Namely, from (7.6) and (7.9), we find

$$\frac{m\rho_0-q^2}{N_0^2\rho_0^2}\leq c<0. \tag{7.34}$$

Computing H_ν from (7.9) and substituting it into (7.8), using also (5.1), we get

$$\begin{aligned}\frac{c^2 N_0^2 \rho_0^6}{2}\,\mathrm{R}^{\partial M} &= c^2 N_0^2 \rho_0^2 q^2 + (2c+1)(m\rho_0 - q^2)^2 \\ &= -kc^2 \rho_0^2 q^2 + (c+1)^2 (m\rho_0 - q^2)^2. \end{aligned} \tag{7.35}$$

To proceed to estimate $\mathrm{R}^{\partial M}$ from below by $\frac{2}{\rho_0^2}$ as before, we now appeal to the assumption that $N_0^2 \geq |1-\Psi_0^2|$ holds if $N_0^2 > (1-|\Psi_0|)^2$, noting that $N_0^2 > (1-|\Psi_0|)^2$ follows from $\Psi_0 > 0$ and $0 < m < q$ via (4.3). Using first $N_0^2 \geq 1 - \Psi_0^2$ and again (4.3), we find that $\Psi_0 \geq \frac{m}{q}$. From this, (5.3), (4.7), we find $\Psi_0 = \frac{q}{\rho_0}$ and thus, using (4.3) to express $\frac{m}{q}$ in terms of N_0 and Ψ_0, we compute

$$1 - \frac{m}{\rho_0} = 1 - \frac{m}{q}\Psi_0 = 1 - \frac{\Psi_0^2 + 1 - N_0^2}{2} = \frac{1 + N_0^2 - \Psi_0^2}{2} \geq 0, \tag{7.36}$$

where we have used the assumption $N_0^2 \geq \Psi_0^2 - 1$ in the last inequality. This tells us that $\rho_0 \geq m$ and hence

$$c + 1 \geq \frac{\rho_0 - m}{N_0^2 \rho_0} \geq 0 \tag{7.37}$$

by (7.37) and (5.1). Hence dividing (7.35) by c^2 and using (7.34), (7.37), and (5.1), we get

$$\frac{\rho_0^2}{2}\,\mathrm{R}^{\partial M} \geq -\frac{kq^2}{N_0^2 \rho_0^2} + \frac{(\rho_0 - m)^2}{N_0^2 \rho_0^2} = 1 \tag{7.38}$$

as desired. From here on, one concludes as usual by integrating and using the Gauss–Bonnet formula as in Sect. 7.1. This concludes the proof of Theorem 3.2. □

7.7. On the Degenerate Horizon Case

In this subsection, we give an alternative proof via the monotonicity argument of the degenerate horizon case, that does not require assuming (6.4).

First of all, we recall (see (5.52)) that $\rho_0 = m = q = \frac{1}{C}$, where ρ_0 is the value of the pseudo-radial function ρ at the boundary and C is the constant appearing in the asymptotic decays (2.3). For convenience, we set

$$|\partial M| := \lim_{s \to -\infty} |\{\varphi = s\}|, \tag{7.39}$$

where $|\cdot|$ denotes the area with respect to g_0. Arguing as at the end of the proof of Corollary 6.4, we deduce that the areas of the level sets of φ and z converge to the same value and in particular

$$|\partial M| = |\mathbb{S}^2| \rho_0^2. \tag{7.40}$$

We now recall that Proposition 6.1 applies to the degenerate case as well. Following Proposition 7.3, this tells us that the function F_β, defined as usual by (7.23), is monotonically non-increasing for $\beta \geq 2$. Furthermore, the rigidity statement in Proposition 7.3 carries through to the degenerate horizon case without modifications: in fact, if $F_\beta(s_1) = F_\beta(s_2)$ for some $s_1 < s_2$, then it would follow that (6.1) holds with equality for all s between s_1 and s_2, thus triggering rigidity.

Since we have chosen φ so that $|\nabla\varphi|_g \to 1$, we have $\int_{\{\varphi=s\}} |\nabla\varphi|_g^{\beta+1} \mathrm{d}\sigma_g \to |\partial M|/\rho_0^2$ as $s \to -\infty$, where $\beta \geq 0$. From this fact, the monotonicity of F_β for $\beta \geq 2$ and the constancy of F_0, it follows that (7.31) and (7.32) are in force also in this case. Furthermore, equality holds in (7.32) if and only if the whole solution is isometric to a Reissner–Nordström solution. We conclude by plugging (7.40) into (7.31) and (7.32), which immediately gives $m = \mu$ and triggers equality in (7.32).

8. Discussion

8.1. On the Technical Condition in Theorem 3.2

Theorem 3.2 assumes the technical condition that $N_0^2 \geq |1 - \Psi_0^2|$ holds if $N_0^2 > (1-|\Psi_0|)^2$. We have used this condition in Sect. 7.6 to ensure that $\rho_0 \geq m$ and thus $\mathrm{R}^{\partial M} \geq \frac{2}{\rho_0^2}$ in the case when $\mathfrak{S} \neq \emptyset$ and $N_0 > \sqrt{-k}$. Let us first point out that this technical condition does not pose any additional restrictions in any of the other cases we have discussed: first of all, $N_0^2 > (1 - |\Psi_0|)^2$ is equivalent to asking that $m < q$ by (4.4) and the assumption $\Psi_0 > 0$ and hence only occurs in the super-extremal case (as we have assumed $q > 0$). Next, $N_0^2 \geq |1 - \Psi_0^2|$ poses no restriction when $m \leq 0$ as can be seen when converting it to $\Psi_0 \geq \frac{m}{q}$ and $1 \geq \frac{m}{q}\Psi_0$ by (4.4) (and recalling $\Psi_0 > 0$). Thus the only case when the technical condition possibly adds a restriction is when $0 < m < q$. On the other hand, when $N_0 = \sqrt{-k}$, we have $\Psi_0 = \frac{m}{q}$ by (4.7) and thus automatically $N_0^2 = |1 - \Psi_0^2|$. Furthermore, when $\mathfrak{S} = \emptyset$, we already know that $\rho \geq \frac{q^2}{m} > m$. From (4.4) and (5.1), we compute that either $\Psi = \frac{q}{\rho}$ or $\Psi = \frac{m}{q} - \frac{q}{\rho}$, with the latter contradicting our asymptotic assumption that $\Psi \to 0$ at infinity and the asymptotic assertion that $\rho \to \infty$ at infinity from Lemma 5.2, using again $m > 0$. Hence $N_0^2 \geq |1 - \Psi_0^2|$ holds automatically when $q > m > 0$ and $\mathfrak{S} = \emptyset$.

Now, let us discuss why we need to rely on this technical condition in Sect. 7.6. To close the argument, we would like to use the lower bound on c derived in (7.34) (which is effectively a lower bound on H_ν) in order to prove that $\mathrm{R}^{\partial M} \geq \frac{2}{\rho_0^2}$. Appealing to (7.35), this requires estimating $f(c) := \frac{(c+1)^2}{c^2}$ from below by $f(c_*)$, using that $c < 0$ and that $c \geq c_*$ for $c_* = \frac{m\rho_0 - q^2}{N_0^2\rho_0^2}$ by (7.34). Considering that $f'(c) = -\frac{2(c+1)}{c^3}$ is strictly decreasing for $c < -1$, has a global minimum at $c = -1$, and is strictly increasing on $(-1, 0)$, we can only expect to get the estimate $f(c) \geq f(c_*)$ when $c_* \geq -1$ which is equivalent to $\rho_0 \geq m$ by (5.1). We close this discussion by stating that the above issue is not visible in the exact Reissner–Nordström case with $q > m > 0$ and $\rho_0 < \frac{q^2}{m^2}$ because in that case the estimate (7.6) on the mean curvature, $\mathrm{H}_\nu \geq \frac{2N_0}{\rho_0}$ is sharp on every coordinate sphere and hence $c = c_*$ as well as $\mathrm{R}^{\partial M} = \frac{2}{\rho_0^2}$

hold for any coordinate sphere. In other words, because the estimate $c \geq c_*$ is sharp in this case, it does not matter that f is strictly decreasing rather than increasing for $c < -1$ (or $\rho_0 < m$).

8.2. Comparison of Notions of Sub-/Super-/Extremality

In Sect. 1, we have compared our uniqueness results to other uniqueness results in the literature. In particular, when doing so for equipotential photon surfaces (and in particular for photon spheres), we used the words "sub-/super-/extremal" as if they were universally defined across the various results in the literature; however, this is not a priori the case, as most authors give their own definitions adapted to their strategy of proof. In this section, we will hence compare the various notions of sub-/super-/extremality for equipotential photon surfaces, restricting to the setting where all results apply, i.e., assuming that the electrostatic electro-vacuum system (M, g_0, N, Ψ) has *connected, non-degenerate* equipotential photon surface boundary ∂M with $\Psi_0 > 0$ and that it is *asymptotic to Reissner–Nordström*, meaning that, outside some compact set and in suitable coordinates, we have

$$\begin{gathered} (g_0)_{ij} = \left(1 + \frac{2\mu}{|x|}\right)\delta_{ij} + O_2(|x|^{-2}), \quad N = 1 - \frac{\mu}{|x|} + O_2(|x|^{-2}), \\ \Psi = \frac{\kappa}{|x|} + O_2(|x|^{-2}) \end{gathered} \tag{8.1}$$

for some $\mu, \kappa \in \mathbb{R}$ as $|x| \to \infty$. We will use the notation from our paper also for the other approaches in order to increase legibility. First, recall that we say in this paper that (M, g_0, N, Ψ) is sub-extremal/extremal/super-extremal if $m > q/m = q \neq 0/q > m$, respectively, with $q > 0$ implied by $\Psi_0 > 0$ and where $\frac{m}{q}$ is fixed by (4.3). Using the stronger asymptotic decay assumptions (8.1) and the relationship between N and Ψ established in (4.4), one immediately sees that $\frac{m}{q} = \frac{\mu}{\kappa}$, hence our definitions of extremality align with a definition based on the asymptotic mass μ and charge κ.

Let us begin our comparison by studying notions of sub-/super-/extremality for photon spheres. In [75, Definition 4.1] by Yazadjiev–Lazov, a photon sphere is *non-extremal* if

$$\frac{\mathrm{H}_\nu^2\,|\partial M|}{4\pi} \neq 1. \tag{8.2}$$

It is then asserted that this equivalent to $\mu \neq \kappa$, where we are using here that $\kappa > 0$ by our assumption $\Psi_0 > 0$ as shown in Sect. 7.2.2. Their uniqueness result [75, Theorem 4.1] then pertains to non-extremal photon spheres as in (8.2), with the notion of non-extremality aligned with the one used in this paper. Arguing somewhat differently, it is first established in [24, Lemma 2.6] (see also [36, Theorem 3.1]) that $\mathrm{H}_\nu > 0$. In [24, Definition 2.9] by Cederbaum–Galloway, a photon sphere is defined to be sub-/super-/extremal when

$$\mathrm{H}_\nu\, r_0 > 1 \;/\; \mathrm{H}_\nu\, r_0 < 1 \;/\; \mathrm{H}_\nu\, r_0 = 1, \tag{8.3}$$

respectively, where $r_0 > 0$ denotes the *area radius* defined by $|\partial M| =: 4\pi r_0^2$. It is then shown in the proof of [24, Theorem 3.1] that these definitions are equivalent to asking that $m_0 > q_0/m_0 = q_0/q_0 > m_0$, with $m_0 > 0$, $q_0 > 0$ defined as

$$q_0 = -\frac{\nu(\Psi)|_{\partial M}\, r_0^2}{N_0}, \tag{8.4}$$

$$m_0 = \frac{r_0}{3} + \frac{2q_0^2}{3r_0}, \tag{8.5}$$

respectively, where we are using that $\Psi_0 > 0$ and $\Psi \to 0$ at infinity imply $\nu(\Psi)|_{\partial M} > 0$ by applying the Hopf lemma[5] to the last equation in (2.1). Note that positive mass is to be expected from the Reissner–Nordström systems when a photon sphere exists. Furthermore, in the case of connected ∂M, it is shown in [24, Appendix A] (extending the corresponding argument from [75]) that these definitions are equivalent to those based on the asymptotic quantities μ and κ and hence to our definitions. In particular, recall that $q_0 = \kappa$ by (4.20), so consequently $m_0 = \mu$ by (4.4).

Next, in [44] by Jahns, which mostly aims at an extension to higher dimensions (but presented here only for 4-dimensions), a photon sphere is called sub-/super-/extremal if

$$\frac{\mathrm{H}_\nu^2}{\mathrm{R}^{\partial M}} > \frac{1}{2} \;/\; \frac{\mathrm{H}_\nu^2}{\mathrm{R}^{\partial M}} < \frac{1}{2} / \; \frac{\mathrm{H}_\nu^2}{\mathrm{R}^{\partial M}} = \frac{1}{2} \tag{8.6}$$

respectively. To see that this is well defined, note that it is asserted in [24, Proposition 2.4] (see also [21,75]) that the photon surface identity (2.8) holds with $c = 1$, relying again on the assertion that $\mathrm{H}_\nu > 0$. Thus, in particular ∂M has constant positive scalar curvature $\mathrm{R}^{\partial M} > 0$ by (2.7). Using again the above assertion that $\mathrm{H}_\nu > 0$ and the fact that $\mathrm{R}^{\partial M} = \frac{8\pi}{|\partial M|} = \frac{2}{r_0^2}$ by the Gauss–Bonnet formula, this turns out to be equivalent to (8.3), and thus equivalent to our definitions. This is also addressed in the proof of [44, Theorem 3] via quasi-local mass and charge quantities as in (8.4), (8.5), except that [44] uses a different sign on q_0 (which does not affect the result as we know that Ψ and $-\Psi$ can be interchanged). Finally, in [28, Proposition and Definition 5.12] by Cederbaum–Jahns–Vičánek Martínez, using again that $\mathrm{R}^{\partial M} > 0$ for photon spheres, the quasi-local charge q_0 as in (8.4) is also used, replacing the area radius r_0 by the scalar curvature radius $\overline{r}_0 > 0$ defined by $\mathrm{R}^{\partial M} = \frac{2}{\overline{r}_0^2}$ (again, in view of higher dimensions). Yet by the Gauss-Bonnet formula we know that $r_0 = \overline{r}_0$ here. Assuming that $r_0 > q_0$ with $q_0 > 0$ by the above, a quasi-local mass $\overline{m}_0 > 0$ is then defined as the unique solution of

$$\frac{|\nu(N)|_{\partial M}|}{N_0}\sqrt{1 - \frac{2\overline{m}_0}{r_0} + \frac{q_0^2}{r_0^2}} = \frac{\overline{m}_0}{r_0^2} - \frac{q_0^2}{r_0^3} \tag{8.7}$$

(in view of addressing equipotential photon surfaces, see below). The reason why $r_0 > q_0$ is assumed is similar to why we need to assume $N_0^2 \geq |1 - \Psi_0^2|$ or in other words $\rho_0 \geq m$, see Sect. 8.1 and [28, Remark 5.13]. As $\mathrm{H}_\nu > 0$, it follows from the photon surface constraint (2.8) with $c = 1$ that $\nu(N)|_{\partial M} > 0$. Plugging in (8.5) into (8.7) shows that $\overline{m}_0 = m_0$. The parameters q_0 and $\overline{m}_0 = m_0$ are then used to define sub-/super-/extremality as usual, so the definitions coincide with ours—provided that $r_0 > q_0$. As [28] only addresses uniqueness in the sub-extremal case where $r_0 > m_0 + \sqrt{m_0^2 - q_0^2} > q_0$, this does not cause any restrictions.

Let us now move on to general equipotential photon surfaces, where we need to compare our notions of sub-/super-/extremality with those used by Cederbaum–Jahns–Vičánek Martínez in [28]. As we have already discussed above, we will assume that the equipotential photon surface under consideration is *not* a photon sphere. This will allow us to appeal to the refined analysis for equipotential photon surfaces performed in [28, Section 5.2]. First, note that [28] only discusses sub-/super-/extremality under the additional assumption $\mathrm{H}_\nu\, \nu(N) > 0$, hence we will also assume this in our discussion. By [24, Lemma 2.6] (see also [36, Theorem 3.1]), we know that $\mathrm{H}_\nu > 0$, so this is equivalent to assuming $\nu(N) > 0$ or in other words $N_0 \leq 1$ and $\mathfrak{S} = \emptyset$ in our approach. Hence, this only excludes some super-extremal cases, according to our definitions. Now, [28, Theorem and Definition 5.16] establishes that a (one-sided) tubular neighborhood U of ∂M in (M, g_0, N, Ψ) must be isometric to a suitable piece of the spatial factor of a generalized Reissner–Nordström spacetime of mass $\widetilde{m}$, charge $\widetilde{q}$, parameter $\lambda \in \{\pm 1, 0\}$, and Einstein manifold base (Σ_*, σ_*) with scalar curvature $\mathrm{R}^{\Sigma_*} = 2\lambda$. Using $\Psi_0 > 0$, this means that

$$g_0 = \frac{1}{N^2_{\lambda,\widetilde{m},\widetilde{q}}} dr^2 + r^2 \sigma_*, \qquad N = \alpha N_{\lambda,\widetilde{m},\widetilde{q}}, \qquad \Psi = \alpha \Psi_{\lambda,\widetilde{m},\widetilde{q}} + \beta \tag{8.8}$$

on U up to isometry, for some constants $\alpha > 0$, $\beta \in \mathbb{R}$, where

$$N_{\lambda,\widetilde{m},\widetilde{q}} = \sqrt{\lambda - \frac{2\widetilde{m}}{r} + \frac{\widetilde{q}^2}{r^2}}, \qquad \Psi_{\lambda,\widetilde{m},\widetilde{q}} = \frac{\widetilde{q}}{r}. \tag{8.9}$$

By [28, Lemma 5.20], $\mathrm{H}_\nu\, \nu(N) > 0$ excludes $\lambda = 0, -1$. On the other hand, using $\lambda = 1$, the uniformization theorem informs us that (Σ_*, σ_*) is a round sphere, so that in particular the area radius r_0 coincides with the scalar curvature radius, moreover, [28, Proposition and Definition 5.12] applies. Using this and observing that (8.4) and (8.7) are invariant under the above rescaling by $\alpha > 0$ and $\beta \in \mathbb{R}$, one finds $\widetilde{q} = q_0$ and $\widetilde{m} = \overline{m}_0 = m_0$. When $\nu(N) > 0$, [28, Corollary 5.17] establishes that sub-/super-/extremality is then defined in [28] as usual via $\widetilde{m} = m_0$ and $\widetilde{q} = q_0$ showing coinciding notions of sub-/super-/extremality, recalling (4.4) and $\frac{m}{q} = \frac{\mu}{\kappa}$ by the above.

We conclude that our notions of sub-/super-/extremality coincide with those existing in the literature (whenever those apply).

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UNIQUENESS OF STATIC VACUUM ASYMPTOTICALLY FLAT BLACK HOLES AND EQUIPOTENTIAL PHOTON SURFACES IN $n+1$ DIMENSIONS À LA ROBINSON

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Abstract. In this paper, we combine and generalize to higher dimensions the approaches to proving the uniqueness of connected $(3+1)$-dimensional static vacuum asymptotically flat black hole spacetimes by Müller zum Hagen–Robinson–Seifert and by Robinson. Applying these techniques, we prove and/or reprove geometric inequalities for connected $(n+1)$-dimensional static vacuum asymptotically flat spacetimes with either black hole or equipotential photon surface or specifically photon sphere inner boundary. In particular, assuming a natural upper bound on the total scalar curvature of the boundary, we recover and extend the well-known uniqueness results for such black hole and equipotential photon surface spacetimes. We also relate our results and proofs to existing results, in particular to those by Agostiniani–Mazzieri and by Nozawa–Shiromizu–Izumi–Yamada.

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1. Introduction and results

Black holes are among the most intriguing objects in nature and have captured the attention of researchers since Schwarzschild provided the first non-trivial solution of Einstein's equation of general relativity. Their properties and shape have since and continue to be thoroughly investigated. In the static case, it is well-established [Isr67, MzHRS73, Rob77, BMuA87, Mia05, AM17, Rau21] that the black hole solution found by Schwarzschild constitutes the only $3+1$-dimensional asymptotically flat static vacuum spacetime with an (a priori possibly disconnected) black hole horizon arising as its inner boundary. This fact is known as "static vacuum black hole uniqueness"; it also goes by the pictorial statement that "static vacuum black holes have no hair". We refer the interested reader to the reviews [Heu96, Rob12] for more information.

In the higher dimensional case with spacetime dimension $n+1 \geq 3+1$, the analogous fact has also been asserted [Hwa98, GIS02, Ced, Ced17, Rau21, AM17, NSIY18]; however, all proofs make extra assumptions. The proofs by Hwang [Hwa98] and by Gibbons, Ida, and Shiromizu [GIS02] extend the method by Bunting and Masood-ul-Alam [BMuA87] allowing to deal with possibly disconnected horizons (see [Ced, Ced17] for a more general

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version of this approach, and see [HI12] for a review of related results). These proofs rely on the rigidity case of the positive mass theorem and hence currently[1] make a spin assumption (using Witten's Dirac operator approach [Wit81]) or impose an upper bound of $n+1 \leq 7+1$ on the spacetime dimension (using the minimal hypersurface approach by Schoen and Yau [SY79a, SY79b]). Building on ideas by Walter Simon, Raulot [Rau21] exploits spinor techniques and thus explicitly makes a spin assumption. Instead, the proof by Agostiniani and Mazzieri [AM17] via potential theory, monotone functions, and a (conformal) splitting theorem assumes connectedness of the horizon as well as an upper bound on the total scalar curvature of the (time-slice) of the horizon, see also Section 7.2. Nozawa, Shiromizu, Izumi, and Yamada [NSIY18] derive the same statement as [AM17] by a combination and generalization to higher dimensions of the divergence theorem based methods by Müller zum Hagen, Robinson, and Seifert [MzHRS73] and by Robinson [Rob77], see Section 7.3 for more details. Related results were recently presented in [HW24, Med24].

The first main goal of this paper is to give a rigorous new proof of static vacuum black hole uniqueness under the same geometric assumptions as Agostiniani–Mazzieri [AM17, Theorem 2.8], but allowing for weaker decay assumptions, see Theorem 1.1 and Remark 2.3. Moreover, we reproduce all geometric inequalities for connected horizons proved in [AM17, Theorem 2.8], extend them to a wider class of parameters, and identify a concrete relationship between our method and the approach taken in [AM17], see Section 7. We do so by combining, extending, and generalizing to higher dimensions the approaches by Müller zum Hagen, Robinson, and Seifert [MzHRS73] and by Robinson [Rob77]. Our proof is rather similar to the derivation of the same statement by Nozawa, Shiromizu, Izumi, and Yamada [NSIY18, Section 5] but allowing for weaker decay assumptions as well as filling in subtle analytic details, closing a gap in the uniqueness argument, and highlighting a connection to the analysis of Ricci solitons, see also Section 7.3.

Theorem 1.1 (Black Hole Uniqueness)**.** *Let (M^n, g, f) be an asymptotically flat static vacuum system of mass $m \in \mathbb{R}$ and dimension $n \geq 3$ with connected static horizon inner boundary ∂M. Let*

$$s_{\partial M} := \left(\frac{|\partial M|}{|\mathbb{S}^{n-1}|}\right)^{\frac{1}{n-1}} \tag{1.1}$$

denote the area radius of ∂M*, where $|\partial M|$ and $|\mathbb{S}^{n-1}|$ denote the surface area of $(\partial M, g_{\partial M})$ with respect to the induced metric $g_{\partial M}$ on ∂M and of $(\mathbb{S}^{n-1}, g_{\mathbb{S}^{n-1}})$, respectively. Then*

$$\frac{(s_{\partial M})^{n-2}}{2}\sqrt{\frac{\int_{\partial M} \mathrm{R}_{\partial M}\, dS}{(n-1)(n-2)\,|\mathbb{S}^{n-1}|\,(s_{\partial M})^{n-3}}} \geq m \geq \frac{(s_{\partial M})^{n-2}}{2}, \tag{1.2}$$

where $\mathrm{R}_{\partial M}$ and dS denote the scalar curvature and the hypersurface area element of ∂M with respect to $g_{\partial M}$, respectively. In particular, ∂M satisfies

$$\int_{\partial M} \mathrm{R}_{\partial M}\, dS \geq (n-1)(n-2)|\mathbb{S}^{n-1}|\,(s_{\partial M})^{n-3} \tag{1.3}$$

and (M, g, f) has positive mass $m > 0$.

[1]but see [SY17, Loh16]

Moreover, equality holds on either side of (1.2) *and/or in* (1.3) *if and only if* (M, g) *is isometric to the Schwarzschild manifold* (M^n_m, g_m) *of mass* m *and* f *corresponds to the Schwarzschild lapse function* f_m *under this isometry.*

Remark 1.2 (Black hole uniqueness follows from Theorem 1.1)**.** *The last statement gives the desired black hole uniqueness result subject to the scalar curvature bound condition*

$$\int_{\partial M} \mathrm{R}_{\partial M}\, dS \leq (n-1)(n-2)|\mathbb{S}^{n-1}|\,(s_{\partial M})^{n-3} \tag{1.4}$$

see also Remark 1.4. Theorem 1.1 implies several other interesting geometric inequalities such as a static version of the Riemannian Penrose inequality, see [AM17, MOS10, NSIY18] *for more information.*

Another recent direction of extending static vacuum black hole uniqueness results is to investigate uniqueness of spacetimes containing "photon spheres" (as introduced in [CVE01]) or, more generally, "photon surfaces" (as introduced in [CVE01, Per05]). Here, *photon surfaces* are timelike hypersurfaces of a spacetime which "capture" null geodesics; in static spacetimes, a photon surface is called *equipotential* if the lapse function along it "only depends on time", and called a *photon sphere* if the lapse function is (fully) constant along it (as introduced in [Ced14]), see Section 2.2 and the references given there for definitions and more information. Photon surfaces are relevant in gravitational lensing and in geometric optics, i.e., for trapping phenomena, and related to dynamical stability questions for black holes.

Photon spheres were first discovered in the $3 + 1$-dimensional Schwarzschild spacetimes of positive mass and persist in their higher dimensional analogs. (Equipotential) photon surfaces also naturally occur in Schwarzschild spacetimes of all dimensions and for all positive and negative masses, see [CG21, CJVM23]. (Asymptotically flat) static vacuum equipotential photon surface uniqueness is fully established in $3 + 1$ spacetime dimensions [Ced14, CG17, CG21, CCF24, Rau21]. In particular, [CG17, CG21, Rau21] allow for combinations of black hole horizons and equipotential photon surfaces, assuming that all equipotential photon surface components are "outward directed", meaning that they have "positive quasi-local mass", see Remark 2.14. In contrast, Cederbaum, Cogo, and Fehrenbach [CCF24] restrict to a connected, not necessarily outward directed equipotential photon surface, establishing uniqueness for the first time also in the negative and zero (total) mass cases. They generalize, exploit, and compare different techniques of proof, namely those from [Isr67, Ced14, AM17] and in particular Robinson's approach [Rob77].

In higher dimensions $n + 1 \geq 3 + 1$, the same result is established by Cederbaum and Galloway [CG21], building on work by Cederbaum [Ced, Ced17] which uses the positive mass theorem; hence the ensuing restrictions discussed above apply. Raulot's spinorial approach [Rau21] also covers higher dimensions, subject to a spin condition.

The *second main goal of this paper* is to demonstrate that the generalized divergence theorem based approach we derive can also be used to prove the expected uniqueness claim for connected equipotential photon surfaces, assuming the same upper bound on the total scalar curvature of the boundary as in the black hole case, see Theorem 1.3. This generalizes the $3+1$-dimensional extension of Robinson's approach to connected equipotential photon surfaces by [CCF24]. Moreover, we prove similar geometric inequalities for connected equipotential photon surfaces as for black holes. Last but not least, we include the negative (total) mass case which has so far only been addressed in [CCF24] in $3+1$ dimensions.

We do not address the zero mass case here. For $n = 3$, the zero mass case and its connection to the Willmore inequality is established in [CCF24]. In higher dimensions, this requires extra considerations and is still work in progress.

Theorem 1.3 (Equipotential Photon Surface Uniqueness)**.** *Let (M^n, g, f) be an asymptotically flat static vacuum system of mass $m \in \mathbb{R}$ and dimension $n \geq 3$ with connected boundary ∂M arising as a time-slice of an equipotential photon surface. Let $f_0 > 0$ denote the constant value of f on ∂M and assume that $f_0 \neq 1$. If $f_0 \in (0,1)$ then*

$$\frac{(1-f_0^2)\,(s_{\partial M})^{n-2}}{2}\sqrt{\frac{(s_{\partial M})^2\left(\mathrm{R}_{\partial M} - \frac{n-2}{n-1}H^2\right)}{(n-1)(n-2)(1-f_0^2)}} \geq m \geq \frac{(1-f_0^2)\,(s_{\partial M})^{n-2}}{2}. \tag{1.5}$$

Here, $\mathrm{R}_{\partial M}$, H, and $s_{\partial M}$ denote the scalar curvature, the mean curvature, and the surface area radius (1.1) *of ∂M with respect to the induced metric $g_{\partial M}$ on ∂M, respectively. In particular, ∂M satisfies*

$$\mathrm{R}_{\partial M} - \tfrac{n-2}{n-1}H^2 \geq \frac{(n-1)(n-2)(1-f_0^2)}{(s_{\partial M})^2} \tag{1.6}$$

and (M, g, f) has positive mass $m > 0$. If $f_0 \in (1, \infty)$ then

$$\frac{(1-f_0^2)\,(s_{\partial M})^{n-2}}{2}\sqrt{\frac{(s_{\partial M})^2\left(\mathrm{R}_{\partial M} - \frac{n-2}{n-1}H^2\right)}{(n-1)(n-2)(1-f_0^2)}} \leq m \leq \frac{(1-f_0^2)\,(s_{\partial M})^{n-2}}{2}. \tag{1.7}$$

In particular, ∂M satisfies

$$\mathrm{R}_{\partial M} - \tfrac{n-2}{n-1}H^2 \leq \frac{(n-1)(n-2)(1-f_0^2)}{(s_{\partial M})^2} \tag{1.8}$$

and (M, g, f) has negative mass $m < 0$. Moreover, for any $f_0 \in (0,1) \cup (1, \infty)$, if

$$\mathrm{R}_{\partial M} \leq \frac{(n-1)(n-2)}{(s_{\partial M})^2} \tag{1.9}$$

then (M, g) is isometric to the piece $[s_{\partial M}, \infty) \times \mathbb{S}^{n-1}$ of the Schwarzschild manifold (M^n_m, g_m) of mass m and f corresponds to the restriction of the Schwarzschild lapse function f_m to $[s_{\partial M}, \infty)$ under this isometry.

The last statement gives the desired equipotential photon surface uniqueness result subject to a scalar curvature bound condition, see also Remark 1.4. Theorem 1.3 implies several other interesting geometric inequalities, see [AM17] for more information.

Remark 1.4 (About Conditions (1.4) and (1.9))**.** *Note that* (1.4) *and* (1.9) *are equivalent in case $\mathrm{R}_{\partial M} = const$ (as it is the case for time-slices of equipotential photon surfaces, see Proposition 2.10). In dimension $n = 3$, conditions* (1.4) *and* (1.9) *are of course automatically satisfied by the Gauß–Bonnet theorem. Hence Theorem 1.1 gives static vacuum black hole uniqueness in* 3 *dimensions without extra assumptions (other than connectedness of the static horizon) and Theorem 1.3 gives static vacuum equipotential photon surface uniqueness in* 3 *dimensions without extra assumptions (other than connectedness of the photon surface), including the negative (total) mass case.*

To prove Theorems 1.1 and 1.3, we proceed as follows: First, in Section 5, we derive the following higher dimensional version of Robinson's identity [Rob77, Equation (2.3)], using the so-called T-*tensor* instead of the Cotton tensor C used by Robinson [Rob77]. We also introduce an additional parameter $p \in \mathbb{R}$ as a power into the identity, with $p = 3$ corresponding to Robinson's identity, and $p = \frac{3}{2}$ corresponding to the approach taken by Müller zum Hagen, Robinson, and Seifert [MzHRS73]. Similar parameters called p and c, respectively, were introduced in [AM17] and in [NSIY18]; we refer the reader to Section 7 for a discussion of the relation between the parameter p and its range and the parameters p and c from [AM17, NSIY18].

Theorem 1.5 (Generalized Robinson identity)**.** *Let (M^n, g, f), $n \geq 3$, be a static vacuum system with $0 < f < 1$ or $f > 1$ in M. Then, for all $c, d, p \in \mathbb{R}$, the* generalized Robinson identity

$$\begin{aligned} &\|\nabla f\|^2 \operatorname{div}\left(\frac{F(f)}{f}\|\nabla f\|^{p-3}\,\nabla\|\nabla f\|^2 + G(f)\|\nabla f\|^{p-1}\,\nabla f\right) \\ &= \|\nabla f\|^{p-3}\,F(f)\left[\frac{(n-2)^2 f}{(n-1)^2}\|T\|^2 + \frac{p-p_n}{2f}\left\|\nabla\|\nabla f\|^2 + \frac{4(n-1)}{(n-2)}\frac{f\|\nabla f\|^2\,\nabla f}{1-f^2}\right\|^2\right] \end{aligned} \tag{1.10}$$

holds on $M \setminus \operatorname{Crit} f$, with $\operatorname{Crit} f := \{q \in M \mid \|\nabla f\|_q = 0\}$ denoting the set of critical points of f and the constant p_n is given by

$$p_n := 2 - \frac{1}{n-1}. \tag{1.11}$$

Here, $F, G\colon [0,1) \cup (1,\infty) \to \mathbb{R}$ are given by

$$F(t) := \frac{ct^2 + d}{|1-t^2|^{\frac{(n-1)(p-1)}{n-2}-1}}, \tag{1.12}$$

$$G(t) := \frac{4\left(\frac{(n-1)(p-1)}{n-2} - 1\right)F(t)}{(p-1)(1-t^2)} - \frac{4c}{(p-1)|1-t^2|^{\frac{(n-1)(p-1)}{n-2}-1}} \tag{1.13}$$

for $t \in [0,1) \cup (1,\infty)$, $\|\cdot\|$, ∇, and div *denote the tensor norm, covariant derivative, and covariant divergence with respect to g. The tensor T is given by*

$$\begin{aligned} T(X,Y,Z) :=& \frac{n-1}{n-2}\left(\operatorname{Ric}(X,Z)\nabla_Y f - \operatorname{Ric}(Y,Z)\nabla_X f\right) \\ &- \frac{1}{n-2}\left(\operatorname{Ric}(X,\nabla f)g(Y,Z) - \operatorname{Ric}(Y,\nabla f)g(X,Z)\right), \end{aligned} \tag{1.14}$$

for $X, Y, Z \in \Gamma(TM)$, where Ric *denotes the Ricci curvature tensor of (M, g).*

Moreover, if $p \geq 3$, the divergence on the left hand side continuously extends to $\operatorname{Crit} f$ *and* (1.10) *holds on M. Furthermore, if $p \geq p_n$, it follows that*

$$\operatorname{div}\left(\frac{F(f)}{f}\|\nabla f\|^{p-3}\,\nabla\|\nabla f\|^2 + G(f)\|\nabla f\|^{p-1}\,\nabla f\right) \geq 0 \tag{1.15}$$

on $M \setminus \operatorname{Crit} f$ provided that $F(f) \geq 0$.

Theorem 1.5 reproduces Robinson's identity [Rob77, (2.3)] when $n = 3$, $p = 3$, and $0 < f < 1$, and its generalization to the negative mass case by Cederbaum, Cogo, and Fehrenbach [CCF24] when $n = 3$, $p = 3$, and $f > 1$. When $n = 3$, $p = p_2 = \frac{3}{2}$, and $0 < f < 1$, (1.10) is very closely related to the divergence identities derived by Müller zum Hagen, Robinson, and Seifert [MzHRS73].

Remark 1.6 (Generalizations)**.** *The divergence identity* (1.10) *may be of independent interest, allowing to prove geometric inequalities for more general boundary geometries than the level set boundaries we are interested in this work. As it is purely local, it may also be of use to prove related results in different asymptotic scenarios such as ALE spaces.*

The T-tensor introduced in (1.14) is specifically adapted to the geometry of static vacuum systems, see Section 3. As $\mathrm{R} = 0$ in static vacuum systems, it formally coincides[2] with the D-tensor introduced for the analysis and classification of Ricci solitons by Cao and Chen [CC12, CC13], inspired by Israel's [Isr67] and in particular Robinson's [Rob77] approaches to proving black hole uniqueness. Both the D-tensor and the T-tensor have seen many applications in classification problems for Ricci solitons and quasi-Einstein manifolds.

As the next step in proving Theorems 1.1 and 1.3, we will exploit Theorem 1.5 to prove some important geometric inequalities on ∂M. These inequalities can be stated in a parametric way (Theorem 1.7), or, equivalently, as two separate inequalities (Theorem 1.8). Both versions of the geometric inequalities and their equivalence will be proven in Section 6. The parametric geometric inequalities in Theorem 1.5 have also been established by Agostiniani and Mazzieri [AM17] for $p \geq 3$. To the best knowledge of the authors, they are new for $3 > p \geq p_n$.

Theorem 1.7 (Parametric geometric inequalities)**.** *Let (M^n, g, f), $n \geq 3$, be an asymptotically flat static vacuum system of mass $m \in \mathbb{R}$ with connected boundary ∂M. Assume that $f|_{\partial M} = f_0$ for a constant $f_0 \in [0,1) \cup (1,\infty)$ and that the normal derivative $\nu(f)|_{\partial M} =: \kappa$ is constant, with unit normal ν pointing towards the asymptotic end. Let F and G be as in Theorem 1.5 for some $p \geq p_n$ and some constants $c, d \in \mathbb{R}$ satisfying $c + d \geq 0$ and $cf_0^2 + d \geq 0$. Set $F_0 := F(f_0)$, $G_0 := G(f_0)$. Then*

$$F_0\, \kappa^{p-2} \int_{\partial M} \left(\mathrm{R}_{\partial M} - \tfrac{n-2}{n-1} H^2 + \|\mathring{h}\|^2 \right) dS - G_0\, \kappa^p |\partial M| \geq \mathcal{F}_p^{c,d}(m), \tag{1.16}$$

and $\kappa, m > 0$ if $f_0 \in [0,1)$ and

$$F_0\, |\kappa|^{p-2} \int_{\partial M} \left(\mathrm{R}_{\partial M} - \tfrac{n-2}{n-1} H^2 + \|\mathring{h}\|^2 \right) dS - G_0\, |\kappa|^p |\partial M| \leq -\mathcal{F}_p^{c,d}(m) \tag{1.17}$$

and $\kappa, m < 0$ if $f_0 \in (1,\infty)$. Here, $\mathrm{R}_{\partial M}$, H, $\mathring{h}$, and dS denote the scalar curvature, the mean curvature, the trace-free part of the second fundamental form, and the area element of ∂M, and $|\partial M|$ and $|\mathbb{S}^{n-1}|$ denote the area of $(\partial M, g_{\partial M})$ and of $(\mathbb{S}^{n-1}, g_{\mathbb{S}^{n-1}})$, respectively. The constant $\mathcal{F}_p^{c,d}(m) \in \mathbb{R}$ is given by

$$\mathcal{F}_p^{c,d}(m) := \frac{4(n-2)^p}{2^{\frac{(n-1)(p-1)}{n-2}}(p-1)} |\mathbb{S}^{n-1}|(c+d)|m|^{p-\frac{(n-1)(p-1)}{n-2}}. \tag{1.18}$$

[2] up to a factor $n-1$, and with a different function f

Unless $c = d = 0$, equality holds in (1.16) or in (1.17) if and only if (M, g) is isometric to a suitable piece of the Schwarzschild manifold (M^n_m, g_m) of mass m and f corresponds to the corresponding restriction of the Schwarzschild lapse function f_m under this isometry.

This can equivalently be expressed as follows.

Theorem 1.8 (Geometric inequalities)**.** *Let (M^n, g, f), $n \geq 3$, be an asymptotically flat static vacuum system of mass $m \in \mathbb{R}$ with connected boundary ∂M. Assume that $f|_{\partial M} = f_0$ for a constant $f_0 \in [0, 1) \cup (1, \infty)$ and that $\nu(f)|_{\partial M} =: \kappa$ is constant. Then*

$$\begin{aligned}\frac{(1-f_0^2)\,(s_{\partial M})^{n-2}}{2}\sqrt{\frac{\int_{\partial M}\left(\mathrm{R}_{\partial M}-\frac{n-2}{n-1}H^2+\|\mathring{h}\|^2\right)dS}{(n-1)(n-2)(1-f_0^2)|\mathbb{S}^{n-1}|\,(s_{\partial M})^{n-3}}}\\ \geq m \geq \frac{(1-f_0^2)\,(s_{\partial M})^{n-2}}{2}.\end{aligned} \tag{1.19}$$

holds if $f_0 \in [0, 1)$ and

$$\begin{aligned}\frac{(1-f_0^2)\,(s_{\partial M})^{n-2}}{2}\sqrt{\frac{\int_{\partial M}\left(\mathrm{R}_{\partial M}-\frac{n-2}{n-1}H^2+\|\mathring{h}\|^2\right)dS}{(n-1)(n-2)(1-f_0^2)|\mathbb{S}^{n-1}|\,(s_{\partial M})^{n-3}}}\\ \leq m \leq \frac{(1-f_0^2)\,(s_{\partial M})^{n-2}}{2}\end{aligned} \tag{1.20}$$

holds if $f_0 \in (1, \infty)$. Equality holds on either side in each of (1.19), (1.20) if and only if (M, g) is isometric to a suitable piece of the Schwarzschild manifold (M^n_m, g_m) of mass m and f corresponds to the restriction of the Schwarzschild lapse function f_m under this isometry.

The equality case assertions in Theorems 1.7 and 1.8 and thus in Theorems 1.1 and 1.3 rely on the following rather general rigidity theorem which we will prove in Sections 3 and 4.

Theorem 1.9 (Rigidity theorem)**.** *Let (M^n, g, f), $n \geq 3$, be an asymptotically flat static vacuum system of mass $m \in \mathbb{R}$ with connected boundary ∂M. Assume that $f|_{\partial M} = f_0$ for a constant $f_0 \in [0, 1) \cup (1, \infty)$. Assume that $T = 0$ on M. Then (M, g) is isometric to the piece $[s_{\partial M}, \infty) \times \mathbb{S}^{n-1}$ of the Schwarzschild manifold (M^n_m, g_m) of mass m and f corresponds to the restriction of f_m to $[s_{\partial M}, \infty)$ under this isometry. In particular ∂M is totally umbilic, has constant mean curvature, and is isometric to a round sphere.*

Theorems 1.1 and 1.3 then follow directly from Theorems 1.7 to 1.9 as we will show towards the end of Section 6.

Remark 1.10 (Independent interest)**.** *Theorems 1.8 and 1.9 may be of independent interest as they assume much less about the properties of ∂M than Theorems 1.1 and 1.3. Thus, similar geometric inequalities may be derived from Theorem 1.5 under different asymptotic and/or inner boundary conditions.*

Having completed the proofs of Theorems 1.1 and 1.3, we will then discuss some geometric implications as well as the relation to the existing strategies of proving Theorem 1.1 implemented by Agostiniani and Mazzieri [AM17] and put forward by Nozawa, Shiromizu, Izumi, and Yamada in [NSIY18] in Section 7. In particular, we will define monotone functions $\mathcal{H}^{c,d}_p$ along the level sets of the lapse function f in the style of the functions U_p introduced in

[AM17] and relate $\mathcal{H}_p^{c,d}$ to U_p (see Section 7.2). This will shed light on the relation of the two proofs and extend the monotonicity results of [AM17] to $3 > p \geq p_n$. In Section 7.3, we will investigate the relationship between the $(0,3)$ T-tensor we use in our approach to the $(0,2)$-tensor H used by [NSIY18]; in particular we will show that vanishing of T does *not* locally imply vanishing of H as is claimed in [NSIY18] and as is necessary to conclude for $p = p_n$. Moreover, we will discuss how our analysis completes the strategy of proof put forward in [NSIY18].

This paper is structured as follows: In Section 2, we introduce our notation and definitions, in particular the precise notion of asymptotic flatness we are using. We also collect some straightforward and/or well-known facts about static horizons and equipotential photon surfaces. In Section 3, we prove useful facts about the T-tensor which could also be of independent interest, while in Section 4, we will show how these facts imply Theorem 1.9. In Section 5, we will give a proof of Theorem 1.5. In Section 6, we will prove Theorems 1.7 and 1.8 and show how they imply Theorems 1.1 and 1.3. The final Section 7 is dedicated to deducing and discussing consequences of Theorems 1.1 and 1.3, in particular to constructing monotone functions along the level sets of the lapse function f and comparing those to the monotone functions introduced and exploited in [AM17] and to a comparison between our tensor T and the tensor H used by [NSIY18].

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2. Preliminaries

In this section, we will collect all relevant definitions as well as some straightforward and/or well-known facts useful for the proofs of Theorems 1.1 and 1.3 and Theorems 1.5 and 1.7 to 1.9. Our sign and scaling convention for the mean curvature H of a smooth, oriented hypersurface of (M,g) is such that the unit round sphere $\mathbb{S}^{n-1}$ in $(\mathbb{R}^n,\delta)$ has mean curvature $H = n-1$ with respect to the unit normal ν pointing towards infinity.

2.1. Static vacuum systems and asymptotic considerations.

Definition 2.1 (Static vacuum systems)**.** *A smooth, connected Riemannian manifold* (M^n, g), $n \geq 3$, *is called a* static system *if there exists a smooth* lapse function $f\colon M \to (0,+\infty)$. *A static system is called a* static vacuum system *if it satisfies the* static vacuum equations

$$\nabla^2 f = f\,\mathrm{Ric}, \tag{2.1}$$

$$\Delta f = 0, \tag{2.2}$$

where ∇^2 *and* Δ *denote the Hessian and Laplacian with respect to* g, *respectively, and* Ric *denotes its Ricci curvature tensor. If* M *has non-empty boundary* ∂M, *it is assumed that* g *and* f *extend smoothly to* ∂M, *with* $f \geq 0$ *on* ∂M.

It follows readily from the trace of (2.1) and from (2.2) that the scalar curvature R of a static vacuum system (M^n, g, f) vanishes,

$$\mathrm{R} = 0. \tag{2.3}$$

It can easily be seen that the warped product *static spacetime* $(\mathbb{R} \times M, \overline{g} = -f^2 dt^2 + g)$ constructed[3] from a static vacuum system (M^n, g, f) satisfies the vacuum Einstein equation $\overline{\mathrm{Ric}} = 0$, with $\overline{\mathrm{Ric}}$ denoting the Ricci curvature tensor of $\overline{g}$. Conversely, a static spacetime $(\mathbb{R} \times M^n, \overline{g} = -f^2 dt^2 + g)$ solving the vacuum Einstein equation has time-slices $\{t = \text{const.}\}$ isometric to (M, g) with lapse function $f\colon M \to (0, \infty)$ such that (M, g, f) is a static vacuum system.

The prime example of a static vacuum system is the *n-dimensional Schwarzschild*[4] *system* (M^n_m, g_m, f_m) *of mass* $m > 0$ *and dimension* $n \geq 3$, given by

$$\begin{aligned} f_m(r) &= \sqrt{1 - \frac{2m}{r^{n-2}}}, \\ g_m &= \frac{dr^2}{f_m(r)^2} + r^2 g_{\mathbb{S}^{n-1}}, \end{aligned} \tag{2.4}$$

on $M^n_m = ((2m)^{\frac{1}{n-2}}, \infty) \times \mathbb{S}^{n-1}$, where $g_{\mathbb{S}^{n-1}}$ denotes the canonical metric on $\mathbb{S}^{n-1}$ and $r \in ((2m)^{\frac{1}{n-2}}, \infty)$ is the *radial coordinate*. It is well-known that by a change of coordinates (e.g. to "isotropic coordinates"), one can assert that g_m and f_m smoothly extend to $\partial M_m = \{r = (2m)^{\frac{1}{n-2}}\} \times \mathbb{S}^{n-1}$, with induced metric $g_{\partial M_m} = (2m)^{\frac{2}{n-2}} g_{\mathbb{S}^{n-1}}$ and $f_m = 0$ on ∂M_m. Moreover, by another change of coordinates (e.g. to "Kruskal–Szekeres coordinates"), one can smoothly extend the associated $(n+1)$-dimensional static Schwarzschild spacetime $(\mathbb{R} \times M_m, \overline{g}_m = -f_m^2 dt^2 + g_m)$ to include (and indeed extend beyond) the boundary of $\mathbb{R} \times M_m$. Similarly, the *n-dimensional Schwarzschild system of mass* $m \leq 0$ is given by (2.4) on $M^n_m = (0, \infty) \times \mathbb{S}^{n-1}$; the associated spacetime cannot be extended when $m < 0$ and isometrically embeds into the Minkowski spacetime when $m = 0$.

We will use the following weak notion of asymptotic flatness.

Definition 2.2 (Asymptotic flatness). *A static system (M^n, g, f), $n \geq 3$, is said to be* asymptotically flat of mass $m \in \mathbb{R}$ (and decay rate $\tau \geq 0$) *if there exist a* mass (parameter) $m \in \mathbb{R}$ *as well as a compact subset $K \subset M$ and a smooth diffeomorphism $x\colon M \setminus K \longrightarrow \mathbb{R}^n \setminus \overline{B}$ for some open ball B such that, in the coordinates (x^i) induced by the diffeomorphism x,*

i) the metric components g_{ij} satisfy the decay conditions

$$(x_* g)_{ij} = \delta_{ij} + o_2(|x|^{-\tau}) \tag{2.5}$$

as $|x| \to \infty$ for all $i, j \in \{1, \ldots, n\}$, and

ii) the lapse function f can be written as

$$f \circ x^{-1} = f_m(|x|) + o_2(|x|^{-(n-2)}) = 1 - \frac{m}{|x|^{n-2}} + o_2(|x|^{-(n-2)}) \tag{2.6}$$

as $|x| \to \infty$.

[3]In case $f = 0$ on ∂M, one usually assumes that $\overline{g}$ smoothly extends to the boundary of $\mathbb{R} \times M$ (although of course the warped product structure breaks down there).

[4]In higher dimensions, the associated static spacetimes are also known as Schwarzschild–Tangherlini spacetimes.

Here and throughout the paper, for a given smooth function $\Psi\colon \mathbb{R}^n \to \mathbb{R}$, the notation $\Psi = o_l(|x|^\alpha)$ for some $l \in \mathbb{N}$, $\alpha \in \mathbb{R}$ means that

$$\sum_{|J|\leq l} |x|^{\alpha+|J|}|\partial^J \Psi| = o(1) \tag{2.7}$$

as $|x| \to \infty$. The meaning of the notation $\Psi = O_l(|x|^\alpha)$ is analogous, substituting $O(1)$ by $o(1)$ in Equation (2.7)*. For improved readability, we will from now on mostly suppress the explicit mention of the diffeomorphism x in our formulas.*

Remark 2.3 (Asymptotic assumptions, decay rates)**.** *Theorems 1.1 and 1.3 and Theorems 1.7 to 1.9 apply for all decay rates $\tau \geq 0$, in particular for $\tau = 0$, which is why we do not explicitly mention the decay rates in their statements. See also Remark 2.15 for further information on possible decay rates.*

In the standard definition of asymptotic flatness for Riemannian manifolds, one usually requires stronger asymptotic conditions, namely $g_{ij} = \delta_{ij} + O_2(|x|^{-\frac{1}{2}-\varepsilon})$ for some $\varepsilon > 0$ and integrability of the scalar curvature R *on M (which is automatic in the static setting). Under these additional assumptions, it can be seen by a standard computation that the mass parameter m from* (2.6) *coincides with the ADM-mass of (M, g). We do not appeal to any facts or properties of the ADM-mass, so we don't need to impose such restrictions.*

Our decay assumptions are also very weak when compared with the other static vacuum uniqueness results discussed in Section 1. Most of these results require that (M, g, f) is asymptotic to the Schwarzschild system of mass m, implying standard asymptotic flatness with $\varepsilon = \frac{1}{2}$ and also faster decay of the error term in (2.6)*. In contrast,* [AM17, CCF24] *make the same assumption on the decay of f as we make in* (2.6)*. On the other hand,* [AM17] *assumes $\tau = \frac{n-2}{2}$; but see* [CCF24, Remark 7.1] *and Section 7.2. Instead,* [NSIY18] *assumes Schwarzschildean decay which gives τ arbitrarily close to* 1 *from below. However, all their asymptotic arguments are adapted to $\tau \geq 0$ here. Finally, it is conceivable that our asymptotic decay assumptions can be boot-strapped to stronger decay assertions as e.g. in* [KOM95]*, using* (2.1) *and* (2.2)*.*

It is well-known and straightforward to see that the Schwarzschild system (M^n_m, g_m, f_m) of mass m is asymptotically flat of mass m for any decay rate $\tau \geq 0$. To see this, one switches from the spherical polar coordinates r and $\eta \in \mathbb{S}^{n-1}$ to the canonically associated Cartesian coordinates $x = r\eta$ outside a suitably large ball.

The following remark will be useful for our strategy of proof, in particular for Theorem 1.7, where we will use it when applying the divergence theorem on M, and for Theorem 1.9, where we will use it to properly study the level set flow of f and conclude isometry to a Schwarzschild system.

Remark 2.4 (Completeness)**.** *Asymptotically flat static systems (M^n, g, f), $n \geq 3$, with boundary ∂M are necessarily metrically and geodesically complete (up to the boundary ∂M) with at most finitely many boundary components, see e.g.* [CGM, Appendix]*. Moreover, the connected components of ∂M are necessarily all closed, see e.g.* [CGM, Appendix]*. Here, to be* geodesically complete up to the boundary *means that any geodesic $\gamma\colon I \to M$ with $I \neq \mathbb{R}$ can be smoothly extended to a geodesic $\widehat{\gamma}\colon J \to M \cup \partial M$ such that either $J = \mathbb{R}$, $J = [a, \infty)$, $J = (-\infty, b]$, or $J = [a, b]$ for some $a, b \in \mathbb{R}$ such that $\widehat{\gamma}(a), \widehat{\gamma}(b) \in \partial M$ (if applicable).*

We will later need the following consequences of our asymptotic assumptions which we formulate for general decay rate $\tau \geq 0$ for convenience of the reader.

Lemma 2.5 (Asymptotics)**.** *Let (M^n, g, f), $n \geq 3$, be an asymptotically flat static system of mass $m \in \mathbb{R}$ and decay rate $\tau \geq 0$ with respect to a diffeomorphism $x\colon M \setminus K \longrightarrow \mathbb{R}^n \setminus \overline{B}$ and denote the induced coordinates by (x^i). Then, for $i, j = 1, \ldots, n$, we have*

$$\begin{aligned}
(\nabla f)^i &= (\nabla_m f_m)^i + o_1(|x|^{-(n-1)}) = \frac{(n-2)\, m x^i}{|x|^n} + o_1(|x|^{-(n-1)}),\\
\|\nabla f\|^2 &= \|\nabla_m f_m\|_m^2 + o_1(|x|^{-2(n-1)}) = \frac{(n-2)^2\, m^2}{|x|^{2(n-1)}} + o_1(|x|^{-2(n-1)}),\\
\nabla^2_{ij} f &= (\nabla^2_m)_{ij} f_m + o_0(|x|^{-n}) = \frac{(n-2) m \delta_{ij}}{|x|^n} - \frac{n(n-2) m x_i x_j}{|x|^{n+2}} + o_0(|x|^{-n}),\\
\|\nabla^2 f\|^2 &= \|\nabla^2_m f_m\|_m^2 + o_0(|x|^{-2n}) = \frac{n(n-1)(n-2)^2\, m^2}{|x|^{2n}} + o_0(|x|^{-2n}),\\
\nabla^i \|\nabla f\|^2 &= \nabla^i_m \|\nabla_m f_m\|_m^2 + o_0(|x|^{-(2n-1)}) = -\frac{2(n-1)(n-2)^2\, m^2 x^i}{|x|^{2n}} + o_0(|x|^{-(2n-1)}),\\
\|\nabla \|\nabla f\|^2\|^2 &= \|\nabla_m \|\nabla_m f_m\|_m^2\|_m^2 + o_0(|x|^{-2(2n-1)}),\\
\operatorname{Ric}(\nabla f, \nabla f) &= \operatorname{Ric}_m(\nabla_m f_m, \nabla_m f_m) + o_0(|x|^{-(\tau+2n)}) = o_0(|x|^{-(\tau+2n)})
\end{aligned}$$

as $|x| \to \infty$. Here, ∇ and $\|\cdot\|$ denote the connection and tensor norm with respect to g and ∇_m, $\|\cdot\|_m$, and Ric_m denote the connection, tensor norm, and Ricci tensor with respect to g_m, respectively. Furthermore, let $r > 0$ be such that $B_r := \{x \in \mathbb{R}^n : |x| < r\} \supset \overline{B}$ and let ν denote the unit normal to $x^{-1}(\partial B_r)$ pointing towards the asymptotically flat end and let H denote the mean curvature of $x^{-1}(\partial B_r)$ with respect to ν. Then

$$\nu^i = \frac{x^i}{|x|} + o(|x|^{-\tau}), \tag{2.8}$$

$$H = \frac{n-1}{|x|} + o(|x|^{-1-\tau}) \tag{2.9}$$

as $|x| \to \infty$. Now let $u, v, u_0, v_0 \colon M \setminus K \to \mathbb{R}$ be continuous functions such that $u = u_0 + o(|x|^{-(n-1)})$, $u_0 = O(|x|^{-(n-1)})$, $v = v_0 + o(|x|^{-n})$, and $v_0 = O(|x|^{-n})$ as $|x| \to \infty$. Then

$$\int_{x^{-1}(\partial B_r)} u\, dS = \int_{\partial B_r} (u_0 \circ x^{-1})\, dS_\delta + o(1), \tag{2.10}$$

$$\int_{x^{-1}(\mathbb{R}^n \setminus \overline{B_r})} v\, dV = \int_{\mathbb{R}^n \setminus \overline{B_r}} (v_0 \circ x^{-1})\, dV_\delta + o(1) \tag{2.11}$$

as $r \to \infty$, where dS and dS_δ denote the area elements induced on $x^{-1}(\partial B_r)$ and ∂B_r and dV and dV_δ denote the volume elements induced on $x^{-1}(\mathbb{R}^n \setminus \overline{B_r})$ and $\mathbb{R}^n \setminus \overline{B_r}$ by g and δ, respectively. In particular, v is integrable on $x^{-1}(\mathbb{R}^n \setminus \overline{B_r})$ with respect to dV.

Proof. The claims in Lemma 2.5 follow from straightforward computations. For addressing (2.8), (2.10), and (2.11), let $(r, \theta^J)_{J=1}^{n-1}$ be standard polar coordinates for $\mathbb{R}^n$ so that $(\partial_{\theta^K})^i = O_\infty(|x|)$ as $|x| \to \infty$ and $\delta_{IJ} = r^2 (g_{\mathbb{S}^{n-1}})_{IJ}$. Here and in what follows, we use the convention that capital latin indices $I, J, K, \cdots = 1, \ldots, n-1$ label the polar coordinates

(θ^K), while small latin indices $i, j, k, \dots = 1, \dots, n$ label the Cartesian coordinates (x^i) as before.

For ν, we make the ansatz

$$\nu^i = (1+\lambda)\frac{x^i}{|x|} - \delta^{ij}(g_{jk} - \delta_{jk})\frac{x^k}{|x|} + \mu^L(\partial_{\theta^L})^i$$

for $\lambda, \mu^L \in C^\infty(\mathbb{R}^n \setminus B_r)$, $L = 1, \dots, n-1$. Then for $K, L = 1, \dots, n-1$, we compute

$$\begin{aligned} 0 = g(\nu, \partial_{\theta^K}) &= -(g_{ij} - \delta_{ij})\frac{x^i}{|x|}(\partial_{\theta^K})^j + \mu^L|x|^2 \, (g_{\mathbb{S}^{n-1}})_{KL} + (g_{ij} - \delta_{ij})\nu^i(\partial_{\theta^K})^j \\ &= \mu^L|x|^2 \left((g_{\mathbb{S}^{n-1}})_{KL} + o(|x|^{-\tau})\right) + \lambda \cdot o(|x|^{-\tau+1}) + o(|x|^{-2\tau+1}), \\ 1 = \quad g(\nu, \nu) &= (1+\lambda)^2(1 + o(|x|^{-\tau})) + \mu^L \cdot o(|x|^{-\tau+1}) + (1+\lambda) \cdot o(|x|^{-\tau}) \\ &\quad + \mu^K\mu^L|x|^2 \left((g_{\mathbb{S}^{n-1}})_{KL} + o(|x|^{-\tau})\right) + (1+\lambda)\mu^L \cdot o(|x|^{-\tau+1}) + o(|x|^{-2\tau}) \end{aligned}$$

as $|x| \to \infty$. We rewrite the first equation as

$$\mu^L = \lambda \cdot o(|x|^{-(\tau+1)}) + o(|x|^{-(2\tau+1)}) \tag{2.12}$$

and plug this into the second equation, obtaining $1 = (1+\lambda)^2 + o(r^{-\tau}) + \lambda \cdot o(r^{-\tau}) + \lambda^2 \cdot o(r^{-\tau})$ and hence by Taylor's formula, this quadratic equation has the two solutions $\lambda_1 = o(|x|^{-\tau})$ and $\lambda_2 = -2 + o(|x|^{-\tau})$ as $|x| \to \infty$. As we are interested in finding the normal pointing towards $|x| \to \infty$, we can exclude λ_2 and obtain $\lambda = o(|x|^{-\tau})$ as desired. Combining this with (2.12), we find $\mu^L = o(|x|^{-(2\tau+1)})$ for $L = 1, \dots, n-1$ as $|x| \to \infty$. This proves (2.8).

For (2.9), we compute as above that the components of the inverse induced metric (σ^{IJ}) on $x^{-1}(\partial B_r)$ satisfy $\sigma^{IJ} = \frac{1}{|x|^2}(g_{\mathbb{S}^{n-1}})^{IJ} + o(|x|^{-\tau-2})$ as $|x| \to \infty$, while the components of the inverse metric satisfy $g^{rr} = 1 + o_2(|x|^{-\tau})$, $g^{rI} = o_2(|x|^{-\tau-1})$, $g^{IJ} = \frac{1}{|x|^2}(g_{\mathbb{S}^{n-1}})^{IJ} + o_2(|x|^{-\tau-2})$ as $|x| \to \infty$. From this, one finds that the Christoffel symbols of g behave as

$$\begin{aligned} \Gamma^r_{IJ} &= -|x|(g_{\mathbb{S}^{n-1}})_{IJ} + o(|x|^{1-\tau}), \\ \Gamma^K_{IJ} &= o(|x|^{-\tau}) \end{aligned}$$

as $|x| \to \infty$ and thus, using (2.8), we obtain

$$H = -\sigma^{IJ}g(\nabla_I\partial_J, \nu) = \frac{n-1}{|x|} + o(|x|^{-1-\tau})$$

as $|x| \to \infty$ as claimed. Next, for (2.11), we note that

$$\sqrt{\det(g_{ij})} = \sqrt{\det(\delta_{ij} + o(|x|^{-\tau}))} = 1 + o(|x|^{-\tau})$$

as $|x| \to \infty$ by Taylor's formula.

Hence

$$\begin{aligned}
\int_{x^{-1}(\mathbb{R}^n\setminus\overline{B_r})} v\,dV &= \int_{\mathbb{R}^n\setminus\overline{B_r}} (v\circ x^{-1})\sqrt{\det(g_{ij})}\,dx^1\cdot\dots\cdot dx^n\\
&= \int_{\mathbb{R}^n\setminus\overline{B_r}} (v\circ x^{-1})\left(1+o(|x|^{-\tau})\right)dx^1\cdot\dots\cdot dx^n\\
&= \int_{\mathbb{R}^n\setminus\overline{B_r}} (v_0\circ x^{-1})\left(1+o(|x|^{-\tau})\right)dV_\delta + \int_{\mathbb{R}^n\setminus\overline{B_r}} o(|x|^{-n})\,dV_\delta\\
&= \int_{\mathbb{R}^n\setminus\overline{B_r}} (v_0\circ x^{-1})\,dV_\delta + \int_{\mathbb{R}^n\setminus\overline{B_r}} o(|x|^{-n})\,dV_\delta = \int_{\mathbb{R}^n\setminus\overline{B_r}} (v_0\circ x^{-1})dV_\delta + o(1)
\end{aligned}$$

as $|x|\to\infty$, where we have used the decay assumption on v and v_0 in the third and second to last, and the L^∞-L^1-Hölder inequality in the last step.

Finally, for (2.10), we argue as before and compute

$$\begin{aligned}
\sqrt{\det(g_{IJ})} &= \sqrt{\det\left(r^2(g_{\mathbb{S}^{n-1}})_{IJ}+o(|x|^{-\tau+2})\right)}\\
&= r^{n-1}\sqrt{\det\left((g_{\mathbb{S}^{n-1}})_{IJ}+o(|x|^{-\tau})\right)}\\
&= r^{n-1}\sqrt{\det\left((g_{\mathbb{S}^{n-1}})_{IJ}\right)}\sqrt{\det\left(\delta_{KL}+((g_{\mathbb{S}^{n-1}})^{-1})_{KL}\cdot o(|x|^{-\tau})\right)}\\
&= r^{n-1}\sqrt{\det\left((g_{\mathbb{S}^{n-1}})_{IJ}\right)}\left(1+o(|x|^{-\tau})\right)
\end{aligned}$$

as $|x|\to\infty$ by the algebraic properties of the determinant and by Taylor's formula. Arguing as before and using the decay assumption on u, this implies

$$\begin{aligned}
\int_{x^{-1}(\partial B_r)} u\,dS &= \int_{\partial B_r} (u\circ x^{-1})\sqrt{\det(g_{IJ})}\,d\theta^1\cdot\dots\cdot d\theta^{n-1}\\
&= \int_{\partial B_r} (u\circ x^{-1})\,r^{n-1}\sqrt{\det\left((g_{\mathbb{S}^{n-1}})_{IJ}\right)}\left(1+o(|x|^{-\tau})\right)d\theta^1\cdot\dots\cdot d\theta^{n-1}\\
&= \int_{\partial B_r} (u\circ x^{-1})\left(1+o(|x|^{-\tau})\right)dS_\delta\\
&= \int_{\partial B_r} (u_0\circ x^{-1})\left(1+o(|x|^{-\tau})\right)dS_\delta + \int_{\partial B_r} o(|x|^{-(n-1)})\,dS_\delta\\
&= \int_{\partial B_r} (u_0\circ x^{-1})\,dS_\delta + o(1)
\end{aligned}$$

as $|x|\to\infty$. This completes the proof. □

Remark 2.6 (Choice of normal, regular boundary, tensor norm)**.** *Let (M^n,g,f), $n\geq 3$, be an asymptotically flat static vacuum system of mass m and decay rate $\tau\geq 0$ with connected boundary ∂M. Let ν denote the unit normal to ∂M pointing towards the asymptotically flat end. Now assume first that $f|_{\partial M}=f_0$ for some $f_0\in[0,1)$. Then since f is harmonic by*

(2.2), *the maximum principle*[5] *ensures that*

$$0 \leq f_0 < f < 1 \tag{2.13}$$

holds on M. *Moreover, by the Hopf lemma*[6], *we can deduce that* $\nu(f) = \|\nabla f\| > 0$ *on* ∂M, *implying that* ∂M *is a regular level set of* f. *Thus*

$$\nu = \frac{\nabla f}{\|\nabla f\|}, \tag{2.14}$$

where here and in what follows, $\|\cdot\|$ *denotes the tensor norm induced by* g *and we slightly abuse notation and denote the gradient of* f *by* ∇f. *Next assume that* $f|_{\partial M} = f_0$ *for some* $f_0 > 1$. *The same arguments imply that*

$$f_0 > f > 1 \tag{2.15}$$

holds on M *and*

$$\nu = -\frac{\nabla f}{\|\nabla f\|}. \tag{2.16}$$

When studying (regular) level sets $\{f = f_0\}$ *of* f, *we will also use the unit normal* ν *pointing towards infinity, so that* (2.14) *respectively* (2.16) *hold when* $f_0 \in (0,1)$ *respectively* $f_0 \in (1,\infty)$. *Finally, assume that* $f|_{\partial M} = 1$. *Then by the maximum principle,* $f \equiv 1$ *holds on* M.

2.2. **Static horizons and equipotential photon surfaces.** Static (black hole) horizons and their surface gravity are defined as follows. For simplicity, we will restrict our attention to connected static horizons already here.

Definition 2.7 (Static horizons). *Let* (M^n, g, f), $n \geq 3$, *be a static system with connected boundary* ∂M. *We say that* ∂M *is a* static (black hole) horizon *if* $f|_{\partial M} = 0$.

In fact, static horizons as defined above can be seen to be Killing horizons in the sense that the static Killing vector field ∂_t smoothly extends to the (extension to the) boundary of the static spacetime $(\mathbb{R} \times M, \overline{g} = -f^2 dt^2 + g)$ but at the same time degenerates along this boundary, namely $-f^2 = \overline{g}(\partial_t, \partial_t) \to 0$. The standard example of a static system with a static horizon is the Schwarzschild system (M^n_m, g_m, f_m) of mass $m > 0$.

Let us now collect some important properties of static horizons in static vacuum systems.

Remark 2.8 (Surface gravity, horizons are totally geodesic). *It is a well-known and straightforward consequence of* (2.1) *that static horizons in static vacuum systems are totally geodesic and in particular minimal surfaces. Moreover, using again* (2.1), *one computes that*

$$\nabla \|\nabla f\|^2 = 2f \operatorname{Ric}(\nabla f, \cdot) \tag{2.17}$$

[5] Indeed, the maximum principle applies under our weak asymptotic flatness conditions from Definition 2.2 which can be seen as follows: Suppose that $\{f \geq 1\} \neq \emptyset$. Since $f = f_0$ on ∂M, $f \to 1$ at infinity, f is continuous, and M is metrically complete up ∂M by Remark 2.4, f must have a positive maximum at a point $q_0 \in M \setminus \partial M$, with $f(q_0) \geq 1$. Now let $U \subset M \setminus \partial M$ be an open neighborhood of q_0 with smooth boundary ∂U, large enough to contain some $q \in U$ with $f(q) < f(q_0)$; such a neighborhood exists because $f = f_0 < 1$ on ∂M. Applying the strong maximum principle to $f|_U$ gives a contradiction. The possibility that $\{f \leq f_0\} \neq \emptyset$ can be handled analogously.

[6] Similarly modified as the maximum principle argument to allow for non-compact M.

which manifestly vanishes on a static horizon ∂M. This implies that the surface gravity κ *defined by*

$$\kappa := \nu(f)|_{\partial M} \tag{2.18}$$

for some unit normal along ∂M is constant on the static horizon ∂M. Combined with Remark 2.6, this shows that the surface gravity of a (connected) static horizon in an asymptotically flat static vacuum system is necessarily non-vanishing, $\kappa \neq 0$ and positive when one chooses ν to point to infinity. This fact is sometimes expressed as saying that such static horizons are "non-degenerate".

Next, let us recall the definition and properties of equipotential photon surfaces and of photon spheres, the central objects studied in Theorem 1.3. We will be very brief as we will only need specific properties and refer the interested reader to [CG21, CJVM23] for more information and references. In particular, we will assume that all photon surfaces are necessarily connected for simplicity of the exposition and as we will only study connected photon surfaces in this paper, anyway. It will temporarily be more convenient to think about static spacetimes rather than static systems.

Definition 2.9 ((Equipotential) photon surface, photon sphere)**.** *A smooth, timelike, embedded, and connected hypersurface in a smooth Lorentzian manifold is called a* photon surface *if it is totally umbilic. A photon surface P^n in a static spacetime $(\mathbb{R} \times M^n, \overline{g} = -f^2 dt^2 + g)$ is called* equipotential *if the lapse function f of the spacetime is constant along each connected component of each* time-slice $\Sigma^{n-1}(t) := P^n \cap (\{t\} \times M^n)$ *of the photon surface. An equipotential photon surface is called a* photon sphere *if the lapse function f is constant (in space and time) on P^n.*

It is well-known that the (exterior) Schwarzschild spacetime of mass $m > 0$ (i.e., the spacetime associated to the Schwarzschild system (M^n_m, g_m, f_m) of mass $m > 0$) possesses a photon sphere at $r = (nm)^{\frac{1}{n-2}}$. Moreover, it follows from a combination of results by Cederbaum and Galloway [CG21, Theorem 3.5, Proposition 3.18] and by Cederbaum, Jahns, and Vičánek Martínez [CJVM23, Theorems 3.7, 3.9, and 3.10] that all Schwarzschild spacetimes possess very many equipotential photon surfaces. In particular, every sphere $\mathbb{S}^{n-1}(r) \subset M^n_m$ arises as a time-slice of an equipotential photon surface. On the other hand, no other closed hypersurfaces of (M^n_m, g_m, f_m) arise as time-slices of equipotential photon surfaces by [CG21, Corollary 3.9].

Let us now move on to study the intrinsic and extrinsic geometry of time-slices of equipotential photon spheres. Time-slices of equipotential photon surfaces and in particular of photon spheres have the following useful properties.

Proposition 2.10 ([CJVM23, Proposition 5.5])**.** *Let (M^n, g, f), $n \geq 3$, be an asymptotically flat static vacuum system and let ∂M be a time-slice of an equipotential photon surface with $f = f_0$ on ∂M for some constant $f_0 > 0$, $f_0 \neq 1$. Then ∂M is totally umbilic in (M, g), has constant scalar curvature $\mathrm{R}_{\partial M}$, constant mean curvature H, and constant $\kappa := \nu(f)|_{\partial M}$, related by the* equipotential photon surface constraint

$$\mathrm{R}_{\partial M} = \frac{2\kappa H}{f_0} + \frac{n-2}{n-1} H^2. \tag{2.19}$$

Here, we are using that $\kappa = \nu(f)|_{\partial M} \neq 0$ by Remark 2.6.

Proposition 2.11 ([CG17, Lemma 2.6], [CJVM23, Theorem 5.22]). *In the setting of Proposition 2.10, we have* $H > 0$.

In fact, both [CG17, Lemma 2.6] and [CJVM23, Theorem 5.22] assume stronger asymptotic decay than we do, and in addition assume $\nu(N) > 0$ resp. $H\nu(N) > 0$ on ∂M. As ∂M is connected here, neither of the second assumptions are needed to conclude as can be seen in the corresponding proofs, as these assumptions are only needed to handle potential other boundary componentss. Concerning the asymptotic decay, it suffices to note that our decay assumptions imply that large coordinate spheres have positive mean curvature by Lemma 2.5.

Remark 2.12. *Formally taking the limit of the equipotential photon surface constraint* (2.19) *as* $f_0 \searrow 0$, *one recovers the twice contracted Gauß equation*

$$\mathrm{R}_{\partial M} = -\frac{2\,\mathrm{Ric}(\nu,\nu)\kappa}{\|\nabla f\|} = -2\,\mathrm{Ric}(\nu,\nu),$$

with κ *denoting the surface gravity of the static horizon* $\{f_0 = 0\}$. *To see this, one uses the well-known fact that* $H = -\frac{\nabla^2 f(\nu,\nu)}{\|\nabla f\|}$ *on regular level sets of* f *(for* $0 < f < 1$*),* (2.1)*, and* (2.3)*. In particular, the first term* $\frac{2\kappa H}{f_0}$ *of* (2.19) *remains well-defined in the case* $f_0 = 0$.

Lemma 2.13 (Smarr formula). *Let* (M^n, g, f), $n \geq 3$, *be an asymptotically flat static vacuum system with mass* $m \in \mathbb{R}$. *Then the* Smarr formula

$$\int_{\{f=z\}} \nu(f)\, dS = (n-2)|\mathbb{S}^{n-1}|\, m \tag{2.20}$$

holds for every regular, connected level set $\{f = z\}$ *of* f, *where* $z \geq 0$ *is a constant. Here,* $|\mathbb{S}^{n-1}|$ *denotes the area of* $(\mathbb{S}^{n-1}, g_{\mathbb{S}^{n-1}})$ *and* ν *denotes the unit normal to* $\{f = z\}$. *Moreover, if* (M, g, f) *has a connected boundary* ∂M *then*

$$\int_{\partial M} \nu(f)\, dS = (n-2)|\mathbb{S}^{n-1}|\, m. \tag{2.21}$$

Furthermore, if in addition $f|_{\partial M} = f_0$ *for some* $f_0 \geq 0$ *then* $m > 0$ *when* $f_0 \in [0,1)$, $m = 0$ *when* $f_0 = 1$, *and* $m < 0$ *when* $f_0 > 1$. *In particular, if* ∂M *is a static horizon or a time-slice of an equipotential photon surface with* $f_0 < 1$ *resp.* $f_0 > 1$ *then* $m > 0$ *resp.* $m < 0$.

Remark 2.14 (Quasi-local mass, outward directed equipotential photon surfaces, and why we avoid the zero mass case). *The Smarr formula* (2.21) *allows one to define a quasi-local mass for* ∂M *by expressing* m *in terms of the other quantities in* (2.21) *(see e.g.* [Ced12]*). Lemma 2.13 hence states that said quasi-local mass of a connected boundary* ∂M *coincides with the asymptotic mass parameter* m *of the static system. Furthermore, it informs us that if* $f = f_0$ *on* ∂M *for some constant* $f_0 \geq 0$, *the sign/vanishing of the mass* m *is fixed by the value of* f_0. *This allows to refer to the case* $f_0 \in [0,1)$ *as the* positive mass case, *to* $f_0 = 0$ *as the* zero mass case, *and to the case* $f_0 \in (1,\infty)$ *as the* negative mass case, *respectively. It also explains why we avoid the zero mass case in this paper altogether: If* $f_0 = 1$, *Remark 2.6 informs us that* $f \equiv 1$ *on* M *and thus* (M, g) *is necessarily Ricci-flat by* (2.1). *In dimension* $n = 3$, *this implies that* (M, g) *is indeed flat; one can conclude that it isometric to Euclidean space without a ball using the asymptotically flatness with decay rate* $\tau \geq 0$, *without assuming any additional properties (see* [CCF24]*). In higher dimensions, proving a similar statement is a problem of a different nature, which is going to be addressed elsewhere.*

As briefly touched upon in Section 1, the existing static vacuum uniqueness results for equipotential photon surfaces all[7] *assume that those are* outward directed, *meaning that* $\kappa = \nu(f)|_{\partial M} > 0$. *In view of Lemma 2.13, this corresponds to a restriction to the positive (quasi-local) mass case.*

Proof of Lemma 2.13. The fact that the left-hand side of (2.20) is independent of the value of z is a direct consequence of (2.2) and the divergence theorem. To see that the constants on the right-hand sides of (2.20), (2.21) equal $(n-2)|\mathbb{S}^{n-1}|\, m$, one argues as follows, using the notation from Lemma 2.5. First, $\nu(f) = \frac{(n-2)m}{|x|^{n-1}} + o(|x|^{-(n-1)})$ as $|x| \to \infty$ by Lemma 2.5 and (2.8). Hence $u := \nu(f)$, $u_0 := \frac{(n-2)m}{|x|^{n-1}}$ are suitable functions for the application of (2.10). Then, by (2.2) and the divergence theorem, we get

$$\begin{aligned}\int_{\partial M} \nu(f)\, dS &= -\int_{\{p \in M\,:\,|x|(p)<r\}} \Delta f\, dV + \int_{x^{-1}(\partial B_r)} \nu(f)\, dS = \int_{x^{-1}(\partial B_r)} \nu(f)\, dS \\ &= \int_{\partial B_r} (u_0 \circ x^{-1})\, dS_\delta + o(1) = (n-2)m \int_{\partial B_r} \frac{1}{|x|^{n-1}}\, dS_\delta + o(1) \\ &= (n-2)|\mathbb{S}^{n-1}|\, m + o(1)\end{aligned}$$

as $r = |x| \to \infty$, where dV denotes the volume element on M. This proves (2.21). In particular, if $f_0 = 1$, Remark 2.6 tells us that $f = 1$ on M and hence there are no regular level sets of f and no claim about (2.20). The asymptotic formula for $\nu(f) = 0$ directly shows that $m = 0$. If $f_0 \neq 1$, regular level sets can exist and (2.20) then follows precisely as (2.21), up to a sign in front of the volume integral over $\triangle f$ if $z > 1$, and with the domain of said volume integral taking the form $\{p \in M \,:\, f(p) > z,\, |x|(p) < r\}$ if $0 \leq z < 1$ and the form $\{p \in M \,:\, f(p) < z,\, |x|(p) < r\}$ if $z > 1$ in view of Remark 2.6. The remaining claims are direct consequences of the Smarr formula and of Remark 2.6, via Proposition 2.10. $\square$

Remark 2.15 (Admissible decay rates)**.** *In Definition 2.2, we have allowed the decay rate* $\tau \geq 0$ *to be arbitrary. In the static vacuum setting,* $\tau \geq n-2$ *implies that* $m = 0$ *via Lemma 2.13, arguing as in the proof of Lemma 2.5, hence our assumption* (2.6) *effectively restrict the range of the decay rate to* $\tau < n-2$.

3. The T-tensor and its properties

In this section, we will discuss properties of the T-tensor introduced in (1.14) which will be essential for establishing our results. We will also give a proof of the rigidity result Theorem 1.9. Remember that, for a Riemannian manifold (M^n, g), $n \geq 3$, the Weyl tensor W is defined as

$$W := \mathrm{Rm} - \frac{1}{n-2}\left(\mathrm{Ric} - \frac{\mathrm{R}}{2} g\right) \owedge g - \frac{\mathrm{R}}{2n(n-1)} g \owedge g, \tag{3.1}$$

where Rm stands for the Riemann curvature operator of (M, g), and $\owedge$ denotes the Kulkarni–Nomizu product. Moreover, the Cotton tensor C of (M, g) is given by

$$\begin{aligned}C(X,Y,Z) &:= (\nabla_X \mathrm{Ric})(Y,Z) - (\nabla_Y \mathrm{Ric})(X,Z) \\ &\quad - \frac{1}{2(n-1)}\left((\nabla_X \mathrm{R}) g(Y,Z) - (\nabla_Y \mathrm{R}) g(X,Z)\right)\end{aligned} \tag{3.2}$$

[7]With the exception of [CCF24] for $n = 3$ and connected ∂M.

for $X, Y, Z \in \Gamma(TM)$. It is well-known that W vanishes for $n = 3$, while for $n \geq 4$, W and C are related via

$$C = -\frac{(n-2)}{(n-3)} (\nabla_{E_i} W)(\cdot, \cdot, \cdot, E_j) \delta^{ij} \tag{3.3}$$

for any local orthonormal frame $\{E_i\}_{i=1}^n$ of M.

For $n = 3$, it is well-known that the Cotton tensor detects (local) conformal flatness in the sense that $C = 0$ if and only if (M^3, g) is locally conformally flat. The same holds true for the Weyl tensor when $n \geq 4$.

The T*-tensor* of a Riemannian manifold (M^n, g), $n \geq 3$, carrying a smooth function $f\colon M \to \mathbb{R}$ is given by (1.14). Due to the symmetry of the Ricci tensor, T is antisymmetric in its first two entries. By a straightforward algebraic computation, its squared norm can be computed to be

$$\|T\|^2 = \frac{2(n-1)}{(n-2)^2} \left[(n-1) \|\mathrm{Ric}\|^2 \, \|\nabla f\|^2 - n \| \mathrm{Ric}(\nabla f, \cdot)\|^2 + 2 \, \mathrm{R} \, \mathrm{Ric}(\nabla f, \nabla f) \right]. \tag{3.4}$$

In particular, if (M^n, g, f), $n \geq 3$, is a static vacuum system, the last term in (3.4) vanishes by (2.3). It is interesting to note the following relation between the Weyl, the Cotton, and the T-tensor.

Lemma 3.1 (Relation between W, C, and T)**.** *Let (M^n, g, f), $n \geq 3$, be a static vacuum system. Then*

$$fC = W(\cdot, \cdot, \cdot, \nabla f) + T \tag{3.5}$$

holds on M.

Proof. For simplicity, we will use abstract index notation in this proof. First, taking the covariant derivative of (2.1), we have

$$\nabla_i \nabla_j \nabla_k f = \nabla_i f \, \mathrm{Ric}_{jk} + f \nabla_i \, \mathrm{Ric}_{jk} .$$

Next, from the Ricci equation we get that

$$\mathrm{Ric}_{jk} \nabla_i f - \mathrm{Ric}_{ik} \nabla_j f + f \left(\nabla_i \mathrm{Ric}_{jk} - \nabla_j \mathrm{Ric}_{ik} \right) = \nabla_i \nabla_j \nabla_k f - \nabla_j \nabla_i \nabla_k f = \mathrm{Rm}_{ijkl} \nabla^l f.$$

By (2.3), we obtain from the definition of C in (3.2) that

$$\mathrm{Ric}_{jk} \nabla_i f - \mathrm{Ric}_{ik} \nabla_j f + f C_{ijk} = \mathrm{Rm}_{ijkl} \nabla^l f.$$

Similarly, from the definition of the Weyl tensor in (3.1), we obtain

$$\mathrm{Rm}_{ijkl} \nabla^l f = W_{ijkl} \nabla^l f + \frac{1}{n-2} \left(\mathrm{Ric}_{ik} \nabla_j f - \mathrm{Ric}_{jk} \nabla_i f + \mathrm{Ric}_{jl} \nabla^l f g_{ik} - \mathrm{Ric}_{il} \nabla^l f g_{jk} \right).$$

Combining the last two equations gives the desired result. □

It is well-known that the Schwarzschild system (M^n_m, g_m, f_m) of mass m can be rewritten in a manifestly conformally flat way by using the above-mentioned isotropic coordinates (this also applies in the negative mass case although not in a global isotropic coordinate chart). Hence its Weyl tensor W_m vanishes for all $n \geq 3$ and its Cotton tensor C_m vanishes for $n = 3$. From (3.3), we deduce that in fact its Cotton tensor C_m and hence by Lemma 3.1 its T-tensor T_m vanish for all $n \geq 3$, that is $C_m = T_m = 0$.

We will later make use of the following lemma which relies on the idea of rewriting T only in terms of f.

Lemma 3.2 (An identity for $\|T\|^2$)**.** *Let (M^n, g, f), $n \geq 3$, be a static vacuum system. Then*

$$\frac{(n-2)^2}{(n-1)} f^2 \|T\|^2 = (n-1)\|\nabla f\|^2 \left(\Delta \|\nabla f\|^2 - \frac{\langle \nabla \|\nabla f\|^2, \nabla f \rangle}{f} \right) - \frac{n}{2} \|\nabla \|\nabla f\|^2\|^2$$

holds on M, where $\langle \cdot, \cdot \rangle$ denotes the metric g.

Proof. Let us rewrite the norm of T only in terms of the function f, not explicitly involving any curvature terms. To that end, taking the divergence of (2.17) divided by $2f$ and using (2.1), (2.2) and the Bianchi identity, we get

$$2\|\mathrm{Ric}\|^2 = \frac{\Delta \|\nabla f\|^2}{f^2} - \frac{\langle \nabla \|\nabla f\|^2, \nabla f \rangle}{f^3}.$$

Combining this identity with (2.17) and (3.4) gives the result. $\square$

Let us also state the following interesting fact which is useful for understanding when T vanishes and will be used to prove the rigidity result Theorem 1.9.

Lemma 3.3. *Let (M^n, g), $n \geq 3$, be a smooth Riemannian manifold carrying a smooth function $f \colon M \to \mathbb{R}$. Then $T = 0$ on $M \setminus \mathrm{Crit}\, f$ if and only if*

$$\|\nabla f\|^2 \,\mathrm{Ric} = -\frac{\lambda \|\nabla f\|^2}{n-1} g + \frac{n\lambda}{n-1} df \otimes df \tag{3.6}$$

on $M \setminus \mathrm{Crit}\, f$ for some smooth function $\lambda \colon M \setminus \mathrm{Crit}\, f \to \mathbb{R}$. Note that (3.6) *implies in particular that*

$$\mathrm{Ric}(\nabla f, \cdot)^{\#} = \lambda \nabla f \tag{3.7}$$

so that ∇f is an eigenvector field of Ric *on $M \setminus \mathrm{Crit}\, f$ with eigenvalue λ.*

Proof. If $T = 0$, one has

$$0 = T(\cdot, \cdot, \nabla f) = \frac{1}{n-2} \left(\mathrm{Ric}(\nabla f, \cdot) \otimes df - df \otimes \mathrm{Ric}(\nabla f, \cdot) \right)$$

on $M \setminus \mathrm{Crit}\, f$ which implies (3.7) for a smooth function λ. To see that (3.6) holds, we use (3.7) to compute

$$0 = \frac{n-2}{n-1} T(\cdot, \nabla f, \cdot) = \|\nabla f\|^2 \,\mathrm{Ric} + \frac{\lambda \|\nabla f\|^2}{n-1} g - \frac{n\lambda}{n-1} df \otimes df$$

on $M \setminus \mathrm{Crit}\, f$ as claimed. Conversely, using (3.6), we find by straightforward computations using linear and multilinear arguments that

$$\begin{aligned} T(X, Y, Z) &= \frac{\lambda}{n-2} \left(-g(X, Z) \nabla_Y f + g(Y, Z) \nabla_X f \right) \\ &\quad - \frac{\lambda}{n-2} \left(g(Y, Z) \nabla_X f - g(X, Z) \nabla_Y f \right) = 0 \end{aligned}$$

on $M \setminus \mathrm{Crit}\, f$ for all $X, Y, Z \in \Gamma(TM)$. $\square$

Next, we prove the following local characterization of static vacuum systems (M^n, g, f) satisfying $T = 0$.

Theorem 3.4 (Local characterization of $T = 0$)**.** *Let (M^n, g, f), $n \geq 3$, be a static vacuum system. Then $T = 0$ on M if and only if each regular point of f has an open neighborhood $V \subseteq M \setminus \operatorname{Crit} f$ such that $(V, g|_V, f|_V)$ belongs to precisely one of the following types of systems, with $\lambda|_V\colon V \to \mathbb{R}$ denoting the eigenvalue of* Ric *from* (3.7)*. Either, in **Type 1**, there is a constant $a > 0$, an interval $I \subseteq \mathbb{R}^+$, and a Ricci flat manifold (Σ^{n-1}, σ) such that*

$$\begin{aligned} V &= I \times \Sigma \ni (h, \cdot), \\ g|_V &= dh^2 + \sigma, \\ f|_V(h, \cdot) &= ah, \\ \lambda &\equiv 0. \end{aligned}$$

*Or, in **Types 2–4**, there are constants $a > 0$, $b \in \mathbb{R}$, an interval $I \subseteq \mathbb{R}^+$, a Riemannian manifold (Σ^{n-1}, σ), and a smooth function $u\colon I \to \mathbb{R}^+$ such that*

$$\begin{aligned} V &= I \times \Sigma \ni (r, \cdot), \\ g|_V &= \frac{1}{u(r)^2} dr^2 + r^2 \sigma, \\ f|_V(r, \cdot) &= au(r), \\ \lambda|_V(r, \cdot) &= \lambda(r) \end{aligned}$$

with

Type 2	$b = 0$, (Σ, σ) *is Ricci flat,* $u(r) = \frac{1}{r^{\frac{n-2}{2}}}$, *and* $\lambda(r) = \frac{(n-1)(n-2)}{2r^n}$
Type 3	$b > 0$, (Σ, σ) *is Einstein with* $\mathrm{R}_\sigma = -(n-1)(n-2)$, $u(r) = \sqrt{\frac{b}{r^{n-2}} - 1}$, $\lambda(r) = \frac{(n-1)(n-2)b}{2r^n}$, *and* $I \subseteq (0, b^{\frac{1}{n-2}})$
Type 4	$b \neq 0$, (Σ, σ) *is Einstein with* $\mathrm{R}_\sigma = (n-1)(n-2)$, $u(r) = \sqrt{\frac{b}{r^{n-2}} + 1}$, $\lambda(r) = \frac{(n-1)(n-2)b}{2r^n}$, *and* $I \subseteq (-b^{\frac{1}{n-2}}, \infty)$ *when* $b < 0$

up to a change of local coordinates.

Remark 3.5 (Quasi-Schwarzschild systems)**.** *All systems of Type 4 in Theorem 3.4 are quasi-Schwarzschild systems (cf.* [Ced]*) of negative* $(b > 0)$ *or positive* $(b < 0)$ *mass* $m = -\frac{b}{2}$*, respectively. They are Schwarzschild systems of negative respectively positive mass precisely when (Σ, σ) is a unit radius round sphere.*

Proof. By continuity, each regular point p of f has a neighborhood $\widetilde{U} \subseteq M$ on which $\nabla f \neq 0$, so that $\widetilde{U} \subseteq M \setminus \operatorname{Crit} f$. Let $\Sigma := \widetilde{U} \cap \{f = f(p)\}$ and choose local coordinates $(\varphi^J)_{J=1}^{n-1}$ on Σ (making Σ smaller if necessary). Now flow the coordinates (φ^J) to a neighborhood of Σ along $\frac{\nabla f}{\|\nabla f\|^2}$, staying inside $\widetilde{U}$. Making Σ even smaller if necessary, this construction gives local coordinates $((f, \varphi^J))_{J=1}^{n-1}$ on a neighborhood $U \subset M \setminus \operatorname{Crit} f$ of p with $U \approx F(f) \times \Sigma$, with $F(f)$ some open interval.

In the coordinates $((f,\varphi^J))_{J=1}^{n-1}$, one finds $\partial_f = \frac{\nabla f}{\|\nabla f\|^2}$ and obtains the usual level set flow formulas

$$g = \frac{df^2}{\|\nabla f\|^2} + g_f, \tag{3.8}$$

$$h_f = \frac{\|\nabla f\|}{2}\,\partial_f g_f \tag{3.9}$$

on U and $\{f\}\times\Sigma =: \Sigma_f$, respectively. Here, g_f denotes the induced metric on the regular level set $\{f\}\times\Sigma =: \Sigma_f$ of f in U and h_f denotes the second fundamental form of Σ_f in U with respect to the unit normal $\|\nabla f\|\,\partial_f$. Using harmonicity of f from (2.2), we obtain the usual formula for the mean curvature H_f of Σ_f with respect to the unit normal $\|\nabla f\|\,\partial_f$, that is

$$H_f = -\partial_f\|\nabla f\| \tag{3.10}$$

on Σ_f.

Now by Lemma 3.3, we know that $T = 0$ on U is equivalent to the existence of a smooth function $\lambda\colon U \to \mathbb{R}$ such that (3.6) holds on U. Rewriting this in our adapted coordinates (f,φ^J) and using the static vacuum equation (2.1), (3.6) implies

$$\partial_f g_f = -\frac{2\lambda f}{(n-1)\|\nabla f\|^2}\,g_f, \tag{3.11}$$

$$\partial_f\|\nabla f\| = \frac{\lambda f}{\|\nabla f\|} \tag{3.12}$$

on all Σ_f. Rewriting (3.11) using (3.9), we obtain

$$h_f = -\frac{\lambda f}{(n-1)\|\nabla f\|}\,g_f \tag{3.13}$$

so that in particular each Σ_f is umbilic when $T = 0$. Moreover, rewriting (3.12) as a vector field expression gives

$$\nabla\|\nabla f\|^2 = 2\lambda f\|\nabla f\|^2\,\nabla f \tag{3.14}$$

which shows that $\|\nabla f\|$ is constant on each level set Σ_f of f as can be seen by inserting all vector fields $X \in \Gamma(U)$ with $X(f) = 0$ on U into (3.14). This allows us to set

$$\psi(f) := \|\nabla f\||_{\Sigma_f} > 0 \tag{3.15}$$

for $f \in F(f)$. Inserting this into (3.12) gives

$$\psi'(f) = \frac{\lambda f}{\psi(f)} \tag{3.16}$$

on U, where $' = \frac{d}{df}$. In particular, λ is constant on each level set Σ_f of f so that we can suggestively write $\lambda = \lambda(f) = \frac{\psi'(f)\psi(f)}{f}$ on $F(f)$. Moreover, each level set Σ_f must have constant mean curvature

$$H_f = -\psi'(f). \tag{3.17}$$

In summary, we have established that $T = 0$ on U if and only if g_f satisfies

$$\partial_f g_f = -\frac{2\psi'(f)}{(n-1)\psi(f)}\, g_f \tag{3.18}$$

on all Σ_f for some smooth, positive function $\psi\colon U \to \mathbb{R}^+$ (which implies (3.15)) and

$$\mathrm{Ric} = \frac{\psi'(f)}{(n-1)f\psi(f)}\left((n-1)df^2 - \psi(f)^2 g_f\right) \tag{3.19}$$

holds on U by (3.6) and (3.8). In particular, this implies that all Σ_f are totally umbilic with constant mean curvature given by (3.17). Also, note that the static vacuum equations (2.1), (2.2) are automatically satisfied by metrics of this type via (3.14) and (3.19).

Using (3.8) and the definition of ψ from (3.15), (3.19) can be seen to be equivalent to

$$0 = \frac{\psi''(f)}{\psi(f)} - \frac{\psi'(f)^2}{(n-1)\psi(f)^2} - \frac{\psi'(f)}{f\psi(f)}, \tag{3.20}$$

$$\mathrm{Ric}_{g_f} = \frac{1}{n-1}\left(-\psi(f)\psi''(f) + \psi'(f)^2 - \frac{\psi(f)\psi'(f)}{f}\right) g_f \tag{3.21}$$

on U by standard computations, where Ric_{g_f} denotes the Ricci tensor of g_f on Σ_f. Standard ODE tricks show that the general solution to (3.20) is given by

$$\psi(f) = \left(\alpha f^2 + \beta\right)^{\frac{n-1}{n-2}} \tag{3.22}$$

for constants $\alpha, \beta \in \mathbb{R}$ satisfying

$$\alpha f^2 + \beta > 0 \tag{3.23}$$

on $F(f)$. Inserting (3.22) into (3.21) gives

$$\mathrm{Ric}_{g_f} = -\frac{4\alpha\beta\psi(f)^{\frac{2}{n-1}}}{n-2}\, g_f \tag{3.24}$$

on U. In particular, this shows that each manifold (Σ_f, g_f) is Einstein. Moreover, (3.18) and (3.22) give

$$\partial_f g_f = -\frac{4\alpha f}{(n-2)(\alpha f^2 + \beta)}\, g_f \tag{3.25}$$

on U and $F(f)$, respectively. Summarizing, we have shown that $T = 0$ on U is equivalent to the combination of (3.8), (3.15), (3.25), and (3.22) and (3.24) holding on U for constants $\alpha, \beta \in \mathbb{R}$ satisfying (3.23). Let us now discuss the different cases arising from picking specific cases for the signs of α, β.

First of all, for $\alpha = 0$, we have by (3.24) that g_f is Ricci flat, and by (3.25) that $\partial_f g_f = 0$. Now set $a := \beta^{\frac{n-1}{n-2}}$ which is well-defined as $\beta > 0$ by (3.23) and note that $a \in \mathbb{R}^+$ is unrestricted by (3.23). Then we can rewrite the static vacuum system (U, g, f) as $U \approx I \times \Sigma =: V$ for the open interval $I := a^{-1}F(f) \subseteq \mathbb{R}^+$, $g = dh^2 + \sigma$ on V for $h := a^{-1}f$ and with $\sigma := g_f$ being a fixed Ricci flat metric on Σ, and $f(h, \cdot) = ah$ on V satisfying $f(V) = F(f)$. Moreover, $\lambda = 0$ in this case by (3.16). This shows that for $\alpha = 0$, the system (U, g, f) is of Type 1 and that systems of Type 1 satisfy $T = 0$ on V as well as the static vacuum equations (2.1), (2.2). The latter statement exploits that σ is unrestricted other than being Ricci flat.

Second, for $\beta = 0$, we find that $\alpha > 0$ is unrestricted by (3.23). By (3.24), we learn that g_f is Ricci flat, while (3.25) gives

$$\partial_f g_f = -\frac{4}{(n-2)f}\, g_f. \tag{3.26}$$

Picking $\widetilde{\sigma} := g_{f_0}$ for any fixed $f_0 \in F(f)$, this gives $g_f = \left(\frac{f_0}{f}\right)^{\frac{4}{n-2}} \widetilde{\sigma}$. Now we set

$$\kappa := \frac{n-2}{2\alpha^{\frac{n-1}{n-2}} f_0^{\frac{n}{n-2}}},$$

$$r(f) := \kappa^{\frac{2}{n}} \left(\frac{f_0}{f}\right)^{\frac{2}{n-2}}$$

on $F(f)$ and find that $r = r(f)$ has the inverse function $f = f(r)$ given by

$$f(r) = \frac{\kappa^{\frac{n-2}{n}} f_0}{r^{\frac{n-2}{2}}} =: \frac{a}{r^{\frac{n-2}{2}}}$$

on $I := r(F(f)) \subseteq \mathbb{R}^+$ with unrestricted $a > 0$ by construction (noticing that $a = \frac{(n-2)^{\frac{n-2}{n}}}{2^{\frac{n-2}{n}} \alpha^{\frac{n-1}{n}}}$ with unrestricted $\alpha \in \mathbb{R}^+$). This gives

$$f'(r) = -\frac{(n-2)f(r)}{2r}$$

on I. Setting $\sigma := \kappa^{-\frac{4}{n}}\, \widetilde{\sigma}$ and recalling (3.8), (3.15), and (3.22), we obtain

$$g = \frac{df^2}{\psi(f)^2} + g_f = r^{n-2}\, dr^2 + r^2 \sigma$$

on $V := I \times \Sigma$, with (Σ, σ) being Ricci flat but otherwise unrestricted by (3.26). Moreover, we find

$$\lambda(r) = \frac{(n-1)(n-2)}{2r^n}$$

for $r \in I$ by (3.16). Consequently, for $\beta = 0$, the system (U, g, f) is of Type 2 and systems of Type 2 satisfy $T = 0$ on V as well as the static vacuum equations (2.1), (2.2).

Third, for $\alpha, \beta > 0$, we find that both $\alpha, \beta > 0$ are unrestricted by (3.23). We now observe that $\mathrm{R}_{g_f} < 0$ by (3.24) which allows us to pick $f_0 \in F(f)$ and set

$$r_0 := \sqrt{-\frac{(n-1)(n-2)}{\mathrm{R}_{g_{f_0}}}}, \tag{3.27}$$

$$\sigma := \frac{1}{r_0^2}\, g_{f_0}. \tag{3.28}$$

By (3.25), we find that $g_f = r(f)^2\, \sigma$ for $r\colon F(f) \to \mathbb{R}^+$ given by

$$r(f) := r_0 \left(\frac{\alpha f_0^2 + \beta}{\alpha f^2 + \beta}\right)^{\frac{1}{n-2}}. \tag{3.29}$$

Plugging this into the trace of (3.24) and exploiting that $\mathrm{R}_{r^2\sigma} = -\frac{(n-1)(n-2)}{r^2}$ gives

$$r_0(\alpha f_0^2 + \beta)^{\frac{1}{n-2}} = \frac{n-2}{2\sqrt{\alpha\beta}}$$

which removes our choice of f_0 and our definition of r_0 from the picture, giving

$$r(f) = \frac{n-2}{2\sqrt{\alpha\beta}\,(\alpha f^2 + \beta)^{\frac{1}{n-2}}}.$$

with inverse function $f\colon r(F(f)) \to \mathbb{R}^+$ given by

$$f(r) = \sqrt{\frac{1}{\alpha}\left(\left(\frac{n-2}{2\sqrt{\alpha\beta}}\right)^{n-2}\frac{1}{r^{n-2}} - \beta\right)},$$

where we note that f is well-defined on the interval $I := r(F(f)) \subseteq \mathbb{R}^+$. Using this, we obtain

$$\begin{aligned} f(r) &= a\sqrt{\frac{b}{r^{n-2}} - 1}, \\ g &= \frac{dr^2}{\frac{b}{r^{n-2}} - 1} + r^2\sigma \end{aligned}$$

on I and $V := I \times \Sigma$, respectively, for constants $a, b > 0$ given by

$$\begin{aligned} a &:= \sqrt{\frac{\beta}{\alpha}}, \\ b &:= \frac{1}{\beta}\left(\frac{n-2}{2\sqrt{\alpha\beta}}\right)^{n-2}. \end{aligned}$$

We note that, other than $a, b > 0$, a, b are unrestricted by (3.23) and that σ is an arbitrary Einstein metric on Σ satisfying $\mathrm{R}_\sigma = -(n-1)(n-2)$. However, it must hold that $I \subseteq (0, b^{\frac{1}{n-2}})$ which is implied by $I = r(F(f))$. Moreover, we find

$$\lambda(r) = \frac{(n-1)(n-2)b}{2r^n} \tag{3.30}$$

for $r \in I$ by (3.16). Consequently for $\alpha, \beta > 0$, the system (U, g, f) is of Type 3 and systems of Type 3 satisfy $T = 0$ on V as well as the static vacuum equations (2.1), (2.2).

Last but not least, for $\alpha\beta < 0$, we find that α, β are restricted by (3.23) such that

$$-\frac{\beta}{\alpha} < f^2 \text{ when } \alpha > 0 \quad \text{and} \quad -\frac{\beta}{\alpha} > f^2 \text{ when } \alpha < 0 \tag{3.31}$$

on U. We now observe that $\mathrm{R}_{g_f} > 0$ by (3.24) which allows us to pick $f_0 \in F(f)$ and set

$$r_0 := \sqrt{\frac{(n-1)(n-2)}{\mathrm{R}_{g_{f_0}}}}, \tag{3.32}$$

$$\sigma := \frac{1}{r_0^2}\, g_{f_0}. \tag{3.33}$$

By (3.25), we again have that $g_f = r(f)^2\,\sigma$ for $r\colon F(f) \to \mathbb{R}^+$ given by (3.29). Plugging this into the trace of (3.24) and exploiting that $\mathrm{R}_{r^2\sigma} = \frac{(n-1)(n-2)}{r^2}$ gives

$$r_0(\alpha f_0^2 + \beta)^{\frac{1}{n-2}} = \frac{n-2}{2\sqrt{-\alpha\beta}},$$

giving

$$r(f) = \frac{n-2}{2\sqrt{-\alpha\beta}\,(\alpha f^2 + \beta)^{\frac{1}{n-2}}}.$$

We note that $I := r(F(f)) \subseteq \mathbb{R}^+$ is unrestricted when $\alpha > 0$ while

$$I \subseteq \Big(\frac{n-2}{2\sqrt{-\alpha\beta}\beta^{\frac{1}{n-2}}}, \infty\Big) \tag{3.34}$$

when $\alpha < 0$ by monotonicity of $r\colon F(f) \to \mathbb{R}^+$ and as $F(f) \subseteq \mathbb{R}^+$ is restricted only by (3.31). The inverse function $f\colon r(F(f)) \to \mathbb{R}^+$ of $r = r(f)$ given by

$$f(r) = \sqrt{\frac{1}{\alpha}\left(\left(\frac{n-2}{2\sqrt{-\alpha\beta}}\right)^{n-2}\frac{1}{r^{n-2}} - \beta\right)}$$

and well-defined on I by the above. Using this, we obtain

$$f(r) = a\sqrt{\frac{b}{r^{n-2}} + 1},$$
$$g = \frac{dr^2}{\frac{b}{r^{n-2}} + 1} + r^2\sigma$$

on I and $V := I \times \Sigma$, respectively, for constants $a, b > 0$ given by

$$a := \sqrt{-\frac{\beta}{\alpha}},$$
$$b := -\frac{1}{\beta}\left(\frac{n-2}{2\sqrt{-\alpha\beta}}\right)^{n-2}.$$

We note that, other than $a > 0$, $b \neq 0$, a, b are not further restricted and that σ is an arbitrary Einstein metric on Σ satisfying $\mathrm{R}_\sigma = (n-1)(n-2)$. There is no restriction on I when $b > 0$ and the only restriction $I \subseteq (-b^{\frac{1}{n-2}}, \infty)$ when $b < 0$ by (3.34) and the definition of b. This is consistent with $\frac{b}{r^{n-2}} + 1 > 0$ on $\mathbb{R}^+$ when $b > 0$ and $\frac{b}{r^{n-2}} + 1 > 0$ precisely on $(-b^{\frac{1}{n-2}}, \infty)$ when $b < 0$. Moreover, we recover (3.30) for $r \in I$ by (3.16). Consequently, for $\alpha\beta < 0$, the system (U, g, f) is of Type 4 and systems of Type 4 satisfy $T = 0$ on V as well as the static vacuum equations (2.1), (2.2). □

Corollary 3.6 (Options for λ and ODEs for $\|\nabla f\|^2$)**.** *It will be useful later to observe that the proof of Theorem 3.4 shows that λ and $\|\nabla f\|^2$ satisfy*

Type 1	$\lambda \equiv 0$	$\nabla \|\nabla f\|^2 = 0$
Type 2	$\lambda = \frac{2(n-1)\|\nabla f\|^2}{(n-2)f^2}$	$\nabla \left(\frac{\|\nabla f\|^2}{f^{\frac{4(n-1)}{(n-2)}}} \right) = 0$
Type 3	$\lambda = \frac{2(n-1)\|\nabla f\|^2}{(n-2)(f^2+a^2)}$	$\nabla \left(\frac{\|\nabla f\|^2}{(f^2+a^2)^{\frac{2(n-1)}{(n-2)}}} \right) = 0$
Type 4	$\lambda = \frac{2(n-1)\|\nabla f\|^2}{(n-2)(f^2-a^2)}$	$\nabla \left(\frac{\|\nabla f\|^2}{\lvert f^2-a^2\rvert^{\frac{2(n-1)}{(n-2)}}} \right) = 0$

on V. Note that $a \notin f(V)$ so the ODE in Type 4 is also well-defined.

Remark 3.7 (The corresponding ODE in the (quasi-)Schwarzschild case)**.** *Let (M^n, g, f) be a static system with $f \neq 1$. Then the identity*

$$\nabla \|\nabla f\|^2 + \frac{4(n-1)}{(n-2)} \frac{f \|\nabla f\|^2 \, \nabla f}{1 - f^2} = 0 \tag{3.35}$$

with left hand side appearing in the divergence identity (1.10) *is equivalent to the ODE*

$$\nabla \left[\frac{\|\nabla f\|^2}{\lvert f^2 - 1\rvert^{\frac{2(n-1)}{n-2}}} \right] = 0 \tag{3.36}$$

on M, a special case of the ODE in Type 4, namely for $a = 1$, see Corollary 3.6. It holds in all quasi-Schwarzschild systems (with nonzero mass) as can be seen by direct computations.

The following global characterization of static vacuum systems with $T = 0$ can be proved by appealing to real analyticity. We choose to prove it 'by hand' as this adds some useful insights.

Corollary 3.8 (Global characterization of $T = 0$)**.** *Let (M^n, g, f), $n \geq 3$, be a static vacuum system. Then $T = 0$ on M if and only if either $f \equiv$ const on M or (M, g) is (globally) isometric to a suitable piece of one of the Riemannian manifolds of Types 1, 2, 3, or 4 in Theorem 3.4 and f is regular on M and corresponds to the corresponding restriction of the lapse function of the same system of Type 1, 2, 3, or 4 under this isometry.*

Proof. First, if (M, g, f) is a piece of a static vacuum system of Type 1, 2, 3, or 4, we know from Corollary 3.6 that f is regular on M. Next, from Theorem 3.4, we know that $T = 0$ on M. Also, if (M, g, f) is a static vacuum system with $f \equiv$ const on M, we trivially know that $T = 0$ by definition of T in (1.14).

On the other hand, if (M, g, f) is a static vacuum system with $T = 0$ and $f \neq$ const on M, we know from Theorem 3.4 that (M, g, f) looks like one of the systems of Types 1, 2, 3, or 4 locally near each regular point. From the ODEs in Corollary 3.6, we can deduce that there exists a constant $\rho > 0$ such that

Type 1	$\|\nabla f\|^2 = \rho$
Type 2	$\|\nabla f\|^2 = \rho f^{\frac{4(n-1)}{(n-2)}}$
Type 3	$\|\nabla f\|^2 = \rho(f^2 + a^2)^{\frac{2(n-1)}{(n-2)}}$
Type 4	$\|\nabla f\|^2 = \rho\lvert f^2 - a^2\rvert^{\frac{2(n-1)}{(n-2)}}$

on this neighborhood. Here, we know that $\rho > 0$ because the neighborhood contains no critical points of f and the factors multiplied by ρ are strictly positive on said neighborhood by the assumptions on the interval I. Now suppose towards a contradiction that f has $\operatorname{Crit} f \neq \emptyset$. First note that by smoothness of all involved quantities and because of the geometric nature of the radial coordinate r, if two such (open) neighborhoods overlap, they must be of the same type, have the same constants a, b, and $\rho > 0$ (where applicable), and the same Einstein metric σ on the intersections of the level sets of r in those neighborhoods. Consequently, each connected component of $M \setminus \operatorname{Crit} f$ has a uniform type, fixed constants a, b, and $\rho > 0$ (where applicable), and a global coordinate r.

As $\operatorname{Crit} f$ is closed by continuity and $\operatorname{Crit} f \neq M$ by assumption, we know that every connected component U of $M \setminus \operatorname{Crit} f \neq \emptyset$ is open in M and has $\partial U \subseteq \operatorname{Crit} f$. If $\partial U = \emptyset$ then U is closed in M and by connectedness of M we have $U = M$ and hence $\operatorname{Crit} f = \emptyset$ as desired. Now suppose towards a contradiction that $\partial U \neq \emptyset$. By continuity of f and $\|\nabla f\|^2$, this leads to the contradiction $\rho = 0$ on U in case U is of Types 1, 2, or 3 because $f > 0$ and $f^2 + a^2 > 0$ by the definition of static systems.

The same contradiction arises if U is of Type 4 unless $f|_{\partial U} = a$. Combining this with the formula for f in Type 4, we learn that $r \to \infty$ when approaching ∂U. Now pick $x \in U$ and $q \in \partial U$ and let $\gamma\colon [0,1] \to M$ denote the geodesic connecting $\gamma(0) = x$ to $\gamma(1) = q$ in M. If $\gamma|_{[0,1)}$ does not run entirely within U or in other words if $\gamma([0,1)) \cap \partial U \neq \emptyset$, there exists a smallest parameter $0 < s_\# < 1$ such that $\gamma(s_\#) \in \partial U$ while $\gamma|_{[0,s_\#)} \subset U$ because $\gamma([0,1])$ is compact and ∂U is closed. Thus, without loss of generality, we can and will assume that indeed $\gamma([0,1)) \subset U$, replacing the original endpoint q by $\gamma(s_\#)$ and reparametrizing γ accordingly with an affine parameter transformation. As we have assumed that $\gamma([0,1)) \subset U$, we can now consider the function $S\colon [0,1) \to \mathbb{R}^+$ given by $S := r \circ \gamma$. Then computing in local neighborhoods $I \times \Sigma$ on which we have coordinates $(r, \varphi^K)_{K=1}^{n-1}$ as constructed above, we have $\dot\gamma = \dot S\, \partial_r|_\gamma + \dot X^K \partial_{\varphi^K}|_\gamma$ on all suitably small intervals $J \subseteq [0,1)$, where $X^K := \varphi^K \circ \gamma$ on J. The radial part of the geodesic equation for γ gives

$$\ddot S - \frac{f'(S)}{f(S)} \dot S^2 = \frac{S f(S)^2}{a^2}\, \sigma_\gamma(\dot X, \dot X)$$

on J. In particular, if $\dot{S}(s_*) = 0$ for some $s_* \in [0,1)$, we learn from $\dot{\gamma}(s_*) \neq 0$ that $\ddot{S}(s_*) > 0$. Hence S has a strict local minimum at s_*. In particular, S can have at most one critical point in $[0,1)$. Without loss of generality, we will assume that there is no critical point of S on $(0,1)$, replacing the original starting point x by $\gamma(s_*)$ and reparametrizing γ accordingly with an affine parameter transformation. Moreover, as we have seen that $S(s) \to \infty$ for $s \to 1$, it follows that $\dot{S} > 0$ on $(0,1)$. For any $0 < \varepsilon < \frac{1}{2}$, this allows us to estimate the length $L[\gamma]$ of γ from below by

$$\begin{aligned} L[\gamma] \geq L[\gamma|_{(\varepsilon,1-\varepsilon)}] &= \int_\varepsilon^{1-\varepsilon} \|\dot{\gamma}(s)\|\, ds \geq \int_\varepsilon^{1-\varepsilon} \frac{a\dot{S}(s)}{f(S(s))}\, ds \\ &= \int_\varepsilon^{1-\varepsilon} \frac{\dot{S}(s)}{\sqrt{1+\frac{b}{S(s)^{n-2}}}}\, ds = \int_{c_\varepsilon}^{d_\varepsilon} \frac{1}{\sqrt{1+\frac{b}{r^{n-2}}}}\, dr =: E_\varepsilon \end{aligned}$$

for suitable constants $S(0) < c_\varepsilon < d_\varepsilon < \infty$ with $c_\varepsilon \to S(0)$ and $d_\varepsilon \to \infty$ as $\varepsilon \to 0$. As is well-known, this shows that $E_\varepsilon \to \infty$ as $\varepsilon \to 0$, giving the desired contradiction. Hence $\operatorname{Crit} f = \emptyset$. This also proves the remaining claims as M is connected and smoothly partitioned by the regular level sets of f and because r is the scalar curvature radius and thus a geometric coordinate. □

4. Recovering the Schwarzschild geometry

In this section, we will prove Theorem 1.9. Its proof heavily relies on Corollary 3.8. Before we start, let us make the following remark.

Remark 4.1 (Simplified rigidity argument). *Note that proving Theorem 1.9 with the additional assumption that* (3.35) *holds on M would be somewhat simpler, readily establishing the absence of critical points as in the proof of Corollary 3.8 as well as ruling out Types 1–3 in Theorem 3.4 and fixing $a = 1$ in Type 4. While assuming* (3.35) *would be fully sufficient for getting rigidity in all claimed geometric inequalities when $p > p_n$, it is important to prove Theorem 1.9 without this extra assumption to be able to include the threshold case $p = p_n$.*

Proof of Theorem 1.9. By Corollary 3.8, we know that f regularly foliates $M \cup \partial M$ as $f =$ const on M contradicts Remark 2.6. Also we know that (M, g, f) is isometric to a suitable piece of a system of either of the Types 1–4 with fixed constants a, b. As in our setup here M is regularly foliated by closed level sets of f up to the boundary e.g. via Lemma 2.5 and Remark 2.4, we know that (M, g, f) is in fact globally isometric to a system of either of the Types 1–4 with fixed constants a, b, a fixed closed Einstein manifold (Σ, σ), and a fixed interval $I \subseteq \mathbb{R}^+$ which satisfies $I \subseteq (-b^{\frac{1}{n-2}}, \infty)$ in the $b > 0$ case of Type 4. In particular, (M, g) is a warped product.

Next, we know from the proof of Corollary 3.8 that $\|\nabla f\|$ is fully determined by f via a precise formula, up to a multiplicative constant $\rho > 0$. Now recall from Lemma 2.5 that $\|\nabla f\| \to 0$ as $|x| \to \infty$ for asymptotically flat static vacuum systems and note that this rules out Types 1–3 as $f \to 1$ as $|x| \to \infty$. For Type 4, the same argument fixes $a = 1$.

Now let $x\colon M \setminus K \longrightarrow \mathbb{R}^n \setminus \overline{B}$ denote a diffeomorphism making (M, g, f) asymptotically flat and denote the induced coordinates by (x^i). For convenience, let us switch to standard polar coordinates $(|x|, \theta^J)_{J=1}^{n-1}$ for $\mathbb{R}^n$ associated to (x^i) so that $(\partial_{\theta^K})^i = O(|x|)$ as $|x| \to \infty$ and $\delta(\partial_{\theta^I}, \partial_{\theta^J}) = r^2 (g_{\mathbb{S}^{n-1}})(\partial_{\theta^I}, \partial_{\theta^J})$ for $I, J, K = 1, \ldots, n-1$. Thinking of the coordinate

r in the Type 4 representation of our system as a function $r\colon M \to \mathbb{R}^+$, the asymptotic assumptions we made on f in (2.6) give

$$r = \left(-\frac{b}{2m}\right)^{\frac{1}{n-2}} |x| + o_2(|x|)$$

as $|x| \to \infty$. On the other hand, recalling that r denotes the scalar curvature radius of the level sets of f and plugging in $\mathrm{R} = 0$ as well as the asymptotic decay assertions from Lemma 2.5 and in particular (2.9), the twice contracted Gauß equation gives

$$\frac{(n-1)(n-2)}{\left(-\frac{b}{2m}\right)^{\frac{2}{n-2}} |x|^2} - \frac{n-2}{n-1}\left(\frac{n-1}{|x|}\right)^2 = o(|x|^{-2})$$

as $|x| \to \infty$, where we have used that we already know that all level sets of f are totally umbilic. This gives $b = -2m$ (in line with Remark 3.5) and thus $r = |x| + o_2(|x|)$ as $|x| \to \infty$. Taking a ∂_{θ^K}-derivative of this identity, one sees that

$$\frac{\partial r}{\partial \theta^K} = \partial_{\theta^K} o_2(|x|) = (\partial_{\theta^K})^i o_1(1) = o_1(|x|)$$

as $|x| \to \infty$ for $K = 1, \ldots, n-1$.

To see that (Σ, σ) is indeed isometric to the standard round $(n-1)$-sphere of radius 1, we need to carefully consider the asymptotic decay of g. As implicitly done above and as usual, we will interpret σ as an r-independent tensor field on M which is applied to tensor fields on M by first projecting them tangentially to the level sets $\Sigma_f \approx \{r(f)\} \times \partial M$ of f. Similarly, we interpret $g_{\mathbb{S}^{n-1}}$ as an $|x|$-independent tensor field on $\mathbb{R}^+ \times \mathbb{S}^{n-1}$ by projection onto round spheres as usual. Exploiting this convention, our asymptotic flatness assumptions in Definition 2.2 translated to spherical polar coordinates and the above insights give

$$|x|^2 (g_{\mathbb{S}^{n-1}})_{\theta^I\theta^J} + o(|x|^2) = g_{\theta^I\theta^J} = o(|x|^2) + |x|^2(1 + o(1))\sigma_{\theta^I\theta^J}$$

as $|x| \to \infty$ for $I, J = 1, \ldots, n-1$. This can easily be rewritten as

$$\sigma_{\theta^I\theta^J} = (1 + o(1))(g_{\mathbb{S}^{n-1}})_{\theta^I\theta^J} = (g_{\mathbb{S}^{n-1}})_{\theta^I\theta^J} + o(r)$$

as $r \to \infty$ for $I, J = 1, \ldots, n-1$. As σ is independent of r, this allows us to conclude (Σ, σ) is isometric to the round $(n-1)$-sphere of radius 1 as desired.

Thus, as $b = -2m$, we deduce that (M, g) must be isometric to the piece $[r_0, \infty) \times \mathbb{S}^{n-1}$ of the Schwarzschild manifold (M^n_m, g_m) of mass m for $f_0 > 0$, with f corresponding to f_m while $(M \setminus \partial M, g)$ must be isometric to the piece $(r_0, \infty) \times \mathbb{S}^{n-1}$ of the Schwarzschild manifold (M^n_m, g_m) of mass m when $f_0 = 0$, with f corresponding to f_m. Switching to isotropic coordinates then also allows us to conclude that the claims extend to ∂M when $f_0 = 0$. Here, $r_0 := r(\partial M)$ denotes the scalar curvature radius of ∂M. $\square$

5. The divergence identity

With the help of Lemma 3.2, we are now in the position to prove Theorem 1.5.

Proof of Theorem 1.5. First, note that $f \neq 1$ on M by assumption. Then clearly

$$\begin{aligned}\left\|\nabla\|\nabla f\|^2+\frac{4(n-1)}{n-2}\frac{f\|\nabla f\|^2\,\nabla f}{1-f^2}\right\|^2 &= \|\nabla\|\nabla f\|^2\|^2+\frac{8(n-1)}{n-2}\frac{f\|\nabla f\|^2}{1-f^2}\langle\nabla\|\nabla f\|^2,\nabla f\rangle\\ &\quad+\frac{16(n-1)^2}{(n-2)^2}\frac{f^2\|\nabla f\|^6}{(1-f^2)^2}\end{aligned}$$

holds on M. Combining this with Lemma 3.2, we get that

$$\begin{aligned}&\frac{(n-2)^2}{(n-1)^2}fF(f)\|T\|^2+F(f)\left(\frac{n}{2(n-1)}+\frac{p-3}{2}\right)\frac{1}{f}\left\|\nabla\|\nabla f\|^2+\frac{4(n-1)}{n-2}\frac{f\|\nabla f\|^2}{1-f^2}\nabla f\right\|^2\\ &\quad=\|\nabla f\|^2\,F(f)\left[\frac{1}{f}\Delta\|\nabla f\|^2-\frac{1}{f^2}\langle\nabla\|\nabla f\|^2,\nabla f\rangle\right.\\ &\qquad\qquad+\left(\frac{4n}{n-2}+\frac{4(n-1)(p-3)}{n-2}\right)\frac{1}{1-f^2}\langle\nabla\|\nabla f\|^2,\nabla f\rangle\\ &\qquad\qquad\left.+\left(\frac{8n(n-1)}{(n-2)^2}+\frac{8(n-1)^2(p-3)}{(n-2)^2}\right)\frac{f}{(1-f^2)^2}\|\nabla f\|^4\right]\\ &\qquad+\frac{p-3}{2}\frac{F(f)}{f}\|\nabla\|\nabla f\|^2\|^2\end{aligned}$$

holds on M for any smooth function $F\colon [0,1)\cup(1,\infty)\to\mathbb{R}$ and any $0<f<1$ or $f>1$. Also, using (2.1), one computes

$$\begin{aligned}&\operatorname{div}\left(\frac{F(f)}{f}\|\nabla f\|^{p-3}\,\nabla\|\nabla f\|^2+G(f)\|\nabla f\|^{p-1}\,\nabla f\right)\\ &\quad=\frac{F(f)}{f}\|\nabla f\|^{p-3}\Delta\|\nabla f\|^2+\left(\frac{F'(f)}{f}-\frac{F(f)}{f^2}+\frac{p-1}{2}G(f)\right)\|\nabla f\|^{p-3}\langle\nabla\|\nabla f\|^2,\nabla f\rangle\\ &\qquad+\frac{p-3}{2}\frac{F(f)}{f}\|\nabla f\|^{p-5}\,\|\nabla\|\nabla f\|^2\|^2+G'(f)\|\nabla f\|^{p+1}\end{aligned}$$

on $M\setminus\operatorname{Crit} f$ for any $p\in\mathbb{R}$ and any smooth functions $F,G\colon [0,1)\cup(1,\infty)\to\mathbb{R}$, where $'=\frac{d}{df}$. Combining these two identities, we find

$$\begin{aligned}&\|\nabla f\|^{p-3}\,F(f)\left[\frac{(n-2)^2 f}{(n-1)^2}\|T\|^2\right.\\ &\qquad\left.+\left(\frac{n}{2(n-1)}+\frac{p-3}{2}\right)\frac{1}{f}\left\|\nabla\|\nabla f\|^2+\frac{4(n-1)}{(n-2)}\frac{f\|\nabla f\|^2}{(1-f^2)}\nabla f\right\|^2\right]\end{aligned}$$

$$
\begin{aligned}
= \|\nabla f\|^2 \Bigg[\operatorname{div} \left(\frac{F(f)}{f} \|\nabla f\|^{p-3} \nabla \|\nabla f\|^2 + G(f) \|\nabla f\|^{p-1} \nabla f \right) \\
+ \Bigg\{ - \left(\frac{F'(f)}{f} - \frac{F(f)}{f^2} + \frac{p-1}{2} G(f) \right) \|\nabla f\|^{p-3} \langle \nabla \|\nabla f\|^2, \nabla f \rangle - G'(f) \|\nabla f\|^{p+1} \\
- \frac{F(f)}{f^2} \|\nabla f\|^{p-3} \langle \nabla \|\nabla f\|^2, \nabla f \rangle \\
+ \left(\frac{4n}{n-2} + \frac{4(n-1)(p-3)}{n-2} \right) \frac{F(f)}{1-f^2} \|\nabla f\|^{p-3} \langle \nabla \|\nabla f\|^2, \nabla f \rangle \\
+ \frac{2(n-1)}{n-2} \left(\frac{4n}{n-2} + \frac{4(n-1)(p-3)}{n-2} \right) \frac{fF(f)}{(1-f^2)^2} \|\nabla f\|^{p+1} \Bigg\} \Bigg]
\end{aligned}
$$

on $M \setminus \operatorname{Crit} f$ for any $p \in \mathbb{R}$ and any smooth functions $F, G \colon [0,1) \cup (1, \infty) \to \mathbb{R}$. Now, plugging in the precise forms of F and G given in (1.12) and (1.13) and observing that they solve the system of ODEs

$$
\begin{aligned}
F'(t) &= 4 \left(\frac{(n-1)(p-1)}{(n-2)} - 1 \right) \frac{tF(t)}{1-t^2} - \frac{p-1}{2} tG(t), \\
G'(t) &= \frac{8(n-1)}{n-2} \left(\frac{(n-1)(p-1)}{(n-2)} - 1 \right) \frac{tF(t)}{(1-t^2)^2}
\end{aligned}
$$

for $t \in [0,1) \cup (1, \infty)$, one detects that the term inside the braces vanishes and obtains (1.10) on $M \setminus \operatorname{Crit} f$ for all $p \in \mathbb{R}$.

Now let us address the claimed continuity of the divergence on M for $p \geq 3$: First, we note that second term in the argument of the divergence is continuously differentiable on M for $p \geq 3$ by smoothness of f and $\|\nabla f\|^2$, as one immediately sees upon rewriting $\|\nabla f\|^{p-1} = (\|\nabla f\|^2)^{\frac{p-1}{2}}$. Considering the first term $\|\nabla f\|^{p-3} \nabla \|\nabla f\|^2$ in the argument of the divergence, we note that it vanishes at all critical points of f as $\|\nabla f\|^2 \geq 0$ attains a minimum there, hence it is continuous on M. Next, its derivative (defined on $M \setminus \operatorname{Crit} f$) extends continuously across critical points because $\nabla \|\nabla f\|^2 = 2\nabla^2 f(\nabla f, \cdot)$, so that the derivative of $\|\nabla f\|^{p-3}$ is bounded from above by $\|\nabla f\|^{p-4} \|\nabla^2 f\|$ by the Cauchy–Schwarz inequality away from $\operatorname{Crit} f$; as it gets multiplied by another factor of $\nabla \|\nabla f\|^2$, we recover a continuous upper bound multiplied by $\|\nabla f\|^{p-3}$ which goes to zero for $p > 3$. This bound has a numerical factor $p - 3$, hence it identically vanishes for $p = 3$. Altogether, noting that the other contributions to the derivative of $\|\nabla f\|^{p-3} \nabla \|\nabla f\|^2$ are smooth anyways, this establishes that the divergence and thus (1.10) continuously extends to M for $p \geq 3$ as claimed. Last but not least, the right hand side of (1.10) is manifestly non-negative if (1.11) holds which gives (1.15). $\square$

It may be worth noting that the free constants $c, d \in \mathbb{R}$ in the statement of Theorem 1.5 correspond to the free constants of integration of the ODEs for F and G arising in its proof. Moreover, it may be useful to note that, for the Schwarzschild system (M^n_m, g_m, f_m) of mass $m \neq 0$, the first term of the right-hand side of (1.10) vanishes by Lemma 3.1, while the second term manifestly vanishes by an explicit computation, see also Remark 3.7. Moreover, $\operatorname{Crit} f_m = \emptyset$. Hence by Theorem 1.5, the vector field inside the divergence of (1.10) is divergence-free in the Schwarzschild case and thus gives rise to a three-parameter family (parametrized by c, d, p) of conserved quantities

$$\int_{\{f_m=z\}} \left[\frac{F(f_m)}{f_m} \|\nabla_m f_m\|_m^{p-3} g_m(\nabla_m \|\nabla_m f_m\|_m^2, \nu_m) + G(f_m)\|\nabla_m f_m\|_m^{p-1} g_m(\nabla_m f_m, \nu_m) \right] dS_m,$$

by the divergence theorem. Here, dS_m, ∇_m, $\|\cdot\|_m$, and ν_m denote the area element on $\{f_m = z\}$, the covariant derivative, and the tensor norm induced by g_m, and the g_m-unit normal to $\{f_m = z\}$ pointing towards infinity. Evaluating this conserved quantity for $z \to 1$, one finds[8] $-\mathcal{F}_p^{c,d}(m)$ from (1.18).

6. Geometric inequalities

In this section, we will prove the geometric inequality in Theorem 1.7 and its equivalent formulation Theorem 1.8. To do so, we will exploit the divergence identity (1.10) by estimating its right-hand side from below by zero and applying the divergence theorem in combination with the asymptotic flatness assumptions. To apply a suitably adapted version of the divergence theorem, we will need to first assert integrability of the left hand side in (1.10); our proof of said integrability is inspired by [CM24, Section 4], see also [AM20]. For the equality claim, we will rely on the rigidity assertion of Theorem 1.9. We will then also show how to derive Theorems 1.1 and 1.3 from Theorems 1.7 and 1.8.

Remark 6.1. *In the setting of Theorem 1.7, consider first the case when $f_0 \in [0,1)$. We have $F \geq 0$ on $[f_0, 1)$ if and only if both $c + d \geq 0$ and $cf_0^2 + d \geq 0$, see also Figure 1. In particular, $F > 0$ holds on $(f_0, 1)$ provided that in addition we do not have $c = d = 0$. Similarly if $f_0 \in (1, \infty)$, we have $F \geq 0$ on $(1, f_0]$ if and only if both $c+d \geq 0$ and $cf_0^2+d \geq 0$ and $F > 0$ on $(1, f_0)$ if in addition $c = d = 0$ does not hold, see also Figure 2. This explains the assumptions made on c, d in Theorem 1.7.*

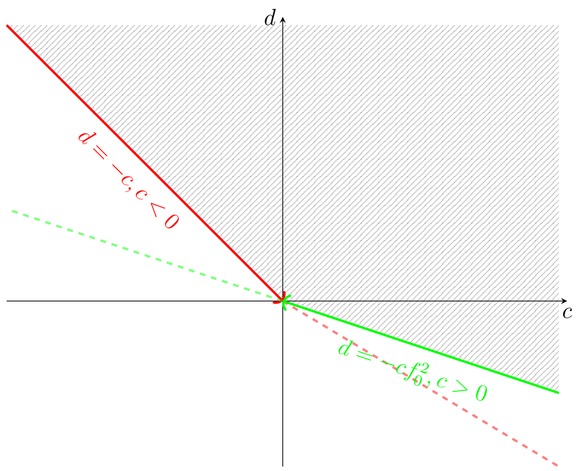

Figure 1. The shaded region together with the red semi-axis represents all $(c, d) \in \mathbb{R}^2$ such that $F > 0$ on $[f_0, 1)$ for some $f_0 \in [0, 1)$. On the green semi-axis, $F(f_0) = 0$ and $F > 0$ on $(f_0, 1)$ so in particular $F \geq 0$.

[8] See also the more general discussion and the computations in Section 6.

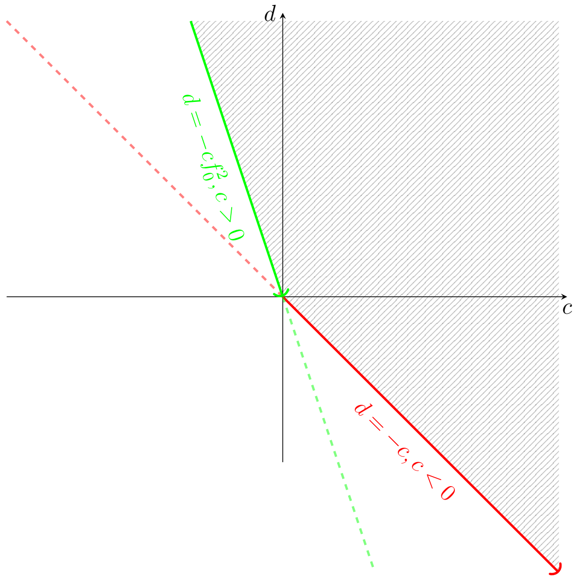

FIGURE 2. The shaded region together with the red semi-axis represents all $(c, d) \in \mathbb{R}^2$ such that $F > 0$ on $(1, f_0]$ for some $f_0 \in (1, \infty)$. On the green semi-axis, $F(f_0) = 0$ and $F > 0$ on $(1, f_0)$ so in particular $F \geq 0$.

Proof of Theorem 1.7. First of all, by Remark 2.6 and Lemma 2.13, we know that $m \neq 0$ and $\kappa \neq 0$, with $m, \kappa > 0$ if $f_0 \in [0, 1)$ and $m, \kappa < 0$ if $f_0 \in (1, \infty)$. Next, note that the right-hand side of (1.10) is non-negative on $M \setminus \operatorname{Crit} f$ if $c, d \in \mathbb{R}$ satisfy $c + d \geq 0$, $cf_0^2 + d \geq 0$ as this gives $F(f) \geq 0$ by Remark 6.1. Now set

$$\mathcal{D} := \operatorname{div}\left(\frac{F(f)}{f} \|\nabla f\|^{p-3} \nabla \|\nabla f\|^2 + G(f) \|\nabla f\|^{p-1} \nabla f\right)$$

on $M \setminus \operatorname{Crit} f$. Then by Theorem 1.5, $\mathcal{D}$ is non-negative and satisfies (1.10) on $M \setminus \operatorname{Crit} f$. Aiming for an application of the divergence theorem to $\mathcal{D}$, let us show that $\mathcal{D}$ can be extended to a dV-integrable function on $M \cup \partial M$, where the volume measure dV naturally extends to ∂M by smoothness of the metric g. Let us first extend $\mathcal{D}$ to ∂M. As f is regular in a neighborhood of ∂M, $\mathcal{D}$ continuously extends to ∂M; this is immediate when $f_0 > 0$ and follows from (2.17) for $f_0 = 0$ via

$$\frac{F(f)}{f} \nabla \|\nabla f\|^2 = 2F(f) \operatorname{Ric}(\nabla f, \cdot).$$

Thus $\mathcal{D}$ is dV-integrable on a regular neighborhood of ∂M. To analyze the behavior of $\mathcal{D}$ towards the asymptotic end of (M, g, f), recall that by the asymptotic decay established in Lemma 2.5, we know that f has no critical points in a suitable neighborhood of infinity. This means we can choose a compact subset $K \subseteq M$ such that $(M \setminus K) \cap \operatorname{Crit} f = \emptyset$ and a diffeomorphism $x\colon M \setminus K \longrightarrow \mathbb{R}^n \setminus \overline{B}$ making (M, g, f) asymptotically flat. The asymptotic assumption (2.6) implies that

$$\begin{aligned}F(f) &= F(f_m) + o(|x|^{(n-1)(p-1)-(n-2)}),\\ F'(f) &= F'(f_m) + o(|x|^{(n-1)(p-1)}),\\ G(f) &= G(f_m) + o(|x|^{(n-1)(p-1)}),\\ G'(f) &= G'(f_m) + o(|x|^{(n-1)(p-1)+2(n-2)})\end{aligned}$$

as $|x| \to \infty$, which can be verified most easily via the ODEs for F and G printed in the proof of Theorem 1.5. To study the asymptotics of the first term in $\mathcal{D}$, we apply the Bochner formula and the static vacuum equation (2.2) to see that

$$\frac{1}{2}\Delta\|\nabla f\|^2 = \|\nabla^2 f\|^2 + \mathrm{Ric}(\nabla f, \nabla f) \tag{6.1}$$

on M, as otherwise we would have to deal with the asymptotic behavior of third derivatives of f about which we have not made any assumptions. Doing so, we find that

$$\begin{aligned}&\operatorname{div}\left(\frac{F(f)}{f}\|\nabla f\|^{p-3}\,\nabla\|\nabla f\|^2\right)\\ &= \left(\frac{F'(f)}{f} - \frac{F(f)}{f^2}\right)\|\nabla f\|^{p-3}\,\langle\nabla f, \nabla\|\nabla f\|^2\rangle + \frac{p-3}{2}\frac{F(f)}{f}\|\nabla f\|^{p-5}\|\nabla\|\nabla f\|^2\|^2\\ &\quad + \frac{2F(f)}{f}\|\nabla f\|^{p-3}\,\left(\|\nabla^2 f\|^2 + \mathrm{Ric}(\nabla f, \nabla f)\right),\end{aligned}$$

on $M \setminus K$. Taken together with Lemma 2.5, we find

$$\mathcal{D} = \mathcal{D}_m + o(|x|^{-n}) = o(|x|^{-n})$$

as $|x| \to \infty$. Here, $\mathcal{D}_m$ denotes the divergence $\mathcal{D}$ for $f = f_m$ and $g = g_m$ and we are using that we have seen that $\mathcal{D}_m = 0$ at the end of Section 5. Hence by (2.11), $\mathcal{D}$ is dV-integrable on $M \setminus K$. It remains to study the dV-integrability of $\mathcal{D}$ near $\mathrm{Crit}\, f$. As the divergence theorem readily applies when $\mathrm{Crit}\, f = \emptyset$, we will assume without loss of generality that $\mathrm{Crit}\, f \neq \emptyset$.

To establish dV-integrability near $\mathrm{Crit}\, f$, let us first recall from the work of Cheeger–Naber–Valtorta [CNV15] and Hardt–Hoffmann-Ostenhof–Hoffmann-Ostenhof–Nadirashvili [HHOHON99] that $\mathrm{Crit}\, f$ is a set of Hausdorff dimension at most $n-2$ as f is harmonic. Now note that dV is absolutely continuous with respect to the n-dimensional Hausdorff measure $\mathcal{H}^n$ and vice versa, with bounded densities, respectively, by (2.11). This gives $dV(\mathrm{Crit}\, f) = 0$. Setting

$$W_\varepsilon := \{\|\nabla f\|^2 < \varepsilon\} \tag{6.2}$$

for all $\varepsilon > 0$, it readily follows that $\mathrm{Crit}\, f \subset W_\varepsilon$, $W_\varepsilon \subset M$ is open, and $W_\varepsilon \cap \partial M = \emptyset$ for suitably small $0 < \varepsilon < \mu$. Moreover, it follows from purely topological arguments[9] as well as from the already established fact that $\mathrm{Crit}\, f$ is compact that, for suitably small $0 < \varepsilon < \overline{\mu}$, W_ε has finitely many connected components which are all bounded except precisely one which is a neighborhood of infinity. Next, we observe that $\partial W_\varepsilon = \{\|\nabla f\|^2 = \varepsilon\}$ is closed and satisfies $\partial W_\varepsilon \cap \mathrm{Crit}\, f = \emptyset$ for all $\varepsilon > 0$.

[9] See [CM24, page 15] for details.

It is well-known that static vacuum systems are real analytic in suitable coordinate systems (see e.g. [Chr05]), hence both f and $\|\nabla f\|^2$ are real analytic functions on M. From the Morse–Sard theorem [SS72, Theorem 1], we can then deduce that $f(\operatorname{Crit} f)$ is finite and that $\|\nabla f\|^2(\operatorname{Crit}\|\nabla f\|^2)$ is discrete. Moreover, we know that $\operatorname{Crit} f \subseteq \operatorname{Crit}\|\nabla f\|^2$ because any critical point of f is a local minimum of $\|\nabla f\|^2$. Hence $0 \in \operatorname{Crit}\|\nabla f\|^2$ as we have assumed $\operatorname{Crit} f \neq \emptyset$ so that, by discreteness of $\operatorname{Crit}\|\nabla f\|^2$, there must be a threshold $\delta > 0$ such that

$$\|\nabla f\|^2 \geq \delta$$

on $\operatorname{Crit}\|\nabla f\|^2 \setminus \operatorname{Crit} f$. Then, for $0 < \varepsilon < \delta$, the implicit function theorem applied to $\|\nabla f\|^2$ asserts that ∂W_ε must be a smooth hypersurface with multiple but finitely many components.

Now let $U \subseteq M$ be an open domain with smooth boundary ∂U such that $U \supset \operatorname{Crit} f$, $\overline{U} \cap \partial M = \emptyset$, and $\overline{U} \subset K$, with K the compact subset of M defined above. We can then extend $\mathcal{D}$ to U (and thus to $M \cup \partial M$ in combination with the above) by setting $\mathcal{D} := 0$ on $\operatorname{Crit} f$ and obtain dV-measurability of $\mathcal{D}$ on U (and thus on $M \cup \partial M$) from the fact that $\operatorname{Crit} f$ has dV-measure zero. To prove dV-integrability of $\mathcal{D}$ on U (and thus on M), we introduce the abbreviation

$$Z := \frac{F(f)}{f}\|\nabla f\|^{p-3}\,\nabla\|\nabla f\|^2 + G(f)\|\nabla f\|^{p-1}\,\nabla f$$

on $U \setminus \operatorname{Crit} f$ and extend Z by 0 on $\operatorname{Crit} f$ so that Z is dV-measurable on U, again because $\operatorname{Crit} f$ has dV-measure zero. To study the dV-integrability of $\mathcal{D}$ on U, we want to apply the monotone convergence theorem. Let us consider a smooth cut-off function $\xi\colon [0,\infty) \to [0,1]$ satisfying

$$\begin{cases} \xi(t) = 0 & \text{if } t \leq \frac{1}{2}, \\ \xi(t) = 1 & \text{if } t \geq \frac{3}{2}, \\ 0 < \dot{\xi}(t) < 2 & \text{if } \frac{1}{2} < t < \frac{3}{2}. \end{cases}$$

For $\varepsilon > 0$, we define $\xi_\varepsilon\colon [0,\infty) \to [0,1]$ by setting $\xi_\varepsilon(t) := \xi(\frac{t}{\varepsilon})$ and observe that

$$\begin{cases} \xi_\varepsilon(t) = 0 & \text{if } t \leq \frac{\varepsilon}{2}, \\ \xi_\varepsilon(t) = 1 & \text{if } t \geq \frac{3\varepsilon}{2}, \\ 0 < \dot{\xi}_\varepsilon(t) < \frac{2}{\varepsilon} & \text{if } \frac{\varepsilon}{2} < t < \frac{3\varepsilon}{2}, \\ \xi_{\varepsilon_0} \leq \xi_{\varepsilon_1} & \text{if } 0 < \varepsilon_1 < \varepsilon_0, \\ \xi_\varepsilon \to 1 & \text{as } \varepsilon \to 0. \end{cases}$$

Next, we cut off $\|\nabla f\|^2$ near $\operatorname{Crit} f$ or in other words analyze the function $\Theta_\varepsilon\colon \mathbb{R}^n \setminus \Omega \to [0,1]$

$$\Theta_\varepsilon := \xi_\varepsilon \circ \|\nabla f\|^2,$$

with $\operatorname{supp}\Theta_\varepsilon \subseteq \overline{W_{\frac{3\varepsilon}{2}}} \setminus W_{\frac{\varepsilon}{2}}$. We consider a strictly decreasing sequence $\{\varepsilon_k\}_{k\in\mathbb{N}}$ with $\varepsilon_k > 0$ satisfying $\frac{3\varepsilon_k}{2} < \min\{\delta, \mu, \overline{\mu}\}$ for all $k \in \mathbb{N}$ and $\varepsilon_k \to 0$ as $k \to \infty$. With this choice of $\{\varepsilon_k\}_{k\in\mathbb{N}}$, $\{\Theta_{\varepsilon_k}\}_{k\in\mathbb{N}} \subset L^1(U, dV)$ is an increasing sequence, and we have $\Theta_{\varepsilon_k} \to 1$ pointwise on U as $k \to \infty$.

We compute

$$\begin{aligned}
&\operatorname{div}(\Theta_{\varepsilon_k} Z)\\
&= \underbrace{\left(\dot{\xi}_{\varepsilon_k} \circ \|\nabla f\|^2\right)\left[\frac{F(f)}{f}\|\nabla f\|^{p-3}\|\nabla\|\nabla f\|^2\|^2 + G(f)\|\nabla f\|^{p-1} g(\nabla f, \nabla\|\nabla f\|^2)\right]}_{=:\mathcal{A}_k} + \underbrace{\Theta_{\varepsilon_k}\operatorname{div} Z}_{=:\mathcal{B}_k}
\end{aligned}$$

on U for all $k \in \mathbb{N}$. For $\mathcal{B}_k$, we note that as Θ_{ε_k} vanishes near $\operatorname{Crit} f$, $\mathcal{B}_k \in L^1(U, dV)$ for all $k \in \mathbb{N}$. Hence, by the monotone convergence theorem and using that $\operatorname{div} Z \geq 0$ dV-almost everywhere on U by Theorem 1.5 because $dV(\operatorname{Crit} f) = 0$, we find that

$$\int_U \mathcal{B}_k \, dV = \int_U \Theta_{\varepsilon_k} \operatorname{div}(Z)\, dV \to \int_U \operatorname{div}(Z)\, dV \in \mathbb{R}_0^+ \cup \{\infty\}$$

as $k \to \infty$. For $\mathcal{A}_k$, note that as ξ_{ε_k} vanishes near $\operatorname{Crit} f$ we know that $\mathcal{A}_k \in L^1(U, dV)$ for all $k \in \mathbb{N}$. Now observe that $\operatorname{supp} \mathcal{A}_k \subseteq \overline{W}_{\frac{3\varepsilon_k}{2}} \setminus W_{\frac{\varepsilon_k}{2}}$ for all $k \in \mathbb{N}$. Also, all involved quantities are continuous on $U \cap \left(\overline{W}_{\frac{3\varepsilon_k}{2}} \setminus W_{\frac{\varepsilon_k}{2}}\right)$ which informs us that the map

$$s \mapsto \int_{U \cap \partial W_s} \left(\dot{\xi}_{\varepsilon_k} \circ \|\nabla f\|^2\right) \frac{F(f)}{f}\|\nabla f\|^{p-3}\|\nabla\|\nabla f\|^2\|^2 \, dS$$

is non-negative and dV-integrable on $[\frac{\varepsilon_k}{2}, \frac{3\varepsilon_k}{2}]$ for all $k \in \mathbb{N}$ and all $p > p_n$. As $\|\nabla\|\nabla f\|^2\|^2$ is smooth on the compact set $\overline{U}$, the coarea formula applies (see e.g. [Eva22, Theorem 5]). Using the Cauchy–Schwarz inequality, the coarea formula, and the mean value theorem for integrals on intervals, we compute

$$\begin{aligned}
&\int_U |\mathcal{A}_k| \, dV\\
&\leq \int_{U \cap \left(\overline{W}_{\frac{3\varepsilon_k}{2}} \setminus W_{\frac{\varepsilon_k}{2}}\right)} \left(\dot{\xi}_{\varepsilon_k} \circ \|\nabla f\|^2\right)\left[\frac{F(f)}{f}\|\nabla f\|^{p-3}\|\nabla\|\nabla f\|^2\|^2 + 2|G(f)|\|\nabla f\|^{p+1}\|\nabla^2 f\|\right] dV\\
&= \int_{\frac{\varepsilon_k}{2}}^{\frac{3\varepsilon_k}{2}} \left(\int_{U \cap \partial W_s} \left(\dot{\xi}_{\varepsilon_k} \circ \|\nabla f\|^2\right) \frac{F(f)}{f}\|\nabla f\|^{p-3}\|\nabla\|\nabla f\|^2\| \, dS\right) ds\\
&\quad + 2\int_{U \cap \left(\overline{W}_{\frac{3\varepsilon_k}{2}} \setminus W_{\frac{\varepsilon_k}{2}}\right)} \left(\dot{\xi}_{\varepsilon_k} \circ \|\nabla f\|^2\right) |G(f)|\|\nabla f\|^{p+1}\|\nabla^2 f\| \, dV\\
&= \int_{\frac{\varepsilon_k}{2}}^{\frac{3\varepsilon_k}{2}} \left(\dot{\xi}_{\varepsilon_k}(s) s^{\frac{p-3}{2}} \int_{U \cap \partial W_s} \frac{F(f)\|\nabla\|\nabla f\|^2\|}{f} \, dS\right) ds\\
&\quad + 2\int_{U \cap \left(\overline{W}_{\frac{3\varepsilon_k}{2}} \setminus W_{\frac{\varepsilon_k}{2}}\right)} \left(\dot{\xi}_{\varepsilon_k} \circ \|\nabla f\|^2\right) |G(f)|\|\nabla f\|^{p+1}\|\nabla^2 f\| \, dV
\end{aligned}$$

$$
\begin{aligned}
&\leq \frac{2}{\varepsilon_k} \max_{\overline{U}}(F(f)) \int_{\frac{\varepsilon_k}{2}}^{\frac{3\varepsilon_k}{2}} s^{\frac{p-3}{2}} \left(\int_{U \cap \partial W_s} \frac{\|\nabla \|\nabla f\|^2\|}{f} \, dS \right) ds + \frac{4}{\varepsilon_k} \max_{\overline{U}} \left(|G(f)| \|\nabla^2 f\| \right) |U| \left(\frac{3\varepsilon_k}{2} \right)^{\frac{p+1}{2}} \\
&= 2 r_k^{\frac{p-3}{2}} \max_{\overline{U}}(F(f)) \int_{U \cap \partial W_{r_k}} \frac{\|\nabla \|\nabla f\|^2\|}{f} \, dS + \underbrace{\left[\frac{3^{\frac{p+1}{2}}}{2^{\frac{p-3}{2}}} \max_{\overline{U}} \left(|G(f)| \|\nabla^2 f\| \right) |U| \right]}_{=: D} \varepsilon_k^{\frac{p-1}{2}} \\
&\leq \underbrace{2 \max_{\overline{U}}(F(f))\, r_k^{\frac{p-3}{2}} \int_{\overline{U} \cap \partial W_{r_k}} \frac{\|\nabla \|\nabla f\|^2\|}{f} \, dS}_{=: \mathcal{C}_k} + \underbrace{D\, \varepsilon_k^{\frac{p-1}{2}}}_{=: \mathcal{D}_k}
\end{aligned}
$$

for some $r_k \in (\frac{\varepsilon_k}{2}, \frac{3\varepsilon_k}{2})$ and all $k \in \mathbb{N}$. Here, $|U|$ denotes the (finite) dV-volume of U. Clearly, $\mathcal{D}_k \to 0$ as $k \to \infty$ as $p \geq p_n > 1$ recalling that $\varepsilon_k \to 0$ as $k \to \infty$. We will now show that $\mathcal{C}_k \to 0$ as $k \to \infty$, asserting by the above that

$$
\int_U \operatorname{div} (\Theta_{\varepsilon_k} Z) \, dV \to \int_U \operatorname{div} Z \, dV \in \mathbb{R}^+ \cup \{\infty\} \tag{6.3}
$$

as $k \to \infty$. For reasons that will become clear later, we first assume $p > p_n$ and will handle the case $p = p_n$ separately towards the end of the proof. To analyze $\mathcal{C}_k$, we set

$$
\rho_U := \min \left\{ \min_{\partial U} \|\nabla f\|^2, \mu, \overline{\mu}, \delta \right\} > 0
$$

and choose $k_0 = k_0(U, g, f) \in \mathbb{N}$ such that $\frac{3\varepsilon_k 2}{<}\rho_U$ for all $k \geq k_0$. This in particular asserts that $r_k < \rho_U$ for all $k \geq k_0$. By definition of ρ_U, we find that $\partial U \cap \overline{W_r} = \emptyset$ and thus $\partial(U \cap W_r) = U \cap \partial W_r$ for all $0 < r < \rho_U$. With this in mind, let us anaylze the auxiliary function $\zeta \colon (0, \rho_U) \to \mathbb{R}$ defined by

$$
\zeta(r) := \int_{U \cap \partial W_r} \frac{\|\nabla \|\nabla f\|^2\|}{f} \, dS.
$$

Clearly, we have $\zeta \in L^\infty(0, \rho_U) \subset L^1(0, \rho_U)$ as $\|\nabla f\|^2$ is continuous on M and $\overline{U} \subset M$ is compact. Recall that we have asserted above that $\partial(U \cap W_r)$ is a smooth hypersurface with finitely many components. Thus, applying the divergence theorem, the static vacuum equation (2.1), and the Bochner formula (6.1), we get

$$
\begin{aligned}
\zeta(r) &= \int_{\partial(U \cap W_r)} g \left(\frac{\nabla \|\nabla f\|^2}{f}, \frac{\nabla \|\nabla f\|^2}{\|\nabla \|\nabla f\|^2\|} \right) dS = \int_{U \cap W_r} \operatorname{div} \left(\frac{\nabla \|\nabla f\|^2}{f} \right) dV \\
&= \int_{U \cap W_r} \left(\frac{\Delta \|\nabla f\|^2}{f} - \frac{g\left(\nabla \|\nabla f\|^2, \nabla f\right)}{f^2} \right) dV = \int_{U \cap W_r} \left(\frac{\Delta \|\nabla f\|^2 - 2 \operatorname{Ric}(\nabla f, \nabla f)}{f} \right) dV \\
&= 2 \int_{U \cap W_r} \frac{\|\nabla^2 f\|^2}{f} \, dV
\end{aligned}
$$

for all $0 < r < \rho_U$. Applying the coarea formula, we find

$$
\zeta(\overline{r}) - \zeta(r) = 2 \int_r^{\overline{r}} \left(\int_{U \cap \partial W_s} \frac{\|\nabla^2 f\|^2}{f \|\nabla \|\nabla f\|^2\|} \, dS \right) ds
$$

for all $0 < r \leq \overline{r} < \rho_U$ because $\|\nabla\|\nabla f\|^2\|$ is bounded from below by a positive constant on $\overline{U} \cap (\overline{W_{\overline{r}}} \setminus W_r)$ and thus $\frac{\|\nabla^2 f\|^2}{f\|\nabla\|\nabla f\|^2\|} \in L^\infty(U \cap (\overline{W_{\overline{r}}} \setminus W_r)) \subset L^1(U \cap (\overline{W_{\overline{r}}} \setminus W_r))$. Similarly, appealing in addition to the fundamental theorem of calculus in the Sobolev space $W^{1,1}(\tau, \rho_U)$, we have $\zeta \in W^{1,1}(\tau, \rho_U)$ for any fixed $0 < \tau < \rho_U$ with weak derivative

$$\zeta'(r) = 2 \int_{U \cap \partial W_r} \frac{\|\nabla^2 f\|^2}{f\|\nabla\|\nabla f\|^2\|} \, dS$$

for almost all $\tau < r < \rho_U$. The 1-dimensional Sobolev embedding theorem then gives us that ζ is continuous on (τ, ρ_U) for all $0 < \tau < \rho_U$ and hence continuous on $(0, \rho_U)$. The refined Kato inequality (see e.g. [SSY75]) implies that

$$\|\nabla^2 f\|^2 \geq \frac{n}{n-1} \|\nabla\|\nabla f\|\|^2 \tag{6.4}$$

on $U \setminus \operatorname{Crit} f$. Thus

$$\zeta'(r) \geq \frac{2n}{n-1} \int_{U \cap \partial W_r} \frac{\|\nabla\|\nabla f\|\|^2}{f\|\nabla\|\nabla f\|^2\|} \, dS = \frac{n}{2(n-1)} \frac{\zeta(r)}{r}$$

for almost all $\tau < r < \rho_U$, using that $\|\nabla\|\nabla f\|\| = \frac{\|\nabla\|\nabla f\|^2\|}{2\|\nabla f\|}$ and $(U \cap \partial W_r) \cap \operatorname{Crit} f = \emptyset$. As $0 < \tau < \rho_U$ is arbitrary, this is equivalent to

$$(\ln \circ \zeta)'(r) \geq \frac{n}{2(n-1)} \ln'(r)$$

for almost all $0 < r < \rho_U$. Picking a fixed $0 < R < \rho_U$ for which this inequality holds, this integrates to

$$\zeta(r) \leq \frac{\zeta(R)}{R^{\frac{n}{2(n-1)}}} \, r^{\frac{n}{2(n-1)}}$$

for all $0 < r < R$ by continuity of ζ. Hence

$$0 < r^{\frac{p-3}{2}} \zeta(r) \leq \frac{\zeta(R)}{R^{\frac{n}{2(n-1)}}} \, r^{\frac{p-p_n}{2}} \tag{6.5}$$

holds for all $0 < r < R$. For $p > p_n$, the exponent of r on the right hand side of (6.5) is strictly positive so that $\mathcal{C}_k = r_k^{\frac{p-3}{2}} \zeta(r_k) \to 0$ as $k \to \infty$. This proves (6.3) for $p > p_n$.

Consider now the surface integral term

$$\int_{\partial U} g(\Theta_{\varepsilon_k} Z, \eta) \, dS,$$

where η denotes the unit normal to ∂U pointing to the outside of U. Recalling that $\partial U \cap \operatorname{Crit} f = \emptyset$, we know that Z is continuous on ∂U and hence by compactness of ∂U and Lebesgue's dominated convergence theorem, we have

$$\int_{\partial U} g(\Theta_{\varepsilon_k} Z, \eta) \, dS \to \int_{\partial U} g(Z, \eta) \, dS \in \mathbb{R}$$

as $k \to \infty$. Together with (6.3) and applying the divergence theorem to $\Theta_{\varepsilon_k} Z$ on U, this establishes dV-integrability of $\mathcal{D} = \operatorname{div} Z$ on U and hence on M when $p > p_n$. Moreover, denoting Z by Z_p to be able to distinguish the above results for different $p > p_n$, we have asserted that

$$\int_U \operatorname{div} Z_p \, dV = \int_{\partial U} g(Z_p, \eta) \, dS \tag{6.6}$$

for all $p > p_n$. To conclude that $\mathcal{D} = \operatorname{div} Z$ is dV-integrable for $p = p_n$, let us consider a strictly decreasing sequence $\{p_l\}_{l\in\mathbb{N}}$ with $p_l > p_n$ and $p_l \to p_n$ as $l \to \infty$. Using again that $\partial U \cap \operatorname{Crit} f = \emptyset$, we find that $Z_{p_l} \to Z_{p_n}$ on ∂U as $l \to \infty$. As $\{Z_{p_l}\}_{l\in\mathbb{N}}$ is uniformly bounded on the compact set ∂U by continuity, Lebesgue's dominated convergence theorem informs us that

$$\int_{\partial U} g(Z_{p_l}, \eta) \, dS \to \int_{\partial U} g(Z_{p_n}, \eta) \, dS$$

as $l \to \infty$. Now recall that $dV(\operatorname{Crit} f) = 0$ and note that this gives $Z_{p_l} \to Z_{p_n}$ pointwise dV-almost everywhere as $l \to \infty$. Splitting U into $U \cap W_1$ and $U \setminus W_1$, Lebesgue's dominated convergence theorem tells us that

$$\int_{U\setminus W_1} \operatorname{div} Z_{p_l} \, dV \to \int_{U\setminus W_1} \operatorname{div} Z_{p_n} \, dV \in \mathbb{R}_0^+$$

as $l \to \infty$. On $U \cap W_1$, we rewrite (1.10) as

$$\begin{aligned}
\operatorname{div} Z_{p_l} &= \underbrace{\|\nabla f\|^{p_l - 5} \frac{(n-2)^2 F(f) f}{(n-1)^2} \|T\|^2}_{=:\mathcal{E}_l} \\
&\quad + (p_l - p_n) \underbrace{\|\nabla f\|^{p_l-5} \frac{F(f)}{2f} \left\| \nabla \|\nabla f\|^2 + \frac{4(n-1)}{(n-2)} \frac{f \|\nabla f\|^2 \, \nabla f}{1 - f^2} \right\|^2}_{=:\mathcal{F}_l}
\end{aligned}$$

and note that $\{\mathcal{E}_l\}_{l\in\mathbb{N}}$, $\{\mathcal{F}_l\}_{l\in\mathbb{N}}$ are non-negative sequences of dV-measurable functions on $U \cap W_1$ by Theorem 1.5 and because $p_l > p_n$ and $F(f) \geq 0$. Moreover, both $\{\mathcal{E}_l\}_{l\in\mathbb{N}}, \{\mathcal{F}_l\}_{l\in\mathbb{N}}$ are monotonically increasing sequences on $U \cap W_1$ as

$$\frac{\partial \|\nabla f\|^{p-5}}{\partial p} = \ln(\|\nabla f\|) \|\nabla f\|^{p-5} < 0$$

holds for all $p \in \mathbb{R}$ dV-almost everywhere on $U \cap W_1$ as $\|\nabla f\| < 1$ on W_1. By the monotone convergence theorem, we obtain[10]

$$\int_{U\cap W_1} \operatorname{div} Z_{p_l} \, dV = \int_{U\cap W_1} \mathcal{E}_l \, dV + (p_l - p_n) \int_{U\cap W_1} \mathcal{F}_l \, dV \to \int_{U\cap W_1} \operatorname{div} Z_{p_n} \, dV \in \mathbb{R}_0^+ \cup \{\infty\}$$

as $l \to \infty$. By (6.6), we can thus deduce that $\mathcal{D} = \operatorname{div} Z$ is dV-integrable on U and thus on M also for $p = p_n$ as claimed.

We now would like to apply the divergence theorem to the vector field Z on M for all $p \geq p_n$. As we have asserted the dV-integrability of $\mathcal{D} = \operatorname{div} Z$ and know that Z is smooth near ∂M and in a neighborhood of infinity and because (M, g) is geodesically complete up to ∂M by Remark 2.4, the divergence theorem applies. It only remains to study the "boundary integral at infinity" and to evaluate the surface integral at the inner boundary. To this end,

[10]We would like to remark that we cannot conclude that the term involving $(p_l - p_n)$ vanishes in the limit $l \to \infty$ as $\lim_{l\to\infty} \int_{U\cap W_1} \mathcal{F}_l \, dV$ may be infinite. This causes no issues as $p_l > p_n$ and $\mathcal{E}_l, \mathcal{F}_l \geq 0$.

let $r > 0$ be such that $B_r := \{x \in \mathbb{R}^n : |x| < r\} \supset \overline{B}$ and thus $x^{-1}(\mathbb{R}^n \setminus B_r) \subseteq M$, where x denotes the asymptotically flat chart and $\overline{B}$ the complement of the image of x in $\mathbb{R}^n$. From the divergence theorem, our choice of unit normal ν pointing towards infinity, and (2.17), we obtain

$$\begin{aligned}
&\int_{M\setminus x^{-1}(\mathbb{R}^n\setminus B_r)} \mathcal{D}\, dV \\
&\quad= \int_{x^{-1}(\partial B_r)} \left(\frac{F(f)}{f} \|\nabla f\|^{p-3} \langle \nabla \|\nabla f\|^2, \nu\rangle + G(f)\|\nabla f\|^{p-1}\langle \nabla f,\, \nu\rangle \right) dS \\
&\quad\quad - \int_{\partial M} \left(\frac{F(f)}{f} \|\nabla f\|^{p-3} \langle \nabla \|\nabla f\|^2, \nu\rangle + G(f)\|\nabla f\|^{p-1}\langle \nabla f,\, \nu\rangle \right) dS.
\end{aligned}$$

Exploiting Lemma 2.5 and the above asymptotics for $F(f)$ and $G(f)$, we find

$$\begin{aligned}
&\int_{x^{-1}(\partial B_r)} \left(\frac{F(f)}{f} \|\nabla f\|^{p-3} \langle \nabla \|\nabla f\|^2, \nu\rangle + G(f)\|\nabla f\|^{p-1}\langle \nabla f,\, \nu\rangle \right) dS \\
&= \int_{x^{-1}(\partial B_r)} \left(\frac{F(f_m)}{f_m} \|\nabla_m f_m\|_m^{p-3} \langle \nabla_m \|\nabla_m f\|_m^2, \nu_m\rangle_m + G(f_m)\|\nabla_m f_m\|_m^{p-1}\langle \nabla_m f_m,\, \nu_m\rangle_m \right) dS_\delta \\
&\quad + o(1) \\
&= -\mathcal{F}_p^{c,d}(m) + o(1)
\end{aligned}$$

as $r \to \infty$, where $\langle \cdot,\cdot\rangle_m = g_m$ and ν_m denotes the unit normal to $x^{-1}(\partial B_r)$ with respect to g_m and pointing to infinity. For the inner boundary integral, we recall from Remark 2.6 that $\nu = \frac{\nabla f}{\|\nabla f\|}$, $\kappa = \|\nabla f\|$ if $f_0 \in [0,1)$ and $\nu = -\frac{\nabla f}{\|\nabla f\|}$, $\kappa = -\|\nabla f\|$ if $f_0 \in (1,\infty)$. Exploiting that $f = f_0$ and κ are constant on ∂M by assumption, we compute

$$\begin{aligned}
&\int_{\partial M} \left(\frac{F(f)}{f} \|\nabla f\|^{p-3} \langle \nabla \|\nabla f\|^2, \nu\rangle + G(f)\|\nabla f\|^{p-1}\langle \nabla f,\, \nu\rangle \right) dS \\
&= \pm \frac{F_0}{f_0} |\kappa|^{p-4} \int_{\partial M} \langle \nabla \|\nabla f\|^2, \nabla f\rangle dS \pm G_0 |\kappa|^p |\partial M|,
\end{aligned}$$

with $\pm = +$ if $f_0 \in [0,1)$ and $\pm = -$ if $f_0 \in (1,\infty)$, respectively. Using (2.1) and (2.3) as well as the Gauß equation, we compute

$$\begin{aligned}
\langle \nabla \|\nabla f\|^2, \nabla f\rangle &= 2\nabla^2 f(\nabla f, \nabla f) = 2 f_0 \operatorname{Ric}(\nabla f, \nabla f) \\
&= 2 f_0 \kappa^2 \operatorname{Ric}(\nu,\nu) = -f_0 \kappa^2 \left(\mathrm{R}_{\partial M} - \tfrac{n-2}{n-1} H^2 + \|\mathring{h}\|^2 \right).
\end{aligned}$$

Combining this with the above, we find

$$\begin{aligned}
&\int_{\partial M} \left(\frac{F(f)}{f} \|\nabla f\|^{p-3} \langle \nabla \|\nabla f\|^2, \nu\rangle + G(f)\|\nabla f\|^{p-1}\langle \nabla f,\, \nu\rangle \right) dS \\
&= \mp F_0 |\kappa|^{p-2} \int_{\partial M} \left(\mathrm{R}_{\partial M} - \tfrac{n-2}{n-1} H^2 + \|\mathring{h}\|^2 \right) dS \pm G_0 |\kappa|^p |\partial M|
\end{aligned}$$

and thus

$$0 \le \int_M \mathcal{D}\, dV = -\mathcal{F}_p^{c,d}(m) \pm F_0 |\kappa|^{p-2} \int_{\partial M} \left(\mathrm{R}_{\partial M} - \tfrac{n-2}{n-1} H^2 + \|\mathring{h}\|^2 \right) dS \mp G_0 |\kappa|^p \, |\partial M|.$$

Consequently, for $c, d \in \mathbb{R}$ satisfying $c+d \geq 0$, $cf_0^2+d \geq 0$, and for all $p \geq p_n$, we find (1.16) and (1.17). Equality holds in (1.16) or in (1.17) if and only if equality holds in (1.15) and thus $\mathcal{D} = 0$ on M. Hence if $c+d \geq 0$, $cf_0^2+d \geq 0$ (but not $c=d=0$), vanishing of both sides in (1.10) gives $T=0$ and, if $p > p_n$, also (3.35). By Theorem 1.9, this implies the equality assertion of Theorem 1.7. □

Let us now discuss the geometric implications of (1.16) and (1.17) or in other words prove Theorem 1.8 and, in passing, its equivalence to Theorem 1.7, see also Corollary 6.2.

Proof of Theorem 1.8. We begin by choosing $f_0 \in [0,1)$, recalling that $\kappa > 0$ and $m > 0$ in this case. Choosing the admissible constants $c = 1$, $d = -f_0^2$ in (1.16) and any $p \geq p_n$, we find from Lemma 2.13 that $\kappa = (n-2)\frac{|\mathbb{S}^{n-1}|}{|\partial M|}\, m$ and thus

$$m \geq \frac{(1-f_0^2)\,(s_{\partial M})^{n-2}}{2} > 0, \tag{6.7}$$

asserting the right hand side inequality in (1.19). Choosing instead the admissible constants $c=-1$, $d=1$ and any $p \geq p_n$, (1.16) reduces to

$$(1-f_0^2)|\partial M| \int_{\partial M} \left(\mathrm{R}_{\partial M} - \tfrac{n-2}{n-1}H^2 + \|\mathring{h}\|^2\right) dS \geq 4(n-1)(n-2)|\mathbb{S}^{n-1}|^2\, m^2. \tag{6.8}$$

This asserts the left-hand side inequality in (1.19) via an algebraic re-arrangement. Via Theorem 1.9, we have hence proved Theorem 1.8 for $f_0 \in [0,1)$. On the other hand, as (1.16) is linear in c, d and the constraints $c+d \geq 0$, $cf_0^2+d \geq 0$ are linear as well, the combination of (6.7), (6.8) asserts (1.16) for any $p > 1$ via the Smarr formula (2.20).

Next, let us consider $f_0 \in (1,\infty)$, recalling that $\kappa, m < 0$ in this case. Choosing the admissible constants $c=-1$, $d=f_0^2$ in (1.17) and any $p \geq p_n$, we again find from Lemma 2.13 that $\kappa = (n-2)\frac{|\mathbb{S}^{n-1}|}{|\partial M|}\, m$ and thus

$$m \leq \frac{(1-f_0^2)\,(s_{\partial M})^{n-2}}{2} < 0,$$

asserting the right hand side inequality in (1.20). Choosing instead the admissible constants $c=1$, $d=-1$ and any $p \geq p_n$, (1.16) reduces to

$$(f_0^2-1)|\partial M| \int_{\partial M} \left(\mathrm{R}_{\partial M} - \tfrac{n-2}{n-1}H^2 + \|\mathring{h}\|^2\right) dS \leq -4(n-1)(n-2)|\mathbb{S}^{n-1}|^2\, m^2.$$

This asserts the left-hand side inequality in (1.20) via an algebraic re-arrangement. Via Theorem 1.9, we have hence proved Theorem 1.8 for $f_0 \in (1,\infty)$. Again, as (1.17) is linear in c, d and the constraints $c+d \geq 0$, $cf_0^2+d \geq 0$ are linear as well, the combination of the two above inequalities asserts (1.17) for any $p > 1$ via the Smarr formula (2.20). □

Corollary 6.2 (Theorem 1.7 holds for $p > 1$)**.** *It follows from the proof of Theorem 1.8 that Theorem 1.7 remains valid for $p > 1$, with F_0 and G_0 formally extended to $p \in (1, p_n)$.*

We now proceed to proving Theorem 1.1 and Theorem 1.3, where we will use Theorem 1.9 which was proven in Section 4.

Proof of Theorem 1.1. To prove Theorem 1.1, we consider the implications of Theorem 1.8 in the setting of Theorem 1.1, i.e., if $f_0 = 0$ and ∂M is a connected static horizon. Then we know that $H=0$, $\mathring{h}=0$ on ∂M by Remark 2.8 and hence (1.19) reduces to (1.2). Moreover, (1.19) implies (1.3) upon dropping the middle term and squaring. If, in addition, assumption (1.4) holds then we have equality in (1.3) and thus in both inequalities in (1.19). By the equality case assertion in Theorem 1.8, this proves Theorem 1.1. ☐

Proof of Theorem 1.3. To see that Theorem 1.3 holds, we consider the implications of Theorem 1.8 in the setting of Theorem 1.3, i.e., if $f_0 \in (0,1) \cup (1,\infty)$ and ∂M is a connected time-slice of a photon surface and thus in particular has constant scalar curvature $\mathrm{R}_{\partial M}$, constant mean curvature H, is totally umbilic ($\mathring{h} = 0$) and obeys the photon surface constraint (2.19). When $f_0 \in (0,1)$, (1.19) gives (1.5). Moreover, dropping the middle term in (1.19) and squaring it gives (1.6). Rewriting (1.6) via the photon surface constraint (2.19) gives

$$\frac{2\kappa H}{f_0} \geq \frac{(n-1)(n-2)(1-f_0^2)}{(s_{\partial M})^2}.$$

Assuming in addition (1.9) and rewriting it via the photon surface constraint (2.19) gives

$$\frac{2\kappa H}{f_0} + \frac{n-2}{n-1}H^2 \leq \frac{(n-1)(n-2)}{(s_{\partial M})^2}.$$

Taken together, this gives

$$\frac{2\kappa H}{f_0} + \frac{n-2}{n-1}H^2 \leq \frac{(n-1)(n-2)}{(s_{\partial M})^2} \leq \frac{2\kappa H f_0}{1-f_0^2}$$

Recalling that $H > 0$ from the above or by Proposition 2.11 implies

$$1 - f_0^2 \leq \frac{2(n-1)\kappa f_0}{(n-2)H}.$$

On the other hand, the squared left-hand side inequality in (1.5) together with the Smarr formula (2.20) leads to

$$1 - f_0^2 \geq \frac{2(n-1)\kappa f_0}{(n-2)H}.$$

Thus, equality holds in all the above inequalities and hence in (1.5), too. By the equality assertion in Theorem 1.8, this proves Theorem 1.3 when $f_0 \in (0,1)$. For $f_0 \in (1,\infty)$, the argument is the same with reversed signs. ☐

7. Discussion and monotone functions

7.1. **Monotone functions along level sets.** In Theorem 1.5 and the proof of Theorem 1.7, we have seen that the divergence of the vector field

$$Z := \frac{F(f)}{f}\|\nabla f\|^{p-3}\,\nabla\|\nabla f\|^2 + G(f)\|\nabla f\|^{p-1}\,\nabla f, \tag{7.1}$$

$\mathcal{D} = \operatorname{div} Z$, is dV-integrable and non-negative dV-almost everywhere for all $n \geq 3$, $p \geq p_n$, $c, d \in \mathbb{R}$ with $c + d \geq 0$, $cf_0^2 + d \geq 0$, and F and G as defined in (1.12), (1.13). We exploited this to prove the parametric geometric inequalities in Theorem 1.7 and, equivalently, the geometric inequalities in Theorem 1.8, by applying a suitably adapted divergence theorem to Z on M and evaluating the corresponding surface integrals at ∂M with $f = f_0$ on ∂M, $f_0 \in [0,1) \cup (1,\infty)$, and at infinity. Of course, one can also apply the adapted divergence theorem to Z on suitable open domains $N \subset M$ and exploit the non-negativity of $\operatorname{div} Z$ to obtain estimates between the different components of ∂N. In view of the fact that we are using an approach based on a potential, and in order to compare our technique of proof to the monotone function approach by Agostiniani and Mazzieri [AM17], it will be most interesting to study such N for which ∂N consists of level sets of the lapse function f. Our arguments are inspired by [CM24, Proposition 4.2], see also [AM20].

Given a static vacuum system (M^n, g, f), $n \geq 3$, with boundary ∂M such that $f = f_0$ on ∂M for some $f_0 \in [0,1)$ or $f_0 \in (1,\infty)$, and given $p \geq p_n$, $c, d \in \mathbb{R}$ satisfying $c + d \geq 0$, $cf_0^2 + d \geq 0$, and F and G as defined in (1.12), (1.13), we define the functions $\mathcal{H}_p^{c,d} \colon ([f_0, 1) \cup (1, f_0]) \setminus f(\operatorname{Crit} f) \to \mathbb{R}$ by

$$\mathcal{H}_p^{c,d}(f) := \int_{\Sigma_f} \langle Z, \nu \rangle \, dS = \pm \int_{\Sigma_f} \left[\frac{F(f)}{f} \|\nabla f\|^{p-4} \langle \nabla \|\nabla f\|^2, \nabla f \rangle + G(f) \|\nabla f\|^p \right] dS, \tag{7.2}$$

where Σ_f denotes the f-level set of the lapse function f and the sign $\pm$ is $+$ for $f_0 \in [0,1)$ and $-$ for $f_0 \in (1,\infty)$. $\mathcal{H}_p^{c,d}(f)$ is clearly well-defined as we restricted its definition to regular values of f and as we have already asserted that the integral under consideration is well-defined for $f = f_0 = 0$. Using the decomposition of Δ along a level set of f and the static vacuum equation (2.2), one obtains that the mean curvature H of any regular level set Σ_f is given by

$$H = \mp \frac{\nabla^2 f(\nabla f, \nabla f)}{\|\nabla f\|^2}, \tag{7.3}$$

where the sign $\mp$ is $-$ for $f \in [f_0, 1)$ and $+$ if $f \in (1, f_0]$. This shows that

$$\mathcal{H}_p^{c,d}(f) = \int_{\Sigma_f} \|\nabla f\|^{p-1} \left[-\frac{2F(f)H}{f} \pm G(f) \|\nabla f\| \right] dS \tag{7.4}$$

holds for all regular values $f \in [f_0, 1) \cup (1, f_0]$, understood at f_0 as the limit $f \to f_0$ in case $F(f_0) = 0$ or $f_0 = 0$. Recalling from the proof of Theorem 1.7 that $f(\operatorname{Crit} f)$ is finite and f_0 is a regular value of f, we will now show that $\mathcal{H}_p^{c,d}$ (in both of its representations (7.2), (7.4)) can be continuously extended to the at most finitely many singular values of f and is monotone. To see this, let $f_* \in (f_0, 1) \cup (1, f_0]$ be a critical value of f which then necessarily has an open neighborhood $(f_* - 2\varepsilon, f_* + 2\varepsilon)$ for some suitably small $\varepsilon > 0$ such that $(f_* - 2\varepsilon, f_* + 2\varepsilon) \setminus \{f_0\}$ contains only regular points. We set

$$\Psi := \pm \mathcal{H}_p^{c,d}|_{(f_* - 2\varepsilon, f_* + 2\varepsilon) \setminus \{f_0\}} \tag{7.5}$$

which is clearly well-defined. Applying the adapted divergence theorem from the proof of Theorem 1.7 to Z on the domains $(\eta, f_* + \varepsilon)$ and $(f_* - \varepsilon, \eta)$ for a fixed $\eta \in (f_* - \varepsilon, f_* + \varepsilon) \setminus \{f_0\}$, we learn that

$$\Psi(\eta) = \Psi(f_* + \varepsilon) - \int_{\{\eta < f < f_* + \varepsilon\}} \operatorname{div} Z \, dV = \Psi(f_* - \varepsilon) + \int_{\{\eta > f > f_* - \varepsilon\}} \operatorname{div} Z \, dV. \tag{7.6}$$

As $\operatorname{div} Z \geq 0$ holds dV-almost everywhere on M by Theorem 1.5, Ψ is monotonically increasing on $(f_* - \varepsilon, f_* + \varepsilon) \setminus \{f_*\}$ and we have

$$\Psi(f_* + \varepsilon) \geq \Psi(\eta) \geq \Psi(f_* - \varepsilon). \tag{7.7}$$

for all $\eta \in (f_* - \varepsilon, f_* + \varepsilon) \setminus \{f_0\}$. This establishes that $\limsup_{\eta \to f_*} \Psi(\eta)$ and $\liminf_{\eta \to f_*} \Psi(\eta)$ are finite. Moreover, we learn from (7.6) that

$$\Psi(f_* + \varepsilon) - \Psi(f_* - \varepsilon) = \int_{\{f_* - \varepsilon < f < f_* + \varepsilon\}} \operatorname{div} Z \, dV \tag{7.8}$$

holds for all suitably small $\varepsilon > 0$. Thus, recalling from the proof of Theorem 1.7 that dV and the n-dimensional Hausdorff measure $\mathcal{H}^n$ are absolutely continuous with respect to each other with bounded densities and that $\operatorname{div} Z$ is dV-integrable, it follows that $\limsup_{\eta \to f_*} \Psi(\eta) = \liminf_{\eta \to f_*} \Psi(\eta)$ so that Ψ can continuously extended to f_*. Extending $\mathcal{H}_p^{c,d}$ to $\operatorname{Crit} f$ continuously in this way, we obtain from (7.7) that $\mathcal{H}_p^{c,d}$ is well-defined, continuous, and monotonically increasing on $[f_0, 1)$ for all $f_0 \in [0, 1)$ / monotonically decreasing on $(1, f_0]$ for all $f_0 \in (1, \infty)$. Moreover, by Theorem 1.7 and its proof, we know

$$\lim_{f \to 1} \mathcal{H}_p^{c,d}(f) = -\mathcal{F}_p^{c,d}(m) \tag{7.9}$$

with $\mathcal{F}_p^{c,d}$ as in (1.18) and m the mass parameter of (M, g, f), recalling that f has no critical points near infinity. Moreover, recall from the end of Section 5 that $\mathcal{H}_p^{c,d} = -\mathcal{F}_p^{c,d}(m)$ on $[0, 1)$ for the Schwarzschild systems (M_m^n, g_m, f_m) of mass m. From Theorems 1.5 and 1.7, we also know that (suitable subsets of) the Schwarzschild systems are the only static vacuum systems satisfying this identity.

Theorem 7.1 (Monotone functions)**.** *Let (M^n, g, f), $n \geq 3$, be an asymptotically flat static vacuum system of mass $m \in \mathbb{R}$ with connected boundary ∂M. Assume that $f|_{\partial M} = f_0$ for a constant $f_0 \in [0, 1) \cup (1, \infty)$ and choose the unit normal ν to ∂M pointing towards the asymptotic end. Let F and G be as in Theorem 1.5 for some $p \geq p_n$ and constants $c, d \in \mathbb{R}$ satisfying $c + d \geq 0$ and $cf_0^2 + d \geq 0$. Then the function $\mathcal{H}_p^{c,d} \colon [f_0, 1) \cup (1, f_0] \to \mathbb{R}$ given by (7.4) is well-defined, continuous, and monotonically increasing when $f_0 \in [0, 1)$ / monotonically decreasing when $f_0 \in (1, \infty)$, with $\lim_{f \to 1} \mathcal{H}_p^{c,d}(f) = -\mathcal{F}_p^{c,d}(m)$. Unless $c = d = 0$, $\mathcal{H}_p^{c,d} \equiv -\mathcal{F}_p^{c,d}(m)$ on $[f_0, 1)$ or $(1, f_0]$ holds if and only if (M, g) is isometric to a suitable piece of the Schwarzschild manifold (M_m^n, g_m) of mass m and f corresponds to the corresponding restriction of f_m under this isometry. Finally, $m > 0$ if $f_0 \in [0, 1)$ while $m < 0$ for $f_0 \in (1, \infty)$.*

Remark 7.2 (Geometric monotonicity of $\mathcal{H}_p^{c,d}$)**.** *Before we move on, we would like to draw the readers' attention to the fact that no matter whether $f_0 \in [0, 1)$ or $f_0 \in (1, \infty)$, the function $\mathcal{H}_p^{c,d}$ is monotonically increasing along the level sets of f from ∂M towards the asymptotic end. This is because for $f_0 \in (1, \infty)$, f decreases along its level sets from ∂M towards the asymptotic end.*

7.2. **Comparison with the proof by Agostiniani and Mazzieri.** Let us now relate the functions $\mathcal{H}_p^{c,d}$ from (7.2) to the functions U_p and their derivatives introduced in [AM17]. In our notation, these functions are given by

$$U_p(f) := \left(\frac{2m}{1 - f^2} \right)^{\frac{(n-1)(p-1)}{n-2}} \int_{\Sigma_f} \|\nabla f\|^p \, dS, \tag{7.10}$$

$$U_p'(f) = -(p-1) \left(\frac{2m}{1 - f^2} \right)^{\frac{(n-1)(p-1)}{n-2}} \int_{\Sigma_f} \|\nabla f\|^{p-1} \left[H - \frac{2(n-1) f \|\nabla f\|}{(n-2)(1 - f^2)} \right] dS \tag{7.11}$$

for $f \in [f_0, 1) \setminus \operatorname{Crit} f$ for $f_0 \in [0,1)$ and $p \geq 1$. From this and (7.4), one readily computes

$$U_p(f) = \frac{\mu_p}{4(1-f_0^2)} \left[\frac{f^2 - f_0^2}{1-f^2} \, \mathcal{H}_p^{-1,1}(f) - \mathcal{H}_p^{1,-f_0^2}(f) \right], \tag{7.12}$$

$$U_p'(f) = \frac{\mu_p f}{2(1-f^2)^2} \, \mathcal{H}_p^{-1,1}(f) \tag{7.13}$$

on $[f_0, 1) \setminus \operatorname{Crit} f$ for $\mu_p := (p-1)(2m)^{\frac{(n-1)(p-1)}{n-2}} > 0$ when $p \geq p_n$ and $f_0 \in [0,1)$.

It is shown in [AM17, Theorem 1.1] that U_p is differentiable on $[f_0, 1)$ for $p \geq 3$ with derivative U_p' and continuous on $[f_0, 1)$ for $p \geq 1$. Moreover, it is shown in [AM17, Theorem 1.2] that U_p is differentiable with derivative U_p' on $[f_0, 1) \setminus f(\operatorname{Crit} f)$ for $p \geq p_n$. A fortiori, it follows from Theorem 7.1 that the functions U_p and U_p' given by (7.12), (7.13) are well-defined as continuous functions on $[f_0, 1)$ not only for $p \geq 3$ but in fact for $p \geq p_n$. As there are only finitely many critical values of f and as U_p, U_p' are continuous also at critical values of f, the fundamental theorem of calculus implies that U_p is continuously differentiable with derivative U_p' on $[f_0, 1)$ for $f_0 \in [0,1)$ for all $p \geq p_n$. Moreover, as $\mathcal{H}_p^{-1,1}$ is monotonically increasing with limit $-\mathcal{F}_p^{-1,1}(m) = 0$ as $f \to 1$ by Theorem 7.1, we have $U_p' \leq 0$ so that U_p is montonically decreasing on $[f_0, 1)$ for all $p \geq p_n$. Moreover, if $U_p'(f) = 0$ for some $f \in [f_0, 1)$, $f \neq 0$, then $\mathcal{H}_p^{-1,1}(f) = 0$ and hence by Theorem 7.1 (M, g, f) is isometric to a suitable piece of the Schwarzschild manifold (M^n_m, g_m) of mass m and f corresponds to the corresponding restriction of f_m under this isometry. Finally, if $f_0 = 0$, $U_p'(0) = 0$ is automatic from (7.13) and one finds

$$\begin{aligned} U_p''(0) &:= \lim_{f \to 0+} \frac{U_p'(f)}{f} = \frac{\mu_p}{2} \lim_{f \to 0+} \mathcal{H}_p^{-1,1}(f) = \frac{\mu_p}{2} \, \mathcal{H}_p^{-1,1}(0) \\ &= -\frac{(p-1)}{2} (2m)^{\frac{(n-1)(p-1)}{n-2}} \int_{\partial M} \|\nabla f\|^{p-2} \left[\mathrm{R}_{\partial M} - \frac{4(n-1)\|\nabla f\|^2}{n-2} \right] dS, \end{aligned} \tag{7.14}$$

where we have used continuity of $\mathcal{H}_p^{-1,1}$, the expression for the mean curvature in terms of the Hessian of the harmonic function f, the static vacuum equation (2.1), the Gauß equation, and Remark 2.8. Also, it follows that $U_p''(0) \leq 0$ because we have already established above that $\mathcal{H}_p^{-1,1} \leq 0$. Finally, $U_p''(0) = 0$ if and only if $\mathcal{H}_p^{-1,1}(0) = 0$ if and only if (M, g, f) is isometric to a suitable piece of the Schwarzschild manifold (M^n_m, g_m) of mass m and f corresponds to the corresponding restriction of f_m under this isometry by Theorem 7.1. This can be re-expressed as follows.

Corollary 7.3 (Monotonicity-Rigidity à la Agostiniani–Mazzieri)**.** *Items (ii) and (iii) of the Monotonicity-Rigidity Theorem* [AM17, Theorem 1.1] *hold for $p \geq p_n$.*

Let us now discuss the case $f_0 \in (1, \infty)$. To do so, let us extend (7.10) as the definition of U_p also to $(1, f_0] \setminus \operatorname{Crit} f$ for any $f_0 \in (1, \infty)$ and any $p \geq p_n$, noting that the pre-factor of the integral is well-defined as $m < 0$ and $f > 1$ in this case. One directly computes that (7.12) gets replaced by

$$U_p(f) = \frac{\mu_p}{4(f_0^2 - 1)} \left[\frac{f_0^2 - f^2}{f^2 - 1} \, \mathcal{H}_p^{1,-1}(f) - \mathcal{H}_p^{-1,f_0^2}(f) \right] \tag{7.15}$$

with $\mu_p := (p-1)(2|m|)^{\frac{(n-1)(p-1)}{n-2}} > 0$. Next, using our above results, we can show continuous differentiability of U_p for $f_0 \in (1, \infty)$ and $p \geq p_n$ and compute its derivative U_p' as follows:

Recall from Section 5 that

$$\operatorname{div}\left(\|\nabla f\|^{p-1}\nabla f\right) = (p-1)\|\nabla f\|^{p-3}\,\nabla^2 f(\nabla f, \nabla f)$$

on $M \setminus \operatorname{Crit} f$. Exploiting (7.3) and arguing as in Sections 5 and 6, in particular using the coarea formula and the adapted divergence theorem, we get

$$(p-1)\int_{f_1}^{f_2}\int_{\Sigma_\tau} \|\nabla f\|^{p-1} H\, dS\, d\tau = \int_{\Sigma_{f_2}} \|\nabla f\|^{p}\, dS - \int_{\Sigma_{f_1}} \|\nabla f\|^{p}\, dS$$

for any $f_0 \geq f_2 > f_1 > 1$. Taking the limit $f_1 \to 1$, this reduces to

$$(p-1)\int_{1}^{f}\int_{\Sigma_\tau} \|\nabla f\|^{p-1} H\, dS\, d\tau = \int_{\Sigma_f} \|\nabla f\|^{p}\, dS = \left(\frac{1-f^2}{2m}\right)^{\frac{(n-1)(p-1)}{(n-2)}} U_p(f)$$

for every $f \in (1, f_0]$ by continuity of U_p. This can be re-expressed as

$$U_p(f) = (p-1)\left(\frac{2m}{1-f^2}\right)^{\frac{(n-1)(p-1)}{(n-2)}} \int_{1}^{f}\int_{\Sigma_\tau} \|\nabla f\|^{p-1} H\, dS\, d\tau$$

for all $f \in (1, f_0]$. Now, arguing as in Section 7.1, the function $f \mapsto \int_{\Sigma_f} \|\nabla f\|^{p-1} H\, dS$ is continuous on $(1, f_0]$ which gives continuous differentiability of U_p and

$$U_p'(f) = (p-1)\left(\frac{2m}{1-f^2}\right)^{\frac{(n-1)(p-1)}{n-2}} \int_{\Sigma_f} \|\nabla f\|^{p-1}\left[H + \frac{2(n-1)f\|\nabla f\|}{(n-2)(1-f^2)}\right] dS \tag{7.16}$$

for all $f \in (1, f_0]$ by the fundamental theorem of calculus. In particular, U_p is continuously differentiable on $(1, f_0]$ for all $f_0 \in (1, \infty)$ and all $p \geq p_n$ and (7.13) gets replaced by

$$U_p'(f) = -\frac{\mu_p f}{2(f^2-1)^2}\,\mathcal{H}_p^{1,-1}(f) \tag{7.17}$$

as can be seen by a direct computation. Moreover, by (7.17) and $\mathcal{H}_p^{-1,1}(f) \to -\mathcal{F}_p^{1,-1}(m) =$ as $f \to 1$ by (1.18) and in view of Remark 7.2, we have $U_p' \geq 0$ on $(1, f_0]$. Again, just as in Remark 7.2, this means that U_p is monotonically decreasing from ∂M to infinity. In analogy with Corollary 7.3, we can summarize our findings as follows.

Corollary 7.4 (Monotonicity-Rigidity à la Agostiniani–Mazzieri for $f_0 \in (1, \infty)$)**.** *Item (ii) of the Monotonicity-Rigidity Theorem* [AM17, Theorem 1.1] *holds for U_p given by* (7.10) *on $(1, f_0]$ with derivative U_p' given by* (7.16) *for all $f_0 \in (1, \infty)$ and all $p \geq p_n$, with the opposite inequality $U_p' \geq 0$ on $(1, f_0]$.*

Last but not least, we would like to point out that we have computed U_p and U_p' only from $\mathcal{H}_p^{c,d}$ using only the extremal values of (c, d) (normalized to $|c| = 1$) as in the proofs of Theorems 1.1 and 1.3, see also Figures 1 and 2. All other functions $\mathcal{H}_p^{c,d}$ are related to the extremal ones by

$$\mathcal{H}_p^{c,d} = \frac{cf_0^2 + d}{1 - f_0^2}\mathcal{H}_p^{-1,1} + \frac{c+d}{1-f_0^2}\mathcal{H}_p^{1,-f_0^2}, \tag{7.18}$$

$$\mathcal{H}_p^{c,d} = \frac{cf_0^2 + d}{f_0^2 - 1}\mathcal{H}_p^{1,-1} + \frac{c+d}{f_0^2 - 1}\mathcal{H}_p^{-1,f_0^2} \tag{7.19}$$

for $f_0 \in [0,1)$ and $f_0 \in (1,\infty)$, respectively. These relations allow us to express all functions $\mathcal{H}_p^{c,d}$ by U_p and U_p', obtaining

$$\mu_p \mathcal{H}_p^{c,d}(f) = \frac{2(cf^2+d)(1-f^2)}{f} U_p'(f) - 4(c+d)U_p(f) \tag{7.20}$$

for all $f \in [f_0,1) \cup (1,\infty)$, in consistency with (7.12), (7.13) and (7.15), (7.17), respectively. As a consequence, for $f_0 = 0$, one has

$$\mu_p \mathcal{H}_p^{c,d}(0) = 2dU_p''(0) - 4(c+d)U_p(0), \tag{7.21}$$

in consistency with (7.14).

To summarize, our comparison of the monotone functions $\mathcal{H}_p^{c,d}$ and U_p shows that our divergence theorem approach leads to (an extension of) the results of the monotone function approach by Agostiniani and Mazzieri [AM17], in some sense lifting the monotonicity from a derivative to the function itself. This circumvents the conformal change to an asymptotically cylindrical picture as introduced in [AM17]. In particular, working directly with the divergence theorem in the static system makes the analysis of the equality case simpler, avoiding the need to appeal to a splitting theorem.

Corollary 7.5 (Relation between governing functionals). *Let (M^n, g, f), $n \geq 3$, be an asymptotically flat static vacuum system of mass m with connected boundary ∂M. Let $p \geq p_n$ and $c, d \in \mathbb{R}$, and suppose that $f|_{\partial M} = f_0$ for some $f_0 \in [0,1) \cup (1,\infty)$. Then the functions $\mathcal{H}_p^{c,d}$ given by* (7.2) *and U_p given by* (7.10) *are related by* (7.20) *as well as by* (7.12), (7.13) *when $f_0 \in [0,1)$ and by* (7.15), (7.17) *when $f_0 \in (1,\infty)$. Here, $\mu_p = (p-1)(2|m|)^{\frac{(n-1)(p-1)}{n-2}}$. Moreover, if $f_0 = 0$, they satisfy the relation* (7.21).

Consequently, our approach gives new proofs for all geometric inequalities for black hole horizons described in [AM17, Section 2.2] and for the Willmore-type inequalities for black hole horizons and other level sets of the lapse function f described in [AM17, Section 2.3]. It also extends their results to $p \geq p_n$, to weaker asymptotic assumptions, and to $f_0 \in (1,\infty)$. In particular, we would like to point out that the right-hand side inequality in (1.3) is the Riemannian Penrose inequality. We hence in particular reprove the Riemannian Penrose inequality for asymptotically flat static vacuum systems with connected black hole inner boundary (but with the extra assumption (1.4) necessary to conclude rigidity) under very weak asymptotic assumptions.

Last but not least, we would like to mention that it should be possible to extend the methods used in [AM17] from $p \geq 3$ to $p \geq p_n$ for $f_0 \in [0,1)$ like it is successfully done by Agostiniani and Mazzieri in a different context in [AM20]. Furthermore, it may be possible to modify the methods used in [AM17] to handle the case $f_0 \in (1,\infty)$, performing a different, but likely similar conformal transformation. Finally, it is conceivable that the methods used in [AM17] may be extended to weaker asymptotic assumptions.

7.3. **Comparison with the proof by Nozawa, Shiromizu, Izumi, and Yamada.** In [NSIY18, Section 5], Nozawa, Shiromizu, Izumi, and Yamada devise a strategy to proving a black hole uniqueness theorem which turns out to coincide with Theorem 1.1. To do so, they devise a Robinson style strategy of proof including a parameter c arising as a power which should be understood as $c =: p-1$, see below. In this section, we will first identify that their strategy of proof almost coincides with our strategy of proving Theorem 1.1. After that, we will discuss which new insights our proof gives and which hurdles we overcame to make this strategy a rigorous proof. We will restrict to $f_0 = 0$ as [NSIY18] only addresses black holes.

The strategy followed by Nozawa, Shiromizu, Izumi, and Yamada is to introduce a divergence identity just like (1.10) based on a vector field J. They then show that the divergence of J is non-negative and can be related to the pointwise tensor norm of certain expressions related to a $(0,2)$-tensor H, in analogy with the proof of Theorem 1.5, see below. They then discuss why the vanishing of the H-tensor should imply isometry to a suitable Schwarzschild system, see below; their approach is somewhat reminiscent of parts of our proof of Theorem 1.9 but bears some issues, see below. To obtain the black hole uniqueness result Theorem 1.1, they then suggest to apply the divergence theorem to J on the static manifold M and use the properties of the static black hole horizon and the asymptotic decay assumptions (g and f are assumed to be asymptotic to g_m and f_m for some $m \in \mathbb{R}$, including at least two derivatives) and then conclude as we do.

First, let us note that the vector field Z from (7.1) used in our approach in fact coincides with the vector field J that was used in [NSIY18] up to a constant factor — despite its seemingly different definition. To see this, recall that $m > 0$ and $0 < f < 1$ on M in the black hole case. Then, adjusting to our notation, in particular choosing their exponent $c =: p-1$, their vector field J can be computed to be the $\frac{(n-2)(p-1)}{2}$-multiple of

$$\overline{J} := \frac{F_J(f)}{f}\|\nabla f\|^{p-1}\,\nabla\|\nabla f\|^2 + G_J(f)\|\nabla f\|^{p-1}\,\nabla f, \tag{7.22}$$

on M, where

$$\begin{aligned} F_J(t) &:= \frac{a + b(1-t^2)}{(1-t^2)^{\frac{(n-1)(p-1)}{(n-2)}-1}}, \\ G_J(t) &:= \frac{\frac{4(n-1)}{n-2}(a + b(1-t^2)) - \frac{4}{p-1}a}{(1-t^2)^{\frac{(n-1)(p-1)}{(n-2)}}} \end{aligned}$$

for parameters $a, b, p \in \mathbb{R}$ with $p \geq p_n$ and variables $0 < t < 1$. Choosing our parameters $c := -b$, $d := a + b$, we find that $F_J = F$ and $G_J = G$ with F and G from (1.12), (1.13) and hence $\overline{J} = Z$. Moreover, the conditions for positivity of F_J identified in [NSIY18], $a \geq 0$ and $a + b \geq 0$, exactly coincide with our (black hole case) conditions $c + d \geq 0$ and $d \geq 0$.

This of course informs us that the divergences of $\overline{J}$ and Z must also coincide. To relate the divergence identity [NSIY18, (5.12)] to (1.10), we spell out [NSIY18, (5.12)] in our notation, obtaining

$$\operatorname{div}\overline{J} = \|\nabla f\|^{p-1}\,\frac{F_J(f)}{f}\left[\|S\|^2 + \frac{2((n-1)(p-1)-(n-2))}{n-1}\|\overline{H}\|^2\right] \tag{7.23}$$

for parameters $a, b \in \mathbb{R}$, where S is given by

$$\begin{aligned} S(X,Y,Z) :=& \frac{1}{\|\nabla f\|^2}(X(f)H(Y,Z) - Y(f)H(X,Z)) \\ &- \frac{1}{n-1}(g(\overline{H},X)g(Y,Z) - g(\overline{H},Y)g(X,Z)) \end{aligned} \tag{7.24}$$

away from $\operatorname{Crit} f$ for $X, Y, Z \in \Gamma(TM)$ in terms of the H-tensor

$$H := \nabla^2 f - \frac{2}{n-2} \frac{f\|\nabla f\|^2}{1-f^2} g + \frac{2n}{n-2} \frac{f}{1-f^2} (df \otimes df) \tag{7.25}$$

and the vector field $\overline{H}$ given by

$$\overline{H} := \frac{\nabla\|\nabla f\|^{-1}}{\|\nabla f\|^{-1}} - \frac{2(n-1)}{(n-2)} \frac{f\,\nabla f}{(1-f^2)} = -\frac{H(\nabla f, \cdot)^{\#}}{\|\nabla f\|^2} \tag{7.26}$$

away from $\operatorname{Crit} f$. Knowing already that $F_J = F$, let us now relate $\overline{H}$ to (3.35) and compare the tensor S with the T-tensor, obtaining

$$\overline{H} = -\frac{1}{2\|\nabla f\|^2} \left(\nabla \|\nabla f\|^2 + \frac{4(n-1)}{(n-2)} \frac{f\|\nabla f\|^2\, \nabla f}{1-f^2} \right), \tag{7.27}$$

$$S = -\frac{(n-2)f}{(n-1)\|\nabla f\|^2} T \tag{7.28}$$

away from $\operatorname{Crit} f$. To find (7.28), we have used (3.5) and the corresponding identity [NSIY18, (5.16)]. This confirms that, away from $\operatorname{Crit} f$, the two divergence identities are in fact identical and only expressed differently, with [NSIY18] building upon the H-tensor and our approach working directly with the T-tensor. However, we would like to point out that T and (3.35) are well-defined on $\operatorname{Crit} f$ while S and $\overline{H}$ are not.

Before we suggest geometric interpretations of the two different viewpoints of the identical divergence identity expressed as (7.23) and (1.10), respectively, let us quickly delve into the strategies of asserting rigidity. Our proof of Theorem 1.9 heavily relies on the local analysis of solutions of $T = 0$ in Section 3 in combination with our asymptotic assumptions. It does *not* exploit (3.35) which is not available when $p = p_n$, see also Remark 4.1. In contrast, Nozawa, Shiromizu, Izumi, and Yamada [NSIY18] first use $\overline{H} = 0$ to obtain the functional relationship [NSIY18, (5.11)] which is equivalent to (3.35) away from $\operatorname{Crit} f$. Using this functional relationship and $H = 0$, they then suggest to proceed very similarly in spirit as we do in the proof of Theorem 3.4 with strong simplifications as most cases we study are excluded by the functional relationship [NSIY18, (5.11)], see again Remark 4.1. However, as we pointed out before, $\overline{H} = 0$ is not readily explicitly deducible when $p = p_n$, a case also included in [NSIY18]. Also, it is not discussed explicitly in [NSIY18] how $H = 0$ follows from $S = 0$ and $\overline{H} = 0$. It is thus worthwhile to investigate the relationship between S, H, and $\overline{H}$ more closely. To see that the H-tensor vanishes when $S = 0$ and $\overline{H} = 0$, we compute

$$0 = S(X, \nabla f, Y) = \frac{1}{\|\nabla f\|^2} \left(X(f) \underbrace{H(\nabla f, Y)}_{-\|\nabla f\|^2 \langle \overline{H}, Y\rangle = 0} - \|\nabla f\|^2 H(X,Y) \right) = -H(X,Y)$$

for all vector fields $X, Y \in \Gamma(TM)$ to conclude that $H = 0$ when $S = 0$, away from $\operatorname{Crit} f$. We would like to point out that this argument is very similar to the one given in the proof of Lemma 3.3. In fact, it turns out that

$$H = \frac{f}{\|\nabla f\|^2} \left(\|\nabla f\|^2 \operatorname{Ric} + \frac{\lambda\|\nabla f\|^2}{n-1} g - \frac{n\lambda}{n-1} df \otimes df \right)$$

follows from $T = 0$ via (3.35), with λ denoting the eigenvalue for the eigenvector ∇f of Ric with the right hand side coming from (3.6) as can be seen by computing λ via (3.35). This is another way of seeing that $S = 0$ implies $H = 0$ when assuming $\overline{H} = 0$ (or equivalently (3.35)). This is a purely local result. However, it does *not* locally follow from $S = 0$ that $H = 0$ *without* assuming $\overline{H} = 0$ — not even for $n = 3$ — which can be seen as follows. From Theorem 3.4, we know that $S = T = 0$ implies that each regular point $p \in M$ has an open neighborhood $p \in V \subseteq M$ such that $(V, g|_V, f|_V)$ has Type 1, 2, 3, or 4. In particular, Corollary 3.6 gives a full characterization of $\lambda|_V$. Thus, supposing that $\overline{H} = 0$ and thus $\lambda = -\frac{2(n-1)|\nabla f|^2}{(n-2)(1-f^2)}$ on M, excludes Types 1-3 and enforces Type 4 with $a = 1$. Note that this can be restated as saying that the vanishing of $\overline{H}$ implies that a system $(V, g|_V, f|_V)$ as in Theorem 3.4 has to be a suitable piece of a quasi-Schwarzschild manifold, see Remarks 3.5 and 4.1. Note furthermore that even the systems of Type 4 with $a = 1$ are not spherically symmetric unless the Einstein manifold (Σ, σ) is the standard round sphere, see Remark 3.5. In other words, any system of Types 1-3 or of Type 4 with $a \neq 1$ is a counter-example to concluding vanishing of $\overline{H}$ and thus of H from vanishing of S.

As discussed above, this insight becomes relevant when choosing the threshold parameter value $p-1 = c = 1-\frac{1}{n-1} = p_n-1$, included in the analysis in [NSIY18], when the divergence (7.23) vanishes as one can then *not* conclude that $\overline{H} = 0$. The threshold case thus needs some more careful treatment as we have given it in Sections 3 and 4. Note that for $n = 3$, the rigidity for the treshold value $c = p - 1 = \frac{1}{2}$ was already handled in [MzHRS73].

Let us now turn to some geometric considerations regarding the various tensor and vector fields discussed in this section. First, from (7.28) and the second part of (7.26), we learn that T is fully determined by H, as was already implicitly observed and exploited in [NSIY18]. However, as argued above, in order to recover (vanishing of) H from (vanishing of) T, we need the functional relationship (3.35) which takes the form $\overline{H} = 0$ in [NSIY18]. The geometric interpretation of T and thus of S is given by the definition (1.14) of T. The geometric interpretation of H can be understood in multiple ways: First (see also [NSIY18, (5.9)]), H can be derived from knowing that, in spherical symmetry with radial direction ∇f, one can express

$$\nabla^2 f = \alpha g + \beta df \otimes df \tag{7.29}$$

for suitable $\alpha, \beta \in C^\infty(M)$ because $\nabla^2 f$ will vanish in tangential-normal directions along level sets of f. Next, assuming (3.35) or $\overline{H} = 0$ and plugging ∇f into (7.29), we obtain

$$\alpha + \beta = \frac{2(n-1)f}{(n-2)(1-f^2)}. \tag{7.30}$$

On the other hand, using the static equation (2.2), we find

$$n\alpha + \|\nabla f\|^2 \beta = 0. \tag{7.31}$$

Taken together, this gives

$$\nabla^2 f = \frac{2f\|\nabla f\|^2}{(n-2)(1-f^2)} g + \beta df \otimes df - \frac{2nf}{(n-2)(1-f^2)} df \otimes df \tag{7.32}$$

which explains the definition of H as measuring the deviation from spherical symmetry, subject to (3.35) or $\overline{H} = 0$. The second interpretation of H (see also [NSIY18, (2.50)]) is exploiting the fact that $r f_m(r)\partial_r$ is a conformal Killing vector field in the Schwarzschild system (or in other words in a spherically symmetric system satisfying (3.35) or $\overline{H} = 0$. Thus,

$$\zeta := \frac{\nabla f}{(1-f^2)^{n/(n-2)}} \tag{7.33}$$

is a conformal Killing vector field in the Schwarzschild case, with

$$\mathcal{L}_\zeta g - \frac{2}{n}(\operatorname{div}\zeta) = \frac{2}{(1-f^2)^{\frac{n}{n-2}}} H \tag{7.34}$$

as can be seen from a straightforward computation. Hence, $H = 0$ if and only if ζ as defined in (7.33) is a conformal Killing vector field in (M, g). Yet another geometric interpretation of H (see [NSIY18, page 22]) is that

$$\overline{\mathrm{Ric}} = \frac{1-f}{f(1+f)} H, \tag{7.35}$$

where $\overline{\mathrm{Ric}}$ denotes the Ricci curvature tensor of the conformally transformed metric

$$\overline{g} = \left(\frac{1+f}{2}\right)^{\frac{4}{n-2}} g \tag{7.36}$$

appearing in the Bunting–Masood-ul-Alam proof [BMuA87] and its higher dimensional versions [GIS02, Hwa98]. In summary, no matter which strategy of proof we follow (i.e., via T as in this paper or via H as in [NSIY18]), we are heavily exploiting conformal flatness – with the conformal factor expressed as a function of f – of Schwarzschild as well as its spherical symmetry made explicit in the functional relationship (1.10) or in the vanishing of $\overline{H}$.

Let us close by highlighting that despite the similarities in the proofs of static vacuum black hole uniqueness subject to the scalar curvature bound (1.4) presented here and in [NSIY18], this paper adds the necessary and subtle analytic details needed to handle critical points of f, notably when asserting integrability of the divergence $\mathcal{D}$ and applicability of the divergence theorem in this weak regularity scenario, see Section 6. This particularly applies to the case $p \in [p_n, 3)$ when $\mathcal{D}$ is not continuous across $\operatorname{Crit} f$. We also demonstrate that much lower decay assumptions (namely asymptotic flatness with decay rate $\tau = 0$) suffice to conclude. And of course, we add the equipotential photon surface uniqueness results of Theorem 1.3 with $f_0 \in (0,1) \cup (1,\infty)$.

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ROTATIONAL SYMMETRY OF ANCIENT SOLUTIONS TO FULLY NONLINEAR CURVATURE FLOWS

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Abstract. We address the classification of ancient solutions to fully nonlinear curvature flows for hypersurfaces. Under natural conditions on the speed of motion we classify ancient solutions which are convex, noncollapsing, uniformly two-convex and noncompact. There are exactly two possibilities—every such solution is either a self-similarly shrinking cylinder, or else is a rotationally symmetric translating soliton. For a large class of flows this yields a complete classification of the blow-up limits that can arise at a singularity of a solution which is compact, embedded and two-convex.

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1. Introduction

The study of singularity formation in geometric flows began with the fundamental work of Hamilton [Ham82]. There it was shown that positively curved three-manifolds become singular in finite time under the Ricci flow, and are deformed to a constant curvature space in the process. This is an example of a phenomenon which has since been observed in many other situations—close to a singularity, solutions to geometric flows tend to become highly symmetric. This allows for a very precise description of the geometric structure of singularities; an essentially complete picture has now been established e.g. for the three-dimensional Ricci flow and for two-convex mean curvature flow. In these settings it has been possible to extend the flow in question beyond singularities using a surgery procedure (see [Per02a, Per02b] and [HS09, BH16, HK17b]), enabling profound insights into questions in geometry and topology.

A solution of a geometric flow is called ancient if it exists on a time interval of the form $(-\infty, T]$. Ancient solutions arise as dilations of singularities, so their classification is of central importance in understanding singularity formation. In this paper we are concerned with the classification of ancient solutions to fully nonlinear geometric flows. Consider a family of hypersurfaces $M_t \subset \mathbb{R}^{n+1}$, $t \in (-\infty, T]$, which move with pointwise normal speed equal to a symmetric function of their principal curvatures. Under natural assumptions on the speed function, we show that if M_t is convex, noncollapsing, uniformly two-convex and noncompact then it is rotationally symmetric. Moreover, M_t is either a self-similarly shrinking round $\mathbb{R} \times S^{n-1}$, or else moves by translation in a constant direction (i.e. it is a 'bowl soliton').

Brendle and Choi proved the corresponding statement for ancient solutions of the mean curvature flow [BC19, BC21] (see also [ADS19, ADS20] and [CHH21a, CHHW22]). One principal difficulty in our more general setting is that we may not appeal to Huisken's monotonicity formula. Huisken's formula asserts that, up to a certain nonlinear change of variables, the mean curvature flow is the gradient flow for the Gaussian area functional. The vast majority of the flows with which we are concerned lack an analogous interpretation. One of our main contributions is to reinterpret a number of deep phenomena, usually derived from the monotonicity formula, as consequences of mild concavity properties for the speed of motion.

In certain situations our results yield a complete classification of the possible ancient solutions that can arise by dilating a singularity. A key example we have in mind is the flow which moves M_t with speed

$$\gamma(\lambda) = \left(\sum_{i<j} \frac{1}{\lambda_i + \lambda_j - 2\kappa} \right)^{-1}, \tag{1}$$

where the λ_i are the principal curvatures. This flow was first considered by Brendle and Huisken [BH17]. They developed a surgery procedure for continuing the flow past singularities, and thereby proved a remarkable classification result for compact Riemannian manifolds. Namely, if (X, g) has boundary satisfying $\lambda_1 + \lambda_2 > 2\kappa$ and its curvature satisfies $R_{ikik} + R_{jkjk} \geq -\kappa^2$, then X is diffeomorphic to a standard ball or 1-handlebody.

The surgery procedure implemented by Brendle–Huisken relies on a priori cylindrical and gradient estimates (cf. [HS09]), which yield a detailed picture of regions of large curvature in the solution. Our results go further, showing that any blow-up at a singularity of the flow is rotationally symmetric, and either a round sphere, cylinder or bowl soliton (see Corollary 1.3 for a precise statement).

A key point in [BH17] is that the flow with speed (1) preserves two-convexity in an arbitrary ambient space. This is not true of the mean curvature flow, for example. Fully nonlinear flows were also employed to study the global geometry of Riemannian manifolds in the earlier works [And94b, And03]. Each of the papers we have mentioned employs a flow which is tailored to the specific geometric problem at hand. This highlights the necessity of developing a theory for the widest possible range of flows, in order to enable a range of geometric applications in the future.

1.1. **Main results.** Let $\Gamma \subset \mathbb{R}^n$ denote an open, symmetric (under permutations), convex cone. Fix a function $\gamma : \Gamma \to \mathbb{R}$ which is smooth, positive, symmetric, strictly increasing in each argument, and homogeneous of degree one. We assume γ is either **convex** or **concave**.

We also assume that Γ contains the positive cone $\mathbb{R}^n_+ := \{\lambda : \min_i \lambda_i > 0\}$ and the cone $\{0\} \times \mathbb{R}^{n-1}_+$, and that both $\gamma|_{\mathbb{R}^n_+}$ and $\gamma|_{\{0\}\times\mathbb{R}^{n-1}_+}$ are strictly inverse-concave functions. (These conditions ensure that we have a splitting theorem [Lyn22b] and a differential Harnack inequality [Lyn].) Note that strict inverse-concavity is automatic if γ is convex.

For a hypersurface M with principal curvatures $\lambda = (\lambda_1, \dots, \lambda_n)$ in Γ we write $G = \gamma(\lambda)$. The principal curvatures are the eigenvalues of the second fundamental form A with respect to the induced metric g. We label these so that $\lambda_1 \leq \cdots \leq \lambda_n$. The outward unit normal to M is denoted ν.

A G-flow is a smooth one-parameter family of embedded hypersurfaces M_t in $\mathbb{R}^{n+1}$ which move inwards with pointwise velocity $-G\nu$. We say that a G-flow is:

(1) convex if $M_t = \partial\Omega_t$, where Ω_t is convex;
(2) uniformly two-convex if there is a constant $\beta > 0$ such that $\lambda_1 + \lambda_2 \geq \beta H$, where H is the mean curvature;
(3) noncollapsing if there is a constant $\alpha > 0$ such that every point $x \in M_t$ admits an inscribed ball of radius at least $\alpha G(x,t)$.

We can now state our main result.

Theorem 1.1. *Let M_t, $t \in (-\infty, T]$, be a convex ancient G-flow which is noncollapsing and uniformly two-convex. Suppose also that M_t is noncompact. Then M_t is either a self-similarly shrinking round $\mathbb{R} \times S^{n-1}$, or else is a rotationally symmetric translating soliton.*

We say that M_t, $t \in (-\infty, T]$, is a translating soliton if there is a constant vector $V \in \mathbb{R}^{n+1}$ such that $M_t = M_T + (t-T)V$. Up to rigid motions of the ambient space and parabolic rescalings, there is a unique rotationally symmetric translating G-flow (the 'bowl' soliton). This solution was constructed rigorously in [Ren21].

As we mentioned above, Theorem 1.1 applies when γ is the speed (1) introduced in [BH17]. It also applies to the speeds introduced in [Lyn22a], and when γ is a ratio of elementary symmetric polynomials σ_k/σ_{k-1} such that $n \geq k+2$.

Under some further conditions on γ, Theorem 1.1 provides a complete classification of the possible limits that can arise when we blow up a singularity of a compact G-flow.

Corollary 1.2. *Suppose γ is concave. Suppose also that Γ is contained in the two-positive cone $\{\lambda : \min_{i<j} \lambda_i + \lambda_j > 0\}$, and that γ admits a continuous extension to $\bar\Gamma$ which vanishes on $\partial\Gamma$. Consider a compact embedded hypersurface M_0 in $\mathbb{R}^{n+1}$ whose principal curvatures lie in Γ (this implies strict two-convexity). Let M_t denote the maximal evolution of M_0 which moves inwards with pointwise velocity $-G\nu$. The hypersurfaces M_t form a singularity in finite time, and every blow-up of M_t at the singular time is a self-similarly shrinking round S^n or $\mathbb{R} \times S^{n-1}$, or else is the bowl soliton.*

Corollary 1.2 applies, for example, when $n \geq 3$, γ is the speed (1), and Γ is the two-positive cone.

Let us indicate how Corollary 1.2 is proven. We write C for positive constant depending on M_0. Assuming γ is concave, the parabolic maximum principle implies that $\min_{M_t} G > C^{-1}$ and $\max_{M_t} H/G \leq C$. Assuming that Γ sits inside the two-positive cone and γ vanishes on $\partial\Gamma$, the bound $H/G \leq C$ implies uniform two-convexity, and also allows us to estimate

$$\left(\frac{\partial}{\partial t} - \frac{\partial\gamma}{\partial A_{ij}}\nabla_i\nabla_j\right)G = \frac{\partial\gamma}{\partial A_{ij}}A_i^k A_{kj} G \geq \frac{1}{C}G^3.$$

Therefore, by the maximum principle, a singularity must form in finite time. The flow M_t is noncollapsing by [ALM13], and satisfies a convexity estimate (implying that blow-ups are convex) by [BH17] (see also [LL20]). The gradient estimates in [BH17] (see also [Lyn20]) imply that if $G(p_k, t_k) \to \infty$ then the sequence

$$G(p_k, t_k)(M_{t/G(p_k,t_k)^2+t_k} - p_k)$$

subconverges in C^∞_{loc} to an ancient G-flow. This ancient flow is either compact, in which case it is a sphere by [And07, And94a], or else is noncompact and satisfies the hypotheses of Theorem 1.1. The claim follows.

Corollary 1.2 also applies when M_t is the boundary of a region of a compact Riemannian $(n+1)$-manifold, provided the ambient space satisfies a natural curvature condition. In particular, it applies in the setting of [BH17].

Corollary 1.3. *Let $\gamma : \Gamma \to \mathbb{R}$ be as in the statement of Corollary 1.2. Additionally, we require that $\sup_\Gamma |\frac{\partial\gamma}{\partial\lambda_i}| < \infty$ for each $1 \leq i \leq n$. Let N^{n+1} be a compact Riemannian manifold, and suppose the curvature tensor R of N satisfies*

$$(R(e_1, e_{n+1}, e_1, e_{n+1}) + \kappa^2, \dots, R(e_n, e_{n+1}, e_n, e_{n+1}) + \kappa^2) \in \bar\Gamma$$

for every orthonormal frame $\{e_i\}_{i=1}^{n+1}$, where $\kappa \geq 0$ is a constant. Let $M_0 = \partial\Omega_0$ be a smooth hypersurface in N whose principal curvatures satisfy

$$(\lambda_1 - \kappa, \dots, \lambda_n - \kappa) \in \Gamma,$$

and let M_t denote the maximal evolution of M_0 which moves inwards with pointwise velocity $-G_\kappa\nu$, where $G_\kappa := \gamma(\lambda_1 - \kappa, \dots, \lambda_n - \kappa)$. The hypersurfaces M_t form a singularity in finite time, and (after pulling back by the exponential map) every blow-up of M_t at the singular time is a self-similarly shrinking round S^n or $\mathbb{R} \times S^{n-1}$, or else is the bowl soliton, in $\mathbb{R}^{n+1}$.

The proof of Corollary 1.3 is very similar to that of Corollary 1.2. All of the adjustments that need to be made can be readily extracted from [BH17].

Based on Theorem 1 and the works [ADS19, ADS20], we are led to the following

Conjecture 1.4. *Up to parabolic rescalings and rigid motions of the ambient space, there is a unique convex ancient G-flow which is noncollapsing, uniformly two-convex and compact. This solution is the ancient 'oval' constructed in* [LZ21] *(see also* [RS23b]*).*

We note that there is an ongoing program [Zhu22, CHH21b, DH21, DH22, CDD$^+$22] aimed at classifying noncollapsing convex ancient solutions to mean curvature flow which lie in $\mathbb{R}^4$, or are uniformly three-convex in $\mathbb{R}^{n+1}$. The present work initiates a corresponding program for fully nonlinear flows.

While all of our results concern noncollapsing ancient solutions, we would like to point out that there are many interesting phenomena to be studied in the collapsing case. Collapsing ancient solutions to curvature flows appear to exhibit far less symmetry, in general, than their noncollapsing cousins [BLT22]. A first step towards understanding this rich class of solutions was recently taken in [RL23], which constructed collapsing ancient 'pancake' solutions in great generality.

1.2. **Overview of the proof.** Let M_t denote a noncompact, convex ancient G-flow which is noncollapsing and uniformly two-convex. We write $\bar{M}_\tau$ for the rescaled family of hypersurfaces $e^{\tau/2}M_{-e^{-\tau}}$, which move with velocity $-(G - \frac{1}{2}\langle x, \nu\rangle)\nu$.

In Section 2 we collect preliminary results to be used throughout the paper. We argue in particular that, unless M_t is strictly convex, it splits off a line and hence is a shrinking cylinder by a result in [LL20]. So for the remainder of this discussion let us assume M_t is strictly convex. The claim is that M_t must then be the bowl soliton.

In Section 3 we begin studying the behaviour of the hypersurfaces M_t at very early times. We show that as $\tau \to -\infty$, up to a choice of ambient Euclidean coordinates, the family $\bar{M}_\tau$ converges in C^∞_{loc} to the cylinder $\Sigma = \mathbb{R} \times S^{n-1}((2\gamma(0, 1, \dots, 1))^{1/2})$. In the case of mean curvature flow, this property would typically be established using Huisken's monotonicity formula, but no such tool is available for fully nonlinear flows; the majority of these certainly lack any kind of variational structure. In [Wan11], Wang found another approach which does not rely on the monotonicity formula. The idea is to extract a (subsequential) blow-down limit, and then translate it infinitely far along its asymptotic cone. The limit of the translated solutions is again an ancient solution, which splits off a line and encloses the blow-down. Moreover, both the translated solution and the blow-down form a singularity at the origin at the same time. Wang then exploits the fact that lower-dimensional slices through a convex mean curvature flow satisfy an avoidance principle to conclude that the translated solution actually coincides with the blow-down. In particular, the blow-down splits off a line. Since it is also uniformly two-convex, the blow-down is a cylinder by [Hui84] (this also follows from [HS15, HH16]).

Unfortunately, for flows by other curvature functions, there is no avoidance principle for lower-dimensional slices. The validity of such a principle for the mean curvature flow traces back to the fact that the graphical mean curvature operator is the Laplacian (and in particular is isotropic in the Hessian) at points with horizontal tangent planes. This does not turn out to be a major issue—the translated solution and blow-down can still be shown to coincide via a barrier argument (this argument appeared in [Lyn22b], where it was used by the second author to classify Type I convex ancient solutions to a large class of fully nonlinear flows). In short, we are able to adapt Xu-Jia Wang's approach to prove subsequential convergence of the family $\bar{M}_\tau$ to a cylinder of radius $(2\gamma(0,1,\dots,1))^{1/2}$ whose axis passes through the origin. This leaves open the possibility that $\bar{M}_\tau$ might slowly rotate, so that along different subsequences we might see cylinders with different axes arising in the limit. We rule out this possibility by showing that the asymptotic cone of $\bar{M}_\tau$ is independent of τ. The subsequential convergence can then be upgraded to full convergence to Σ.

The next step is to prove a sharp asymptotic estimate for the rate at which $\bar{M}_\tau$ converges to Σ. For this we make use of certain barrier solutions, which are constructed in Section 4. For each sufficiently large $a > 0$ we construct an embedded disc Σ_a which is strictly convex, rotationally symmetric, and generates a self-similarly shrinking G-flow. Equivalently, each Σ_a is a stationary solution of the rescaled flow. We also establish rather precise asymptotics for these solutions as $a \to \infty$, which allow us to use them as barriers for $\bar{M}_\tau$. To describe these asymptotics let us express Σ_a as the set of points of the form

$$(z, \Psi_a(z)\theta), \qquad z \in [L_0, a],\ \theta \in S^{n-1},$$

where Ψ_a is a nonnegative concave function. We show that L_0 can be fixed independently of a. The tip of Σ_a is located at $(a, 0)$, and we have $\Psi_a(z)^2 < 2\gamma(0,1,\dots,1)$, meaning Σ_a lies inside the region bounded by Σ. In addition, there is a constant $C = C(n,\gamma)$ such that, for every bounded L, when a is sufficiently large we have

$$\Psi_a(z)^2 \le 2\gamma(0,1,\dots,1)\left(1 - \left(1 - C\frac{\log a}{a^2}\right)\frac{z^2 - C}{a^2}\right).$$

for $L_0 \le z \le L$. We also have the lower bound

$$\Psi_a(z)^2 \ge 2\gamma(0,1,\dots,1)\left(1 - \frac{z^2}{a^2}\right)$$

for $L_0 \le z \le a$. Analogous self-similarly shrinking solutions to the mean curvature flow were constructed in [ADS19] and played a central role in [BC19, BC21, ADS20]. Our construction is similar to that in [ADS19], but is made substantially more difficult by the fact that the profile function Ψ_a now solves a fully nonlinear ODE.

By the results of Section 3, we may express $\bar{M}_\tau$ as the graph of a function $u(z,\theta,\tau)$ over the z-axis, and we have $u \to 0$ in C^∞_{loc} as $\tau \to -\infty$. In Section 5 we prove the following sharp asymptotic estimate for u: for each $L < \infty$, we have

$$\sup_{|z|\le L} |u(\cdot,\tau)| \le O(e^{-\tau/2}). \tag{2}$$

This is achieved by performing a spectral decomposition of u with respect to the linearised rescaled G-flow about the cylinder Σ, and determining that this decomposition is dominated by unstable modes as $\tau \to -\infty$. The argument is one of the key steps in our analysis, so let us describe it in more detail.

We show that u satisfies

$$\frac{\partial u}{\partial \tau} = a\frac{\partial^2 u}{\partial z^2} + \frac{1}{2(n-1)}\Delta_{S^{n-1}} u - \frac{1}{2} z \frac{\partial u}{\partial z} + u + E, \tag{3}$$

where $a := \dot{\gamma}^1(0,1,\ldots,1)$ and $E = O(|u|^2 + |\nabla u|^2 + |\nabla^2 u|^2)$. For each large $-\tau$ this holds in a large but bounded subset of space. The linear elliptic operator on the right-hand side of (3) admits a complete set of L^2-eigenfunctions with respect to the measure $e^{-z^2/4a}\,dzd\theta$, and its eigenvalues are $1 - \frac{k}{2} - \frac{1}{2(n-1)}l(l+n-2)$, where $k, l \geq 0$ are integers. After cutting off in space we decompose u into pieces lying in the positive, neutral and negative eigenspaces. These evolve as one would expect for a solution to the linearised equation, up to error terms depending on E and the cutoff function. We show that all of these errors decay rapidly as $\tau \to -\infty$ by employing the hypersurfaces Σ_a as barriers. Here the asymptotics for Σ_a established in Section 4 play a decisive role. A standard argument (due to Merle and Zaag) shows that either positive or neutral modes dominate as $\tau \to -\infty$. The latter possibility is ruled out using the fact that $\bar{M}_\tau$ is noncompact—this forces $u(z,\theta,\tau)$ to be monotone in z, whereas no nonzero linear combination of zero modes has this property. The estimate (2) follows.

The proof of (2) is broadly similar to the spectral analysis of convex ancient solutions to the mean curvature flow carried out in [BC19, BC21] and [ADS19, ADS20]. In these works sufficient decay of errors could be proven by using Huisken's monotonicity formula to derive a powerful reverse-Poincaré inequality for the profile function u. Our approach using barriers is instead informed by Brendle's work on ancient κ-solutions to the 3-dimensional Ricci flow [Bre20]. We note however that error terms involving the Hessian $|\nabla^2 u|$ do not arise when considering the mean curvature flow or Ricci flow, as these equations are quasilinear, whereas for fully nonlinear equations such higher-order errors are inevitable.

In our setting we are able to control these higher-order errors, roughly speaking, as follows. Using a gradient bound furnished by the barriers, we can appeal to the interior C^2-estimate for radial graph solutions due to Brendle–Huisken [BH17], followed by Krylov's $C^{2,\alpha}$-estimate for convex parabolic PDE [Kry82], and then use Schauder estimates to get uniform bounds on higher derivatives. Next we use interpolation inequalities and interior estimates for linear parabolic equations to obtain an improved bound for the supremum of $|u| + |\nabla u| + |\nabla^2 u|$. Crucially, this bound is only slightly worse than linear in the L^2-norm of u with respect to the measure $e^{-z^2/4a}dzd\theta$ (see Lemma 5.8). This is sufficient to carry out the Merle–Zaag argument. Note that all of the estimates just described hold in a subset of space which grows at a definite rate as $\tau \to -\infty$.

With the sharp asymptotic (2) in hand, the remainder of the proof of Theorem 1.1 is quite similar to its counterpart for solutions of mean curvature flow [BC19, BC21]. In Section 6 we prove the Neck Improvement Theorem for G-flows (cf. [BC19, Theorem 4.4] and [BC21, Theorem 4.4]). Using this, we show that M_t is rotationally symmetric (Section 7) and is therefore a translating soliton (Section 8). This completes the proof of Theorem 1.1.

The arguments in Section 7 and Section 8 make use of a differential Harnack inequality for noncompact convex solutions to fully nonlinear flows, which was recently proven in [Lyn]. For solutions of mean curvature flow this inequality is due to Hamilton [Ham95], and for compact convex solutions to fully nonlinear flows it is due to Andrews [And94c]. The Harnack inequality lets us show that near its 'tip', our solution looks like a bowl soliton. This step also relies on work of Bourni–Langford [BL16], who showed that every convex, noncollapsing, uniformly two-convex G-translator is the bowl soliton.

2. Preliminaries

In this section we collect auxhiliary results which will be needed in the proof of Theorem 1.1.

We may view γ as a symmetric function defined in Γ, or as an $O(n)$-invariant function on the space of symmetric matrices with eigenvalues in Γ. We alternate between these viewpoints. We write $\dot\gamma^i(\lambda)$ and $\ddot\gamma^{ij}(\lambda)$ for derivatives with respect to eigenvalues, so that

$$\frac{d}{ds}\bigg|_{s=0}\gamma(\lambda+s\mu)=\dot\gamma^i(\lambda)\mu_i,\qquad \frac{d}{ds}\bigg|_{s=0}\dot\gamma^i(\lambda+s\mu)=\ddot\gamma^{ij}(\lambda)\mu_j,$$

and write $\dot\gamma^{ij}(A)$ and $\ddot\gamma^{ij,kl}(A)$ for derivatives with respect to matrix entries, so that

$$\frac{d}{ds}\bigg|_{s=0}\gamma(A+sB)=\dot\gamma^{ij}(A)B_{ij},\qquad \frac{d}{ds}\bigg|_{s=0}\dot\gamma^{ij}(A+sB)=\ddot\gamma^{ij,kl}(A)B_{kl}.$$

If A is a diagonal matrix with entries λ, $\dot\gamma^{ij}(A)$ is also diagonal, with entries $\dot\gamma^i(\lambda)$.

Recall that we assume γ is convex or concave. We also assume

(4) the functions $\gamma|_{\mathbb{R}^n_+}$ and $\gamma|_{\{0\}\times\mathbb{R}^{n-1}_+}$ are strictly inverse-concave.

That is,

$$\lambda\mapsto-\gamma(\lambda_1^{-1},\dots,\lambda_n^{-1})$$

is strictly concave for $\lambda\in\mathbb{R}^n_+$, and

$$\mu\mapsto-\gamma(0,\mu_1^{-1},\dots,\mu_{n-1}^{-1})$$

is strictly concave for $\mu\in\mathbb{R}^{n-1}_+$. When γ is convex the hypotheses (4) are redundant.

We often use the notation

$$F(\lambda,\mu)=\gamma(\lambda,\mu,\dots,\mu).$$

2.1. **Uniform ellipticity.** Let M_t, $t\in I$, denote a G-flow. We say that γ is uniformly elliptic on M_t if there is a constant $\delta>0$, independent of time, such that

$$\inf_{M_t}\operatorname{dist}\left(\tfrac{\lambda}{|\lambda|},\partial\Gamma\right)\geq\delta. \tag{5}$$

This implies that for any symmetric $(0,2)$-tensor B we have

$$C^{-1}|B|\leq\dot\gamma^{ij}(A)B_{ij}\leq C|B|$$

and

$$|\ddot\gamma^{ij,kl}(A)B_{ij}B_{kl}|\leq CG^{-1}|B|^2,$$

where C is a positive constant depending only on n, γ and δ.

Recall that we assume Γ contains the cones $\mathbb{R}^n_+$ and $\{0\}\times\mathbb{R}^{n-1}_+$. Given these hypotheses, weak convexity and uniform two-convexity imply uniform ellipticity. That is, if $\lambda_1\geq0$ and $\lambda_1+\lambda_2\geq\beta H$ at every point on a G-flow, then there is a constant $\delta=\delta(n,\gamma,\beta)$ such that (5) holds for $t\in I$.

Lemma 2.1. *Let M_t be a convex, uniformly two-convex G-flow. Then γ is uniformly elliptic on M_t.*

Proof. If not, let (x_k, t_k) be a sequence such that $\lambda(x_k, t_k)/|\lambda(x_k, t_k)| \to \partial\Gamma$. Given that Γ contains $\mathbb{R}^n_+$ and $\{0\} \times \mathbb{R}^{n-1}_+$, it must be the case that both $\lambda_1(x_k, t_k)$ and $\lambda_2(x_k, t_k)$ tend to zero. But convexity implies

$$\frac{1}{n} H(x_k, t_k)^2 \leq |\lambda(x_k, t_k)|^2 \leq H(x_k, t_k)^2,$$

so we have $\lambda_1(x_k, t_k) + \lambda_2(x_k, t_k)/H(x_k, t_k) \to 0$, which contradicts uniform two-convexity. ☐

2.2. **Compactness of noncollapsing convex ancient solutions.** The following interior curvature estimate was proven in [BH17, Proposition 5.1] for the speed $\gamma(\lambda) = (\sum_{i<j}(\lambda_i + \lambda_j)^{-1})^{-1}$. However the same proof works for any γ satisfying our hypotheses (see [Lyn22b] for further discussion).

Lemma 2.2. *Let $M_t = \partial\Omega_t, t \in [-T, 0]$, be a convex G-flow which is δ-uniformly parabolic. If $B(0, r)$ is contained in Ω_0, then*

$$\sup_{B(0,R)\times[-T/2,0]} |A| \leq C r^{-3} R^3 \Big(r^{-1} + T^{-1/2} \Big),$$

where C depends only on n, γ, δ and R/r.

As a consequence of Lemma 2.2 and the avoidance principle, noncollapsing G-flows satisfy the following curvature estimate (cf. [HK17a]). For the proof we refer to [Lyn20, Theorem 4.17] (note however that the argument there simplifies considerably when M_t is convex).

Here and throughout the paper we only consider ancient G-flows which exist for $t \in (-\infty, 0]$. This is purely for convenience, and can always be arranged by shifting the time variable.

Proposition 2.3. *Let $M_t = \partial\Omega_t$, $t \in (-\infty, 0]$, be a convex ancient G-flow which is δ-uniformly parabolic and α-noncollapsing. Suppose $x_0 \in M_{t_0}$. We then have*

$$C^{-1} G(x_0, t_0) \leq G(x, t) \leq C G(x_0, t_0),$$

for $(x, t) \in B(x_0, LG(x_0, t_0)^{-1}) \times [t_0 - L^2 G(x_0, t_0)^{-2}, t_0]$, where the constant C depends only on n, γ, δ, α and L.

As a consequence of the curvature estimate we have the following compactness theorem for sequences of convex ancient G-flows which are uniformly parabolic and noncollapsing.

Proposition 2.4. *Let M^k_t, $t \in (-\infty, 0]$, be a sequence of convex ancient G-flows. Suppose each M^k_t is δ-uniformly parabolic and α-noncollapsing, where δ and α are independent of k. Suppose in addition that there exists an $R < \infty$ such that $M^k_0 \cap B(0, R)$ is nonempty for every k and*

$$\liminf_{k\to\infty} \sup_{M^k_0 \cap B(0,R)} (G_k + G_k^{-1}) > 0.$$

Then there is a convex ancient G-flow M_t, $t \in (-\infty, 0]$, such that after passing to a subsequence

$$M^k_t \to M_t \ \ in\ \ C^\infty_{\mathrm{loc}}(\mathbb{R}^{n+1} \times (-\infty, 0]).$$

Proof. The assumptions imply that, for each k, there is a point p_k satisfying $|p_k| \leq R$ and a ball

$$B(p_k, r) \subset \Omega_0^k,$$

where $r > 0$ is independent of k. We may therefore apply the interior curvature bound of Lemma 2.2 to conclude that, in every compact subset of spacetime, $|A_k|$ is bounded independently of k. Moreover, by Proposition 2.3, in every compact subset of spacetime G_k is bounded from below by a positive constant which is independent of k.

With the uniform upper bound for $|A_k|$, we can deduce uniform $C^{2,\alpha}$-estimates in compact subsets of spacetime by expressing M_t^k locally as a graph and (since γ is convex or concave) appealing to the Hölder estimate for for concave/convex parabolic PDE proven in [Kry82]. Using the upper bound for $|A_k|$ and the uniform lower bound for G_k, we can bootstrap using the Schauder estimates to get uniform derivative bounds up to any order in compact subsets of spacetime. Since the hypersurfaces M_t^k are radial graphs over $\partial B(p_k, r)$, a standard application of the Arzelá–Ascoli Theorem shows that the flows subconverge in C^∞_{loc}. □

2.3. **Neck-cap decomposition.** We have the following consequence of the splitting theorem in [Lyn22b].

Lemma 2.5. *Let M_t, $t \in (-\infty, 0]$, be a convex, noncollapsing, uniformly two-convex ancient G-flow. If there is a time $t \leq 0$ and a point in M_t at which $\lambda_1 = 0$, then M_t is a self-similarly shrinking $\mathbb{R} \times S^{n-1}$.*

Proof. Suppose λ_1 vanishes at some point in spacetime. By the splitting theorem [Lyn22b, Proposition A.2],

$$M_t = \mathbb{R} \times M_t^\perp$$

for $t \leq 0$, where $M_t^\perp$ is a family of hypersurfaces in $\mathbb{R}^n$. If $n - 1 \geq 2$ then $M_t^\perp$ is uniformly convex, and hence compact by [Ham94], so Theorem 1.7 in [LL20] (see also [RS19]) implies that $M_t^\perp$ is a self-similarly shrinking sphere. If $n - 1 = 1$, since we are assuming noncollapsing, the classification result in [BLT20] implies that $M_t^\perp$ is a self-similarly shrinking circle. This completes the proof. □

Let M_t be a G-flow. For $\bar{x} \in M_{\bar{t}}$ we define

$$\begin{aligned}\hat{\mathcal{P}}(\bar{x}, \bar{t}, L, \theta) := \Big\{(x,t) : x \in B_{g(\bar{t})}(\bar{x}(t), \gamma(0,1,\dots,1)LG(\bar{x},\bar{t})^{-1}),\\ t \in [\bar{t} - \gamma(0,1,\dots,1)^2\theta G(\bar{x},\bar{t})^{-2}, \bar{t}]\Big\},\end{aligned}$$

where $\bar{x}(t)$ is the unique curve satisfying

$$\frac{d}{dt}\bar{x}(t) = -G(\bar{x}(t), t)\nu(\bar{x}(t), t), \qquad \bar{x}(\bar{t}) = \bar{x}.$$

Definition 2.6. *A point $(\bar{x}, \bar{t})$ is said to lie at the center of an ε-neck if, after shifting $(\bar{x}, \bar{t})$ to $(0,0)$, parabolically rescaling to arrange $G(0,0) = (\gamma(0,1,\dots,1)/2)^{1/2}$, and applying a rigid motion, the region $\hat{\mathcal{P}}(\bar{x}, \bar{t}, 100, 100)$ is a normal graph over $\mathbb{R} \times S^{n-1}((2\gamma(0,1,\dots,1)(1-t))^{1/2})$, and the C^{10}-norm of the height function is at most ε.*

Let M_t be a noncompact, strictly convex ancient G-flow. Then M_t has a single end at each time. The following result asserts that if M_t is also noncollapsing and uniformly two-convex, then this end consists of necks. For solutions of the mean curvature flow this was proven in [HK17b, Proposition 3.4].

Lemma 2.7. *Let M_t, $t \in (-\infty, 0]$, be a convex ancient G-flow which is α-noncollapsing and β-uniformly two-convex. For every ε_1, $\varepsilon_2 > 0$ there is a positive constant $R = R(n, \gamma, \alpha, \beta, \varepsilon_1, \varepsilon_2)$ with the following property. If x_1 and x_2 are points in M_0, neither of which lies at the center of an ε_1-neck, then one of the following holds:*

(1) $\max\{G(x_1, 0), G(x_2, 0)\}|x_1 - x_2| \leq R$.

(2) M_t is compact and bounds a region of diameter at most $(1 + \varepsilon_2)|x_1 - x_2|$.

Proof. Suppose the claim is false for some ε_1 and ε_2. Then there is a sequence of convex ancient G-flows $M_t^k = \partial\Omega_t^k$, $t \in (-\infty, 0]$, which are α-noncollapsing and β-uniformly two-convex, and have the following property. There are points x_1^k and x_2^k in M_0^k, neither of which lies at the center of an ε_1-neck, such that

$$\max\{G(x_1, 0), G(x_2, 0)\}|x_1 - x_2| \to \infty$$

and

$$\mathrm{diam}(\Omega_0^k) \geq (1 + \varepsilon_2)|x_1 - x_2|.$$

After performing a parabolic rescaling, we may assume that $|x_1^k - x_2^k| = 1$. We then have $\max\{G(x_1^k, 0), G(x_2^k, 0)\} \to \infty$. Consequently, after passing to a subsequence, the sets $\bar{\Omega}_0^k$ Hausdorff-converge to a closed convex set of dimension at most n (this follows from Lemma 2.2). Let us denote this set by K.

We claim that the dimension of K is 1. If K has dimension at least 2 then there is a 'cross' in K consisting of intervals $\overline{y_1 y_2}$ and $\overline{y_3 y_4}$ which intersect transversally at a point y_0. Approximate the points y_i by sequences y_i^k in M_0^k, and let T_k denote the convex hull of the set $\{y_i^k\}$. We have $T_k \subset \bar{\Omega}_0^k$. After performing a parabolic rescaling to arrange $G(y_0^k, 0) = 1$, we may apply Proposition 2.4 to extract a limiting G-flow which is uniformly two-convex for $t \in (-\infty, 0]$. But after rescaling the sets ∂T_k converge to a 2-plane which makes interior contact with the limiting flow at time $t = 0$. This contradicts uniform two-convexity. So we conclude that K has dimension at most 1. Moreover, since $|x_1^k - x_2^k| = 1$, K is not a point. Therefore, the dimension of K is 1.

If x_1^k and x_2^k converge to the endpoints of K then for sufficiently large k we have $\mathrm{diam}(\Omega_0^k) < 1 + \varepsilon_2$, which is impossible by hypothesis. Therefore, we may assume without loss of generality that x_2^k converges to an interior point of K. Let us now parabolically rescale so that $G(x_2^k, 0) = 1$ and extract a limiting G-flow using Proposition 2.4. Arguing as above, we see that the limit makes interior contact with a line at time $t = 0$. Therefore, by Lemma 2.5, x_2^k lies at the center of an ε_1-neck for all large k. This is a contradiction. ∎

Lemma 2.7 implies that we have bounded curvature on bounded time intervals.

Lemma 2.8. *Let $M_t = \partial\Omega_t$, $t \in (-\infty, 0]$, be a noncompact, convex ancient G-flow which is noncollapsing and uniformly two-convex. Then for each $T < \infty$, $\sup_{M_t} G$ is bounded from above independently of $t \in [-T, 0]$.*

Proof. Let B be an open ball in Ω_0. Then B is contained in Ω_t for all $t \leq 0$. Suppose there is a sequence of times $t_k \in [-T, 0]$ and points $x_k \in M_{t_k}$ such that $G(x_k, t_k) \to \infty$. Then $|x_k| \to \infty$, so for any $\varepsilon > 0$ we may apply Lemma 2.7 to conclude that (x_k, t_k) lies at the center of an ε-neck for all large k. Since $G(x_k, t_k) \to \infty$, the radius of the neck must tend to zero as $k \to \infty$. On the other hand, by convexity and noncompactness, Ω_{t_k} contains a ray emanating from the center of B. As we slide B along this ray it must remain inside Ω_{t_k} and eventually pass through the neck containing x_k. This is impossible if the neck is too thin, so we have reached a contradiction. □

2.4. **Differential Harnack inequality.** Every strictly convex ancient G-flow which is noncollapsing and uniformly two-convex satisfies the following differential Harnack inequality. In addition, if equality is ever attained then the G-flow is a translating soliton. These results require the inverse-concavity hypotheses (4).

Proposition 2.9. *Let M_t be a strictly convex ancient G-flow which is uniformly two-convex and noncollapsing. We then have*

$$\partial_t G + 2\langle \nabla G, V\rangle + A(V, V) \geq 0 \tag{6}$$

at every point $x \in M_t$ and for every $V \in T_x M_t$. Equivalently,

$$\partial_t G - A^{-1}(\nabla G, \nabla G) \geq 0 \tag{7}$$

for every $x \in M_t$. The last inequality is strict at every point in spacetime unless M_t is a translating soliton.

Proof. We first observe that (6) is equivalent to (7); indeed, we have

$$2\langle \nabla G, V\rangle + A(V, V) \geq -A^{-1}(\nabla G, \nabla G)$$

with equality exactly when $V = -A^{-1}(\nabla G)$. When M_t is compact, (7) was proven in [And94c]. When M_t is noncompact we have bounded curvature on bounded time intervals by Lemma 2.8. Therefore, Theorem 1.2 in [Lyn] gives the inequality (6). If equality is attained in (7) then M_t is a translating soliton by [Lyn, Corollary 1.3]. □

As a consequence of the Harnack inequality we obtain a curvature bound which is valid for all times.

Lemma 2.10. *Let M_t be a strictly convex ancient G-flow which is uniformly two-convex and noncollapsing. Then $\sup_{M_t} G$ is bounded from above independently of t.*

Proof. If λ_1 vanishes somewhere then M_t is a shrinking cylinder by Lemma 2.5, in which case the claim is immediate, so assume $A > 0$.

We have $\sup_{M_0} G < \infty$. This is immediate if M_0 is compact, and follows from Lemma 2.8 if M_t is noncompact. On the other hand, the Harnack inequality says that G is nondecreasing at each point in spacetime. With this the claim is proven. □

2.5. **Expansions for hypersurfaces close to a cylinder.** Let

$$\Sigma = \mathbb{R} \times S^{n-1}(r)$$

and consider a smooth function $u : \Sigma \to \mathbb{R}$. We study the hypersurface

$$M = \operatorname{graph} u = \{x + u(x)\nu_\Sigma(x) : x \in \Sigma\},$$

where ν_Σ is the outward unit normal to Σ. We assume M is close to Σ in the sense that

$$\sup_\Sigma \,(r^{-1}|u| + |\nabla^\Sigma u| + r|\nabla^2_\Sigma u|)$$

is small, and compute expansions for certain geometric quantities on M in terms of their counterparts on Σ. We write $a = b + O(c)$ when $|a-b| \leq K|c|$ for a positive constant K depending only on n and γ.

Let g and A denote the induced metric and second fundamental form of M. Let g_Σ and A_Σ denote the induced metric and second fundamental form of Σ. We will not distinguish between tensors on M and their pullbacks to Σ via the map $x \mapsto x + u(x)\nu_\Sigma(x)$.

The proof of the following lemma is straightforward, so we omit it.

Lemma 2.11. *We have the expansions*

$$\begin{aligned} g &= g_\Sigma + 2uA_\Sigma + O(r^{-2}|u|^2 + |\nabla^\Sigma u|^2) \\ \nu &= \nu_\Sigma - \nabla^\Sigma u + O(r^{-2}|u|^2 + |\nabla^\Sigma u|^2) \\ A &= A_\Sigma - \nabla^2_\Sigma u + uA^2_\Sigma + O(r^{-3}|u|^2 + r^{-1}|\nabla^\Sigma u|^2). \end{aligned}$$

From now on we work with a fixed orthonormal frame of eigenvectors for A_Σ with respect to the metric g_Σ, denoted $\{e_i\}$. Moreover, we assume e_1 is orthogonal to S^{n-1}. This frame will typically fail to be orthonormal with respect to g. The following lemma establishes that we can, in a controlled fashion, smoothly transform $\{e_i\}$ to a frame which is g-orthonormal. We will find it convenient to use this transformed frame when we expand G in Lemma 2.13 below.

Lemma 2.12. *There is a smooth field of endomorphisms $\alpha : T\Sigma \to T\Sigma$ such that $\{\alpha(e_i)\}$ is orthonormal with respect to g, and*

$$\alpha = I - uA_\Sigma + O(r^{-2}|u|^2 + |\nabla^\Sigma u|^2),$$

where $I : T\Sigma \to T\Sigma$ is the identity.

Proof. With respect to $\{e_i\}$, g has entries

$$g_{ij} = \delta_{ij} + 2u(A_\Sigma)_{ij} + u^2(A^2_\Sigma)_{ij} + \nabla^\Sigma_i u \nabla^\Sigma_j u.$$

For $s \in [0,1]$ we define

$$h_{ij}(s) := \delta_{ij} + 2u(A_\Sigma)_{ij} + u^2(A^2_\Sigma)_{ij} + s\nabla^\Sigma_i u \nabla^\Sigma_j u.$$

We then define $\alpha^k_i(s)$ to be the solution of the initial value problem

$$h_{jk}(s)\frac{d}{ds}\alpha^k_i(s) = -\frac{1}{2}\alpha^m_i(s)\nabla^\Sigma_m u \nabla^\Sigma_j u, \qquad (\delta^i_k + u(A_\Sigma)^i_k)\alpha^k_j(0) = \delta^i_j.$$

This ensures that

$$\frac{d}{ds}(h_{kl}(s)\alpha^k_i(s)\alpha^l_j(s)) = 0 \qquad \text{and} \qquad h_{kl}(0)\alpha^k_i(0)\alpha^l_j(0) = \delta_{ij},$$

and hence $\alpha^k_i(1)$ a smooth field of matrices on Σ satisfying

$$\delta_{ij} = h_{kl}(1)\alpha^k_i(1)\alpha^l_j(1) = g_{kl}\alpha^k_i(1)\alpha^l_j(1).$$

That is, the frame $\{\alpha(1)(e_i)\}$ is g-orthonormal. Straightforward computations show that

$$\alpha_i^k(1) = \alpha_i^k(0) + O(|\nabla^\Sigma u|^2) = \delta_i^k - u(A_\Sigma)_i^k + O(r^{-2}|u|^2 + |\nabla^\Sigma u|^2).$$

Therefore, $\alpha(1)$ is a smooth field of endomorphisms with the properties claimed. □

Let λ and λ_Σ denote the principal curvatures of M and Σ, respectively. Recall that we assume Γ contains $\{0\} \times \mathbb{R}_+^{n-1}$, and hence $\lambda_\Sigma \in \Gamma$. We define $G_\Sigma(x) = \gamma(\lambda_\Sigma(x))$. We may also write $G_\Sigma = \gamma(A_\Sigma)$, where on the right we evaluate γ at the matrix with entries $(A_\Sigma)_{ij} = A_\Sigma(e_i, e_j)$. Suppose λ lies in Γ and define $G(x) = \gamma(\lambda(x))$. We also define $\tilde{A}_{ij} = A_{kl}\alpha_i^k\alpha_j^l$, so that $\tilde{A}_{ij}$ is the matrix representation of A with respect to the g-orthonormal frame $\{\alpha(e_i)\}$. We may then express $G = \gamma(\tilde{A})$.

Lemma 2.13. *We have*

$$G = G_\Sigma - \dot{\gamma}^{ij}(A_\Sigma)(\nabla_\Sigma^2 u)_{ij} - \dot{\gamma}^{ij}(A_\Sigma)(A_\Sigma^2)_{ij}u + O(r^{-3}|u|^2 + r^{-1}|\nabla^\Sigma u|^2 + r|\nabla_\Sigma^2 u|^2).$$

This may also be written as

$$G = G_\Sigma - \dot{\gamma}^1(0,1,\dots,1)\frac{\partial^2 u}{\partial z^2} - \frac{1}{n-1}\frac{\gamma(0,1,\dots,1)}{r^2}r^2\Delta_{S_r^{n-1}}u - \frac{\gamma(0,1,\dots,1)}{r^2}u$$
$$+ O(r^{-3}|u|^2 + r^{-1}|\nabla^\Sigma u|^2 + r|\nabla_\Sigma^2 u|^2),$$

where z denotes a Euclidean coordinate on the x_1-axis.

Proof. The function

$$h(t) := \gamma(A_\Sigma + t(\tilde{A} - A_\Sigma))$$

is well defined and smooth in t. Performing a Taylor expansion at $t = 0$ yields

$$|h(t) - G_\Sigma - \dot{\gamma}^{ij}(A_\Sigma)(\tilde{A} - A_\Sigma)_{ij}| \leq \frac{1}{2}\left(\sup_{t\in[0,1]}|h''(t)|\right)t^2$$

for $t \in [0,1]$. Evaluating at $t = 1$, and estimating the right-hand side, we obtain

$$|G - G_\Sigma - \dot{\gamma}^{ij}(A_\Sigma)(\tilde{A} - A_\Sigma)_{ij}| \leq O(r|\tilde{A} - A|^2).$$

Now we use Lemma 2.11 and Lemma 2.12 to deduce that

$$\tilde{A} - A = -\nabla_\Sigma^2 u - A_\Sigma^2 u + O(r^{-3}|u|^2 + r^{-1}|\nabla^\Sigma u|^2 + r|\nabla_\Sigma^2 u|^2),$$

and so obtain

$$|G - G_\Sigma + \dot{\gamma}^{ij}(A_\Sigma)((\nabla_\Sigma^2 u)_{ij} + (A_\Sigma^2)_{ij}u)| = O(r^{-3}|u|^2 + r^{-1}|\nabla^\Sigma u|^2 + r|\nabla_\Sigma^2 u|^2).$$

This proves the first claim.

Since A_Σ is diagonal with respect to $\{e_i\}$, $\dot{\gamma}^{ij}(A_\Sigma)$ is also diagonal, and we have

$$\dot{\gamma}^{ij}(A_\Sigma)(\nabla_\Sigma^2 u)_{ij} = \dot{\gamma}^1(0, r^{-1}, \dots, r^{-1})\frac{\partial^2 u}{\partial z^2}$$
$$+ \frac{1}{n-1}\sum_{i=2}^n \dot{\gamma}^i(0, r^{-1}, \dots, r^{-1})\Delta_{S_r^{n-1}}u + O(|\nabla^\Sigma u|^2).$$

Here we have also used the symmetry of γ in its arguments to express

$$\dot{\gamma}^j(0, r^{-1}, \dots, r^{-1}) = \frac{1}{n-1}\sum_{i=2}^n \dot{\gamma}^i(0, r^{-1}, \dots, r^{-1})$$

for $j \geq 2$. Since γ is homogeneous of degree one we have

$$\dot{\gamma}^i(0, r^{-1}, \ldots, r^{-1}) = \dot{\gamma}^i(0, 1, \ldots, 1),$$

and hence

$$\sum_{i=2}^n \dot{\gamma}^i(0, r^{-1}, \ldots, r^{-1}) = \gamma(0, 1, \ldots, 1).$$

The same reasoning yields

$$\dot{\gamma}^{ij}(A_\Sigma)(A_\Sigma^2)_{ij} u = r^{-2} \sum_{i=2}^n \dot{\gamma}^i(0, r^{-1}, \ldots, r^{-1}) u = r^{-2} \gamma(0, 1, \ldots, 1) u.$$

The second claim follows. □

Let S be a symmetric $(0,2)$-tensor on M. After fixing a g-orthonormal frame, so that A and S can be identified with symmetric matrices, we define

$$\operatorname{tr}_\gamma(S) := \frac{d}{dt}\Big|_{t=0} \gamma(A + tS).$$

The right-hand side is independent of the chosen frame, so $\operatorname{tr}_\gamma(S)$ is a well defined smooth function on M.

Lemma 2.14. *Given a symmetric $(0,2)$-tensor S on M, we have*

$$\operatorname{tr}_\gamma(S) = \dot{\gamma}^{ij}(A_\Sigma) S_{ij} + O(r^{-1}|u| + |\nabla^\Sigma u| + r|\nabla^2_\Sigma u|)|S|.$$

In particular, when $S = \nabla^2 f$ for some smooth function f we have

$$\begin{aligned}\operatorname{tr}_\gamma(\nabla^2 f) &= \dot{\gamma}^1(0, 1, \ldots, 1) \frac{\partial^2 f}{\partial z^2} + \frac{1}{n-1} \frac{\gamma(0, 1, \ldots, 1)}{r^2} r^2 \Delta_{S_r^{n-1}} f \\ &\quad + O(r^{-1}|u| + |\nabla^\Sigma u| + r|\nabla^2_\Sigma u|)|\nabla^2 f| + O(|\nabla^\Sigma u|)|\nabla f|\end{aligned}$$

where z denotes a Euclidean coordinate on the x_1-axis.

Proof. We let $\tilde{S}_{ij} = S_{kl} \alpha_i^k \alpha_j^l$, so that $\tilde{S}_{ij}$ is the matrix representation of S with respect to the g-orthonormal frame $\{\alpha(e_i)\}$. We may then compute

$$\operatorname{tr}_\gamma(S) = \frac{d}{dt}\Big|_{t=0} \gamma(\tilde{A} + t\tilde{S}) = \dot{\gamma}^{ij}(\tilde{A}) \tilde{S}_{ij}.$$

We define an auxiliary function

$$h(t) := \dot{\gamma}^{ij}(A_\Sigma + t(\tilde{A} - A_\Sigma)) \tilde{S}_{ij},$$

which is smooth in t. We have

$$|\operatorname{tr}_\gamma(S) - \dot{\gamma}^{ij}(A_\Sigma) \tilde{S}_{ij}| = |h(1) - h(0)| \leq \sup_{t \in [0,1]} |h'(t)| = O(r|\tilde{A} - A|)|\tilde{S}|,$$

and hence, by Lemma 2.12,

$$|\operatorname{tr}_\gamma(S) - \dot{\gamma}^{ij}(A_\Sigma) S_{ij}| \leq O(r^{-1}|u| + |\nabla^\Sigma u| + r|\nabla^2_\Sigma u|)|S|.$$

The proves the first claim. The second follows as in the proof of Lemma 2.13. □

2.6. **Asymptotic cones.** Let Ω be an open convex subset of $\mathbb{R}^{n+1}$. We define the asymptotic cone $C(\Omega)$ of Ω to be the Hausdorff-limit of the sequence $k^{-1}\bar{\Omega}$. When Ω is bounded, $C(\Omega) = \{0\}$. When Ω is unbounded, $C(\Omega)$ is a closed cone. If we translate Ω so that it contains the origin, then $C(\Omega)$ is precisely the union of all the rays emanating from the origin which are contained in $\bar{\Omega}$.

Lemma 2.15. *Let $M_t = \partial\Omega_t$, $t \in (-\infty, 0]$, be a noncompact, convex ancient G-flow. The asymptotic cone $C(\Omega_t)$ is independent of t.*

Proof. Since the hypersurfaces M_t evolve inwards, we have $\Omega_t \subset \Omega_s$ whenever $s < t$. It follows that $C(\Omega_t) \subset C(\Omega_s)$. In particular, $C(\Omega_0) \subset C(\Omega_t)$ for $t \leq 0$.

Now consider an arbitrary time $\bar{t} < 0$, and an arbitrary ray R contained in $C(\Omega_{\bar{t}})$. We may assume without loss of generality that $0 \in \Omega_0$. We then have $C(\Omega_{\bar{t}}) \subset \bar{\Omega}_{\bar{t}}$, and hence $R \subset \Omega_{\bar{t}}$. Let $\tilde{t}$ denote the supremum of all the times t for which $R \subset \Omega_t$. Let B be an open ball with center at 0 such that $B \subset \Omega_0$. Sliding B along R and appealing to the avoidance principle, we see that unless $\tilde{t} = 0$, there is a short time period following $\tilde{t}$ in which $R \subset \Omega_t$. So $\tilde{t} = 0$, and hence $R \subset \bar{\Omega}_0$. It follows that $R \subset C(\Omega_0)$. Since R was arbitrary, we conclude that $C(\Omega_{\bar{t}}) \subset \Omega_0$.

We have shown that

$$C(\Omega_t) \subset C(\Omega_0) \subset C(\Omega_t)$$

for every $t \leq 0$. This proves the claim. □

3. The parabolic blow-down limit

Let $M_t = \partial\Omega_t$, $t \in (-\infty, 0]$, be a convex ancient G-flow which is noncollapsing and uniformly two-convex. In this section we study the possible parabolic blow-down limits of M_t. We show that the only possibilities are self-similarly shrinking sphere and cylinder solutions.

We first prove that the region Ω_t contains a shrinking ball with radius on the order of $(-t)^{1/2}$ (cf. [Wan11, Theorem 2.2] and [BLL21, Proposition 4.3]). Without loss of generality we may assume $0 \in M_0$.

Lemma 3.1. *There is a constant $\eta > 0$ such that $B(0, \eta(-t)^{1/2}) \subset \Omega_t$ whenever $-t$ is sufficiently large.*

Proof. Supposing the claim is false, we can find a sequence of times $t_k \to -\infty$ such that the rescaled flows

$$\tilde{M}^k_t := (-t_k)^{-1/2} M_{(-t_k)t}$$

satisfy $\operatorname{dist}(0, \tilde{M}^k_{-1}) \to 0$. The function $f_k(t) := \operatorname{dist}(0, \tilde{M}^k_t)$ is locally Lipschitz, and for almost every t we have

$$\frac{df_k}{dt}(t) \leq -\sup\{\tilde{G}_k(x,t) : |x| = f_k(t)\}.$$

Therefore, since

$$-\int_{-1}^{0} \frac{df_k}{dt}(t)\, dt = \operatorname{dist}(0, \tilde{M}^k_{-1}) \to 0,$$

there are sequences $s_k \in [-1, 0]$ and $x_k \in \tilde{M}^k_{s_k}$ such that $|x_k| = f_k(s_k) \to 0$ and $\tilde{G}_k(x_k, s_k) \to 0$. By the noncollapsing property, $\tilde{M}^k_{s_k}$ admits an inscribed ball at x_k whose radius becomes arbitrarily large as $k \to \infty$. In particular, given any $R > 0$,

$\tilde{M}^k_{s_k}$ encloses the ball $B(y_k, R)$ for all large k, where $y_k := x_k - R\tilde{\nu}_k(x_k, s_k)$ and $\tilde{\nu}_k$ is the outward unit normal to $\tilde{M}^k_t$. By the avoidance principle, $\tilde{M}^k_0$ encloses $B(y_k, (R^2 - 2\gamma(1,\dots,1))^{1/2})$ for all large k. Choosing $R^2 = 1 + 2\gamma(1,\dots,1)$, we conclude that $B(y_k, 1)$ is contained in the region bounded by $\tilde{M}^k_t$ for all $t \le 0$ and large k. This contradicts Lemma 2.2, since

$$\tilde{G}_k(0,0) = (-t_k)^{1/2} G(0,0) \to \infty.$$

□

Let us fix an arbitrary sequence $a_k \to 0$. We show that the rescaled flows $M^k_t := a_k M_{a_k^{-2} t}$ converge smoothly along a subequence.

Lemma 3.2. *Possibly after passing to a subsequence, the rescaled flows M^k_t converge in $C^\infty_{\mathrm{loc}}(\mathbb{R}^{n+1} \times (-\infty, 0))$ to a limiting flow $\hat{M}_t = \partial\hat{\Omega}_t$.*

Proof. Consider a fixed $s < 0$. As a consequence of Lemma 3.1, we have $B(0, \eta(-t)^{1/2}) \subset \Omega^k_t$ when $t < s$ and k is sufficiently large depending on s.

Using this fact we will show that

$$\inf_{B(0,R)} G_k(\cdot, s) > 0 \tag{8}$$

for every $R < \infty$. If this is not the case then, by Proposition 2.3, $G_k \to 0$ uniformly in every compact subset of $\mathbb{R}^{n+1} \times (-\infty, s]$. Using the noncollapsing assumption we see that Ω^k_s converges to a halfspace in the Hausdorff topology. But since $G_k \to 0$ locally uniformly in $\mathbb{R}^{n+1} \times (-\infty, s]$, Ω^k_t converges to the same halfspace for every $t \le s$. This contradicts $B(0, \eta(-t)^{1/2}) \subset \Omega^k_t$.

To complete the argument we use (8) and Proposition 2.4 to establish subconvergence in $C^\infty_{\mathrm{loc}}(\mathbb{R}^{n+1} \times (-\infty, s])$ for each $s < 0$. The claim follows by sending $s \to 0$ and passing to a diagonal subsequence. □

We have the following characterisation of $\hat{M}_t$ in case it is compact. (We do not make use of this result, it is only included for the sake of completeness.)

Lemma 3.3. *Suppose $\hat{M}_t$ is compact. Then M_t is a self-similarly shrinking sphere. In particular, $\hat{M}_t$ is a self-similarly shrinking sphere.*

Proof. Since $\hat{M}_t$ is compact, the splitting theorem [Lyn22b, Proposition A.2] implies $\hat{\lambda}_1 > 0$ on $\hat{M}_t$ for every $t \in (-\infty, 0)$. In particular, there is a positive constant ε such that $\hat{\lambda}_1 \ge \varepsilon \hat{H}$ on $\hat{M}_{-1}$. Therefore, for every sufficiently large k, $\lambda_1 \ge \frac{\varepsilon}{2} H$ on the hypersurface $M_{-a_k^{-2}}$. If γ is concave and inverse-concave, this inequality is preserved by the flow [And07], and hence $\lambda_1 \ge \frac{\varepsilon}{2} H$ on M_t for all $t \le 0$. Using [LL20, Theorem 1.7] (see also [RS19]) we conclude that M_t is a shrinking sphere solution.

We proceed similarly if γ is convex rather than concave, but work with λ_1/G in place of λ_1/H. A maximum principle argument shows that positive lower bounds for this ratio are preserved by the flow (see [Lan14]). □

Suppose now that $\hat{M}_t$ is noncompact. Applied to the solution $\hat{M}_t$, Proposition 4.5 in [Lyn22b] yields the following statement.

Lemma 3.4. *Suppose that, possibly after applying a rotation, $\hat{M}_t$ satisfies*
$$\hat{M}_t \subset \{x \in \mathbb{R}^{n+1} : x_1^2 + \cdots + x_n^2 \leq -2\gamma(0,1,\ldots,1)t\}$$
for every $t < 0$. We then have
$$\hat{M}_t = \{x \in \mathbb{R}^{n+1} : x_1^2 + \cdots + x_n^2 = -2\gamma(0,1,\ldots,1)t\}$$
for $t < 0$.

Using this result, we show that $\hat{M}_t$ is a self-similarly shrinking cylinder.

Lemma 3.5. *Suppose $\hat{M}_t$ is noncompact. Up to a rotation, we have*
$$\hat{M}_t = \{x \in \mathbb{R}^{n+1} : x_1^2 + \cdots + x_n^2 = -2\gamma(0,1,\ldots,1)t\}$$
for $t < 0$.

Proof. Each of the asymptotic cones $C(\hat{\Omega}_t)$ is equal to $C := C(\hat{\Omega}_{-1})$ by Lemma 2.15. Since $\hat{M}_t$ is noncompact, C is noncompact. Moreover, Lemma 2.7 implies that the dimension of C is one, so C is either a ray or a line. If C is a line then $\hat{M}_t$ splits off a line, and the claim follows from Lemma 2.5. Suppose then that C is a ray. Applying a rotation if necessary, we may assume C is contained in the x_{n+1}-axis.

Consider the set $\hat{M}_0 := \cap_{t<0}\hat{\Omega}_t$. This set is closed and convex. A straightforward argument using Lemma 2.2 and the avoidance principle shows that
$$\sup_{\hat{M}_t \cap B(0,1)} \hat{G} \to \infty$$
as $t \to 0$. Therefore, using Lemma 2.7, we conclude that the dimension of $\hat{M}_0$ is at most one. On the other hand $C \subset \hat{\Omega}_t$ for $t < 0$. It follows that $C \subset \hat{M}_0$. From this we conclude that the dimension of $\hat{M}_0$ is equal to one, and that every point in C is reached by $\hat{M}_t$ as $t \to 0$

Let $x_j \in C$ be a sequence of points going to ∞, and consider the shifted flows $\hat{M}_t - x_j$. We know that $\hat{M}_t - x_j$ reaches the origin as $t \to 0$. Therefore, by the avoidance principle, for each $t < 0$ there is a point in $\hat{M}_t - x_j$ whose distance to the origin is at most $\sqrt{-2\gamma(1,\ldots,1)t}$. Moreover, the ball $B(0, \eta(-t)^{1/2})$ is contained in $\hat{\Omega}_t$, and hence in $\hat{\Omega}_t - x_j$, for every $t < 0$. This is due to Lemma 3.1.

Using these facts, we can appeal to Proposition 2.4 to extract a smooth limit of the shifted flows in $C^\infty_{\mathrm{loc}}(\mathbb{R}^{n+1} \times (-\infty, 0))$. Denote the limiting flow by $\tilde{M}_t$. By construction, $\tilde{M}_t$ encloses the line through the origin parallel to C. Therefore, $\tilde{M}_t$ splits off a line and (since we rotated to align C with the x_{n+1}-axis) Lemma 2.5 implies that
$$\tilde{M}_t = \{x \in \mathbb{R}^{n+1} : x_1^2 + \cdots + x_n^2 = -2\gamma(0,1,\ldots,1)t\}.$$
But $\tilde{M}_t$ also encloses $\hat{M}_t$, so using Lemma 3.4 we conclude that
$$\hat{M}_t = \{x \in \mathbb{R}^{n+1} : x_1^2 + \cdots + x_n^2 = -2\gamma(0,1,\ldots,1)t\}.$$
This completes the proof. $\square$

Let $\bar{M}_\tau$ denote the rescaled family of hypersurfaces $e^{\tau/2}M_{-e^{-\tau}}$. These move with velocity $-(G - \frac{1}{2}\langle x, \nu\rangle)\nu$. We now prove the main result of this section, which asserts that in case M_t is noncompact we have full (as opposed to subsequential) convergence of $\bar{M}_\tau$ to a cylinder of radius $(2\gamma(0,1,\ldots,1))^{1/2}$ as $\tau \to -\infty$.

Proposition 3.6. *Suppose M_t is noncompact. Up to a rotation, the rescaled hypersurfaces $\bar{M}_\tau$ converge in $C^\infty_{\mathrm{loc}}(\mathbb{R}^{n+1})$ to the cylinder*

$$\Sigma := \{x \in \mathbb{R}^{n+1} : x_1^2 + \cdots + x_n^2 = 2\gamma(0, 1, \ldots, 1)\}$$

as $\tau \to -\infty$.

Proof. Each of the asymptotic cones $C(\Omega_t)$ is equal to $C := C(\Omega_0)$ by Lemma 2.15. Moreover, Lemma 2.7 implies that the dimension of C is one. After rotating if necessary, we may assume C lies in the x_{n+1}-axis.

It suffices to show that, for every sequence $\tau_k \to \infty$, $\bar{M}_{\tau_k}$ admits a subsequence which converges in $C^\infty_{\mathrm{loc}}(\mathbb{R}^{n+1})$ to Σ. Lemma 3.5 guarantees subconvergence to a cylinder of radius $\sqrt{2\gamma(0, 1, \ldots, 1)}$ with axis through the origin. But since $C \subset \Omega_t$ for $t \leq 0$, the axis of every such limiting cylinder is the x_{n+1}-axis. The claim follows. □

4. A barrier construction

In this section we prove the following statement.

Proposition 4.1. *There is a positive constant $L_0 = L_0(n, \gamma)$ such that, for each sufficiently large $a > 0$, there exists a function $\Psi_a : [L_0, a] \to \mathbb{R}$ with the following properties:*

(i) *$\Psi_a(z) \geq 0$ with equality at $z = a$ and $\Psi_a(z)^2 < 2\gamma(0, 1, \ldots, 1)$ for $z \in [L_0, a]$.*
(ii) *Ψ_a is strictly decreasing and strictly concave.*
(iii) *The set of points*

$$\{(z, \Psi_a(z)\theta) : z \in (L_0, a],\ \theta \in S^{n-1}\}$$

is a smooth hypersurface which solves the G-shrinker equation, $G = \frac{1}{2}\langle x, \nu\rangle$.
(iv) *For each $z \in [L_0, a]$ we have*

$$\Psi_a(z)^2 \geq 2\gamma(0, 1, \ldots, 1)\left(1 - \frac{z^2}{a^2}\right).$$

(v) *There is a positive constant $C = C(n, \gamma)$ with the following property. For any given $L > L_0$, the inequality*

$$\Psi_a(z)^2 \leq 2\gamma(0, 1, \ldots, 1)\left(1 - \left(1 - C\frac{\log a}{a^2}\right)\frac{z^2 - C}{a^2}\right).$$

holds for $z \in [L_0, L]$, provided that a is sufficiently large depending on L.

Consider a strictly concave smooth function $h : (0, R) \to \mathbb{R}$. The hypersurface

$$\{(h(r), r\theta) : r \in (0, R),\ \theta \in S^{n-1}\}$$

solves $G = \frac{1}{2}\langle x, \nu\rangle$ if and only if h solves the ODE

$$\gamma\left(-\frac{h_{rr}}{1 + h_r^2}, -\frac{h_r}{r}, \ldots, -\frac{h_r}{r}\right) = \frac{1}{2}(h - rh_r).$$

For a given $a > 0$, consider the function $\psi : (0, Ra) \to \mathbb{R}$ defined by

$$h(r) = a - \frac{1}{a}\psi(ar),$$

and set $\rho = ar$. Then h solves (4) if and only if ψ solves

$$\gamma\left(\frac{\psi_{\rho\rho}}{1+\psi_\rho^2}, \frac{\psi_\rho}{\rho}, \dots, \frac{\psi_\rho}{\rho}\right) = \frac{1}{2} + \frac{1}{2a^2}(\rho\psi_\rho - \psi). \tag{9}$$

As a first step towards proving Proposition 4.1, for each $a > 0$ we will construct a solution of (9) which satisfies the initial conditions $\psi(0) = 0$ and $\psi_\rho(0) = 0$.

Proposition 4.2. *For each $a > 0$ there is a strictly convex function*

$$\psi \in C^\infty([0, \sqrt{2F(0,1)}a))$$

which solves (9) *with the initial conditions*

$$\psi(0) = 0, \qquad \psi_\rho(0) = 0.$$

Moreover, there is a constant $C > 0$ depending only on n and γ such that $\psi_\rho \geq C\rho$ for every $\rho \in (0, \sqrt{2F(0,1)}a)$, and $\psi_\rho \leq C\varepsilon^{-1}\rho$ for every $\rho \in (0, \sqrt{(2-\varepsilon)F(0,1)}a)$.

The solution ψ in Proposition 4.2 is constructed as a limit of solutions to a sequence of initial value problems, in which we pose carefully chosen initial conditions at ρ_k, where ρ_k is a sequence of positive numbers tending to 0. Some of our arguments take inspiration from [Ren21], where (9) was studied in the case $a = \infty$ (which corresponds to translating, rather than shrinking, G-flows). To be able to take a limit of the approximating solutions using the Arzelá–Ascoli theorem, we need to establish uniform derivative estimates. We first prove a priori estimates showing that, for a convex solution of (11), appropriate control on the gradient implies a C^2-estimate and hence higher derivative bounds via bootstrapping. This reduces the entire construction to the derivation of gradient bounds for the approximating solutions, which we achieve using sub- and supersolutions and a comparison principle.

It will be convenient to rewrite (9). Consider the function $F : \mathbb{R}^2_+ \to \mathbb{R}_+$ given by

$$F(x,y) := \gamma(x, y, \dots, y).$$

Since γ is increasing in each argument we have $\frac{\partial F}{\partial x} > 0$, and consequently the set

$$\{F(x,y) - z = 0 : (x,y,z) \in \mathbb{R}^3_+\}$$

is a smooth hypersurface. Moreover, by the implicit function theorem there is an open set $U \subset \mathbb{R}^2_+$ and a smooth function $f : U \to \mathbb{R}$ such that $F(x,y) - z = 0$ if and only if $(y,z) \in U$ and $x = f(y,z)$. In particular, $F(f(y,z),y) = z$ for all $(y,z) \in U$. We have

$$\frac{\partial f}{\partial y}(y,z) = -\frac{\frac{\partial F}{\partial y}(f(y,z),y)}{\frac{\partial F}{\partial x}(f(y,z),y)}, \qquad \frac{\partial f}{\partial z}(y,z) = \frac{1}{\frac{\partial F}{\partial x}(f(y,z),y)},$$

so f is strictly decreasing in its first argument and strictly increasing in its second. If ψ is such that

$$\left(\frac{\psi_\rho}{\rho}, \frac{1}{2} + \frac{1}{2a^2}(\rho\psi_\rho - \psi)\right) \in U, \tag{10}$$

then ψ solves (9) if and only if

$$\psi_{\rho\rho} = (1+\psi_\rho^2) f\left(\frac{\psi_\rho}{\rho}, \frac{1}{2} + \frac{1}{2a^2}(\rho\psi_\rho - \psi)\right). \tag{11}$$

Let $Q \in (0, \infty]$ denote $\lim_{x\to\infty} F(x,1)$. For example, when $\gamma(\lambda) = \lambda_1 + \cdots + \lambda_n$ we have $Q = \infty$, whereas for the speed $\gamma(\lambda) = (\sum_{i<j}(\lambda_i + \lambda_j)^{-1})^{-1}$ we have $Q = \frac{1}{4(n-1)(n-2)}$. Observe that $f(y,z) < \infty$ if and only if $z/y = F(f(1,z/y),1) < Q$. Moreover, $f(y,z) > 0$ if and only if $z/y = F(f(1,z/y),1) > F(0,1)$. Therefore, U is the open cone

$$U = \{(y,z) \in \mathbb{R}^2_+ : F(0,1) < z/y < Q\}.$$

If ψ is a C^2 solution of (9) such that $\psi_\rho > 0$ and $\psi_{\rho\rho} > 0$ then

$$F\left(\frac{\rho\psi_{\rho\rho}}{\psi_\rho(1+\psi_\rho^2)}, 1\right) = \frac{\rho}{\psi_\rho}\gamma\left(\frac{\psi_{\rho\rho}}{1+\psi_\rho^2}, \frac{\psi_\rho}{\rho}, \dots, \frac{\psi_\rho}{\rho}\right) = \frac{\rho}{\psi_\rho}\left(\frac{1}{2} + \frac{1}{2a^2}(\rho\psi_\rho - \psi)\right),$$

and the left-hand side takes values in $(F(0,1), Q)$, so (10) holds, and hence we are free to pass between the two forms of the equation (9) and (11).

4.1. **A priori estimates.** The possibility that $Q < \infty$ is one reason why the construction we carry out in this section is more subtle than its counterpart for the mean curvature flow. Geometrically, if γ is such that $Q < \infty$, a convex hypersurface on which G is bounded can have principal curvatures which are arbitrarily large. This cannot happen if γ is uniformly elliptic on the hypersurface. The following lemma provides conditions under which γ is uniformly elliptic on a strictly convex solution of (9). Given a solution, let us define

$$\Lambda := \frac{\rho\psi_{\rho\rho}}{\psi_\rho(1+\psi_\rho^2)}.$$

Thinking of ψ as the profile of a surface of rotation, Λ is equal to the radial principal curvature divided by the principal curvature in the direction of rotation.

Lemma 4.3. *Let ψ be a solution of* (9) *on the interval $[\rho_0, \rho_1]$ such that $\psi_\rho > 0$ and $\psi_{\rho\rho} > 0$. Suppose $\rho_1 < \sqrt{2F(0,1)}a$. There is a constant C depending only on n and γ such that*

$$\Lambda(\rho) \le \max\{C, \Lambda(\rho_0)\}$$

for each $\rho \in [\rho_0, \rho_1]$.

Proof. Let $B(\rho) := \rho\psi_\rho^{-1}(\frac{1}{2} + \frac{1}{2a^2}(\rho\psi_\rho - \psi))$. Writing (9) in the form (11) gives

$$\Lambda = f(1, B).$$

We compute

$$\rho B_\rho = \left(\frac{\rho^2}{2a^2} - B\right)(1+\psi_\rho^2)f(1,B) + B.$$

Let us define $2\varepsilon_0 := 1 - \frac{F(0,1)}{Q}$. If $\rho < \sqrt{2F(0,1)}a$ is such that

$$B(\rho) > \max\{\tfrac{F(0,1)}{1-\varepsilon_0}, F(1/\varepsilon_0, 1)\}$$

then $f(1, B(\rho)) > 1/\varepsilon_0$, and hence

$$\rho B_\rho < (F(0,1) - (1-\varepsilon_0)B - \varepsilon_0 B)(1+\psi_\rho^2)f(1,B) + B < 0.$$

It follows that

$$\max_{\rho_0 \le \rho \le \rho_1} B(\rho) \le \max\{B(\rho_0), \tfrac{F(0,1)}{1-\varepsilon_0}, F(1/\varepsilon_0, 1)\}.$$

Since $\frac{F(0,1)}{1-\varepsilon_0}$ and $F(1/\varepsilon_0, 1)$ are both less than Q, we may define a finite constant

$$C := \max\left\{f\left(1, \tfrac{F(0,1)}{1-\varepsilon_0}\right), f(1, F(1/\varepsilon_0, 1))\right\}.$$

Since f is increasing in its second argument,

$$\max_{\rho_0\le\rho\le\rho_1} f(1, B(\rho)) \le \max\{C, f(1, B(\rho_0))\}.$$

The claim follows. $\square$

In addition to the upper bound in Lemma 4.3, we have the following lower bound for Λ.

Lemma 4.4. *Let ψ be a solution of (9) on the interval $[\rho_0, \rho_1]$ such that $\psi_\rho > 0$ and $\psi_{\rho\rho} > 0$. There is a constant c depending only on n and γ such that*

$$\Lambda(\rho) \ge \min\{c(1 + \psi_\rho(\rho_1)^2)^{-1}, \Lambda(\rho_0)\},$$

for each $\rho \in [\rho_0, \rho_1]$.

Proof. We have

$$\rho B_\rho = \left(\frac{\rho^2}{2a^2} - B\right)(1 + \psi_\rho^2) f(1, B) + B > (1 - Mf(1, B))B,$$

where $M := (1 + \psi_\rho(\rho_1)^2)$. This implies that $B_\rho > 0$ whenever $B < F(M^{-1}, 1)$. Therefore,

$$\min_{\rho_0\le\rho\le\rho_1} B \ge \min\{B(\rho_0), F(M^{-1}, 1)\}$$

and hence

$$\min_{\rho_0\le\rho\le\rho_1} f(1, B) \ge \min\{f(1, B(\rho_0)), f(1, F(M^{-1}, 1))\}.$$

There is a constant c depending only on n and γ such that

$$f(1, F(M^{-1}, 1)) \ge cM^{-1},$$

which gives the claim when we insert $\Lambda = f(1, B)$. $\square$

Lemma 4.3 can be employed to prove higher derivative bounds for solutions to (9), via the following result. For our purposes a C^3-estimate will suffice, but a bound on the C^k-norm can be proven along the same lines.

Lemma 4.5. *Let ψ be a solution of (9) on the interval $[\rho_0, \rho_1]$ such that $\psi_\rho > 0$ and $\psi_{\rho\rho} > 0$. Define*

$$\bar{\Lambda} := \sup_{\rho_0\le\rho\le\rho_1} \Lambda(\rho) < \infty.$$

We then have

$$\sup_{\rho_0\le\rho\le\rho_1} |\psi_{\rho\rho\rho}(\rho)| \le Ca^{-2}\rho_1 + C(1 + \rho_0^{-1})\left(1 + \sup_{\rho_0\le\rho\le\rho_1} \frac{\psi_\rho}{\rho}\right)$$

where C depends only on n, γ, $\bar{\Lambda}$, and $\|\psi\|_{C^2([\rho_0,\rho_1])}$.

Proof. Differentiating 11, we obtain

$$\begin{aligned}\psi_{\rho\rho\rho} &= 2\psi_\rho\,\psi_{\rho\rho}\; f\left(\frac{\psi_\rho}{\rho}, \frac{1}{2}+\frac{1}{2a^2}(\rho\psi_\rho-\psi)\right)\\ &\quad+(1+\psi_\rho^2)\frac{\partial f}{\partial y}\left(\frac{\psi_\rho}{\rho}, \frac{1}{2}+\frac{1}{2a^2}(\rho\psi_\rho-\psi)\right)\left(\frac{\psi_{\rho\rho}}{\rho}-\frac{\psi_\rho}{\rho^2}\right)\\ &\quad+(1+\psi_\rho^2)\frac{\partial f}{\partial z}\left(\frac{\psi_\rho}{\rho}, \frac{1}{2}+\frac{1}{2a^2}(\rho\psi_\rho-\psi)\right)\frac{1}{2a^2}\rho\psi_{\rho\rho}.\end{aligned}$$

Recalling the notation $B := \rho\psi_\rho^{-1}(\frac{1}{2}+\frac{1}{2a^2}(\rho\psi_\rho-\psi))$, and given that

$$F(\Lambda,1)=F(f(1,B),1)=B,$$

our hypotheses ensure that

$$\left(\frac{\psi_\rho}{\rho}, \frac{1}{2}+\frac{1}{2a^2}(\rho\psi_\rho-\psi)\right)\in U',$$

where $U' := \{(y,z)\in U : F(0,1) < z/y \leq F(\bar\Lambda,1)\}$. Using the fact that f is homogeneous of degree one, it is straightforward to show that its first derivatives are bounded by some constant C in the cone U', where $C = C(n,\gamma,\bar\Lambda)$. The claim follows. □

4.2. **Local existence.** As mentioned above, to prove Proposition 4.2, we will solve a sequence of initial value problems and take a limit. To do so we will need appropriate sub- and supersolutions.

The following two lemmas establish that $w = \frac{\theta}{4F(1,1)}\rho^2$ is a lower barrier for solutions of (11) when $\theta \in (\frac{F(1,1)}{Q}, 1)$.

Lemma 4.6. *Fix a constant* $\frac{F(1,1)}{Q} < \theta < 1$ *and let* $w = \frac{\theta}{4F(1,1)}\rho^2$. *We then have*

$$w_{\rho\rho} < (1+w_\rho^2) f\left(\frac{w_\rho}{\rho}, \frac{1}{2}\right)$$

for all $\rho \geq 0$.

Proof. Since

$$F(0,1)\frac{w_\rho}{\rho} = \frac{F(0,1)}{2F(1,1)} < \frac{1}{2} < \frac{Q}{2F(1,1)} = Q\frac{w_\rho}{\rho},$$

we have $(\frac{w_\rho}{\rho}, \frac{1}{2}) \in U$ for all $\rho \geq 0$. Observe that

$$F\left(f\left(\frac{w_\rho}{\rho}, \frac{1}{2}\right), \frac{w_\rho}{\rho}\right) = \frac{1}{2}$$

and

$$F\left(\frac{w_{\rho\rho}}{1+w_\rho^2}, \frac{w_\rho}{\rho}\right) = \frac{\theta}{2F(1,1)} F\left(\frac{1}{1+w_\rho^2}, 1\right).$$

Since F is increasing in each of its arguments, we see that

$$w_{\rho\rho} < (1+w_\rho^2) f\left(\frac{w_\rho}{\rho}, \frac{1}{2}\right)$$

if and only if

$$\frac{\theta}{2F(1,1)}F\left(\frac{1}{1+w_\rho^2},1\right) < \frac{1}{2},$$

or equivalently

$$F\left(\frac{1}{1+w_\rho^2},1\right) < \theta^{-1}F(1,1).$$

The last inequality is always true when $\theta < 1$. $\square$

Lemma 4.7. *Let $\psi \in C^2([\rho_0,\rho_1))$ be a solution of* (11) *satisfying $\psi_\rho > 0$ and $\psi_{\rho\rho} > 0$. Let $w = \frac{\theta}{4F(1,1)}\rho^2$ for some constant $\theta \in (\frac{F(1,1)}{Q},1)$. If at $\rho = \rho_0$ we have*

$$\psi \geq w, \qquad \psi_\rho \geq w_\rho, \qquad \rho\psi_\rho - \psi \geq 0,$$

then each of these inequalities also holds for $\rho \in (\rho_0,\rho_1)$.

Proof. Let $s := \sup\{\rho \in (\rho_0,\rho_1) : \psi_\rho(\rho) > w_\rho(\rho)\}$. At $\rho = \rho_0$ we have

$$w_{\rho\rho} < (1+w_\rho^2)f\left(\frac{w_\rho}{\rho},\frac{1}{2}\right) \leq (1+\psi_\rho^2)f\left(\frac{\psi_\rho}{\rho},\frac{1}{2}+\frac{1}{2a^2}(\rho\psi_\rho - \psi)\right) = \psi_{\rho\rho},$$

so since $\psi_\rho(\rho_0) \geq w_\rho(\rho_0)$, we deduce that $s > \rho_0$. With the aim of deriving a contradiction, suppose $s < \rho_1$. We then have

$$\psi_\rho(\rho) > w_\rho(\rho)$$

for $\rho < s$, and

$$\psi_\rho(s) = w_\rho(s), \qquad \psi_{\rho\rho}(s) \leq w_{\rho\rho}(s).$$

In addition, since

$$(\rho\psi_\rho - \psi)_\rho = \rho\psi_{\rho\rho} > 0$$

for $\rho \in [\rho_0, s)$, and $\rho\psi_\rho - \psi \geq 0$ at $\rho = \rho_0$, we have

$$\rho\psi_\rho - \psi_\rho \geq 0$$

for all $\rho \in [\rho_0,\rho_1)$. We conclude that, at $\rho = s$,

$$\psi_{\rho\rho} = (1+\psi_\rho^2)f\left(\frac{\psi_\rho}{\rho},\frac{1}{2}+\frac{1}{2a^2}(\rho\psi_\rho - \psi)\right) \geq (1+w_\rho^2)f\left(\frac{w_\rho}{\rho},\frac{1}{2}\right) > w_{\rho\rho}.$$

This is a contradiction, so $s = \rho_1$. The claim follows. $\square$

Next we identify upper barriers for solutions of (11).

Lemma 4.8. *Let $\psi \in C^2([\rho_0,\rho_1))$ be a solution of* (11) *satisfying $\psi_\rho > 0$ and $\psi_{\rho\rho} > 0$. Fix a constant $\Theta > \frac{F(1,1)}{F(0,1)}$. We assume*

$$\rho_1^2 < 2a^2(F(0,1) - F(1,1)\Theta^{-1}),$$

and define $W = \frac{\Theta}{4F(1,1)}\rho^2$. If at $\rho = \rho_0$ we have

$$\psi \leq W, \qquad \psi_\rho < W_\rho,$$

then these inequalities also hold for $\rho \in (\rho_0,\rho_1)$.

Proof. First observe that

$$s := \sup\{\rho \in [\rho_0, \rho_1) : W_\rho(\rho) > \psi_\rho(\rho)\}$$

satisfies $s > \rho_0$. With the aim of deriving a contradiction, suppose that $s < \rho_1$. We then have

$$W_\rho(\rho) - \psi_\rho(\rho) > 0$$

for $\rho \in (\rho_0, s)$, and

$$W_\rho(s) = \psi_\rho(s).$$

Since $\psi_{\rho\rho} > 0$ we have $\big(\frac{\psi_\rho}{\rho}, \frac{1}{2} + \frac{1}{2a^2}(\rho\psi_\rho - \psi)\big) \in U$ and, in particular,

$$\frac{1}{2} + \frac{1}{2a^2}(\rho\psi_\rho - \psi) > F(0,1)\frac{\psi_\rho}{\rho}.$$

At $\rho = s$ this yields

$$\frac{1}{2} + \frac{1}{2a^2}(sW_\rho(s) - \psi(s)) > F(0,1)\frac{W_\rho(s)}{s},$$

which we rearrange to obtain

$$s^2 > 2a^2(F(0,1) - F(1,1)\Theta^{-1}).$$

This contradicts our assumption that $\rho_1^2 < 2a^2(F(0,1) - F(1,1)\Theta^{-1})$. The claim follows. ☐

We now have all the tools needed to prove Proposition 4.2.

Proof of Proposition 4.2. Fix a constant $\theta \in (\frac{F(1,1)}{Q}, 1)$ and set $w = \frac{\theta}{4F(1,1)}\rho^2$. Let ρ_k be a decreasing sequence of positive numbers such that $\rho_k \to 0$ as $k \to \infty$. By the Picard–Lindelöf theorem, for each k there exists some maximal $R_k > \rho_k$ such that (11) admits a solution $\psi^k \in C^2([\rho_k, R_k))$ which is strictly convex and satisfies the initial conditions

$$\psi^k(\rho_k) = w(\rho_k), \qquad \psi^k_\rho(\rho_k) = w_\rho(\rho_k).$$

By Lemma 4.7 we have

$$\psi^k \geq \frac{\theta}{4F(1,1)}\rho^2, \qquad \psi^k_\rho \geq \frac{\theta}{2F(1,1)}\rho, \qquad \rho\psi^k_\rho - \psi^k \geq 0$$

for every $\rho \in [\rho_k, R_k)$. Moreover, by Lemma 4.8, given a constant $\Theta > \frac{F(1,1)}{F(0,1)}$ we have

$$\psi^k \leq \frac{\Theta}{4F(1,1)}\rho^2, \qquad \psi^k_\rho \leq \frac{\Theta}{2F(1,1)}\rho,$$

for every $\rho \in [\rho_k, R_k)$ such that $\rho^2 < 2a^2(F(0,1) - F(1,1)\Theta^{-1})$.

At $\rho = \rho_k$ we have

$$F(\Lambda^k, 1) = \frac{\rho}{\psi^k_\rho}\left(\frac{1}{2} + \frac{1}{2a^2}(\rho\psi^k_\rho - \psi^k)\right) = \frac{F(1,1)}{\theta} + \frac{\rho_k^2}{4a^2} \to \frac{F(1,1)}{\theta}$$

as $k \to \infty$. Since $F(1,1) < \frac{F(1,1)}{\theta} < Q$, we conclude that $\lim_{k\to\infty} \Lambda^k(\rho_k)$ is some finite number in $(1, \infty)$ which depends only on n, γ and θ. Therefore, for sufficiently large k, Lemma 4.3 yields the bound

$$\Lambda^k(\rho) \leq C$$

for every $\rho \in [\rho_k, R_k)$ satisfying $\rho < \sqrt{2F(0,1)}a$, where $C = C(n, \gamma, \theta)$. Consequently, we have

$$\psi^k_{\rho\rho} \leq C(1 + |\psi^k_\rho|^2)\frac{\psi^k_\rho}{\rho} \leq C\frac{\Theta}{2F(1,1)}\left(1 + \frac{\Theta^2}{4F(1,1)^2}\rho^2\right)$$

for every $\rho \in [\rho_k, R_k)$ such that $\rho^2 < 2a^2(F(0,1) - F(1,1)\Theta^{-1})$. In addition, Lemma 4.4 gives

$$\Lambda^k(\rho) \geq c\left(1 + \frac{\Theta^2}{4F(1,1)^2}\rho^2\right)^{-1},$$

and hence

$$\psi^k_{\rho\rho} \geq \frac{c\theta}{2F(1,1)}\left(1 + \frac{\Theta^2}{4F(1,1)^2}\rho^2\right)^{-1}$$

for every $\rho \in [\rho_k, R_k)$ such that $\rho^2 < 2a^2(F(0,1) - F(1,1)\Theta^{-1})$, where c depends only on n and γ.

Suppose $R_k < \sqrt{2F(0,1)}a$. By the estimates above, when k is sufficiently large we have

$$\sup_{\rho_k \leq \rho < R_k} |\psi^k| + |\psi^k_\rho| + |\psi^k_{\rho\rho}| < \infty, \qquad \sup_{\rho_k \leq \rho < R_k} \Lambda^k < \infty, \qquad \inf_{\rho_k \leq \rho < R_k} \psi^k_{\rho\rho} > 0.$$

Therefore, the Picard-Lindelöf theorem can be used to extend ψ^k beyond R_k so that it is still a strictly convex C^2 solution of (11). This contradicts maximality, so we must have $R_k \geq \sqrt{2F(0,1)}a$.

We now take a limit of the solutions ψ^k as $k \to \infty$. We have shown that $\|\psi^k\|_{C^2(I)}$ is bounded independently of k for every compact subinterval $I \subset (0, \sqrt{2F(0,1)}a)$, and using Lemma 4.5 we can bound $\|\psi^k\|_{C^3(I)}$ in terms of $\|\psi^k\|_{C^2(I)}$. Therefore, by a standard diagonal argument, ψ^k subconverges to some limiting function ψ in $C^2_{\text{loc}}((0, \sqrt{2F(0,1)}a))$. The function ψ is strictly convex, and solves (11), so it is smooth. Moreover, we have

$$\psi \geq \frac{\theta}{4F(1,1)}\rho^2, \qquad \psi_\rho \geq \frac{\theta}{2F(1,1)}\rho$$

for $\rho \in (0, \sqrt{2F(0,1)}a)$, and

$$\psi \leq \frac{\Theta}{4F(1,1)}\rho^2, \qquad \psi_\rho \leq \frac{\Theta}{2F(1,1)}\rho$$

for $\rho \in (0, \sqrt{2(F(0,1) - F(1,1)\Theta^{-1})}a)$. In particular, $(\psi, \psi_\rho) \to 0$ as $\rho \to 0$.

It remains to show that ψ can be extended smoothly to $\rho = 0$. Let $\beta(\rho) = \frac{\psi_\rho}{\rho}$. We have

$$\rho\beta_{\rho\rho} = (\rho\beta)_{\rho\rho} - 2\beta_\rho = \psi_{\rho\rho\rho} - 2\beta_\rho$$

and hence

$$\begin{aligned}\rho\beta_{\rho\rho} &= 2\psi_\rho\,\psi_{\rho\rho}\; f\left(\frac{\psi_\rho}{\rho}, \frac{1}{2} + \frac{1}{2a^2}(\rho\psi_\rho - \psi)\right)\\ &\quad + (1+\psi_\rho^2)\frac{\partial f}{\partial y}\left(\frac{\psi_\rho}{\rho}, \frac{1}{2} + \frac{1}{2a^2}(\rho\psi_\rho - \psi)\right)\beta_\rho\\ &\quad + (1+\psi_\rho^2)\frac{\partial f}{\partial z}\left(\frac{\psi_\rho}{\rho}, \frac{1}{2} + \frac{1}{2a^2}(\rho\psi_\rho - \psi)\right)\frac{1}{2a^2}\rho\psi_{\rho\rho} - 2\beta_\rho.\end{aligned}$$

The right-hand side is positive at any point where $\beta_\rho < 0$. Therefore, β_ρ is either negative for all sufficiently small ρ or positive for all sufficiently small ρ. In particular, β converges as $\rho \to 0$, and hence ψ extends to a C^2 function on the interval $[0, \sqrt{2F(0,1)}a)$. Using (9) we conclude that $\psi_{\rho\rho}(0) = \frac{1}{2F(1,1)}$.

We thus have that the function

$$u(x) := a - a^{-1}\psi(a|x|)$$

is of class C^2 in $B^{n-1}_{\sqrt{2F(0,1)}}(0)$. Moreover, u solves the shrinker equation,

$$\gamma\bigg(- \left(\delta^{ik} - \frac{D^i u D^k u}{1+|Du|^2}\right)\frac{D_k u D_j u}{\sqrt{1+|Du|^2}}\bigg)\bigg) = \frac{1}{2}\,(x,u)\cdot\frac{(-Du,1)}{\sqrt{1+|Du|^2}}.$$

Standard theory for fully nonlinear elliptic PDE now implies that u is of class C^∞ (see for example Lemma 17.16 in [GT01]). In particular, ψ extends to a function in $C^\infty([0, \sqrt{2F(0,1)}a))$. This completes the proof. □

Next we extend the solutions constructed in Proposition 4.2 to the origin.

4.3. **Global behaviour.** For each $a > 0$, let ψ_a be a solution to (9) as in Proposition 4.2. We will typically suppress the dependence on a and simply write $\psi = \psi_a$. We define

$$h(r) := a - a^{-1}\psi(ar)$$

and write $v(z)$ for the inverse of h, so that $r = v(h(r))$. Note that since ψ is defined for $\rho \in [0, \sqrt{2F(0,1)}a)$ we have $0 \le v < \sqrt{2F(0,1)}$. For each $a > 0$ the function v is positive, strictly decreasing, strictly concave, and satisfies

$$\gamma\bigg(- \frac{v_{zz}}{1+v_z^2}, \frac{1}{v}, \ldots, \frac{1}{v}\bigg) = \frac{v}{2} - \frac{z}{2}v_z, \qquad v(a) = 0.$$

We are interested in the asymptotic behaviour of v as $a \to \infty$. Following [ADS19], we prove upper and lower bounds for the quantity

$$w(z) := z\frac{d}{dz}\log(2F(0,1) - v^2) = \frac{-2zvv_z}{2F(0,1) - v^2},$$

and then integrate to obtain estimates for v.

Lemma 4.9. *We have $w(z) > 2$.*

Proof. We first show that $w(z) \to 2\frac{F(1,1)}{F(0,1)}$ as $z \to a$. Observe that when $z = h(\rho/a)$ we have

$$w(z) = -\frac{zv(z)v_z(z)}{F(0,1) - \frac{v(z)^2}{2}} = \frac{\left(1 - \frac{\psi(\rho)}{a^2}\right)}{F(0,1) - \frac{\rho^2}{2a^2}} \frac{\rho}{\psi_\rho(\rho)}.$$

Using $\psi_{\rho\rho}(0) = \frac{1}{2F(1,1)}$, we conclude that

$$\lim_{z\to a} w = \frac{1}{F(0,1)} \lim_{\rho\to 0} \frac{\rho}{\psi_\rho} = \frac{1}{F(0,1)} \frac{1}{\psi_{\rho\rho}(0)} = 2\frac{F(1,1)}{F(0,1)}.$$

Since $2\frac{F(1,1)}{F(0,1)} > 2$, if the claim is false then there is a value z_1 such that $w(z_1) = 2$ and $w_z(z_1) \geq 0$. But we have

$$zw_z = w - w^2\left(\frac{1}{2} + \frac{F(0,1)}{v^2}\right) - (1 + v_z^2)\frac{2z}{v_z(z)} f\left(\frac{1}{v}, \frac{1}{2}(v - zv_z)\right),$$

and $w(z_1) = 2$ implies that $\frac{1}{2}(v - zv_z) = \frac{F(0,1)}{v}$, which in turn gives

$$f\left(\frac{1}{v}, \frac{1}{2}(v - zv_z)\right) = \frac{1}{v} f(1, F(0,1)) = 0$$

at $z = z_1$. It follows that

$$w_z = -\frac{4F(0,1)}{zv^2} < 0$$

at $z = z_1$, which is a contradiction. □

Lemma 4.9 immediately implies the following estimate for v.

Lemma 4.10. *We have*

$$v(z)^2 \geq 2F(0,1)\left(1 - \frac{z^2}{a^2}\right).$$

Proof. The estimate $w > 2$ says exactly that

$$\frac{d}{dz}\log(2F(0,1) - v^2) > \frac{2}{z}.$$

Integrating this inequality from z to a gives the claim. □

Next we derive an upper bound for w, using the following inequality.

Lemma 4.11. *There is a constant $c > 0$ depending only on n and γ such that*

$$zw_z \geq w - w^2\left(\frac{1}{2} + \frac{F(0,1)}{v^2}\right) + cz^2(1 + v_z^2)(w - 2). \tag{12}$$

Proof. We first note that

$$-\frac{wz}{v_z} f\left(\frac{1}{v}, \frac{1}{2}(v - zv_z)\right) = \frac{2z^2}{2F(0,1) - v^2} f\left(1, \frac{2F(0,1) - v^2}{2}\left(\frac{w}{2} - 1\right) + F(0,1)\right).$$

To estimate the right-hand side we write

$$f(1, b + F(0,1)) = f(1, b + F(0,1)) - f(1, F(0,1)) = \int_0^b \frac{\partial f}{\partial z}(1, t + F(0,1))\,dt$$

and observe that since γ is one-homogeneous,

$$\begin{aligned}\inf_{t\in[0,\infty)} \frac{\partial f}{\partial z}(1, t+F(0,1)) &= \inf_{t\in[0,\infty)} \dot\gamma^1(t,1,\dots,1)^{-1}\\ &= \inf_{t\in[0,\infty)} \dot\gamma^1\left(\frac{t}{n-1+t}, \frac{1}{n-1+t}, \dots, \frac{1}{n-1+t}\right)^{-1}.\end{aligned}$$

The last infimum is over a compact subset of Γ (the segment connecting $(0, \frac{1}{n-1}, \dots, \frac{1}{n-1})$ with $(1,0,\dots,0)$), so

$$c := \inf_{t\in[0,\infty)} \frac{\partial f}{\partial z}(1, t+F(0,1))$$

is a positive constant depending only on n and γ, and we have

$$f\left(1, \frac{2F(0,1)-v^2}{2}\left(\frac{w}{2}-1\right)+F(0,1)\right) \geq c\frac{2F(0,1)-v^2}{2}\left(\frac{w}{2}-1\right).$$

Therefore,

$$-\frac{wz}{v_z} f\left(\frac{1}{v}, \frac{1}{2}(v - zv_z)\right) \geq cz^2(w-2),$$

and the claim follows when we combine this inequality with

$$zw_z = w - w^2\left(\frac{1}{2} + \frac{F(0,1)}{v^2}\right) - (1+v_z^2)\frac{2z}{v_z} f\left(\frac{1}{v}, \frac{1}{2}(v - zv_z)\right)$$

and $w > 2$. □

By [RS23a], there is a unique strictly convex solution $\zeta : \mathbb{R}_+ \to \mathbb{R}_+$ to the equation

$$\zeta_{\rho\rho} = (1+\zeta_\rho^2) f\left(\frac{\zeta_\rho}{\rho}, \frac{1}{2}\right)$$

which satisfies $(\zeta, \zeta_\rho) \to 0$ as $\rho \to 0$. This is the profile function of the bowl soliton which translates with speed $1/2$. We will make use of the following asymptotic expansions for ζ and ζ_ρ, which are derived in Appendix A:

$$\zeta(M) = \frac{1}{4F(0,1)}M^2 - 2\dot F^1(0,1)\log M + o(\log(M))$$

and

$$\zeta_\rho(M) = \frac{1}{2F(0,1)}M - 2\dot F^1(0,1)M^{-1} + o(M^{-1})$$

as $M \to \infty$, where $\dot F^1(0,1) = \dot\gamma^1(0,1,\dots,1)$.

Lemma 4.12. *As $a \to \infty$ we have $\psi \to \zeta$ in $C^\infty_{\mathrm{loc}}(\mathbb{R}_+)$. In particular, if M is sufficiently large,*

$$\psi(M) \to \frac{1}{4F(0,1)}M^2 - \dot F^1(0,1)\log M + o(\log(M))$$

and

$$\psi_\rho(M) \to \frac{1}{2F(0,1)}M - 2\dot F^1(0,1)M^{-1} + o(M^{-1})$$

as $a \to \infty$.

Proof. The claim follows if we can show that every sequence $a_k \to \infty$ admits a subsequence a_{k_i} along which $\psi_{a_{k_i}} \to \zeta$ in $C^\infty_{\mathrm{loc}}(\mathbb{R}_+)$. So consider an arbitrary sequence $a_k \to \infty$. Recall that there is a positive constant C depending only on n and γ such that $C^{-1}\rho \le (\psi_{a_k})_\rho \le C\rho$ holds in $(0, \sqrt{F(0,1)}a_k)$. Moreover, as in the proof of Proposition 4.2, we have $(\psi_{a_k})_{\rho\rho}(\rho) \to \frac{1}{2F(1,1)}$ as $\rho \to 0$ for every k. Using Lemma 4.3, Lemma 4.4 and Lemma 4.5, we conclude that

$$\sup_k \|\psi_{a_k}\|_{C^3(I)} < \infty$$

for every compact interval $I \subset \mathbb{R}_+$. Therefore, after passing to a subsequence, we have that ψ_{a_k} converges to some limit $\bar\psi$ in $C^{2,\alpha}_{\mathrm{loc}}(\mathbb{R}_+)$. The function $\bar\psi$ is strictly convex by Lemma 4.4, and satisfies

$$\bar\psi_{\rho\rho} = (1+\bar\psi_\rho^2) f\left(\frac{\bar\psi_\rho}{\rho}, \frac{1}{2}\right).$$

In particular, $\bar\psi$ is smooth. Moreover, $\bar\psi$ satisfies the initial conditions $(\bar\psi, \bar\psi_\rho) \to 0$ as $\rho \to 0$. But ζ is the unique function with these properties, so $\bar\psi = \zeta$, as required. □

Let us fix a large positive constant M and set $z_{M,a} = a - a^{-1}\psi(M)$. By Lemma 4.12, if M is sufficiently large we have $a^{-1}z_{M,a} \to 1$ as $a \to \infty$. The following lemma concerns the asymptotic behaviour of $w(z_{M,a})$ as $a \to \infty$. (Note that since $z_{M,a} \sim a$, this is really a statement about the behaviour of our self-shrinking solutions near their tips.)

Lemma 4.13. *If M is sufficiently large then we have*

$$w(z_{M,a}) \to \frac{1}{F(0,1)}\frac{M}{\zeta_\rho(M)} = 2 + 8F(0,1)^2M^{-2} + O(M^{-4})$$

as $a \to \infty$.

Proof. We first note that $z_{M,a} = h(M/a)$ and hence $v(z_{M,a}) = M/a$. It follows that

$$z_{M,a}\, v(z_{M,a}) \to M, \qquad v(z_{M,a})^2 \to 0$$

as $a \to \infty$. Also,

$$v_z(z_{M,a}) = -\psi_\rho(M)^{-1} \to -\zeta_\rho(M)^{-1}$$

as $a \to \infty$. Combining these facts we see that

$$w(z_{M,a}) = \frac{-2z_{M,a}v(z_{M,a})v_z(z_{M,a})}{2F(0,1) - v(z_{M,a})^2} \to \frac{1}{F(0,1)}\frac{M}{\zeta_\rho(M)}$$

as $a \to \infty$. We conclude using the expansion

$$\frac{1}{\zeta_\rho(M)} = \frac{1}{\frac{1}{2F(0,1)}M - 2\dot F^1(0,1)M^{-1} + O(M^{-2})} = \frac{2F(0,1)}{M} + 8F(0,1)^3M^{-3} + O(M^{-4}).$$

□

We define

$$w_1(z) = \frac{1}{z^2} + \frac{1}{a^2 - z^2}$$

and set $\bar w = 2 + Kw_1$, where $K := \max\{1, 6F(0,1), 17c^{-1}\}$ and c is the constant appearing in Lemma 4.11.

Lemma 4.14. *If M is sufficiently large, and a is sufficiently large depending on M, then we have*

$$w(z_{M,a}) < \bar{w}(z_{M,a}).$$

Proof. Recalling that $z_{M,a} = a - a^{-1}\psi(M)$, we observe that

$$a^2 - z_{M,a}^2 \to 2\zeta(M)$$

as $a \to \infty$, and hence

$$\bar{w}(z_{M,a}) \to 2 + \frac{K}{2\zeta(M)} = 2 + 2F(0,1)KM^{-2} + O(M^{-4})\log(M)$$

as $a \to \infty$. On the other hand,

$$w(z_{M,a}) \to 2 + 8F(0,1)^2M^{-2} + O(M^{-4}).$$

Since $K > 4F(0,1)$, the claim follows. □

Proposition 4.15. *If M is sufficiently large, and a is sufficiently large depending on M, then we have*

$$w(z) \leq \bar{w}(z)$$

for $\sqrt{K} < z < z_{M,a}$.

Proof. We claim that

$$z\overline{w}_z < \bar{w} - \overline{w}^2\left(\frac{1}{2} + \frac{F(0,1)}{v^2}\right) + cv^2(1+v_z^2)(\overline{w}-2)$$

for $\sqrt{K} < z < z_{M,a}$.

If a is sufficiently large then we have

$$z\bar{w}_z = -\frac{2K}{z^2} + \frac{2Kz^2}{(a^2-z^2)^2} \leq \frac{2K}{a^2 - z_{M,a}^2} z^2 w_1 \leq \frac{2K}{\zeta(M)} z^2 w_1.$$

If M is large enough then we have $2K < \zeta(M)$, and so conclude that

$$z\bar{w}_z < z^2 w_1,$$

for $z < z_{M,a}$ and all sufficiently large a.

Next we use $\bar{w} = 2 + Kw_1$ to obtain

$$cz^2(1+v_z^2)(\overline{w}-2) \geq cKz^2w_1,$$

and use $v^2 \geq 2F(0,1)(1 - z^2/a^2)$ to obtain

$$\frac{1}{2} + \frac{F(0,1)}{v^2} \leq \frac{1}{2} + \frac{a^2}{2(a^2-z^2)} \leq z^2 w_1.$$

Together these inequalities give

$$\overline{w} - \left(\frac{1}{2} + \frac{F(0,1)}{v^2}\right)\overline{w}^2 + cz^2(1+v_z^2)(\overline{w}-2) \geq -z^2\bar{w}^2w_1 + cKz^2w_1,$$

and so since, for large M, we have

$$\bar{w} = 2 + \frac{K}{z} + \frac{K}{a^2-z^2} \leq 2 + \frac{K}{z^2} + \frac{K}{\zeta(M)} \leq 4$$

for $\sqrt{K} \le z \le z_{M,a}$, we may conclude that

$$\overline{w} - \left(\frac{1}{2} + \frac{F(0,1)}{v^2}\right)\overline{w}^2 + cz^2(1+v_z^2)(\overline{w}-2) \ge -16z^2 w_1 + cKz^2 w_1 \ge z^2 w_1$$

for $\sqrt{K} \le z \le z_{M,a}$.

Putting all of this together, we see that

$$z\overline{w}_z < \bar{w} - \overline{w}^2\left(\frac{1}{2} + \frac{F(0,1)}{v^2}\right) + cv^2(1+v_z^2)(\overline{w}-2)$$

for $\sqrt{K} \le z \le z_{M,a}$. In the previous lemma we showed that, if a is sufficiently large, we have

$$w(z_{M,a}) < \bar{w}(z_{M,a}).$$

The claim now follows from a simple comparison principle argument. ◻

Proposition 4.16. *If M is sufficiently large, and a is sufficiently large depending on M, then we have*

$$v^2 \le 2F(0,1)\left(1 - \left(1 - C\frac{\log a}{a^2}\right)\frac{z^2 - K/2}{a^2}\right)$$

for $\sqrt{K} < z < z_{M,a}$, where C depends only on K and M.

Proof. We use $w = z\frac{d}{dz}\log(2F(0,1) - v^2)$ and the estimate in Lemma 4.15 to obtain

$$\begin{aligned}
\frac{\log(2F(0,1) - M^2/a^2)}{\log(2F(0,1) - v(z)^2)} &= \int_z^{z_{M,a}} \frac{w(\eta)}{\eta}\,d\eta \\
&\le \log\left(\frac{z_{M,a}^2}{z^2}\right) + \frac{K}{2z^2} + \frac{K}{2a^2}\log\left(\frac{z_{M,a}^2(a^2 - z^2)}{z^2(a^2 - z_{M,a}^2)}\right) \\
&\le \log\left(\frac{z_{M,a}^2}{z^2}\right) + \frac{K}{2z^2} + \frac{C}{a^2}\log a - \frac{K}{2a^2}\log(z^2(a^2 - z_{M,a}^2)),
\end{aligned}$$

where C depends only on K and M. If M is so large that $\zeta(M) > 1/2$ then we have

$$a^2 - z_{M,a}^2 = 2\psi(M) - a^{-2}\psi(M)^2 > 1$$

for all sufficiently large a, and hence the last term on the right is negative for $z > \sqrt{K} > 1$. We thus have

$$\frac{\log(2F(0,1) - M^2/a^2)}{\log(2F(0,1) - v(z)^2)} \le \log\left(\frac{z_{M,a}^2}{z^2}\right) + \frac{K}{2z^2} + \frac{C}{a^2}\log a$$

for $\sqrt{K} < z \le z_{M,a}$ whenever a is sufficiently large. We rearrange and use $e^{-x} \ge 1 - x$ to obtain

$$2F(0,1) - v^2 \ge \left(2F(0,1) - C\frac{\log a}{a^2}\right)\frac{z^2 - K/2}{a^2}$$

for $\sqrt{K} < z < z_{M,a}$, where C depends only on M and K. This implies the claim. ◻

We can finally prove the main result of this section, Proposition 4.1.

Proof of Proposition 4.1. Let M be large enough and fixed so that Proposition 4.16 applies. It then suffices to take $\Psi_a(z) = v(z)$ and $L_0 = \sqrt{K} + 1$. ◻

5. Sharp asymptotics for the rescaled flow

Let $M_t = \partial\Omega_t$, $t \in (-\infty, 0]$, be a strictly convex ancient G-flow which is noncollapsing and uniformly two-convex. In addition, we assume that M_t noncompact. We define a family of rescaled hypersurfaces $\bar{M}_\tau = e^{\tau/2} M_{-e^{-\tau}}$. These move with pointwise normal speed $-(G - \frac{1}{2}\langle x, \nu\rangle)$. Proposition 3.6 implies that, as $\tau \to -\infty$, $\bar{M}_\tau$ converges in $C^\infty_{\mathrm{loc}}(\mathbb{R}^{n+1})$ to a cylinder Σ of radius $\sigma := \sqrt{2\gamma(0, 1, \dots, 1)}$ whose axis passes through the origin. Without loss of generality we may assume the axis of Σ is the x_1-axis, which we equip with the coordinate z. That is,

$$\Sigma := \{(z, \sigma\theta) : z \in \mathbb{R},\ \theta \in S^{n-1}\}.$$

Recall that the asymptotic cone of M_t (and hence of $\bar{M}_\tau$) is constant, by Lemma 2.15, and of dimension 1, by Lemma 2.7. Given that $\bar{M}_\tau \to \Sigma$, the asymptotic cone is a subset of the x_1-axis. Since M_t is strictly convex, we may assume without loss of generality that the asymptotic cone is $\{(z, 0) : z \leq 0\}$. For each τ we define

$$\bar{z}(\tau) := \sup\{z \in \mathbb{R} : (z, 0) \in e^{\tau/2}\Omega_{-e^{-\tau}}\}.$$

Observe that $\bar{z}(\tau) \to \infty$ as $\tau \to -\infty$. We may define a function $u(z, \theta, \tau)$ such that $\bar{M}_\tau$ coincides with the set of points $(z, \sigma\theta) + u(z, \theta, \tau)\theta$ in the region of space where $z < \bar{z}(\tau)$. Since $\bar{M}_\tau$ is strictly convex and noncompact we have

$$u > 0, \qquad \frac{\partial u}{\partial z} < 0, \qquad \frac{\partial^2 u}{\partial z^2} < 0.$$

In this section we establish the following sharp asymptotic estimate for u.

Proposition 5.1. *For every $L < \infty$ we have*

$$\sup_{|z| \leq L} |u(\cdot, \tau)| \leq O(e^{\tau/2})$$

as $\tau \to -\infty$.

In other words, there is a positive constant K such that

$$M_t \cap \{|z| \leq L(-t)^{1/2}\} \subset \{(z, y) : ||y| - (-2\gamma(0, 1, \dots 1)t)^{1/2}| \leq K\}$$

whenever $-t$ is sufficiently large.

We first prove that the maximum of $|u|$ for $|z| \leq L$ controls u and its derivatives in a subset which grows as $\tau \to -\infty$. To obtain these 'inner-outer' estimates, we employ the hypersurfaces Σ_a constructed in Section 4 as barriers. We then use the inner-outer estimates to conduct a spectral analysis of the linearised equation for u about 0 as $\tau \to -\infty$. To do so we must cut off in space. Both linearising and cutting off introduce error terms, but we are be able to show that these decay rapidly as $\tau \to -\infty$. Proposition 5.1 is then proven by showing that the spectral decomposition is dominated by unstable modes.

We take inspiration from [ADS19, BC19, BC21], which concern ancient solutions to the mean curvature flow, and Brendle's work on ancient solutions to the Ricci flow [Bre20] (see also [BN23]). Similar ideas also played a role in Colding and Minicozzi's work on uniqueness of cylindrical tangent flows [CM12]. Our analysis is the first of its kind for a fully nonlinear geometric flow. The main new difficulty which we need to overcome in this setting is the presence of error terms depending on the Hessian of u. For an overview of the analysis in this section we refer back to the introduction of the paper.

5.1. **Inner-outer estimates.** For each sufficiently large $a > 0$ we write

$$\Psi_a : [L_0, a] \to \mathbb{R}$$

for the function referred to in Proposition 4.1. Since $u(\cdot, \tau) \to 0$ on compact subsets as $\tau \to -\infty$, for every $a > 0$ the inequality

$$\Psi_a(z) < \sqrt{2\gamma(0, 1, \dots, 1)} + u(z, \theta, \tau)$$

holds for $L_0 \le z \le a$ and sufficiently large $-\tau$. Since Ψ_a is the profile of a self-similarly shrinking G-flow, which is a stationary solution to the rescaled flow, the parabolic maximum principle implies the following statement.

Lemma 5.2. *Suppose $z_0 \ge L_0$ and $\tau_0 < 0$ are such that*

$$\Psi_a(z_0) < \sqrt{2\gamma(0, 1, \dots, 1)} + u(z_0, \theta, \tau)$$

for every $\tau \le \tau_0$. We then have

$$\Psi_a(z) \le \sqrt{2\gamma(0, 1, \dots, 1)} + u(z, \theta, \tau)$$

for $z \in [z_0, a]$ and $\tau \le \tau_0$.

Let $L > L_0$ be an arbitrary large constant. For each integer $k \ge 0$ we define

$$\delta_k := \sup_{\tau \le -k} \sup_{|z| \le L} |u(\cdot, \tau)|.$$

Since $\bar{M}_\tau$ is strictly convex we have $\delta_k > 0$. Since $\delta_k \to 0$ as $k \to \infty$, after passing to a subsequence we may assume $\delta_k \le 10^{-6}$. We fix a small constant $r > 0$. Choosing $r = 10^{-4}$ will prove to be sufficient.

In this section we write ∇ for the Levi-Civita connection on $\mathbb{R} \times S^{n-1}$. We use C to denote a positive constant which is independent of k. The value of C may increase as we proceed.

Lemma 5.3. *When k is sufficiently large we have the estimates*

$$|u| + \left|\frac{\partial u}{\partial z}\right| \le C\delta_k^{1-2r}, \qquad |\nabla^{S^{n-1}} u| \le C\delta_k^{1/2-2r}$$

for $|z| \le 8\delta_k^{-r}$ and $\tau \le -k$. Moreover, we have

$$|u| + \left|\frac{\partial u}{\partial z}\right| \le C\delta_k$$

for $|z| \le L + 2$ and $\tau \le -k$.

Proof. Recall from Proposition 4.1 that when a is large we have

$$\Psi_a(z) \le \sqrt{2\gamma(0, 1, \dots, 1)\left(1 - \left(1 - C\frac{\log a}{a^2}\right)\frac{z^2 - C}{a^2}\right)}$$

for $z \in [L_0, L]$, where $C = C(n, \gamma)$. Let $z_0 = L_0 + \sqrt{C}$ and $a = \Lambda\delta_k^{-1/2}$, where

$$\Lambda^2 := \sqrt{\frac{\gamma(0, 1, \dots, 1)}{8}}(L_0^2 + 2L_0\sqrt{C}).$$

We may assume without loss of generality that $L > z_0$. The definition of z_0 ensures that

$$\sqrt{2\gamma(0,1,\dots,1)} - \delta_k > \Psi_a(z_0)$$

whenever k is sufficiently large. Since $|u| \le \delta_k$ for $|z| \le L$ and $\tau \le -k$,

$$\sqrt{2\gamma(0,1,\dots,1)} + u(z_0,\theta,\tau) \ge \sqrt{2\gamma(0,1,\dots,1)} - \delta_k > \Psi_a(z_0)$$

for $\tau \le -k$. Using Lemma 5.2 and Proposition 4.1 we obtain

$$\sqrt{2\gamma(0,1,\dots,1)} + u(z,\theta,\tau) \ge \Psi_a(z) \ge \sqrt{2\gamma(0,1,\dots,1)\Big(1 - \frac{z^2}{a^2}\Big)}$$

for $z \in [z_0, a]$ and $\tau \le -k$. In particular, for all sufficiently large k, we have

$$u \ge -C\delta_k^{1-2r}$$

for $z \in [z_0, 9\delta_k^{-r}]$ and $\tau \le -k$. Moreover,

$$u \ge -C\delta_k$$

for $z \in [L+2, L+3]$ and $\tau \le -k$.

Since $\frac{\partial u}{\partial z} < 0$ we conclude that

$$-C\delta_k^{1-2r} \le u(z,\theta,\tau) \le u(0,\theta,\tau) \le \delta_k$$

for $z \in [0, 9\delta_k^{-r}]$ and $\tau \le -k$. The mean value theorem then gives

$$\inf_{8\delta_k^{-r} \le z \le 9\delta_k^{-r}} \frac{\partial u}{\partial z}(z,\theta,\tau) \ge -C\delta_k^{1-r}$$

for $\tau \le -k$, and so since $\frac{\partial u}{\partial z} < 0$ and $\frac{\partial^2 u}{\partial z^2} < 0$,

$$\left|\frac{\partial u}{\partial z}\right| \le C\delta_k^{1-r}$$

for $z \le 8\delta_k^{-r}$ and $\tau \le -k$. Using this inequality and the fundamental theorem of calculus we obtain

$$|u| \le C\delta_k^{1-2r}$$

for $|z| \le 8\delta_k^{-r}$ and $\tau \le -k$.

Similarly, since $u \ge -C\delta_k$ for $z \in [L+2, L+3]$, we have

$$\left|\frac{\partial u}{\partial z}\right| \le C\delta_k$$

for $z \le L+2$ and $\tau \le -k$. It follows that

$$|u| + \left|\frac{\partial u}{\partial z}\right| \le C\delta_k$$

for $|z| \le L+2$ and $\tau \le -k$.

Since $\bar{M}_\tau$ is convex, a straightforward geometric argument using the bound $|u| \le C\delta_k^{1-2r}$ gives

$$|\nabla^{S^{n-1}} u| \le C\delta_k^{1/2-r}$$

for $|z| \le 8\delta_k^{-r}$ and $\tau \le -k$. This completes the proof. □

Next we obtain a stronger bound for $\nabla^{S^{n-1}} u$, and establish an estimate for the Hessian of u, in the region where $|z| \le 5\delta_k^{-r}$ and $\tau \le -k$.

Lemma 5.4. *When k is sufficiently large we have $|\nabla u| \leq C\delta_k^{1-6r}$ and $|\nabla^2 u| \leq C\delta_k^{1-6r}$ for $|z| \leq 5\delta_k^{-r}$ and $\tau \leq -k$.*

Proof. Consider a time $\tau_0 \leq -k$, and set $t_0 = -e^{-\tau_0}$. Lemma 5.3 implies that
$M_{t_0} \cap \{|z| \leq 8\delta_k^{-r}(-t_0)^{1/2}\} \subset \{(z, y) : ||y| - (-2\gamma(0, 1, \ldots, 1)t_0)^{1/2}| \leq \delta_k^{1-2r}(-t_0)^{1/2}\}$.
In particular, if k is large and $|z| \leq 7\delta_k^{-r}(-t_0)^{1/2}$, then

$$B((z, 0), \tfrac{99}{100}(-2\gamma(0, 1, \ldots, 1)t_0)^{1/2}) \subset \Omega_{t_0}.$$

Using the interior curvature estimate of Proposition 2.2, we conclude that

$$(-t_0)^{1/2}|A_{M_{t_0}}| \leq C$$

at every point in $M_{t_0} \cap \{|z| \leq 7\delta_k^{-r}\sqrt{-t_0}\}$. Moreover, the noncollapsing property implies

$$(-t_0)^{1/2}G_{M_{t_0}} \geq \frac{1}{C}$$

at every point in $M_{t_0} \cap \{|z| \leq 7\delta_k^{-r}(-t_0)^{1/2}\}$ It follows that $|A_{\bar{M}_{\tau_0}}| \leq C$ and $G_{\bar{M}_{\tau_0}} \geq 1/C$ on $\bar{M}_{\tau_0} \cap \{|z| \leq 7\delta_k^{-r}\}$. Since $\tau_0 \leq -k$ was chosen arbitrarily we conclude that $|A_{\bar{M}_\tau}| \leq C$ and $G_{\bar{M}_\tau} \geq 1/C$ for $|z| \leq 7\delta_k^{-r}$ and $\tau \leq -k$ whenever k is sufficiently large.

Using $|A| \leq C$, a straightforward computation shows that when k is sufficiently large we have

$$|\nabla^2 u| \leq C$$

for $|z| \leq 7\delta_k^{-r}$ and $\tau \leq -k$. Recall that the hypersurfaces $\bar{M}_\tau$ move with normal speed $-(G(x, \tau) - \frac{1}{2}\langle x, \nu(x, \tau)\rangle)$. This implies that u solves a fully nonlinear parabolic PDE which is convex or concave in $\nabla^2 u$ (depending on whether γ is convex or concave) and uniformly parabolic by Lemma 2.1. We also have

$$|u_\tau| + |\nabla^2 u| + |\nabla u| + |u| \leq C$$

for $|z| \leq 7\delta_k^{-r}$ and $\tau \leq -k$. Therefore, the parabolic Hölder estimate for convex/concave fully nonlinear parabolic equations due to [Kry82] and the parabolic Schauder estimates imply that

$$|\nabla^m u| \leq C(m)$$

for each integer $m \geq 2$ in the region where $|z| \leq 7\delta_k^{-r} - m$ and $\tau \leq -k$. In this step we also need the bound $\bar{G} \geq 1/C$, since this implies bounds on the higher derivatives of γ.

Given a smooth function $\varphi(z, \theta)$ which is compactly supported in the region $|z - z_0| < 2$, we have the interpolation inequality

$$\sup_{|z-z_0|\leq 2} |\nabla\varphi| \leq C(m)\bigg(\sup_{|z-z_0|\leq 2} |\varphi|\bigg)^{1-1/m}\bigg(\sup_{|z-z_0|\leq 2} |\nabla^m\varphi|\bigg)^{1/m}.$$

We apply this with $\varphi(z, \theta) = \eta(z)u(z, \theta, \tau)$, where η is an appropriate cutoff function, and use $|\nabla^j u| \leq C(m)$ for $j \leq m$ to obtain

$$\sup_{|z-z_0|\leq 1} |\nabla u| \leq C(m) \sup_{|z-z_0|\leq 3} |u| + C(m)\bigg(\sup_{|z-z_0|\leq 3} |u|\bigg)^{1-1/m}\bigg(\sup_{|z-z_0|\leq 3} |\nabla^m u|\bigg)^{1/m}.$$

Fix an integer $m_0 \geq 1/r$. Consider a point $|z_0| \leq 7\delta_k^{-r} - m_0 - 3$ and a time $\tau \leq -k$. Applying the last inequality with $m = m_0$, and using Lemma 5.3 and $|\nabla^j u| \leq C(m_0)$ for $j \leq m_0$, we obtain

$$\sup_{|z-z_0|\leq 1} |\nabla u|(z,\theta,\tau) \leq C\delta_k^{1-2r} + C\delta_k^{(1-r)(1-2r)} \leq C\delta_k^{1-4r}.$$

It follows that for k sufficiently large we have $|\nabla u| \leq C\delta_k^{1-4r}$ for $|z| \leq 6\delta_k^{-r}$.

Now consider a point $|z_0| \leq 6\delta_k^{-r} - m_0 - 3$ and a time $\tau \leq -k$. The same kind argument as above yields

$$\begin{aligned}&\sup_{|z-z_0|\leq 1} |\nabla^2 u|(z,\theta,\tau) \\ &\quad\leq \sup_{|z-z_0|\leq 3} |\nabla u| + \Big(\sup_{|z-z_0|\leq 3} |\nabla u|\Big)^{1-1/m_0} \Big(\sup_{|z-z_0|\leq 4} |\nabla^{m_0+1} u|\Big)^{1/m_0}.\end{aligned}$$

Inserting $|\nabla u| \leq C\delta_k^{1-4r}$ on the right-hand side, we conclude that when k is sufficiently large, $|\nabla^2 u| \leq C\delta_k^{1-6r}$ for $|z| \leq 5\delta_k^{-r}$ and $\tau \leq -k$. □

We have shown that u and its first two spatial derivatives are small in the region where $|z| \leq 5\delta_k^{-r}$ and $\tau \leq -k$. It follows that u solves a linear parabolic equation in this region, up to quadratic error terms.

Lemma 5.5. *When k is sufficiently large we have*

$$|u_\tau - \mathcal{L}u| \leq C(|u|^2 + |\nabla u|^2 + |\nabla^2 u|^2)$$

for $|z| \leq 5\delta_k^{-r}$ and $\tau \leq -k$, where

$$\mathcal{L}u := a\frac{\partial^2 u}{\partial z^2} + \frac{1}{2(n-1)}\Delta_{S^{n-1}} u - \frac{1}{2}z\frac{\partial u}{\partial z} + u$$

and $a := \dot{\gamma}^1(0,1,\ldots,1)$.

Proof. Recall the notation $\sigma = \sqrt{2\gamma(0,1,\ldots,1)}$. Given $(z,\theta) \in \mathbb{R} \times S^{n-1}$, let us write $x = (z,\sigma\theta)$ for the corresponding point in Σ and $y = x + u(z,\theta,\tau)\theta$ for the corresponding point in $\bar{M}_\tau$. We have

$$u_\tau(z,\theta,\tau) = -\frac{1}{\nu_\Sigma(x)\cdot\nu(y)}G(y,\tau) + \frac{1}{2}\frac{1}{\nu_\Sigma(x)\cdot\nu(y)} y\cdot\nu(y).$$

For $\tau \leq -k$ and $|z| \leq 5\delta_k^{-r}$, Lemma 2.11 gives

$$\nu(y) = \nu_\Sigma(x) - \nabla u(z,\theta,\tau) + O(|u|^2 + |\nabla u|^2).$$

We thus have

$$\begin{aligned} u_\tau(z,\theta,\tau) = {} & -G(y,\tau) - \frac{1}{2}x\cdot\nabla u(z,\theta,\tau) + \frac{1}{2}u(z,\theta,\tau) \\ & + \frac{1}{2}x\cdot\nu_\Sigma(x) + O(|u|^2 + |\nabla u|^2).\end{aligned}$$

Lemma 2.13 now gives

$$\begin{aligned} u_\tau = {} & a\frac{\partial^2 u}{\partial z^2} + \frac{1}{2(n-1)}\Delta_{S^{n-1}} u - \frac{1}{2}x\cdot\nabla u + u \\ & + \frac{1}{2}x\cdot\nu_\Sigma - G_\Sigma(x) + O(|u|^2 + |\nabla u|^2 + |\nabla^2 u|^2),\end{aligned}$$

for $\tau \leq -k$ and $|z| \leq 5\delta_k^{-r}$. The claim follows upon insertion of the identities

$$x \cdot \nabla u = z\frac{\partial u}{\partial z} \qquad \text{and} \qquad \frac{1}{2}x \cdot \nu_\Sigma - G_\Sigma = 0.$$

□

Using Lemma 5.5, we obtain the following improved estimate for ∇u in the region $|z| \leq L+1$.

Lemma 5.6. *When k is sufficiently large we have $|\nabla u| \leq C\delta_k$ for $|z| \leq L+1$ and $\tau \leq -k$.*

Proof. Let us define

$$\tilde{\mathcal{L}}u := a\frac{\partial^2 u}{\partial z^2} + \frac{1}{2(n-1)}\Delta_{S^{n-1}}u.$$

We then have

$$|u_\tau - \tilde{\mathcal{L}}u| \leq \frac{1}{2}\left|z\frac{\partial u}{\partial z}\right| + |u| + C(|u|^2 + |\nabla u|^2 + |\nabla^2 u|^2)$$

for $|z| \leq 5\delta_k^{-r}$ and $\tau \leq -k$. Using Lemma 5.3 and Lemma 5.4 we conclude that

$$|u_\tau - \tilde{\mathcal{L}}u| \leq C\delta_k$$

for $|z| \leq L+2$ and $\tau \leq -k$. Given any (z_0, θ_0, τ_0) such that $|z_0| \leq L+1$ and $\tau \leq -k$, there exists a small $d = d(n)$ such that

$$|\nabla u|(z_0, \theta_0, \tau_0) \leq Cd^{-1} \sup_{\tau_0 - d^2 \leq \tau \leq \tau_0} \sup_{|z-z_0| \leq d} |u| + Cd \sup_{\tau_0 - d^2 \leq \tau \leq \tau_0} \sup_{|z-z_0| \leq d} |u_\tau - \tilde{\mathcal{L}}u|.$$

This is a standard estimate in the theory of linear parabolic PDE. Since d depends only on n, we obtain

$$|\nabla u|(z_0, \theta_0, \tau_0) \leq C\delta_k.$$

Since z_0 was chosen arbitrarily in the region $|z| \leq L+1$, we conclude that $|\nabla u| \leq C\delta_k$ for $|z| \leq L+1$ and $\tau \leq -k$. □

5.2. **Spectral analysis.** Let us denote by $\mathcal{H}$ the space of functions $v : \mathbb{R} \times S^{n-1} \to \mathbb{R}$ such that

$$\|v\|_{\mathcal{H}}^2 := \int_{\mathbb{R} \times S^{n-1}} e^{-z^2/4a} v^2 \, dz d\theta < \infty.$$

The operator $\mathcal{L}$ is symmetric with respect to the inner product $\langle \cdot, \cdot \rangle_{\mathcal{H}}$. Therefore, there exists an orthogonal basis for $\mathcal{H}$ consisting of eigenfunctions of $\mathcal{L}$. Let us write H_k for the k-th Hermite polynomial, and fix a basis Y_l^m of eigenfunctions for $\Delta_{S^{n-1}}$ such that

$$\Delta_{S^{n-1}} Y_l^m = -l(l+n-2)Y_l^m.$$

A basis of eigenfunctions for $\mathcal{L}$ is then given by $H_k(z/2\sqrt{a})Y_l^m(\theta)$, where k and l range over the nonnegative integers. We have

$$\left(a\frac{\partial^2}{\partial z^2} - \frac{1}{2}z\frac{\partial}{\partial z}\right)H_k\left(\frac{z}{2\sqrt{a}}\right) = -\frac{k}{2}.$$

Therefore, the eigenvalues of $\mathcal{L}$ are given by $1 - \frac{k}{2} - \frac{l(l+n-2)}{2(n-1)}$, where k and l are nonnegative integers. Up to scaling, the eigenfunctions with positive eigenvalues are of the form 1, $z/\sqrt{a}$ and θ^i, $1 \leq i \leq n$. The eigenfunctions with eigenvalue equal to zero are of the form $z^2/a - 1$ and $z\theta^i/\sqrt{a}$, $1 \leq i \leq n$. Let P_+, P_0 and P_- denote the orthogonal projection operators onto the positive, zero and negative eigenspaces. These projections satisfy

$$\begin{aligned} &\langle \mathcal{L}P_+v, P_+v\rangle_{\mathcal{H}} \geq \frac{1}{2}\|P_+v\|_{\mathcal{H}}^2, \\ &\langle \mathcal{L}P_0v, P_0v\rangle_{\mathcal{H}} = 0, \\ &\langle \mathcal{L}P_-v, P_-v\rangle_{\mathcal{H}} \leq -\frac{1}{2}\|P_-v\|_{\mathcal{H}}^2. \end{aligned} \tag{13}$$

We are going to show that P_+u dominates P_0u and P_-u in the region where $|z| \leq \delta_k^{-r}$ and $\tau \leq -k$. To this end, let $\xi : \mathbb{R} \to \mathbb{R}$ be a smooth cutoff function such that $\xi(s) = 1$ for $s \in [-1/2, 1/2]$, $\xi(s) = 0$ for $|s| \geq 1$, and $s\xi'(s) \leq 0$ for $s \in \mathbb{R}$. We define

$$\begin{aligned} \gamma_j &:= \sup_{\tau \in [-j-1,-j]} \int_{\mathbb{R}\times S^{n-1}} e^{-z^2/4a} |u(z,\theta,\tau)\chi(\delta_j^r z)|^2 \, dz d\theta \\ \gamma_j^+ &:= \sup_{\tau \in [-j-1,-j]} \int_{\mathbb{R}\times S^{n-1}} e^{-z^2/4a} |P_+(u(z,\theta,\tau)\chi(\delta_j^r z))|^2 \, dz d\theta \\ \gamma_j^0 &:= \sup_{\tau \in [-j-1,-j]} \int_{\mathbb{R}\times S^{n-1}} e^{-z^2/4a} |P_0(u(z,\theta,\tau)\chi(\delta_j^r z))|^2 \, dz d\theta \\ \gamma_j^- &:= \sup_{\tau \in [-j-1,-j]} \int_{\mathbb{R}\times S^{n-1}} e^{-z^2/4a} |P_-(u(z,\theta,\tau)\chi(\delta_j^r z))|^2 \, dz d\theta. \end{aligned}$$

We have $C^{-1}\gamma_j \leq \gamma_j^+ + \gamma_j^0 + \gamma_j^- \leq C\gamma_j$, and Lemma 5.3 gives $\gamma_j \leq C\delta_j^{2-4r}$. In particular, $\gamma_j \to 0$ as $j \to \infty$.

The following result is similar to [Bre20, Lemma 3.12].

Lemma 5.7. *We have*

$$\begin{aligned} \gamma_{j+1}^+ &\leq e^{-1}\gamma_j^+ + C(\gamma_j + \gamma_{j+1})^{1/2}\varepsilon_j^{1/2} + C\exp(-\delta_j^{-2r}/64a), \\ |\gamma_{j+1}^0 - \gamma_j^0| &\leq C(\gamma_j + \gamma_{j+1})^{1/2}\varepsilon_j^{1/2} + C\exp(-\delta_j^{-2r}/64a), \\ \gamma_{j+1}^- &\geq e\gamma_j^- - C(\gamma_j + \gamma_{j+1})^{1/2}\varepsilon_j^{1/2} - C\exp(-\delta_j^{-2r}/64a), \end{aligned}$$

where

$$\varepsilon_j := \sup_{\tau \in [-j-2,-j]} \int_{|z| \leq \delta_j^{-r}} e^{-z^2/4a} |u_\tau - \mathcal{L}u|^2.$$

Proof. Let us write $\hat{u}(z,\theta,\tau) = u(z,\theta,\tau)\,\chi(\delta_j^r z)$. We first note that

$$\sup_{\tau \in [-j-2,-j]} \int_{\mathbb{R}\times S^{n-1}} e^{-z^2/4a} |\hat{u}|^2 \leq \gamma_j + \gamma_{j+1}.$$

Combining this with (13) and using Hölder's inequality we obtain

$$\frac{d}{d\tau}\int e^{-z^2/4a}|P_+\hat{u}|^2 \geq \int e^{-z^2/4a}|P_+\hat{u}|^2 - 2(\gamma_j+\gamma_{j+1})^{1/2}\bigg(\int e^{-z^2/4a}|\hat{u}_\tau - \mathcal{L}\hat{u}|^2\bigg)^{\frac{1}{2}},$$
$$\bigg|\frac{d}{d\tau}\int e^{-z^2/4a}|P_0\hat{u}|^2\bigg| \leq 2(\gamma_j+\gamma_{j+1})^{1/2}\bigg(\int e^{-z^2/4a}|\hat{u}_\tau - \mathcal{L}\hat{u}|^2\bigg)^{\frac{1}{2}},$$
$$\frac{d}{d\tau}\int e^{-z^2/4a}|P_-\hat{u}|^2 \leq -\int e^{-z^2/4a}|P_-\hat{u}|^2 + 2(\gamma_j+\gamma_{j+1})^{1/2}\bigg(\int e^{-z^2/4a}|\hat{u}_\tau - \mathcal{L}\hat{u}|^2\bigg)^{\frac{1}{2}}$$

for $\tau \in [-j-2,-j]$.

Since $\chi(s)$ is constant for $|s| \leq 1/2$ and $|s| \geq 1$, and $|\chi(s)| \leq 1$ for $s \in \mathbb{R}$, we have

$$\begin{aligned}\int e^{-z^2/4a}|\hat{u}_\tau - \mathcal{L}\hat{u}|^2 &\leq C\int_{|z|\leq \delta_j^{-r}} e^{-z^2/4a}|u_\tau - \mathcal{L}u|^2 + C\int_{\delta_j^{-r}/2\leq|z|\leq\delta_j^{-r}} e^{-z^2/4a}(|u|^2+|\nabla u|^2)\\ &\leq C\varepsilon_j + C\exp(-\delta_j^{-2r}/32a)\end{aligned}$$

for $\tau \in [-j-2,-j]$. It follows that

$$\frac{d}{d\tau}\int e^{-z^2/4a}|P_+\hat{u}|^2 \geq \int e^{-z^2/4a}|P_+\hat{u}|^2 - C(\gamma_j+\gamma_{j+1})^{1/2}\varepsilon_j^{1/2} - C\exp(-\delta_j^{-2r}/64a),$$
$$\bigg|\frac{d}{d\tau}\int e^{-z^2/4a}|P_0\hat{u}|^2\bigg| \leq 2(\gamma_j+\gamma_{j+1})^{1/2}\varepsilon_j^{1/2} + C\exp(-\delta_j^{-2r}/64a),$$
$$\frac{d}{d\tau}\int e^{-z^2/4a}|P_-\hat{u}|^2 \leq -\int e^{-z^2/4a}|P_-\hat{u}|^2 + C(\gamma_j+\gamma_{j+1})^{1/2}\varepsilon_j^{1/2} + C\exp(-\delta_j^{-2r}/64a)$$

for $\tau \in [-j-2,-j]$. Integrating over $[\tau-1,\tau]$ we obtain

$$\begin{aligned}&\int e^{-z^2/4a}|P_+\hat{u}(\cdot,\tau-1)|^2\\ &\leq e^{-1}\int e^{-z^2/4a}|P_+\hat{u}(\cdot,\tau)|^2 + C(\gamma_j+\gamma_{j+1})^{1/2}\varepsilon_j^{1/2} + C\exp(-\delta_j^{-2r}/64a),\\ &\bigg|\int e^{-z^2/4a}|P_0\hat{u}(\cdot,\tau-1)|^2 - \int e^{-z^2/4a}|P_0\hat{u}(\cdot,\tau)|^2\bigg|\\ &\leq C(\gamma_j+\gamma_{j+1})^{1/2}\varepsilon_j^{1/2} + C\exp(-\delta_j^{-2r}/64a),\\ &\int e^{-z^2/4a}|P_-\hat{u}(\cdot,\tau-1)|^2\\ &\geq e\int e^{-z^2/4a}|P_-\hat{u}(\cdot,\tau)|^2 - C(\gamma_j+\gamma_{j+1})^{1/2}\varepsilon_j^{1/2} - C\exp(-\delta_j^{-2r}/64a).\end{aligned}$$

for $\tau \in [-j-1,-j]$.

Define $\tilde{u}(z,\tau) = u(z,\tau)\chi(\delta_{j+1}^r z)$. Observing that

$$\int e^{-z^2/4a}|\tilde{u}(z,\tau-1) - \hat{u}(z,\tau-1)|^2 \leq C\exp(-\delta_j^{-2r}/64a)$$

for all $\tau \in [-j-1, j]$, we conclude that

$$\begin{aligned}
&\int e^{-z^2/4a}|P_+\tilde{u}(\cdot,\tau-1)|^2 - e^{-1}\int e^{-z^2/4a}|P_+\hat{u}(\cdot,\tau)|^2 \\
&\qquad \leq C(\gamma_j+\gamma_{j+1})^{1/2}\varepsilon_j^{1/2} + C\exp(-\delta_j^{-2r}/64a), \\
&\left|\int e^{-z^2/4a}|P_0\tilde{u}(\cdot,\tau-1)|^2 - \int e^{-z^2/4a}|P_0\hat{u}(\cdot,\tau)|^2\right| \\
&\qquad \leq C(\gamma_j+\gamma_{j+1})^{1/2}\varepsilon_j^{1/2} + C\exp(-\delta_j^{-2r}/64a), \\
&\int e^{-z^2/4a}|P_-\tilde{u}(\cdot,\tau-1)|^2 - e\int e^{-z^2/4a}|P_-\hat{u}(\cdot,\tau)|^2 \\
&\qquad \geq -C(\gamma_j+\gamma_{j+1})^{1/2}\varepsilon_j^{1/2} - C\exp(-\delta_j^{-2r}/64a).
\end{aligned}$$

for $\tau \in [-j-1, -1]$. Taking the supremum over $\tau \in [-j-1, -j]$, we arrive at the claim. □

We now define

$$\Gamma_k := \sup_{j\geq k}\gamma_j, \qquad \Gamma_k^+ := \sup_{j\geq k}\gamma_j^+, \qquad \Gamma_k^0 := \sup_{j\geq k}\gamma_j^0, \qquad \Gamma_k^- := \sup_{j\geq k}\gamma_j^-.$$

We have $C^{-1}\Gamma_k \leq \Gamma_k^+ + \Gamma_k^0 + \Gamma_k^- \leq C\Gamma_k$ and $\Gamma_k \to 0$ as $k \to \infty$. Since

$$\begin{aligned}
\varepsilon_j &= \sup_{\tau\in[-j-2,-j]}\int_{|z|\leq\delta_j^{-r}} e^{-z^2/4a}|u_\tau - \mathcal{L}u|^2 \\
&\leq C\sup_{\tau\in[-j-2,-j]}\Big(\sup_{|z|\leq\delta_j^{-r}}|u|^4 + |\nabla u|^4 + |\nabla^2 u|^4\Big),
\end{aligned}$$

as a consequence of Lemma 5.3 and Lemma 5.4 we have

$$\varepsilon_j \leq C\delta_j^{4-24r}$$

whenever j is sufficiently large. In particular,

$$E_k := \sup_{j\geq k}\varepsilon_j$$

satisfies $E_k \leq CD_k^{4-24r}$, where $D_k := \sup_{j\geq k}\delta_j$. Note that $D_k \to 0$ as $k \to \infty$. Lemma 5.7 immediately implies

$$\begin{aligned}
\Gamma_{k+1}^+ &\leq e^{-1}\Gamma_k^+ + C\Gamma_k^{1/2}E_k^{1/2} + C\exp(-\delta_k^{-2r}/64a), \\
|\Gamma_{k+1}^0 - \Gamma_k^0| &\leq C\Gamma_k^{1/2}E_k^{1/2} + C\exp(-\delta_k^{-2r}/64a), \\
\Gamma_{k+1}^- &\geq e\Gamma_k^- - C\Gamma_k^{1/2}E_k^{1/2} - C\exp(-\delta_k^{-2r}/64a).
\end{aligned}$$

In the next lemma we improve our estimate for the error term E_k.

Lemma 5.8. *For sufficiently large k we have $\delta_k \leq C\Gamma_k^{1/2}$ and*

$$E_k \leq CD_k^{2-24r}\Gamma_k.$$

Proof. Since

$$\sup_{|z|\leq L}|u(z,\theta,\tau)| \to 0$$

as $\tau \to -\infty$, for each k there is a point (z_0, θ_0, τ_0) such that $|z_0| \leq L$, $\tau_0 \leq -k$, and

$$|u(z_0, \theta_0, \tau_0)| = \sup_{\tau \leq -k} \sup_{|z| \leq L} |u(z, \theta, \tau)| = \delta_k.$$

Lemma 5.6 provides a constant C_0 such that $|\nabla u| \leq C_0 \delta_k$ for $|z| \leq L+1$ and $\tau \leq -k$. Using this estimate we obtain

$$\inf \left\{ |u(z, \theta, \tau_0)| : \mathrm{dist}_{\mathbb{R} \times S^{n-1}}((z, \theta), (z_0, \theta_0)) \leq \tfrac{1}{100} \min\{1, C_0^{-1}\} \right\} \geq \frac{1}{C} \delta_k.$$

It follows that

$$\delta_k^2 \leq C \int_{|z| \leq L+1} |u|^2(\cdot, \tau_0)\, dz d\theta.$$

There exists an integer $j \geq k$ such that $\tau_0 \in [-j-1, -j]$. For this j, as a result of the last inequality we have

$$\delta_k^2 \leq C \sup_{\tau \in [-j-1, -j]} \int_{\mathbb{R} \times S^{n-1}} e^{-z^2/4a} |u(z, \theta, \tau) \chi(\delta_j^r z)|^2 \, dz d\theta = C \gamma_j,$$

at least if k is sufficiently large. Therefore,

$$\delta_k^2 \leq C \sup_{j \geq k} \gamma_j \leq C \Gamma_k$$

for every sufficiently large k. Combining this inequality with $E_k \leq C D_k^{4-24r}$, and assuming k is sufficiently large, we obtain

$$E_k \leq C D_k^{4-24r} = C D_k^{2-24r} \sup_{j \geq k} \delta_j^2 \leq C D_k^{2-24r} \sup_{j \geq k} \Gamma_j \leq C D_k^{2-24r} \Gamma_k.$$

Here we have used the fact that Γ_k is nonincreasing in k. This completes the proof. □

Lemma 5.8 tells us that $\delta_k \leq C \Gamma_k^{1/2}$ and $E_k \leq C D_k \Gamma_k$. Moreover, for sufficiently large k we have

$$\exp(-\delta_k^{-2r}/64a) \leq \delta_k^{5/2} \leq C \delta_k^{1/2} \Gamma_k \leq C D_k^{1/2} \Gamma_k.$$

Consequently, for large k we have

$$\begin{aligned} \Gamma_{k+1}^+ &\leq e^{-1} \Gamma_k^+ + C D_k^{1/2} \Gamma_k \\ |\Gamma_{k+1}^0 - \Gamma_k^0| &\leq C D_k^{1/2} \Gamma_k \\ \Gamma_{k+1}^- &\geq e \Gamma_k^- - C D_k^{1/2} \Gamma_k, \end{aligned}$$

and in particular

$$\begin{aligned} \Gamma_{k+1}^+ &\leq e^{-1} \Gamma_k^+ o(1) \Gamma_k \\ |\Gamma_{k+1}^0 - \Gamma_k^0| &\leq o(1) \Gamma_k \\ \Gamma_{k+1}^- &\geq e \Gamma_k^- - o(1) \Gamma_k \end{aligned}$$

as $k \to \infty$. This system of inequalities implies that the decomposition $\Gamma_k = \Gamma_k^+ + \Gamma_k^0 + \Gamma_k^-$ is dominated either by Γ_k^+ or Γ_k^0 as $k \to \infty$. For the proof of this statement, which is a variation on an argument by Merle and Zaag [MZ98], we refer to Lemma 3.1 in [Bre20].

Lemma 5.9. *As $k \to \infty$ we either have $\Gamma_k^0 + \Gamma_k^- \leq o(1) \Gamma_k^+$ or $\Gamma_k^+ + \Gamma_k^- \leq o(1) \Gamma_k^0$.*

We now establish that $\Gamma_k^0 + \Gamma_k^- \leq o(1)\Gamma_k^+$. The argument makes use of the fact that $\bar{M}_\tau$ is noncompact—recall that this allowed us to assume $\frac{\partial u}{\partial z} < 0$.

Proposition 5.10. *As $k \to \infty$ we have $\Gamma_k^0 + \Gamma_k^- \leq o(1)\Gamma_k^+$.*

Proof. By Lemma 5.9, if the claim is false then $\Gamma_k^+ + \Gamma_k^- \leq o(1)\Gamma_k^0$. Suppose this is the case. Fix a sequence $\tau_k \leq -k$ such that

$$\hat{u}_k(z,\theta) := \chi(\delta_k^r z)u(z,\theta,\tau_k)$$

satisfies

$$\|\hat{u}_k\|_{\mathcal{H}} = \Gamma_k^{1/2}.$$

Since $\Gamma_k^+ + \Gamma_k^- \leq o(1)\Gamma_k^0$, the sequence $\hat{u}_k/\|\hat{u}_k\|_{\mathcal{H}}$ converges to a limit $\hat{u}(z,\theta)$ with respect to the norm $\|\cdot\|_{\mathcal{H}}$. Passing to a further subsequence, we may assume that $\hat{u}_k/\|\hat{u}_k\|_{\mathcal{H}} \to \hat{u}$ pointwise almost everywhere in $\mathbb{R}\times S^{n-1}$. Since $\frac{\partial u}{\partial z} < 0$ we conclude that for each fixed θ the function $z \mapsto \hat{u}(z,\theta)$ is nonincreasing. On the other hand, the limit $\hat{u}$ lies in the zero-eigenspace of $\mathcal{L}$, which is to say that

$$\hat{u}(z,\theta) = \alpha\left(\frac{z^2}{a} - 2\right) + \beta_i\frac{z}{\sqrt{a}}\theta^i,$$

for some constants α and β_i. The right-hand side is monotone in z if and only if $\alpha = 0$ and $\beta_i = 0$ for each $1 \leq i \leq n$. With this we have reached a contradiction, since $\|\hat{u}\|_{\mathcal{H}} = 1$. □

We are now able to prove Proposition 5.1.

Proof of Proposition 5.1. Using Proposition 5.10 we conclude that

$$\Gamma_{k+1}^+ \leq e^{-1}\Gamma_k^+ + CD_k^{1/2}\Gamma_k^+ \leq e^{-1/2}\Gamma_k^+$$

whenever k is sufficiently large. Iterating this estimate we find that $\Gamma_k^+ \leq O(e^{-k/2})$, and hence $\Gamma_k \leq O(e^{-k/2})$. Using the estimate

$$D_k = \sup_{j\geq k}\delta_j \leq C\sup_{j\geq k}\Gamma_j^{1/2} \leq C\Gamma_k^{1/2}$$

we obtain $D_k \leq O(e^{-k/4})$ and hence

$$\Gamma_{k+1}^+ \leq e^{-1}\Gamma_k^+ + Ce^{-k/8}\Gamma_k^+.$$

Iterating this estimate we obtain $\Gamma_k^+ \leq O(e^{-k})$ and hence $\Gamma_k \leq O(e^{-k})$.

We have

$$\sup_{\tau\leq -k}\sup_{|z|\leq L}|u| = \delta_k.$$

Using $\delta_k \leq C\Gamma_k^{1/2}$ we conclude that

$$\sup_{\tau\leq -k}\sup_{|z|\leq L}|u| \leq O(e^{-k/2}).$$

The claim follows. □

We conclude this section with an important consequence of Proposition 5.1—namely, that $\max_{M_t} G$ is bounded from below by a positive constant which is independent of time.

A straightforward computation using Proposition 4.1 shows that if $L > L_0$ is sufficiently large depending on n and γ, and a is sufficiently large depending on L, then we have

$$\Psi_a(L) - \Psi_a(L-1) < -a^{-2}. \tag{14}$$

Moreover, Proposition 5.1 implies there is a positive constant $K = K(L)$ such that

$$M_t \cap \{|z| \leq L(-t)^{1/2}\} \subset \{(z,y) : ||y| - (-2\gamma(0,1,\dots 1)t)^{1/2}| \leq K\}$$

whenever $-t$ is sufficiently large.

Proposition 5.11. *For K as above we have* $\max_{M_t} G \geq -1/8K$ *for every* $t \leq 0$.

Proof. The proof is a direct adaptation of [BC19, Proposition 3.3]. We make use of the hypersurfaces

$$\Sigma_a = \{(y,z) : y = \Psi_a(z)\theta,\ \theta \in S^{n-1},\ L_0 \leq z \leq a\}$$

constructed in Section 4. The hypersurfaces

$$\begin{aligned}\Sigma_{a,t} &:= (-t)^{1/2}\Sigma_a - (0, Ka^2)\\ &= \{(y,z) : y = (-t)^{1/2}\Psi_a((-t)^{-1/2}(z+Ka^2))\theta,\ \theta \in S^{n-1},\\ &\qquad\qquad (-t)^{1/2}L_0 - Ka^2 \leq z \leq (-t)^{1/2}a - Ka^2\}\end{aligned}$$

form a G-flow. As $t \to -\infty$ the rescaled hypersurfaces $(-t)^{-1/2}M_t$ converge in C^∞_{loc} to the cylinder $\{|y|^2 = 2\gamma(0,1,\dots,1)\}$, and the hypersurfaces

$$(-t)^{-1/2}(\Sigma_{a,t} \cap \{z \geq (-t)^{1/2}L_0\})$$

converge to a compact subset of $\{|y|^2 < 2\gamma(0,1,\dots,1)\}$. So when $-t$ is sufficiently large depending on a we have

$$\Sigma_{a,t} \cap \{z \geq (-t)^{1/2}L_0\} \subset \Omega_t,$$

where Ω_t is the open region bounded by M_t.

Since $\Psi_a(z)$ is concave, as a consequence of (14) we have

$$\begin{aligned}\Psi_a(L + Ka^2(-t)^{-1/2}) &\leq \Psi_a(L) + (\Psi_a(L) - \Psi_a(L-1))Ka^2(-t)^{-1/2}\\ &< (2\gamma(0,1,\dots,1))^{1/2} - K(-t)^{-1/2}\end{aligned}$$

for $-t \geq 4K^2a^2$ and a sufficiently large. On the other hand,

$$\{(y,z) : |y| \leq (-2\gamma(0,1,\dots,1)t)^{1/2} - K,\ z = (-t)^{1/2}L\} \subset \Omega_t \cap \{z = (-t)^{1/2}L\}$$

when $-t$ is sufficiently large. It follows that for $-t \geq 4K^2a^2$ and a sufficiently large we have

$$\Sigma_{a,t} \cap \{z = (-t)^{1/2}L\} \subset \Omega_t \cap \{z = (-t)^{1/2}L\}.$$

Therefore, the parabolic maximum principle implies that

$$\Sigma_{a,t} \cap \{z \geq (-t)^{1/2}L\} \subset \Omega_t$$

for $-t \geq 4K^2a^2$ and a sufficiently large. When $t = -4K^2a^2$ the tip of $\Sigma_{a,t}$ is located at $(0, Ka^2) = (0, -t/4K)$. From this we conclude that $M_t \cap \{z \geq -t/4K\}$ is nonempty for all sufficiently large $-t$.

It follows that $\sup_{M_t} G \geq -1/8K$ at some sequence of times tending to $-\infty$. On the other hand, the differential Harnack inequality implies that G is strictly increasing in time at every point. The claim follows. □

6. The Neck Improvement Theorem

In this section we generalise the Neck Improvement Theorem for solutions of mean curvature flow [BC21] to hypersurfaces moving with speed G.

To begin with we recall the following definition from [BC21].

Definition 6.1. *A set of vector fields $\mathcal{K} = \{K_\alpha : 1 \leq \alpha \leq \frac{1}{2}n(n-1)\}$ on $\mathbb{R}^{n+1}$ is a normalised set of rotation vector fields if there exists an orthonormal basis $\{J_\alpha : 1 \leq \alpha \leq \frac{1}{2}n(n-1)\}$ of $\mathrm{so}(n) \subset \mathrm{so}(n+1)$, a matrix $S \in O(n+1)$ and a point $q \in \mathbb{R}^{n+1}$ such that*

$$K_\alpha(x) = SJ_\alpha S^{-1}(x-q).$$

Next we define a notion of quantitative almost-rotational symmetry.

Definition 6.2. *Consider a G-flow M_t. A point $\bar{x} \in M_{\bar{t}}$ is called ε-symmetric if there exists a normalised set of rotation vector fields $\mathcal{K} = \{K_\alpha : 1 \leq \alpha \leq \frac{1}{2}n(n-1)\}$ such that*

$$\max_\alpha |K_\alpha|\, G \leq 10\gamma(0, 1, \dots, 1)$$

at the point $(\bar{x}, \bar{t})$ and

$$\max_\alpha |\langle K_\alpha, \nu\rangle|\, G \leq \varepsilon$$

in the neighbourhood $\hat{\mathcal{P}}(\bar{x}, \bar{t}, 10, 100)$.

We now establish the Neck Improvement Theorem for G-flows. In fact, the proof reduces to the proof for mean curvature flow, since up to a change of variables the linearised G-flow over a shrinking neck is identical to the linearised mean curvature flow over a shrinking neck. We compute the linearisation and then refer to [BC21] for the rest of the argument.

Theorem 6.3. *Let M_t be a uniformly parabolic G-flow. There exists a large constant L (depending on n, γ and $\mathrm{dist}(\lambda/|\lambda|, \partial\Gamma)$), and a small constant ε_1 depending on L, with the following property. Let $(\bar{x}, \bar{t})$ be a point in spacetime such that every point in the parabolic neighbourhood $\hat{\mathcal{P}}(\bar{x}, \bar{t}, L, L^2)$ lies at the center of an ε_1-neck and is ε-symmetric for some $\varepsilon \leq \varepsilon_1$. Then $(\bar{x}, \bar{t})$ is $\frac{\varepsilon}{2}$-symmetric.*

Proof. Without loss of generality, we assume $\bar{t} = -1$. After parabolically rescaling we can arrange that $G(\bar{x}, -1) = (\gamma(0, 1, \dots, 1)/2)^{1/2}$. Since every point in the parabolic neighbourhood $\hat{\mathcal{P}}(\bar{x}, \bar{t}, L, L^2)$ lies at the center of an ε_1-neck, we can then approximate M_t by the cylinder

$$S^{n-1}((-2\gamma(0, 1, \dots, 1)t)^{1/2}) \times \mathbb{R}$$

in $\hat{\mathcal{P}}(\bar{x}, \bar{t}, L, L^2)$, up to errors which are bounded by $C(L)\varepsilon_1$ in the C^{10}-norm.

For every $(x_0,t_0)\in\hat{\mathcal{P}}(\bar{x},-1,L,L^2)$ there exists a normalised set of rotation vector fields $\{K_\alpha^{(x_0,t_0)} : 1\le\alpha\le\frac{1}{2}n(n-1)\}$ such that $\max_\alpha|K_\alpha|G\le 10\gamma(0,1,\dots,1)$ at (x_0,t_0) and

$$\max_\alpha|\langle K_\alpha^{(x_0,t_0)},\nu\rangle|G\le\varepsilon$$

in $\hat{\mathcal{P}}(x_0,t_0,10,100)$. As in [BC21, Theorem 4.4], we may assume without loss of generality that

$$\sup_{B(0,10\gamma(0,1\dots,1)L)}\max_\alpha|K_\alpha^{(\bar{x},-1)}-K_\alpha^{(x_0,t_0)}|\le C(L)\varepsilon \tag{15}$$

for each $(x_0,t_0)\in\hat{\mathcal{P}}(\bar{x},-1,L,L^2)$. This implies that

$$|\langle K^{(\bar{x},-1)},\nu\rangle|\le C(L)\varepsilon$$

at each point in $\hat{\mathcal{P}}(\bar{x},-1,L,L^2)$. We may also assume that the common axis of rotation of the vector fields $K_\alpha^{(\bar{x},-1)}$ is the x_{n+1}-axis (that is, $K_\alpha=J_\alpha x$ for some orthonormal basis $\{J_\alpha : 1\le\alpha\le\frac{1}{2}n(n-1)\}$ of $\mathrm{so}(n)\subset\mathrm{so}(n+1)$) and that $\bar{x}$ lies in the hyperplane $\{x_{n+1}=0\}$. Let us define $\bar{K}_\alpha:=K_\alpha^{(\bar{x},-1)}$.

Consider M_t as a graph over the x_{n+1}-axis, which we equip with the coordinate z, so that

$$\Big\{\big(r(\theta,z,t)\theta,z\big):\theta\in S^{n-1},\ z\in\big[-\tfrac{3}{4}L,\tfrac{3}{4}L\big]\Big\}\subset M_t.$$

Exactly as in [BC21], we have

$$|\langle\bar{K}_\alpha,\nu\rangle+\mathrm{div}_{S^{n-1}}(r(\theta,z,t)J_\alpha\theta)|\le C(L)\varepsilon_1\varepsilon \tag{16}$$

in $\hat{\mathcal{P}}(\bar{x},-1,L,L^2)$.

Fix α and set $\bar{u}_\alpha:=\langle\overline{K}_\alpha,\nu\rangle$. With respect to any orthonormal frame on M_t, $\bar{u}_\alpha$ satisfies

$$\frac{\partial\bar{u}_\alpha}{\partial t}=\dot{\gamma}^{ij}(\nabla_i\nabla_j\bar{u}_\alpha+A_i^kA_{kj}\bar{u}_\alpha).$$

Define $a:=\dot{\gamma}^1(0,1,\dots,1)$. By Lemma 2.14 we have

$$\left|\dot{\gamma}^{ij}\nabla_i\nabla_j\bar{u}_\alpha-a\frac{\partial^2\bar{u}_\alpha}{\partial z^2}-\frac{1}{(-2(n-1)t)}\Delta_{S^{n-1}}\bar{u}_\alpha\right|\le C(L)\varepsilon_1(|\nabla\bar{u}_\alpha|+|\nabla^2\bar{u}_\alpha|).$$

Standard interior estimates for parabolic equations imply $|\nabla\bar{u}_\alpha|+|\nabla^2\bar{u}_\alpha|\le C(L)\varepsilon$, and hence

$$\left|\dot{\gamma}^{ij}\nabla_i\nabla_j\bar{u}_\alpha-a\frac{\partial^2\bar{u}_\alpha}{\partial z^2}-\frac{1}{(-2(n-1)t)}\Delta_{S^{n-1}}\bar{u}_\alpha\right|\le C(L)\varepsilon_1\varepsilon$$

for $|z|\le\frac{L}{4}$ and $t\in[-\frac{L^2}{16},-1]$. Lemma 2.11 and Lemma 2.14 imply

$$\left|\dot{\gamma}^{ij}A_i^kA_{kj}\bar{u}_\alpha-\frac{1}{(-2t)}\bar{u}_\alpha\right|\le C(L)\varepsilon_1\varepsilon.$$

Putting this all together, we obtain

$$\left|\frac{\partial\bar{u}_\alpha}{\partial t}-a\frac{\partial^2\bar{u}_\alpha}{\partial z^2}-\frac{1}{(-2(n-1)t)}\Delta_{S^{n-1}}\bar{u}_\alpha-\frac{1}{(-2t)}\bar{u}_\alpha\right|\le C(L)\varepsilon_1\varepsilon$$

for $|z|\le\frac{L}{4}$ and $t\in[-\frac{L^2}{16},-1]$.

Let us define $u_\alpha(\theta, z, t) := \bar{u}_\alpha(\theta, a^{-1/2}z, t)$. We then have

$$\left|\frac{\partial u_\alpha}{\partial t} - \frac{\partial^2 u_\alpha}{\partial z^2} - \frac{1}{(-2(n-1)t)}\Delta_{S^{n-1}} u_\alpha - \frac{1}{(-2t)} u_\alpha\right| \leq C(L)\varepsilon_1\varepsilon \tag{17}$$

for $|z| \leq \frac{a^{1/2}}{4}L$ and $t \in [-\frac{L^2}{16}, -1]$. In addition, (16) gives

$$|u_\alpha(\theta, z, t) + \mathrm{div}_{S^{n-1}}(r(\theta, z, t)J_\alpha\theta)| \leq C(L)\varepsilon_1\varepsilon \tag{18}$$

for $|z| \leq a^{1/2}L$ and $t \in [-L^2, -1]$.

Notice that the linear operator on the left-hand side of (17) is completely independent of the speed γ. We may therefore appeal to Step 5 of [BC21, Theorem 4.4] to conclude that, for each $1 \leq \alpha \leq \frac{1}{2}n(n-1)$, there are vectors A_α and $\tilde{B}_\alpha$ such that $|A_\alpha| \leq C(L)\varepsilon$, $|\tilde{B}_\alpha| \leq C(L)\varepsilon$ and

$$|u_\alpha(\theta, z, t) - \langle A_\alpha, \theta\rangle - \langle \tilde{B}_\alpha, \theta\rangle z| \leq CL^{-1/(n-1)}\varepsilon + C(L)\varepsilon_1\varepsilon$$

for $|z| \leq 20\gamma(0, 1, \ldots, 1)a^{1/2}$ and $t \in [-400\gamma(0, 1, \ldots, 1)^2, -1]$. Undoing the rescaling $z \to a^{-1/2}z$ and setting $B_\alpha := a^{1/2}\tilde{B}_\alpha$, we obtain

$$|\langle \bar{K}_\alpha, \nu\rangle - \langle A_\alpha, \theta\rangle - \langle B_\alpha, \theta\rangle z| \leq CL^{-1/(n-1)}\varepsilon + C(L)\varepsilon_1\varepsilon$$

for $|z| \leq 20\gamma(0, 1, \ldots, 1)$ and $t \in [-400\gamma(0, 1, \ldots, 1)^2, -1]$. Step 7 and Step 8 of [BC21] show that if L is sufficiently large and ε_1 is sufficiently small (depending on L), then $(\bar{x}, -1)$ is $\frac{\varepsilon}{2}$-symmetric. This completes the proof. ∎

7. Rotational symmetry

Let $M_t = \partial\Omega_t$, $t \in (-\infty, 0]$, be a strictly convex ancient G-flow which is noncollapsing and uniformly two-convex. Suppose in addition that M_t is noncompact. In this section we prove that M_t is rotationally symmetric. The arguments follow [BC19, Section 5] and [BC21, Section 5]. We make use of the Neck Improvement Theorem and the lower bound for the maximum of G over M_t established in Proposition 5.11.

Throughout this section C denotes some large constant which is independent of t.

Lemma 7.1. *For sufficiently large $-t$ the maximum of G over M_t is attained at some unique point p_t. The Hessian of G is negative-definite at p_t, and hence p_t is smooth as a function of t.*

Proof. When $-t$ is sufficiently large, $M_t \cap B(0, (-10\gamma(0, 1, \ldots, 1)t)^{1/2})$ is close to the cylinder $\mathbb{R} \times S^{n-1}((-2\gamma(0, 1, \ldots, 1)t)^{1/2})$. As a consequence, in the complement of $B(0, (-10\gamma(0, 1, \ldots, 1)t)^{1/2})$, M_t consists of two connected components. One of these is compact and the other is noncompact. By Lemma 2.7, on the noncompact component we have $G \leq C(-t)^{-1/2}$. Since we have shown (in Proposition 5.11) that $\max_{M_t} G$ is bounded from below by a positive constant, it follows that $\max_{M_t} G$ is attained by some p_t in the compact component.

Now consider an arbitrary sequence $t_k \to \infty$. After passing to a subsequence, the sequence of shifted flows $M_{t+t_k} - p_{t_k}$ converges in C^∞_{loc} to an ancient G-flow. Moreover, on the limit, the spacetime maximum of G is attained at $(0, 0)$. Therefore, by Proposition 2.9, the limit is a translating soliton, and hence is the bowl soliton by [BL16] (cf. [Has15]). The claim follows. ∎

Let ε_1 and L be the constants in the Neck Improvement Theorem. Given that $\max_{M_t} G$ is bounded from below, Proposition 2.3 implies that $G(x,t)^{-1} = o(|x - p_t|)$ uniformly in t. Therefore, by Lemma 2.7, there is a large constant Λ with the following property: every point $x \in M_t$ satisfying $|x - p_t| \geq \Lambda$ lies at the center of an ε_1-neck and satisfies $G(x,t)|x - p_t| \geq 10^6\gamma(0, 1, \dots, 1)L$.

Lemma 7.2. *There is a time $T < 0$ with the following property. If $\bar{t} \leq T$ and $\bar{x} \in M_{\bar{t}}$ is such that $|\bar{x} - p_{\bar{t}}| \geq \Lambda$, then $|\bar{x} - p_t| \geq |\bar{x} - p_{\bar{t}}|$ for all $t \leq \bar{t}$.*

Proof. We have seen that, when $-t$ is large, M_t is extremely close to the bowl soliton near p_t. In particular, $\frac{d}{dt}p_t$ is almost parallel to $-\nu(p_t, t)$. It follows that there is a time $T \leq 0$ such that

$$\left\langle x - p_t, \frac{d}{dt}p_t \right\rangle > 0$$

whenever $t \leq T$, $x \in M_t \cup \Omega_t$ and $|x - p_t| \geq \Lambda$. Consequently,

$$\frac{d}{dt}|x - p_t| = -\left\langle \frac{x - p_t}{|x - p_t|}, \frac{d}{dt}p_t \right\rangle < 0$$

whenever $t \leq T$, $x \in M_t \cup \Omega_t$ and $|x - p_t| \geq \Lambda$.

Consider a time $\bar{t} \leq T$ and a point $\bar{x} \in M_{\bar{t}}$ such that $|\bar{x} - p_{\bar{t}}| \geq \Lambda$. We claim that $|\bar{x} - p_t| \geq |\bar{x} - p_{\bar{t}}|$ for all $t \leq \bar{t}$. If this is not the case then

$$\tilde{t} := \sup\{t \leq \bar{t} : |\bar{x} - p_t| < |\bar{x} - p_{\bar{t}}|\}$$

is finite. We have $\tilde{t} < \bar{t}$ and $|\bar{x} - p_t| \geq |\bar{x} - p_{\bar{t}}| \geq \Lambda$ for $t \in [\tilde{t}, \bar{t}]$. But our choice of T ensures that $\frac{d}{dt}|\bar{x} - p_t| < 0$ for $t \in [\tilde{t}, \bar{t}]$, and hence $|\bar{x} - p_{\tilde{t}}| > |\bar{x} - p_{\bar{t}}|$. This contradicts the definition of $\tilde{t}$, so T has the desired property. $\square$

Next we employ the Neck Improvement Theorem to show that the end of M_t becomes increasingly symmetric at large distances.

Proposition 7.3. *If $t \leq T$, and $x \in M_t$ satisfies $|x - p_t| \geq 2^{j/400}\Lambda$, then (x,t) is $2^{-j}\varepsilon_1$-symmetric.*

Proof. We argue by induction on j. The assertion is true for $j = 0$, by virtue of the fact that if $t \leq T$ and $|x - p_t| \geq \Lambda$, then (x,t) lies at the center of an ε_1-neck. Suppose that $j \geq 1$ and the assertion is true for $j - 1$. If the assertion fails for j then there is a time $\bar{t} \leq T$ and a point $\bar{x} \in M_{\bar{t}}$ such that $|\bar{x} - p_{\bar{t}}| \geq 2^{j/400}\Lambda$ and yet $(\bar{x}, \bar{t})$ fails to be $2^{-j}\varepsilon_1$-symmetric. Appealing to the Neck Improvement Theorem, we conclude that there is a point $(x,t) \in \hat{\mathcal{P}}(\bar{x}, \bar{t}, L, L^2)$ such that either (x,t) is not $2^{-j+1}\varepsilon_1$-symmetric, or else (x,t) does not lie at the center of an ε_1-neck. In the former case the induction hypothesis implies $|x - p_t| \leq 2^{\frac{j-1}{400}}\Lambda$, and in the latter we have $|x - p_t| \leq \Lambda$. So in both cases $|x - p_t| \leq 2^{\frac{j-1}{400}}\Lambda$. Since $t \leq \bar{t} \leq T$, Lemma 7.2 implies that $|\bar{x} - p_{\bar{t}}| \leq |\bar{x} - p_t|$.

Combining all of this we obtain

$$\begin{aligned}|\bar{x}-p_{\bar{t}}| &\leq |\bar{x}-p_t| \\ &\leq |x-p_t|+|\bar{x}-x| \\ &\leq 2^{\frac{j-1}{400}}\Lambda + 10\gamma(0,1,\dots,1)LG(\bar{x},\bar{t})^{-1} \\ &\leq 2^{-\frac{1}{400}}|\bar{x}-p_{\bar{t}}| + 10^{-5}|\bar{x}-p_{\bar{t}}| \\ &< |\bar{x}-p_{\bar{t}}|.\end{aligned}$$

This is a contradiction, so the assertion holds for j, and the claim follows by induction. $\square$

We now proceed with the main result of this section.

Theorem 7.4. *For each $t \leq 0$ the hypersurface M_t is rotationally symmetric.*

Proof. Fix an arbitrary time $\bar{t} \leq T$. For each j, let

$$\Omega^{(j)} = \{(x,t) : t \leq \bar{t},\ x \in M_t,\ |x-p_t| \leq 2^{\frac{j}{400}}\Lambda\}.$$

Let Q be a large constant to be chosen later. By Proposition 2.3, when j is sufficiently we have

$$G(x,t) \geq Q\Lambda|x-p_t|^{-1} \geq 2^{-j/400}Q$$

for each $(x,t) \in \Omega^{(j)}$.

By Proposition 7.3, every point in $\partial\Omega^{(j)}$ is $2^{-j}\varepsilon_1$-symmetric. That is, for each $(x,t) \in \partial\Omega^{(j)}$ there exists a normalised set of rotation vector fields

$$\mathcal{K}^{(x,t)} = \{K_\alpha^{(x,t)} : 1 \leq \alpha \leq \tfrac{1}{2}n(n-1)\}$$

such that $\max_\alpha |K_\alpha^{(x,t)}|G \leq 10\gamma(0,1,\dots,1)$ at (x,t) and $\max_\alpha |\langle K_\alpha^{(x,t)}, \nu\rangle|G \leq 2^{-j}\varepsilon_1$ in $\hat{\mathcal{P}}(x,t,10,100)$. Arguing as in [BC21, Theorem 5.4], we find that for each j there exists a normalised set of rotation vector fields $\mathcal{K}^{(j)} = \{K_\alpha^{(j)} : 1 \leq \alpha \leq \frac{1}{2}n(n-1)\}$ with the following property. If $(x,t) \in \partial\Omega^{(j)}$ is such that $\bar{t} - 2^{j/100} \leq t \leq \bar{t}$, then

$$\inf_{\omega \in O(\frac{1}{2}n(n-1))} \max_\alpha \left| K_\alpha^{(j)} - \sum_{\beta=1}^{\frac{1}{2}n(n-1)} \omega_{\alpha\beta} K_\beta^{(x,t)} \right| \leq C2^{-j/2}$$

at the point (x,t). It follows that if $(x,t) \in \partial\Omega^{(j)}$ is such that $\bar{t} - 2^{j/100} \leq t \leq \bar{t}$ then

$$\max_\alpha |\langle K_\alpha^{(j)}, \nu\rangle| \leq C2^{-j/2}$$

at (x,t). Finally, for $\bar{t} - 2^{j/100} \leq t \leq \bar{t}$ we have

$$|p_t - p_{\bar{t}}| \leq C2^{j/100},$$

and hence

$$\max_\alpha |\langle K_\alpha^{(j)}, \nu\rangle| \leq C2^{j/100}$$

for every $(x,t) \in \Omega^{(j)}$ such that $\bar{t} - 2^{j/100} \leq t \leq \bar{t}$.

For each $1 \leq \alpha \leq \frac{1}{2}n(n-1)$ we define a function $f_\alpha^{(j)} : \Omega^{(j)} \to \mathbb{R}$ by

$$f_\alpha^{(j)} := \exp(-2^{-j/200}(\bar{t}-t))\frac{\langle K_\alpha^{(j)}, \nu\rangle}{G - 2^{-j/400}}.$$

When j is large each of these functions satisfies

$$|f_\alpha^{(j)}(x,t)| \leq C2^{-j/4},$$

for every $(x,t) \in \partial\Omega^{(j)}$ such that $\bar t - 2^{j/100} \leq t \leq \bar t$, and for every $(x,t) \in \Omega^{(j)}$ such that $t = \bar t - 2^{j/100}$. Moreover, we have

$$\begin{aligned}(\partial_t - \dot\gamma^{pq}\nabla_p\nabla_q)f_\alpha^{(j)} &= \frac{2}{G - 2^{-j/400}}\dot\gamma^{pq}\nabla_p G\nabla_q f_\alpha^{(j)} \\ &\quad - 2^{-j/400}\bigg(\frac{\dot\gamma^{pq}A_{pq}^2}{G-2^{-j/400}} - 2^{-j/400}\bigg)f_\alpha^{(j)}.\end{aligned}$$

We estimate the final term on the right by

$$\frac{\dot\gamma^{pq}A_{pq}^2}{G-2^{-j/400}} - 2^{-j/400} \geq \frac{1}{C}\frac{G^2}{G-2^{-j/400}} - 2^{-j/400} \geq \frac{1}{C}G - 2^{-j/400}.$$

Choosing $Q > C$ ensures that the right-hand side is positive when j is large. We may then apply the the parabolic maximum principle to conclude that

$$|f_\alpha^{(j)}(x,t)| \leq C2^{-j/4}$$

for every $(x,t) \in \Omega^{(j)}$ such that $\bar t - 2^{j/100} \leq t \leq \bar t$.

As $j \to \infty$, $\mathcal{K}^{(j)}$ converges to a normalised set of rotation vector fields $\bar{\mathcal{K}}$ such that $\langle \bar K_\alpha, \nu\rangle$ vanishes identically on M_t for every $1 \leq \alpha \leq \frac{1}{2}n(n-1)$ and $t \leq \bar t$. In particular, $M_{\bar t}$ is rotationally symmetric. Since $\bar t \leq T$ was chosen arbitrarily, we conclude that M_t is rotationally symmetric for $t \leq T$.

It is now straightforward to show that M_t is rotationally symmetric for all $t \leq 0$. The quantity $u_\alpha := \langle \bar K_\alpha, \nu\rangle$ satisfies

$$\partial_t u_\alpha = \dot\gamma^{ij}\nabla_i\nabla_j u_\alpha + \dot\gamma^{ij}A_i^k A_{kj}u_\alpha.$$

Moreover, the function $h(x,t) = e^{2Ct}(|x|^2+1)$ satisfies

$$\partial_t h = \dot\gamma^{ij}\nabla_i\nabla_j h + 2Ch - 2e^{2Ct}\dot\gamma^{pq}g_{pq} > \dot\gamma^{ij}\nabla_i\nabla_j h + \dot\gamma^{ij}A_i^k A_{kj}h$$

for $t \geq T$ if C is sufficiently large. By the maximum principle, $\sup_{M_t}\frac{|u_\alpha|}{h}$ is nonincreasing for $t \leq 0$. Since $\langle \bar K_\alpha, \nu\rangle = 0$ vanishes at $t = T$, we have $\langle \bar K_\alpha, \nu\rangle = 0$ on M_t for all $t \leq 0$. ☐

8. Uniqueness of ancient solutions with rotational symmetry

Let M_t, $t \in (-\infty, 0]$, be a strictly convex ancient G-flow which is noncollapsing and uniformly two-convex. We assume in addition that M_t is noncompact and rotationally symmetric about the x_{n+1}-axis in $\mathbb{R}^{n+1}$. Therefore, there is a convex function $f(r,t)$ such that M_t is the set of points $(r\theta, f(r,t))$ for $\theta \in S^{n-1}$. The function f satisfies

$$f_t = G\langle -e_{n+1}, \nu\rangle^{-1}$$

and hence

$$f_t = \gamma\bigg(\frac{f_{rr}}{1+f_r^2}, \frac{f_r}{r}, \dots, \frac{f_r}{r}\bigg).$$

We can also express M_t in terms of the radius function $r(z,t)$, which is determined by

$$f(r(z,t),t) = z.$$

We have

$$r_t = -\gamma\Big(-\frac{r_{zz}}{1+r_z^2}, \frac{1}{r}, \dots, \frac{1}{r}\Big).$$

Since M_t is strictly convex we may assume that

$$r > 0, \qquad r_z > 0, \qquad r_t < 0 \qquad \text{and} \qquad r_{zz} < 0.$$

Without loss of generality, we may assume $r(0,0) = 0$.

We adapt the arguments in Section 6 of [BC19] (see also Section 6 of [BC21]) to prove that M_t moves by translation. It follows that M_t is the bowl soliton.

Proposition 8.1. *The hypersurfaces M_t move by translation.*

Let $q_t = (0, f(0,t))$ denote the tip of M_t, and set $G_{\mathrm{tip}}(t) = G(q_t, t)$. As a consequence of the differential Harnack inequality we know that G_{tip} is nondecreasing with time, and hence

$$\mathcal{G} := \lim_{t\to-\infty} G_{\mathrm{tip}}$$

is well defined. Proposition 5.11 implies that $\mathcal{G} > 0$.

Next we establish that $f_t(r,t)$ is a nondecreasing function of time.

Lemma 8.2. *We have $f_{tt}(r,t) \geq 0$ everywhere.*

Proof. Proposition 2.9 asserts that

$$\frac{\partial}{\partial t}G + 2\langle V, \nabla G\rangle + A(V,V) \geq 0$$

for every vector field V on M_t. Setting $V = G\langle -e_{n+1}, \nu\rangle^{-1} e_{n+1}^\top$, where $\top$ denotes the component tangent to M_t, we obtain

$$\Big(\frac{\partial}{\partial t} + V^i \nabla_i\Big)(G\langle -e_{n+1}, \nu\rangle^{-1}) \geq 0.$$

On the other hand

$$f_{tt} = \Big(\frac{\partial}{\partial t} + V^i \nabla_i\Big)(G\langle -e_{n+1}, \nu\rangle^{-1}),$$

so this implies the claim. $\square$

We conclude that $f_t(r,t)$ is bounded from below by $\mathcal{G}$.

Lemma 8.3. *We have $f_t(r,t) \geq \mathcal{G}$ everywhere. Moreover, for each $r_0 > 0$,*

$$\lim_{t\to-\infty} \sup_{r\leq r_0} f_t(r,t) = \mathcal{G}.$$

Proof. Fix a sequence $t_k \to -\infty$ and consider the sequence of shifted solutions $M_t^k = M_{t+t_k} - q_{t_k}$. By Proposition 2.4 this sequence converges to a rotationally symmetric limiting solution $\bar{M}_t$ which is defined for all $t \in (-\infty, \infty)$ and is strictly convex. Moreover, the speed of $\bar{M}_t$ at its tip is constant in time and equal to $\mathcal{G}$. Therefore, by Proposition 2.9, the hypersurfaces $\bar{M}_t$ move by translation. We conclude that

$$\lim_{k\to\infty} \sup_{r\leq r_0} |f_t(r,t_k) - \mathcal{G}| = 0,$$

and since $f_{tt}(r,t) \geq 0$, it follows that $f_t(r,t) \geq \mathcal{G}$ at every point in spacetime. $\square$

Next we show that $f_t(r,t)$ is nondecreasing as a function of r.

Lemma 8.4. *We have $f_{tr}(r,t) \geq 0$ everywhere.*

Proof. Fix a time $t_0 \leq 0$ and a radius $r_0 > 0$ at which $f(r_0, t_0)$ is defined. We consider the parabolic region

$$Q_T := \{(x,t) \in \mathbb{R}^n \times [T, t_0] : x_1^2 + \cdots + x_{n-1}^2 \leq r_0^2\}.$$

Using the evolution equations

$$\frac{\partial}{\partial t} G = \dot{\gamma}^{ij} \nabla_i \nabla_j G + \dot{\gamma}^{ij} A_i^k A_{kj} G$$

and

$$\frac{\partial}{\partial t} \langle -e_{n+1}, \nu \rangle = \dot{\gamma}^{ij} \nabla_i \nabla_j \langle -e_{n+1}, \nu \rangle + \dot{\gamma}^{ij} A_i^k A_{kj} \langle -e_{n+1}, \nu \rangle$$

we conclude that

$$\sup_{Q_T} G \langle -e_{n+1}, \nu \rangle^{-1} = \max \left\{ \sup_{r \leq r_0,\, t=T} G \langle -e_{n+1}, \nu \rangle^{-1}, \sup_{r=r_0,\, T \leq t \leq t_0} G \langle -e_{n+1}, \nu \rangle^{-1} \right\}.$$

Since $f_t = G\langle -e_{n+1}, \nu \rangle^{-1}$ it follows that

$$\sup_{Q_T} f_t = \max \left\{ \sup_{r \leq r_0,\, t=T} f_t, \sup_{r=r_0,\, T \leq t \leq t_0} f_t \right\} \leq \max \left\{ \sup_{r \leq r_0,\, t=T} f_t,\ f_t(r_0, t_0) \right\},$$

where we have made use of Lemma 8.2. Sending $T \to -\infty$ and appealing to Lemma 8.3 we obtain

$$f_t(0, t_0) \leq \max\{\mathcal{G}, f_t(r_0, t_0)\} \leq f_t(r_0, t_0).$$

Since r_0 was arbitrary this completes the proof. □

There exists a small constant ε_0 such that q_t does not lie at the center of an ε_0-neck. Since G is at least $\mathcal{G}$ at the point q_t, by Lemma 2.7, there exists a decreasing function $\Lambda : (0, \varepsilon_0] \to \mathbb{R}$ such that if $x \in M_t$ satisfies $|x - q_t| \geq \Lambda(\varepsilon)$ then x lies at the center of an ε-neck.

Recall the notation $F(0,1) = \gamma(0, 1, \ldots, 1)$.

Lemma 8.5. *On every ε_0-neck we have $rr_z = r/f_r \leq 4(F(0,1) + C_0\varepsilon_0)\mathcal{G}^{-1}$, where C_0 depends only on n and γ.*

Proof. On an ε_0-neck we have $\frac{1}{f_r} = r_z \leq \varepsilon_0$. Moreover, the principal curvature in the radial direction is bounded from above by ε_0/r. This gives

$$\frac{f_{rr}}{(1+f_r^2)^{3/2}} \leq \frac{\varepsilon_0}{r},$$

and hence

$$\frac{r f_{rr}}{(1+f_r^2) f_r} \leq \varepsilon_0 (1 + \varepsilon_0^2)^{1/2}.$$

Using Lemma 8.3 we obtain

$$\mathcal{G} \leq f_t = \gamma\left(\frac{f_{rr}}{1+f_r^2}, \frac{f_r}{r}, \ldots, \frac{f_r}{r} \right) = \gamma\left(\frac{r f_{rr}}{(1+f_r^2) f_r}, 1, \ldots, 1 \right) \frac{f_r}{r} \leq (F(0,1) + C_0 \varepsilon_0) \frac{f_r}{r}.$$

□

The following lemma establishes scaling-invariant derivative bounds for $r(z,t)$ on any ε_0-neck.

Lemma 8.6. *There is a constant $C_1 > 1$ such that on every ε_0-neck with $r \geq 1$ we have $r^m|\frac{d^m}{dz^m}r| \leq C_1$, $m = 1, 2, 3$.*

Proof. For $m = 1$ we may appeal to Lemma 8.5.

Set $u = \langle -e_{n+1}, \nu\rangle$. Since $f_t = Gu^{-1}$, Lemma 8.3 implies $u \leq G\mathcal{G}^{-1}$. Suppose (x, t) lies at the center of an ε_0-neck. We may then use the evolution equation

$$\frac{\partial}{\partial t}u = \dot\gamma^{ij}\nabla_i\nabla_j u + \dot\gamma^{ij}A_i^k A_{kj}u$$

and standard interior estimates for linear parabolic equations to conclude that

$$|\nabla^m u|^2 \leq CG^{2m+2}$$

at (x, t).

The claim follows from this estimate by writing the left-hand side in terms of r and rearranging. Indeed, we have $u = r_z(1 + r_z^2)^{-1/2}$, and the induced metric is

$$g = (1 + r_z^2)dz^2 + r^2 g_{S^{n-1}}.$$

Using these formulae the rest of the argument is straightforward. □

Next we prove a sharp estimate for $r_{zz}(z, t)$ which is valid when $r(z, t)$ is sufficiently large.

Lemma 8.7. *Let $C_2 = 2 + 2\Lambda(\varepsilon_0) + \frac{64}{9}(F(0, 1) + C_0\varepsilon_0)^2\mathcal{G}^{-2}$. If $r(z, t) \geq C_2$ then*

$$|(rr_z)_z(z, t)| \leq C_3 r(z, t)^{-3/2} \qquad \text{and} \qquad 0 \leq -r_{zz}(z, t) \leq Cr(z, t)^{-5/2}.$$

Proof. Let us fix a point $(\bar r, \bar t)$ such that $\bar r \geq C_2$ and let $\bar z = f(\bar r, \bar t)$. We then have that every point $x = (r\theta, f(r, t))$ in M_t with $r \geq \bar r$ lies at the center of an ε_0-neck.

Using $f_r \geq (F(0, 1) + C_0\varepsilon_0)^{-1}\mathcal{G}r$ we estimate

$$\bar z - f\left(\frac{\bar r}{2}, \bar t\right) = \int_{\bar r/2}^{\bar r} f_r(r, \bar t)\, dr \geq \frac{3}{8}(F(0, 1) + C_0\varepsilon_0)^{-1}\mathcal{G}\bar r^2.$$

Since $\bar r \geq \frac{64}{9}(F(0, 1) + C_0\varepsilon_0)^2\mathcal{G}^{-2}$ this implies that $f(\frac{\bar r}{2}, \bar t) \leq \bar z - \bar r^{3/2}$. In other words, $r(z, \bar t) \geq \bar r/2$ for $z \geq \bar z - \bar r^{3/2}$. Since $r(z, t)$ is nonincreasing in time it follows that $r(z, t) \geq \bar r/2$ for $t \leq \bar t$ and $z \geq \bar z - \bar r^{3/2}$. In particular, $r \geq \bar r/2$ holds in the set $Q := [\bar z - \bar r^{3/2}, \bar z + \bar r^{3/2}] \times [\bar t - \bar r^3, \bar t]$, and hence every point (x, t) with $(x_{n+1}, t) \in Q$ lies at the center of an ε_0-neck.

We compute

$$\begin{aligned}(rr_z)_t = &-r_z\gamma\left(-\frac{r_{zz}}{1 + r_z^2}, \frac{1}{r}, \dots, \frac{1}{r}\right) + \dot\gamma^1\left(-\frac{r_{zz}}{1 + r_z^2}, \frac{1}{r}, \dots, \frac{1}{r}\right)\left(\frac{rr_{zzz}}{1 + r_z^2} - \frac{2rr_zr_{zz}^2}{(1 + r_z^2)^2}\right)\\ &+ \frac{r_z}{r}\sum_{i=2}^n \dot\gamma^i\left(-\frac{r_{zz}}{1 + r_z^2}, \frac{1}{r}, \dots, \frac{1}{r}\right).\end{aligned}$$

Inserting

$$\begin{aligned}\gamma\left(-\frac{r_{zz}}{1 + r_z^2}, \frac{1}{r}, \dots, \frac{1}{r}\right) &= -\dot\gamma^1\left(-\frac{r_{zz}}{1 + r_z^2}, \frac{1}{r}, \dots, \frac{1}{r}\right)\frac{r_{zz}}{1 + r_z^2}\\ &\quad + \frac{1}{r}\sum_{i=2}^n \dot\gamma^i\left(-\frac{r_{zz}}{1 + r_z^2}, \frac{1}{r}, \dots, \frac{1}{r}\right)\end{aligned}$$

we obtain

$$\begin{aligned}(rr_z)_t &= \dot\gamma^1\Big(-\frac{r_{zz}}{1+r_z^2},\frac{1}{r},\dots,\frac{1}{r}\Big)\Big(\frac{r_zr_{zz}}{1+r_z^2}+\frac{rr_{zzz}}{1+r_z^2}-\frac{2rr_zr_{zz}^2}{(1+r_z^2)^2}\Big)\\ &= \dot\gamma^1\Big(-\frac{r_{zz}}{1+r_z^2},\frac{1}{r},\dots,\frac{1}{r}\Big)\Big((rr_z)_{zz}-\frac{(2+3r_z^2)r_zr_{zz}+rr_z^2r_{zzz}}{1+r_z^2}-\frac{2rr_zr_{zz}^2}{(1+r_z^2)^2}\Big).\end{aligned}$$

Let us define a smooth function $a = a(z,t)$ by

$$a(z,t) = \dot\gamma^1\Big(-\frac{r_{zz}(z,t)}{1+r_z(z,t)^2},\frac{1}{r(z,t)},\dots,\frac{1}{r(z,t)}\Big),$$

so that we may write

$$(rr_z)_t - a(rr_z)_{zz} = -a\frac{(2+3r_z^2)r_zr_{zz}+rr_z^2r_{zzz}}{1+r_z^2} - a\frac{2rr_zr_{zz}^2}{(1+r_z^2)^2}.$$

We have $C^{-1}\le a\le C$, so using Lemma 8.6 we find that

$$\sup_Q |rr_z| \le C \qquad \text{and} \qquad \sup_Q |(rr_z)_t - a(rr_z)_{zz}| \le C\bar r^{-3}.$$

Standard interior estimates for linear parabolic equations then give

$$|(rr_z)_z| \le C\bar r^{-3/2}\sup_Q |rr_z| + C\bar r^{3/2}\sup_Q|(rr_z)_t - a(rr_z)_{zz}|$$

at $(\bar z,\bar t)$. So we conclude that

$$|r_{zz}| \le C\bar r^{-5/2}$$

at $(\bar z,\bar t)$. □

For each $z<0$ we define $\mathcal{T}(z)$ such that $r(z,t)>0$ for $t<\mathcal{T}(z)$ and

$$\lim_{t\to\mathcal{T}(z)} r(z,t) = 0.$$

Lemma 8.8. *Whenever $z<0$ and $t<\mathcal{T}(z)$ we have*

$$2F(0,1)(\mathcal{T}(z)-t) \le r(z,t)^2.$$

If in addition $r(z,t) > 2C_2$ then we also have

$$r(z,t)^2 \le 2F(0,1)(\mathcal{T}(z)-t) + C(\mathcal{T}(z)-t)^{1/4} + C_2^2.$$

Proof. Let us fix a point $(\bar z,\bar t)$ such that $\bar z<0$. Since $r_{zz}<0$ we have

$$\frac{d}{dt}(r(\bar z,t)^2 + 2F(0,1)t) = -2\gamma\Big(-\frac{r(\bar z,t)r_{zz}(\bar z,t)}{1+r_z(\bar z,t)^2},1,\dots,1\Big) + 2F(0,1) < 0,$$

and hence

$$r(\bar z,\bar t)^2 \ge 2F(0,1)(\mathcal{T}(\bar z)-\bar t).$$

From now on we assume $r(\bar z,\bar t) > 2C_2$. Let $\bar t<\tilde t<\mathcal{T}(\bar z)$ denote the time at which $r(\bar z,\tilde t) = C_2$. We have

$$2F(0,1) \ge 2\gamma\Big(-\frac{r(\bar z,t)r_{zz}(\bar z,t)}{1+r_z(\bar z,t)^2},1,\dots,1\Big) - C|r(\bar z,t)r_{zz}(\bar z,t)|,$$

and hence

$$\begin{aligned}\frac{d}{dt}(r(\bar z,t)^2+2F(0,1)t) &= -2\gamma\Big(-\frac{r(\bar z,t)r_{zz}(\bar z,t)}{1+r_z(\bar z,t)^2},1,\dots,1\Big)+2F(0,1)\\ &\geq -C|r(\bar z,t)r_{zz}(\bar z,t)|\end{aligned}$$

for $\bar t < t < \tilde t$. Lemma 8.7 now gives

$$\frac{d}{dt}(r(\bar z,t)^2+2F(0,1)t) \geq -Cr(\bar z,t)^{-3/2}.$$

Integrating this inequality over $\bar t < t < \tilde t$ gives

$$\begin{aligned}r(\bar z,\bar t)^2-r(\bar z,\tilde t)^2 &= 2F(0,1)(\tilde t-\bar t)-\int_{\bar t}^{\tilde t}\frac{d}{dt}(r(\bar z,t)^2+2F(0,1)t)\,dt\\ &\leq 2F(0,1)(\tilde t-\bar t)+C\int_{\bar t}^{\tilde t} r(\bar z,t)^{-3/2}\,dt\\ &\leq 2F(0,1)(\tilde t-\bar t)+C\int_{\bar t}^{\tilde t}(\mathcal T(\bar z)-t)^{-3/4}\,dt\\ &\leq 2F(0,1)(\mathcal T(\bar z)-\bar t)+C(\mathcal T(\bar z)-\bar t)^{1/4}.\end{aligned}$$

Inserting $r(\bar z,\tilde t)^2 = C_2^2$ gives the claim. □

Using the estimates for $r(z,t)$ established in Lemma 8.8, we can now prove a sharp lower bound for $r(z,t)r_z(z,t)$ at $z=0$.

Lemma 8.9. *Let us fix $\delta > 0$ arbitrarily. We have*

$$r(0,t)r_z(0,t) \geq F(0,1)(\mathcal G^{-1}-\delta)$$

whenever $-t$ is sufficiently large.

Proof. In the following we assume $-t$ is so large that $R = r(0,t) \geq 2C_2$. Recall that $C_2 \geq \Lambda(\varepsilon_0)$. Consequently, every point (x,t) with $x_{n+1}=0$ lies at the center of an ε_0-neck. This implies $|r(z,t)-R| \leq \varepsilon_0 R$ for $|z| \leq 2R$. Then we have $R \geq C_2$ for $|z| \leq 2R$, so by Lemma 8.7 we have

$$|(rr_z)_z| \leq C_3 R^{-3/2}.$$

Lemma 8.8 implies $r(0,t)\to\infty$ as $t\to-\infty$, so we may assume $-t$ is so large that $R^{1/2} \geq 4C_3\delta^{-1}$. The fundamental theorem of calculus then gives

$$|r(z,t)r_z(z,t)-r(0,t)r_z(0,t)| \leq 2C_3R^{-1/2} \leq \delta/2$$

for $|z| \leq 2R$. Lemma 8.8 gives

$$r(-R,t)^2 \geq 2F(0,1)(\mathcal T(-R)-t) \qquad \text{and} \qquad r(-2R,t)^2 \geq 2F(0,1)(\mathcal T(-2R)-t).$$

Assuming $-t$ is so large that $r(-2R,t) > 2C_2$, the second of these inequalities and Lemma 8.8 imply

$$\begin{aligned}r(-2R,t)^2 &\leq 2F(0,1)(\mathcal T(-2R)-t)+C(\mathcal T(-2R)-t)^{1/4}+C_2^2\\ &\leq 2F(0,1)(\mathcal T(-2R)-t)+Cr(-2R,t)^{1/2}+C_2^2\\ &\leq 2F(0,1)(\mathcal T(-2R)-t)+CR^{1/2}+C_2^2.\end{aligned}$$

We thus have

$$r(-R,t)^2 - r(-2R,t)^2 \geq 2F(0,1)(\mathcal{T}(-R) - \mathcal{T}(-2R)) - C_4 R^{1/2} - C_2^2.$$

On the other hand, Lemma 8.3 implies

$$\mathcal{T}(-R,t) - \mathcal{T}(-2R,t) \geq (\mathcal{G}^{-1} - \delta/4)R$$

for sufficiently large $-t$. We may assume $-t$ is so large that

$$C_2^2 + C_4 R^{1/2} \leq F(0,1)\delta R/2,$$

so that combining these estimates gives

$$r(-R,t)^2 - r(-2R,t)^2 \geq 2F(0,1)(\mathcal{G}^{-1} - \delta/2)R.$$

Therefore, if $-t$ is sufficiently large,

$$\sup_{z\in[-2R,-R]} r(z,t) r_z(z,t) \geq F(0,1)(\mathcal{G}^{-1} - \delta/2).$$

Recalling that $r(0,t)r_z(0,t) \geq r(z,t)r_z(z,t) - \delta/2$ for $|z| \leq 2R$, we finally obtain

$$r(0,t)r_z(0,t) \geq F(0,1)(\mathcal{G}^{-1} - \delta)$$

for $-t$ sufficiently large. □

We recall [BC19, Proposition 6.9].

Lemma 8.10. *We define a smooth function $\psi : (0,\infty)\times(0,\infty) \to \mathbb{R}$ by*

$$\psi(z,t) = \frac{1}{\sqrt{4\pi t}} \int_0^\infty (e^{-\frac{(z-y)^2}{4t}} - e^{-\frac{(z+y)^2}{4t}})\, dy.$$

Then ψ is a solution to the heat equation $\psi_t = \psi_{zz}$. Moreover, for each $z > 0$ and $t > 0$ we have $\psi_{zz}(z,t) < 0$, and

$$\lim_{z\to 0} \psi(z,t) = 0, \qquad \lim_{z\to\infty} \psi(z,t) = 1, \qquad \lim_{t\to 0} \psi(z,t) = 1, \qquad \lim_{t\to\infty} \psi(z,t) = 0.$$

Using the function $\psi(z,t)$ to construct an appropriate barrier, we extend the lower bound for $r(0,t)r_z(0,t)$ proven in Lemma 8.9 to $z \geq 0$.

Lemma 8.11. *Given $\delta > 0$ there is a time $\bar{t}$ dependig on δ such that*

$$r(z,t)r_z(z,t) \geq F(0,1)(\mathcal{G}^{-1} - \delta)$$

for all $z \geq 0$ and $t \leq \bar{t}$.

Proof. By Lemma 8.7, we have $1 + rr_{zz} \geq 0$ for $r \geq C_1 + C_2$. This implies

$$(rr_z)_t = a\frac{(rr_z)_{zz}}{1+r_z^2} - a\frac{2r_z r_{zz}(1 + r_z^2 + rr_{zz})}{(1+r_z^2)^2} \leq a\frac{(rr_z)_{zz}}{1+r_z^2}$$

for $r \geq C_1 + C_2$. By Lemma 8.11 we may choose $\bar{t}$ so that

$$r(0,t)r_z(0,t) \geq F(0,1)(\mathcal{G}^{-1} - \delta)$$

for every $t \leq \bar{t}$. Moreover, for a suitable choice of $\bar{t}$ we can arrange that $r(z,t) \geq C_1 + C_2$ for all $z \geq 0$ and $t \leq \bar{t}$. For each $s < \bar{t}$ we define a barrier function $\psi^{\delta,s}(z,t)$ by

$$\psi^{\delta,s}(z,t) = F(0,1)(\mathcal{G}^{-1} - 2\delta - \mathcal{G}^{-1}\psi(qz, t-s)), \qquad (z,t) \in (0,\infty)\times(s,\bar{t}],$$

where q is a positive constant. We claim that if q is chosen correctly then $rr_z \geq \psi^{\delta,s}$ for all $z \geq 0$ and $t \in (s, \bar{t}]$.

By our choice of $\bar{t}$,

$$r(0,t)r_z(0,t) \geq F(0,1)(\mathcal{G}^{-1} - \delta) > \limsup_{z\to 0} \psi^{\delta,s}(z,t)$$

for each $t \in (s, \bar{t}]$. Moreover,

$$\liminf_{z\to\infty} r(z,t)r_z(z,t) \geq 0 > \limsup_{z\to\infty} \psi^{\delta,s}(z,t)$$

for each $t \in (s, \bar{t}]$. Finally,

$$r(z,s)r_z(z,s) \geq 0 > \limsup_{t\to s} \psi^{\delta,s}(z,t).$$

Thus, if the inequality $rr_z > \psi^{\delta,s}$ fails, then there is a point $(\tilde{z}, \tilde{t}) \in (0,\infty) \times (s, \bar{t}]$ such that $r(\tilde{z},\tilde{t})r_z(\tilde{z},\tilde{t}) = \psi^{\delta,s}(\tilde{z},\tilde{t})$ and $r(\tilde{z},t)r_z(\tilde{z},t) > \psi^{\delta,s}(\tilde{z},t)$ for $t \in (s, \tilde{t})$. At $(\tilde{z},\tilde{t})$ we have

$$a\frac{(\psi^{\delta,s})_{zz}}{1+r_z^2} \leq a\frac{(rr_z)_{zz}}{1+r_z}^2 \leq (rr_z)_t \leq \psi_t^{\delta,s} = q^{-2}\psi_{zz}^{\delta,s}.$$

Since $a \geq C^{-1}$ and $r_z^2 \leq \varepsilon_0$, recalling that $\psi_{zz}^{\delta,s} > 0$, we have a contradiction if $q > 100 + C$.

We thus conclude that $r(z,t)r_z(z,t) > \psi^{\delta,s}(z,t)$ for all $z > 0$ and $t \in (s, \bar{t}]$, where s can take any value such that $s < \bar{t}$. Sending $s \to -\infty$, this gives

$$r(z,t)r_z(z,t) \geq F(0,1)(\mathcal{G}^{-1} - 2\delta)$$

for all $z \geq 0$ and $t \leq \bar{t}$. □

Lemma 8.11 lets us conclude that, at sufficiently early times, $r(z,t)^2$ grows at least linearly in z as $z \to \infty$.

Lemma 8.12. *We can find a time $T \in (-\infty, 0]$ such that $r(z,t)^2 \geq \mathcal{G}^{-1}z$ whenever $z \geq 0$ and $t \leq T$. In particular, if $t \leq T$ then the function $f(r,t)$ is defined for all $r \geq 0$.*

Proof. By Lemma 8.11 we can choose T so that $r(z,t)r_z(z,t) \geq \frac{1}{2}\mathcal{G}^{-1}$ for all $z \geq 0$ and $t \leq T$. The assertion then follows from the fundamental theorem of calculus. □

With these preparations out of the way, we can show that the limit of $r(z,t)r_z(z,t)$ as $z \to \infty$ is constant in time.

Lemma 8.13. *For each $t \leq T$ we have $\lim_{z\to\infty} r(z,t)r_z(z,t) = F(0,1)\mathcal{G}^{-1}$.*

Proof. Lemma 8.5 gives

$$\limsup_{z\to\infty} r(z,t)r_z(z,t) = F(0,1)\mathcal{G}^{-1}$$

for $t \leq 0$. So it suffices to show that

$$\liminf_{z\to\infty} r(z,t)r_z(z,t) \geq F(0,1)\mathcal{G}^{-1}$$

for $t \leq T$. Given any δ, Lemma 8.11 provides a $\bar{t} \leq T$ such that

$$\liminf_{z\to\infty} r(z,\bar{t})r_z(z,\bar{t}) \geq F(0,1)(\mathcal{G}^{-1} - 2\delta).$$

Moreover, Lemma 8.6 gives

$$|(rr_z)_t| \leq a\left|\frac{r_z r_{zz}}{1+r_z^2} + \frac{r r_{zzz}}{1+r_z^2} + \frac{2r r_z r_{zz}^2}{(1+r_z^2)^2}\right| \leq \frac{C}{r^2}$$

for $r \geq C_2$. Using Lemma 8.12 we obtain

$$\liminf_{z\to\infty} r(z,t) r_z(z,t) = \liminf_{z\to\infty} r(z,\bar{t}) r_z(z,\bar{t}) \geq F(0,1)(\mathcal{G}^{-1} - 2\delta)$$

for $t \leq T$. Since δ can be chosen arbitrarily small, this completes the proof. □

We finally conclude that the hypersurfaces M_t move by translation.

Proof of Proposition 8.1. Since $rr_z = \frac{r}{f_r}$, Proposition 8.13 implies

$$\lim_{r\to\infty} \frac{f_r}{r} = F(0,1)^{-1}\mathcal{G}$$

for $t \leq T$. Using this fact, the evolution equation for f, and the fact that

$$\lim_{r\to\infty} \frac{f_r r}{1+f_r^2} = 0,$$

we obtain

$$\lim_{r\to\infty} f_t(r,t) = F(0,1)\cdot \lim_{r\to\infty} \frac{f_r}{r} = \mathcal{G}$$

for $t \leq T$. Using Lemma 8.4 we find that $f_t(r,t) \leq \mathcal{G}$ for $r \geq 0$ and $t \leq T$. Therefore, Lemma 8.3 implies that $f_t(r,t) = \mathcal{G}$ for $r \geq 0$ and $t \leq T$. That is, the hypersurfaces M_t move by translation with velocity $\mathcal{G}e_{n+1}$ for $t \leq T$.

It is now straightforward to show that M_t moves by translation for all $t \leq 0$. The quantity $v := G + \langle \mathcal{G}e_{n+1}, \nu\rangle$ satisfies

$$\frac{\partial}{\partial t} v = \dot{\gamma}^{ij}\nabla_i\nabla_j v + \dot{\gamma}^{ij} A_i^k A_{kj} v.$$

Moreover, the function $h(x,t) = e^{2Ct}(|x|^2+1)$ satisfies

$$\frac{\partial}{\partial t} h > \dot{\gamma}^{ij}\nabla_i\nabla_j h + \dot{\gamma}^{ij} A_i^k A_{kj} h$$

for $t \geq T$ if the constant C is sufficiently large. By the maximum principle, $\sup_{M_t} \frac{|v|}{h}$ is nonincreasing for $t \leq 0$. Since v vanishes at $t = T$, we have $G = -\langle \mathcal{G}e_{n+1}, \nu\rangle$ on M_t for all $t \leq 0$. That is, M_t moves by translation with velocity $\mathcal{G}e_{n+1}$ for $t \leq 0$. □

We now have all the ingredients required to prove Theorem 1.1.

Proof of Theorem 1.1. Let M_t, $t \in (-\infty, 0]$, be a convex ancient G-flow which is noncollapsing and uniformly two-convex. Suppose also that M_t is noncompact. If M_t fails to be strictly convex, then it is a self-similarly shrinking cylinder by Lemma 2.5. If M_t is strictly convex then it is rotationally symmetric by Theorem 7.4. Therefore, M_t is a translating soliton by Proposition 8.1. This completes the proof. □

Appendix A. Asymptotics for the bowl soliton

A G-flow M_t moves by translation with velocity V if and only if

$$G = -\langle V, \nu \rangle.$$

Suppose $V = \frac{1}{2}e_{n+1}$ and $M_0 = \operatorname{graph} u$. Then this equation can be expressed as

$$\gamma\left(\left(\delta^{ik} - \frac{D^i u D^k u}{1+|Du|^2}\right)\frac{D_k u D_j u}{\sqrt{1+|Du|^2}}\right)\right) = \frac{1}{2}\frac{1}{\sqrt{1+|Du|^2}}. \tag{19}$$

In case $u(x) = \zeta(\rho)$ where $\rho = |x|$, we have

$$\gamma\left(\frac{\zeta_{\rho\rho}}{1+\zeta_\rho^2}, \frac{\zeta_\rho}{\rho}, \dots, \frac{\zeta_\rho}{\rho}\right) = \frac{1}{2}. \tag{20}$$

We use the notation $F(x_1, x_2) = \gamma(x_1, x_2, \dots, x_2)$ and $\dot{F}^i(x) = \frac{\partial F}{\partial x_i}(x)$. As we explain in Section 4, there is a function $f : U \to \mathbb{R}$ such that $F(x_1, x_2) = x_3$ if and only if $x_1 = f(x_2, x_3)$, where

$$U = \{(x_2, x_3) \in \mathbb{R}^2_+ : 0 < x_3/x_2 < Q\}$$

and $Q := \lim_{x_1\to\infty} F(x_1, 1)$. In case ζ is strictly increasing and strictly convex we may equivalently express (20) in the form

$$\zeta_{\rho\rho} = (1+\zeta_\rho^2) f\left(\frac{\zeta_\rho}{\rho}, \frac{1}{2}\right). \tag{21}$$

Up to parabolic rescalings and rigid motions of the ambient space there is a unique translating G-flow which is convex and rotationally symmetric. We refer to this solution as the bowl soliton. Existence and uniqueness of the bowl soliton were established in [Ren21]. We note that existence also follows from Proposition 4.2—simply set $a = \infty$ in the statement and proof. Uniqueness is proven by a standard argument—given two bowl solitons with velocity $V = \frac{1}{2}e_{n+1}$, shift one of them vertically until both agree at some $\bar{\rho} > 0$. The maximum principle then ensures that the two solutions coincide for $\rho \le \bar{\rho}$, and so by uniqueness of solutions to ODEs they also coincide for $\rho \ge 1$.

Let us denote by M_t the bowl soliton which translates with velocity $V = \frac{1}{2}e_{n+1}$ and whose tip lies at the origin. Let $\zeta(\rho)$ denote the profile function of M_0. In [Ren21] it was shown that under the assumption $(0, 1, \dots, 1) \in \Gamma$ (which we impose throughout this paper), the hypersurface M_0 is an entire graph. That is, $\zeta(\rho)$ is defined for all $\rho \ge 0$. This can also be deduced from Lemma 4.2, which implies the bounds

$$C^{-1}\rho \le \zeta_\rho \le C\rho,$$

where $C = C(n, \gamma)$ is a positive constant. Our goal is to establish the following asymptotic expansion for $\zeta(\rho)$ as $\rho \to \infty$.

Proposition A.1. *As $\rho \to \infty$, the function ζ satisfies*

$$\zeta_\rho(\rho) = \frac{1}{2\gamma(0,1,\dots,1)}\rho - 2\dot{\gamma}^1(0,1,\dots,1)\rho^{-1} + o\left(\rho^{-1}\right).$$

It follows that as $\rho \to \infty$ we have

$$\zeta(\rho) = \frac{1}{4\gamma(0,1,\dots,1)}\rho^2 - 2\dot{\gamma}^1(0,1,\dots,1)\log\rho + o(\log\rho).$$

For the mean curvature flow, Proposition A.1 was proven in [CSS07]. We refer also to [RS23a], which contains an independent proof of Proposition A.1 (in addition to bowl soliton asymptotics for some other classes of flows).

Proof of Proposition A.1. Let $\vartheta = \frac{\zeta_\rho}{\rho}$. We compute

$$\vartheta_\rho = \frac{1}{\rho}\left((1+\rho^2\vartheta^2)f\left(\vartheta, \frac{1}{2}\right) - \vartheta\right).$$

Since $f(\frac{1}{2F(0,1)}, \frac{1}{2}) = 0$ and f is strictly decreasing in its first argument, we see that $\vartheta_\rho < 0$ whenever $\vartheta > \frac{1}{2F(0,1)}$. On the other hand, if $\vartheta < \frac{1}{2F(0,1)}$ and ρ is large, we have $\vartheta_\rho > 0$. It follows that $\vartheta \to \frac{1}{2F(0,1)}$ as $\rho \to \infty$. Equivalently, we have

$$\zeta_\rho = \frac{1}{2F(0,1)}\rho + o(\rho)$$

as $\rho \to \infty$.

Next we claim that $\zeta_\rho = \frac{1}{2F(0,1)}\rho + o(1)$ as $\rho \to \infty$. Let us define $\xi := \zeta_\rho - \frac{1}{2F(0,1)}\rho$. We have

$$\xi_\rho = \left(1 + \xi^2 + \frac{1}{F(0,1)}\rho\xi + \frac{1}{4F(0,1)^2}\rho^2\right) f\left(\frac{\xi}{\rho} + \frac{1}{2F(0,1)}, \frac{1}{2}\right) - \frac{1}{2F(0,1)}.$$

Since $\frac{\xi}{\rho} = o(1)$, we may perform a Taylor expansion of f about the point $(1, F(0,1))$ to obtain

$$\begin{aligned} f\left(2F(0,1)\frac{\xi}{\rho} + 1, F(0,1)\right) &= 2F(0,1)\frac{\partial f}{\partial x_2}(1, F(0,1))\frac{\xi}{\rho} + o(1)\frac{\xi}{\rho} \\ &= -\frac{2F(0,1)^2}{\dot{F}^1(0,1)}\frac{\xi}{\rho} + o(1)\frac{\xi}{\rho}, \end{aligned}$$

and hence

$$\xi_\rho = \left(\frac{1}{4F(0,1)^2}\rho^2 + o(\rho)\right)\left(-\frac{F(0,1)}{\dot{F}^1(0,1)}\frac{\xi}{\rho} + o(1)\frac{\xi}{\rho}\right) - \frac{1}{2F(0,1)}.$$

We conclude that $\xi_\rho < 0$ whenever $\xi > 0$ and ρ is large, and $\xi_\rho > 0$ whenever $\xi < 0$ and ρ is large. It follows that $\xi \to 0$ as $\rho \to \infty$.

Now consider the function $\lambda := \rho\xi$. Since $\xi = o(1)$ we have $\lambda = o(\rho)$ as $\rho \to \infty$. We claim that $\lambda \to -2\dot{F}^1(0,1)$ as $\rho \to \infty$. To begin with we compute

$$\begin{aligned} \lambda_\rho = \frac{\rho}{2F(0,1)}&\left(1 + \frac{\lambda^2}{\rho^2} + \frac{\lambda}{F(0,1)} + \frac{\rho^2}{4F(0,1)^2}\right) f\left(2F(0,1)\frac{\lambda}{\rho^2} + 1,\ F(0,1)\right) \\ &- \frac{\rho}{2F(0,1)} + \frac{\lambda}{\rho}, \end{aligned}$$

and so conclude that

$$\lambda_\rho = \frac{\rho}{2F(0,1)}\left(\frac{\rho^2}{4F(0,1)^2} + o(\rho^2)\right) f\left(2F(0,1)\frac{\lambda}{\rho^2} + 1,\ F(0,1)\right) - \frac{\rho}{2F(0,1)} + o(1).$$

as $\rho \to \infty$. Performing a Taylor expansion as above, we obtain

$$\begin{aligned}\lambda_\rho &= \frac{\rho}{2F(0,1)}\left(\frac{\rho^2}{4F(0,1)^2} + o(\rho^2)\right)\left(-\frac{2F(0,1)^2}{\dot{F}^1(0,1)}\frac{\lambda}{\rho^2} + o(1)\frac{\lambda}{\rho^3}\right) - \frac{\rho}{2F(0,1)} + o(1)\\ &= \left(\frac{\rho^2}{4F(0,1)^2} + o(\rho^2)\right)\left(-\frac{1}{\dot{F}^1(0,1)}\frac{\lambda}{\rho} + o(1)\frac{\lambda}{\rho^2}\right) - \frac{\rho}{2F(0,1)} + o(1).\end{aligned}$$

From this we observe that $\lambda_\rho < 0$ whenever $\lambda > -2\dot{F}^1(0,1)$ and ρ is large, and $\lambda_\rho > 0$ whenever $\lambda < -2\dot{F}^1(0,1)$ and ρ is large. It follows that $\lambda \to -2\dot{F}^1(0,1)$ as $\rho \to \infty$.

In summary, we have shown that

$$\rho\left(\zeta_\rho - \frac{1}{2F(0,1)}\rho\right) = -2\dot{F}^1(0,1) + o(1)$$

as $\rho \to \infty$. From this the claim follows. □

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