Formal Language Characterizations of P, NP, and PSPACE

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Abstract

Giammarresi & Restivo (1992) define locality and recognizability for 2-dimensional languages. Based on these notions, generalized to the n-dimensional case, *n-dimensionally colorable* 1-dimensional languages are introduced. It is shown: A language L is in NP if and only if L is n-dimensionally colorable for some n. An analogous characterization in terms of deterministic n-dimensional colorability, based on a definition of 2-dimensional deterministic recognizability from Reinhardt (1998), is obtained for P. For an analogous characterization of PSPACE one unbounded dimension for coloring is added.

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1 Introduction

McNaughton & Papert [MP71] show that a 1-dimensional language is regular iff it is recognizable, i.e. if it consists of the words which positions can be colored so that the coloring respects the letters and obeys a given finite set of neighborhood constraints. Giammarresi & Restivo [GR92] define 2-dimensional recognizable languages the same way, now with 2-dimensional neighborhood constraints. Their definition can be generalized to n-dimensional words. Based on that definition in this paper *n-dimensionally colorable* 1-dimensional languages are defined as the languages consisting of the words which n-tupels of positions (instead of just positions) can be colored so that the coloring respects the letters and obeys a given set of neighborhood constraints. It is shown as the main result: A language L is in NP if and only if L is n-dimensionally colorable for some n.

For the proof the following equivalent characterization of the n-dimensionally colorable languages is used: They can be shown to be the languages consisting of the frontiers (cf. [LS97b]) of recognizable n-dimensional cubes. Latteux & Simplot [LS97b] define this notion of frontier and show that the context-sensitive languages are the languages given by frontiers of recognizable 2-dimensional words (any 2-dimensional words, not just 2-dimensional cubes), rediscovering an unpublished result of Sperber [Sp85]. Giammarresi [Gi03] modifies the definition of Latteux & Simplot [LS97b] and introduces the *bounded grid context sensitive languages* Bgrid-CS. The definition of Bgrid-CS is equivalent to that of being the set of frontiers of 2-dimensional cubes (mentioned in [Gi03][p.312]), and therefore equivalent to the definition of the 2-dimensionally colorable languages. This shows that not only first level of the colorability hierarchy is well-known (the 1-dimensionally colorable languages are the regular languages) but also the second level was studied before.

The main result and its proof is related to the result of Fagin [Fa74] which says that NP equals the set of problems definable in existential second-order logic. In Section 5 the n-dimensionally colorable languages will be shown, as another characterization, to be equal to the following segment of existential second-order logic on words: second-order arity bounded to n, only one first-order quantifier which is universal, and signature [min, max, S] where S is the successor function. In Section 6 *n-dimensionally deterministically colorable* 1-dimensional languages, based on the notion of deterministically recognizable 2-dimensional languages from Reinhard [Re98], are considered and an analogous result to the main result is proven: A language L is in P if and only if L is n-dimensionally deterministically colorable for some n. In Section 7 similar characterizations of some counting classes are given. In Section 8 unbounded dimensions are added to the bounded dimensions in the colorability characterization via frontiers of cubes. It will be shown that with one additional unbounded dimension one gets a charaterization of NPSPACE = PSPACE, generalizing the above mentioned characterization of the context sensitive languages by Sperber (1985) and Latteux & Simplot (1997), while more than one additional unbounded dimensions lead to the recursively enumerable languages.

2 Preliminaries

An n-dimensional word is basically an n-dimensional array of letters - in the sense the term *array* is used in programming languages. A survey for $n = 2$ is given by Giammarresi & Restivo in the Handbook of Formal Languages, Part III [GR96]. Giammarresi & Restivo also transfered the notions of *locality* and *recognizability* from 1-dimensional to 2-dimensional languages [GR92, GR96]. First, the definitions are repeated (with modifications) and generalized from the 2-dimensional to the n-dimensional case.

An *alphabet* is a finite non-empty set. An *n-dimensional word* x over an alphabet Σ is a mapping from $\{1, \ldots, l_1\} \times \cdots \times \{1, \ldots, l_n\}$ to Σ, where $l_1, \ldots, l_n \geq 1$. The empty word(s) for 1-dimensional and n-dimensional languages will be ignored in this paper. The elements of the domain of x are called *positions*, and the tuple $(l_1, \ldots, l_n)$ is called the *size* of x. Let Σ^{n+} be the set of all n-dimensional words over Σ. An *n-dimensional cube* over an alphabet Σ is an n-dimensional word having size $(m, \ldots, m)$, m is called the *edge length* of the cube. A 2-dimensional word is called a *picture* and a 2-dimensional cube is called a *square*.

Let # be a symbol not in Σ. The *boundary extension* $\widehat{x}$ of a n-dimensional word x of size $s = (l_1, \ldots, l_n)$ over Σ is the following n-dimensional word over $\widehat{\Sigma}$ of size $s = (l_1 + 2, \ldots, l_n + 2)$: $\widehat{x}(t) := x(t - (1, \ldots, 1))$ for all positions t in $\{2, \ldots, l_1 + 1\} \times \cdots \times \{2, \ldots, l_n + 1\}$ (these positions are called the *inner positions* of $\widehat{x}$), and $\widehat{x}(t) := \#$ for all other positions t of $\widehat{x}$ (these positions are called the *boundary positions* of $\widehat{x}$). The #'s mark the boundary of the word, see Figures 1 and 2 for a 1-dimensional and 2-dimensional example, respectively. Call x the *kernel* of $\widehat{x}$.

An n-dimensional word q of size $(l_1, \ldots, l_n)$ is a *subword* of another n-dimensional word p if there is a position $x = (x_1, \ldots, x_n)$ in p (call it the *anchor position*) such that the position $(x_1 + l_1 - 1, \ldots, x_n + l_n - 1)$ is still a position in p and for all positions y in q it holds $q(y) = p(x + y - (1, \ldots, 1))$. q can be visually imagined as the word of size $(l_1, \ldots, l_n)$ cut out from p at the anchor position.

Call two positions of an n-dimensional word *neighbored* if they differ in just one dimension j and in that dimension only by 1. Example: $(3, 2, 5, 4)$ and $(3, 2, 4, 4)$ are neighbored (in the 3rd dimension). A position in an n-dimensional word has at most 2^n neighbored positions.

An n-dimensional language L over an alphabet Σ is *recognizable* if there is a finite set Π (called the set of *colors*) with a fixed assignment $\pi : \Pi \rightarrow \Sigma$ of colors to letters (the *alphabet projection*) and a finite set Θ of n-dimensional words over the alphabet $\Pi \cup \{\#\}$ (the set of *forbidden subwords*) such that an n-dimensional word x is in L if and only if all inner positions i of $\widehat{x}$ can be assigned a color $c(i)$ from $\pi^{-1}(x(i))$ (an *appropriate color for* $x(i)$) and the colored word $\widehat{c}$ does not contain a forbidden subword from Θ. The set of n-dimensional words c over the alphabet Π such that $\widehat{c}$

#	a 1	a 1	b 0	b 0	b 0	b 0	c 1	#
1	2	3	4	5	6	7	8	9

1	1	0	0	0	0	1
1	2	3	4	5	6	7

Figure 1: 1-dimensional coloring of a 1-dimensional word

does not contain a forbidden subword from Θ is called a *local language (given by Θ)*. This way, the recognizable n-dimensional languages are by definition the alphabet projections of the local n-dimensional languages.

First an example for a 1-dimensional recognizable language L_1 is given. Consider the 1-dimensional local language on the alphabet $\{a, b, c, \#\}$ given by the set of forbidden words $\Theta = \{ba, cb, ca, \#b, \#c\}$. Figure 1 shows as an example $\widehat{x} = \#aabbbbc\#$. The word x is in the local language given by Θ because none of the forbidden words of Θ appears in $\widehat{x}$. The word $aabbabc$ is for example not in the local language given by Θ because the forbidden word ba appears in $\#aabbabc\#$. Together with the mapping $\pi : \{a, b, c\} \to \{0, 1\}$ defined by $\pi(a) = 1, \pi(b) = 0, \pi(c) = 1$ this local language defines the recognizable language $L_1 = 1^+0^*1^*$, see again Figure 1 where the word 1100001 is the image of $\#aabbbbc\#$ under the alphabet projection π. Verify that L_1 is not a local language (unlike 1^+0^* which is local) – but still it is the alphabet projection (via π) of a local language, i.e. it is a recognizable language. This recognizable language L_1 being a regular language is no coincidence. It is a classical result by McNaughton & Papert 1971 [MP71] that the recognizable 1-dimensional languages are the regular languages. This means that for $n = 1$ the recognizable languages coincide – besides many other characterization of the regular languages – with the languages accepted by deterministic finite automata.

An example of a 2-dimensional recognizable language is the set L_2 of squares of odd length size such that the letter in the center of the square is a 1. The local language for this is given by the alphabet $\{x, y, a, b\}$ and a set of forbidden subwords Θ which gurantee that the only way to color a picture is by assigning the positions of the two diagonals through the picture colors x or y and their crossing point a color y while all other positions have color a or b. If the picture is not a square of odd length size the picture is not colorable because the diagonals and therefore also their crossing point do not exists. After defining the alphabet projection π as $\pi(x) = 0, \pi(y) = 1, \pi(a) = 0, \pi(b) = 1$ one get as the recognizable language the language consisting of the squares of odd length size such that the letter in the center of the square is a 1 (because the color is guaranteed to be y there). Surprisingly, for $n = 2$ there are recognizable languages which are not accepted by a deterministic finite automata acting on the n-dimensional word, see [GR92, GR96]. Actually, the language L_2 was already the witness in the paper of Blum & Hewitt 1967 [BH67] for the proof that nondeterministic automata are more powerful on 2-dimensional words than deterministic ones. The recognizable 2-dimensional languages are btw. also different from the languages accepted by nondeterministic automata, see [GR92, GR96].

Locality is defined in this paper by a finite set of forbidden subwords (of any size). Usually, locality is defined the other way round: A shape of the subwords is fixed, say for example they have to be cubes of edge length k, and then a finite set of *allowed* subwords of that shape is given. The local language is now the set of words x such that every subword in $\widehat{x}$ of the given shape is an

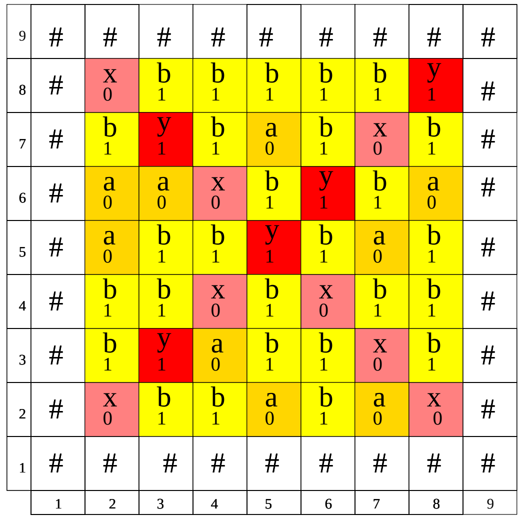

Figure 2: 2-dimensional coloring of a 2-dimensional word

allowed one. See for example [GR92, GR96] where for $n = 2$ the fixed shape is that of squares of edge length 2. A picture is called a *tile* when being an allowed subword because one looks at the tiles (like roof tiles) as a set of small words with which one covers the large word (the roof). Another example are the *domino tiles* of size $(2, 1)$ and $(1, 2)$ studied in [LS97a]. The definition of a local language possibly depends on the given shape, for example locality defined by finite sets of domino tiles is different from locality defined by finite sets of squares of edge length 2, which again is different from locality defined by squares of edge length 3. Nevertheless, the notion of recogizability will turn out to be equivalent for all "reasonable" shapes, see [GR92, GR96, LS97a, PV97], and is equivalent to the definition of recognizability via forbidden subwords. The definition of locality via forbidden subwords is choosen here because it seems to be the shortest and also the most general definition in the sense that every local set given by a finite set of allowed subwords of a certain shape is also a local language definable via a finite set of forbidden subwords. Moreover, this definition will allow a simple logical characterization of the n-dimensional local languages, see Lemma 5.1.

3 The Colorability Hierarchy

The main new notion of this paper is that of n-dimensional colorability. The idea is the following. A 1-dimensional language is recognizable if the positions can be colored so that a given finite set of neighborhood constraints is obeyed. The step from recognizability to that of n-dimensional colorability is that of going from positions of a 1-dimensional word x to n-tupels of positions of x: Instead of coloring the positions now the n-tupels of positions are colored, seen as an n-dimensional word (note that it is a cube), and x is in the n-dimensionally colorable language if a coloring of this n-dimensional cube of n-tupels exists which obeys a given set of n-dimensional neighborhood constraints. Note that for $n = 1$ this remains the definition of recognizability. The idea is repeated in the following formal definition.

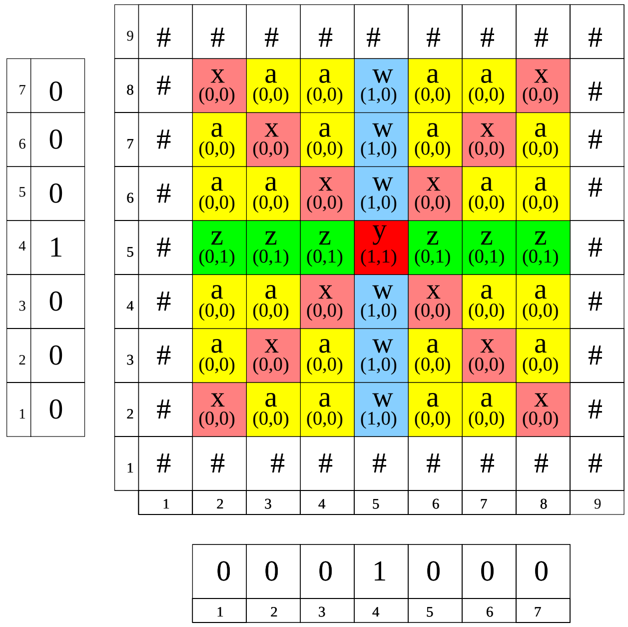

Figure 3: 2-dimensional coloring of a 1-dimensional word

Definition 3.1 (n-dimensionally colorable languages) *An 1-dimensional language L over an alphabet Σ is* n-dimensionally colorable *if there is an alphabet Π (called the set of* colors*) together with a fixed assignment $\pi : \Pi \rightarrow \Sigma^n$ of colors to n-tupels of letters (called the* alphabet projection*) and a set Θ of n-dimensional words over alphabet $\Pi \cup \{\#\}$ (called the* forbidden subwords*) such that a 1-dimensional word x is in L if and only if all n-tupels $i = (i_1, \ldots, i_n)$ of positions of x can be assigned a color $c(i)$ from $\pi^{-1}((x(i))$ (call such a color* appropriate for $x(i)$*) and the colored word $\widehat{c}$ does not contain a forbidden subword from Θ. Let COL^n denote the set of n-dimensionally colorable languages, and let COL be the set of languages which are n-dimensionally colorable for some n.*

As an example it will be shown that the non-regular but context-free language $L = \{0^n 1 0^n \mid n \geq 0\}$ is 2-dimensionally colorable according to the definition above. The colors for the coloring will be $\{a, x, y, w, z\}$, together with the alphabet projection $\pi(a) = (0,0)$, $\pi(x) = (0,0)$, $\pi(y) = (1,1)$, $\pi(w) = (1,0)$, and $\pi(z) = (0,1)$. The set of forbidden subwords Θ can be given in a way so that the diagonal is colored with x's, besides the center which is colored with y. The coloring of the diagonals with x's is done in order to find the center of the square (in the example before it was also done for guaranteeing that the picture is a square - here it is a square anyway). From that center a horizontal line is colored with z's and a vertikal column with w's. This is done because for a word in L the pairs of letters at these positions will be $(0,1)$ and $(1,0)$, respectively. It holds: If a word x is of the form $0^n 1 0^n$ then the 2-tupels of positions can be colored obeying the constraints from Θ, see Figure 3 where a coloring of the 2-tupels of the positions in the word 0001000 is given, and if the word is not of that form any coloring will fail, i.e. will not assign every 2-tupel an appropriate color or it will contain a forbidden subword.

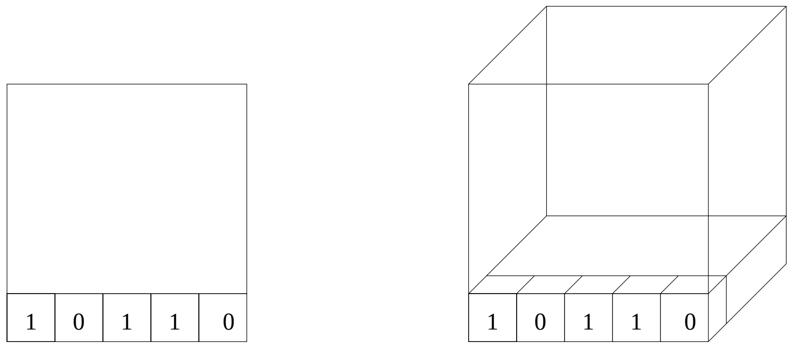

Figure 4: 2- and 3-dimensional cubes with their frontiers

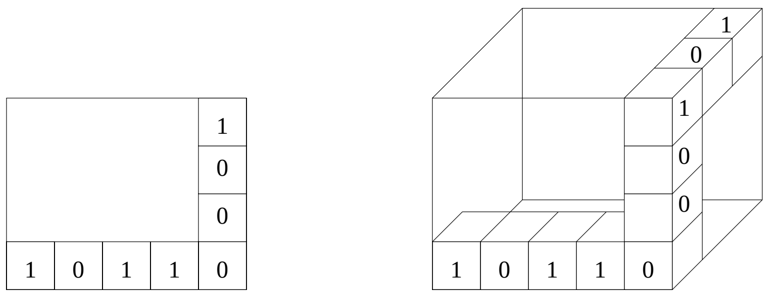

Figure 5: 2- and 3-dimensional words with their circumferential frontiers

In the following Lemma 3.2 equivalent – and possibly easier to understand – characterizations of n-dimensional colorability will be given. The following notion was introduced by Latteux & Simplot in [LS97b]. Let the *frontier* $\text{fr}(x)$ of an n-dimensional word x of size $(l_1, \dots, l_n)$ be its lowest row, i.e. the 1-dimensional word of length l_1 which is the concatenation of the letters $x(1, 1, \dots, 1)$, $x(2, 1, \dots, 1)$, $\dots x(l_1, 1, \dots, 1)$. See Figure 4 for frontiers of a 2- and 3-dimensional cube. Let n-padded-cube(w) for some 1-dimensional word $w \in \Sigma^+$ of length m be the n-dimensional cube with edge size m having frontier w and a blank symbol $B \notin \Sigma$ at all other positions, see also Figure 4 (the frontier notion and the padded-cube notion are kind of inverse). The following definitions are generalizations of a 2-dimensional notion from [Gi03]. Let for $j = 2, \dots, n$ the *frontier in the j-th dimension* $\text{fr}_j(x)$ of an n-dimensional word x be the the 1-dimensional word of length $l_j - 1$ consisting of the concatenation of the letters $x(l_1, \dots, l_{j-1}, 2, 1, \dots, 1)$, $x(l_1, \dots, l_{j-1}, 3, 1, \dots, 1)$, $\dots$, $x(l_1, \dots, l_{j-1}, l_j, 1, \dots, 1)$. Let the *circumferential frontier* of an n-dimensional word be the 1-dimensional concatenation $\text{fr}(x)\text{fr}_2(x) \cdots \text{fr}_n(x)$. See Figure 5 for the circumferential frontiers of a 2- and a 3-dimensional word.

Lemma 3.2 *For a* 1*-dimensional language* L *over* Σ *and* $n \geq 2$ *the following are equivalent:*

(a) L *is* n*-dimensionally colorable,*

(b) there exists an n*-dimensional local (or recognizable) language* L' *such that* L *consists of the frontiers of the cubes in* L'*,*

(c) n-padded-cube(L) *is recognizable (as an* n*-dimensional language),*

(d) there exists an n*-dimensional local (or recognizable) language* L' *such that* L *consists of the circumferential frontiers of the words in* L'*.*

Note that all characterizations are still from Formal Languages Theory, no concept of computation is used in the respective definitions.

Proof. (a) $\Rightarrow$ (b): Let L be n-dimensionally colorable via a local language L given by Θ and $\pi : \Pi \rightarrow \Sigma^n$. In order to build a local language on the n-dimensional cube (which does not contain the n-tupels of letters) - a local language is constructed which "shows" the n-tupels of letters in the frontier in its colors. As the first step build a new Θ_1 over a new alphabet Π_1 which ensures that only n-dimensional cubes are in L, this can be done by building a diagonal, see examples above. As the next step make $\Sigma \cup \Pi_2$ for an extended alphabet $\Pi_2 \supseteq \Pi_1$ the new alphabet and let a Θ_2 guarantee that the letters from Σ only appear in the frontier of the cube. As the next step make $\Sigma \cup \Pi_3 \times \Sigma^n$ for $\Pi_3 \supseteq \Pi_2$ the new alphabet for a new n-dimensional local language given by Θ_3 where Θ_3 ensures that each position $i = (i_1, \ldots, i_n)$ not belonging to the frontier has color $(d, (l_{i_1}, \ldots, l_{i_n}))$ such that letter l_{i_j} is equal to the i_j-th letter of the frontier. This can be done by guaranteeing this first for the frontiers in the other dimensions via 2-dimensional diagonals, and then forwarding this information from neighbor to neighbor in every dimension j. Now that the n-tupels of the frontier can basically be "seen" at all positions, combine the original set of forbidden words Θ and its alphabet projection $\pi : \Pi \rightarrow \Sigma^n$ with Θ_3 in order to get a local language which contains a cube with frontier x iff x is n-dimensionally colorable with Θ and π. (b) $\Leftrightarrow$ (c) is immediate. (c) $\Rightarrow$ (a): Let Π be the of colors of the recognizable set and Θ be the set of forbidden subwords. Construct the following new local language: Make $\Pi \times \Sigma^n$ the new alphabet, let the set of forbidden subwords be like Θ, ignoring the n-tupels of letters, and let the alphabet projection π map a color (c, t) to the n-tupel of letters t. The equivalence (b) $\Leftrightarrow$ (d) is mentioned in [Gi03][p. 312] for $n = 2$, and can also for $n > 2$ be shown in both directions with elementary tiling "programming" techniques. **q.e.d.**

It is obvious that the n-dimensionally colorable languages are a subset of the $(n+1)$-dimensionally colorable languages. Therefore one gets a hierarchy of language classes. Non-collapsing properties of the hierarchy will be concluded in the next paragraph. The 1-dimensionally colorable languages are by definition the 1-dimensional recognizable language which are the regular languages (McNaughton & Papert 1971 [MP71]). An example of a 2-dimensionally recognizable language which is not 1-dimensionally recognizable (because it is not regular) is the language L from the example above consisting of words w over $\{0, 1\}$ such that $w = 0^n 1 0^n$ for some n. In the next section it will be observed that COL^2 already contains NP-complete languages. The languages which correspond to part (d) of the above Lemma 3.2 were introduced for $n = 2$ by Giammarresi [Gi03] as the *bounded-grid context sensitive languages*, short *Bgrid-CS*. *Bgrid-CS* contains the set $\mathrm{LINEAR}_{\mathrm{CS}}$ from Book [Bo71] which contains for example the contextfree languages. The observations of this paragraph are summarized.

Observation 3.3

(a) For all $n \geq 1 : \mathrm{COL}^n \subseteq \mathrm{COL}^{n+1}$*,*

(b) $\mathrm{COL}^1 \subset \mathrm{COL}^2$*,*

(c) $\mathrm{COL}^1 = \mathrm{REC} = \mathrm{REG}$ ([MP71])*,*

(d) $\mathrm{COL}^2 = \mathrm{BgridCS}$ ([Gi03]*).*

At the end of this chapter it will be observed that by looking at neighborship requirements of n-tupels of positions while coloring not the tupels but only the letters one does not get out of the regular languages. Consider a 1-dimensional language from Σ^+, an alphabet Π, a function $\pi : \Pi \to \Sigma$ and a finite set Θ of n-dimensional words over the alphabet $\Pi^n \cup \{\#\}$. Assume that L consists of the words $w = w_1 \cdots w_m$ such that there exists a word e from Π^+ such that $w = \pi(e)$ and the n-dimensional cube $\widehat{c}$ with edge size $m+2$ defined by $c(x_1, \ldots, x_n) = (e(x_1), \ldots, e(x_n))$ does not contain a subword from Θ. Then L is regular. This can be shown by turning the local language given by Θ into an equivalent domino local language [LS97a] with a possibly new alphabet Π' and new π', and arguing that L is the π'-image of the 1-dimensional local language given by the conjunction of the n sets (for each dimension) of domino local constraints.

4 A Characterization of NP

It will be shown that the n-dimensionally colorable languages are the languages accepted by nondeterministic Turing machines, see for example the textbook [Pa94]. More specifically, NP is the union (over all n) of the classes NTIME($|x|^n$) which is defined to be the set of language from Σ^+ accepted by a nondeterministic Turing machine having for every input x run time $c|x|^n$ or less on every nondeterministic path, for some constant c. From now on let Σ always be $\{0, 1\}$. First the following simple result is shown.

Lemma 4.1 $\mathrm{COL}^n \subseteq \mathrm{NTIME}(|x|^n)$.

Proof. Let a language L from COL^n be given via a set Θ of forbidden words over an alphabet $\Pi \cup \{\#\}$, according to characterization (b) from Lemma 3.2, note that Σ is contained in Π. Given an input x of size k the Turing machine builds an n-dimensional cube (= array) p of size $(k+2, \ldots, k+2)$ which cells can hold a code for the letters of $\Pi \cup \{\#\}$. It places the code for letter $\#$ at all boundary positions, writes the word x into the frontier of the cube, and for each inner position it guesses nondeterministically a code for one of the letters from Π. Then it checks deterministically by going through all positions i as anchor positions whether one of the forbidden words in Θ is a subword of p with anchor position i. If it finds such a subword the Turing machine rejects the input x, otherwise it accepts x after the search. By construction the machine accepts the input x if x is in L, and its runtime is $O(|x|^n)$. **q.e.d.**

The above lemma shows COL $\subseteq$ NP. In order to show the other direction (Theorem 4.4) first the following Lemma 4.2 is recalled. Its main idea - namely the simulation of a general, i.e., not ressource-bounded, Turing machine computation by a 2-dimensional tiling system - goes finally back to Wang [Wa61, Wa62] who used an infinite $\omega \times \omega$ area. Lewis [Le77, Le78] modified the idea for the simulation of a ressource-bounded Turing machine by a tiling system for a finite area, resulting in a “tiling” master problem for NP-completeness as an alternative to SAT (resolutely done in the textbook [LP81]), see the papers of van Emde Boas [vEm82, SE84, vEm97] for a survey.

Let $C(x, s, t)$ for a word $x \in \{0, 1\}^+$ and numbers $s \geq |x|$ and t be the 2-dimensional word of size (s, t) over the alphabet $\{0, 1, B\}$ such that the lowest row is a word $xB^{s-|x|}$ and all other positions have the "blank" letter B, see Figure 6. For simplicity it is assumed that a Turing machine has only a halftape. i.e. it cannot move left to the initial cell; complexity classes like NTIME($|x|^n$) are robust under this restriction.

Lemma 4.2 (cf. Wang [Wa61, Wa62], Lewis [Le78]) *Let M be a nondeterministic Turing machine. Then the following 2-dimensional language is recognizable: The set of squares $C(x, t, t)$ such that M accepts x within time t.*

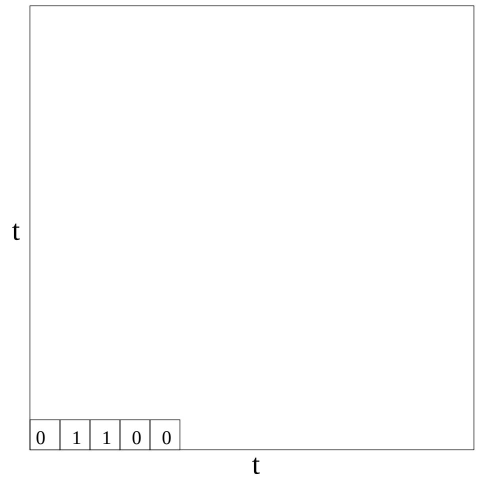

Figure 6: Turing computation, cf. Lemma 4.2

Proof Sketch. The idea of the construction is the following: The local language will ensure that on every row a configuration of the Turing machine is encoded, the cell below the head will be colored with the state the Turing machine is in. The content of the tape will be copied from one row to the upper row, besides a possible change below the head caused by writing. It will be ensured that the upper right corner can only be tiled if the Turing machine has accpeted on some row. **q.e.d.**

Lemma 4.3 $\mathrm{NTIME}(|x|^n) \subseteq \mathrm{COL}^{2n}$.

Proof. First the case $n = 1$ is shown. Let L be accepted by a nondeterministic Turing machine having time bound in $c(|x|)$. Consider $c = 1$. Then Lemma 4.2 gives immediately that L is in COL^2. For $c > 1$ one has to consider tiles which combine $c \times c$ adjacent tiles into one - neighborship requirements of the smaller tiles are translated into neighborship requirements for the $c \times c$ tiles.

Second, the case $n \geq 2$ is shown. Let M be a nondeterministic Turing machine time-bounded by $c|x|^n$. Assume again w.l.o.g. that $c = 1$ because in case $c > 1$ one combines $c \times c$ adjacent tiles into one, see case $n = 1$ above.. Let L be the 2-dimensional local language for this M according to Lemma 4.2 given by the set of forbidden subwords Θ. Θ can according to [LS97a] be assumed to consist of domino tiles. This 2-dimensional local language L will be turned into a 2n-dimensional local language. The first n dimensions are used to represent a configuration of M, i.e. a line in the computation square of Lemma 4.2, while the other n dimensions are used to represent the sequence of configurations of the computation of M. One line of the local language for M given by Lemma 4.2 has length $|x|^n$. This line will be embedded into an n-dimesnional cube of length size n in a way so that two positions which were neighbored in the line are still neighbored in the n-dimensional cube. One way to do this is the "snakelike" embedding, see Figure 7. The horizontal part of the 2-dimensional domino local language L can therefore be translated into a n-dimensional local language which ensures that the n-dimensional cube of length size $|x|$ is in this local n-dimensional language iff the line of length $|x|^n$ is in the local horizontal domino-local language L. This procedure is done the same way in order to now folding the columns of the computation square via the vertical part of the local domino language, using again another n dimensions, see Figure 8. Note that the two

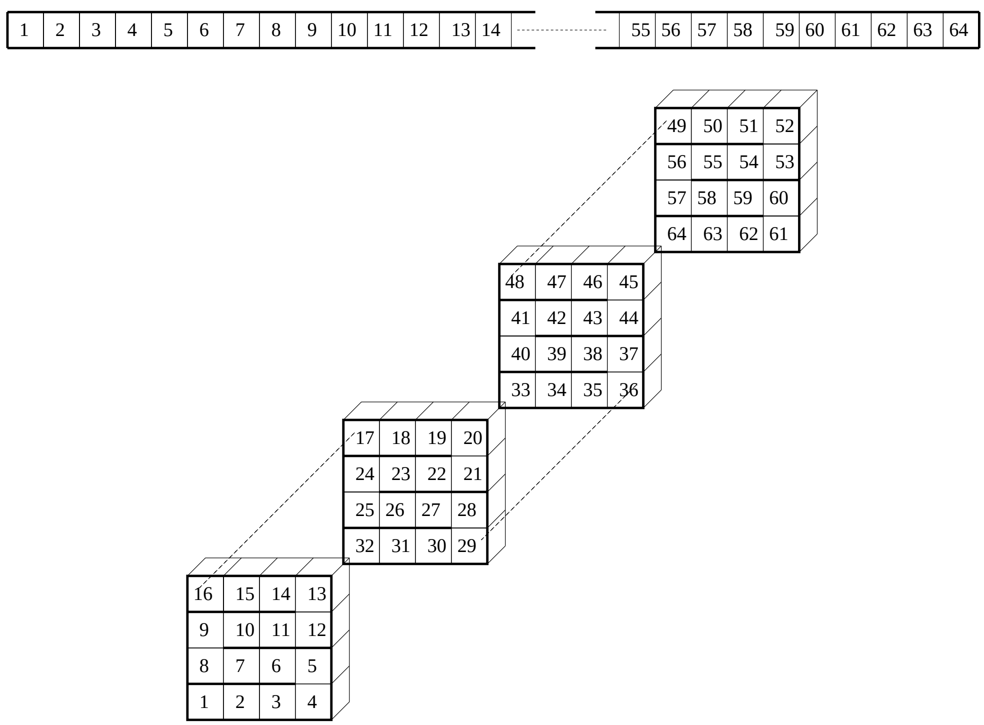

Figure 7: Folding a cubic length word into a cube, keeping neighborship relations

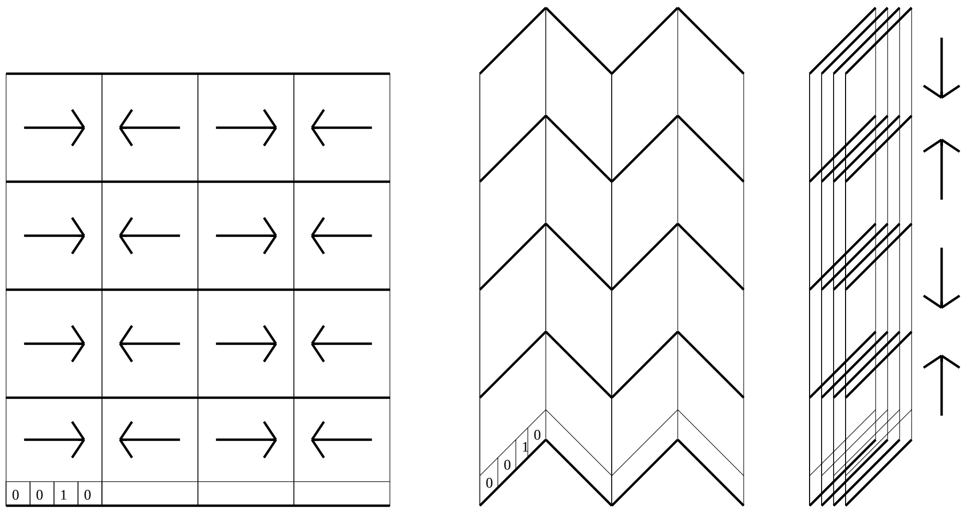

Figure 8: Folding a configuation/time computation

foldings, first the horizontal and then the vertical, can be done independently. Neighbored positions in the 2-dimensional square of Figure 6 are still neighbored in the $2n$-dimensional cube. Finally, this $2n$-dimensional local language has the property that an input x is in the frontier of a cube of this local language iff x is accepted by M. **q.e.d.**

The following theorem combines Lemmata 4.1 and 4.3, and can be seen as a characterization of NP in terms of formal language theory.

Theorem 4.4 NP = COL.

Note that the two hierarchies

$$\mathrm{COL}^1 \subseteq \mathrm{COL}^2 \subseteq \mathrm{COL}^3 \subseteq \ldots$$

and

$$\mathrm{NTIME}(|x|^1) \subseteq \mathrm{NTIME}(|x|^2) \subseteq \mathrm{NTIME}(|x|^3) \subseteq \ldots$$

have both NP as their union limit but seem not to be too closely related: The n-th level of the first hierarchy can be shown to be included in the n-th level of the latter, but the other direction needs a factor of 2. It is unknown to the author whether one can improve this factor of 2. Moreover, the following answer to the question of the previous section concerning the properness of the colorability hierarchy, possible by the result of Cook [Co73] who showed that $\mathrm{NTIME}(|x|^k)$ is a proper subset of $\mathrm{NTIME}(|x|^{k+1})$ for every k ("nondeterministic time hierarchy theorem"), could be made stronger when that factor of 2 would be improved.

Corollary 4.5 (cf. Cook [Co73]) *The colorability hierarchy does not collapse to some level: For every n COL^n is a proper subset of COL^{2n}.*

Lemma 4.3 for $n = 1$ can be combined with the result of Michel[Mi91], stating that $\mathrm{NTIME}(|x|)$ contains NP-complete problems, to conclude the following.

Corollary 4.6 (cf. Michel [Mi91]) $\mathrm{COL}^2 = \mathrm{BgridCS}$ *contains* NP*-complete problems.*

5 The Relation to the Characterization of NP by Fagin

First, the local languages are described logically, using the logical system for 2-dimensional words from Giammarresi et al. [GRST96], generalized to the n-dimensional case. S_i in the following signature $[\min, \max, \mathrm{S}_1, \ldots, \mathrm{S}_\mathrm{n}, (\mathrm{P}_c)_{c\in\Pi}]$ is the successor function for dimension i, and P_c is the predicate which is TRUE for a position iff the letter at that position is c.

Lemma 5.1 *For all $n \geq 1$ and all alphabets Π it holds: The n-dimensional local languages over alphabet Π are the languages expressible with one first-order universal quantifier over signature* $[\min, \max, \mathrm{S}_1, \ldots, \mathrm{S}_\mathrm{n}, (\mathrm{P}_c)_{c\in\Pi}]$.

Proof. Let an n-dimensional local language over an alphabet Π be given by a finite set of forbidden subwords $\Theta = \{w_1, \ldots, w_f\}$ over alphabet $\Pi \cup \{\#\}$. Expressing that a word $\widehat{x}$ does not contain a forbidden subword is done via a universal quantification over every position i in $\widehat{x}$ seen as a potential anchor position and expressing: $\forall i : \neg e_{w_1}(i) \wedge \ldots \wedge \neg e_{w_f}(i)$, where $e_{w_j}(i)$ expresses that w_j is a subword of x with anchor position i. The subexpression $e_{w_j}(i)$ can be build using the successor functions and the letter predicates P_c, and in case the bounding letter # is contained in w_j it uses the min and max predicates appropriately. If on the other hand an expression $\forall i e(i)$ with only one universal quantifier is given, let k be an upper bound for the length of a chain of successor functions occuring in $e(i)$. For all n-dimensional words x with sizes $(l_1, \ldots, l_n)$ over alphabet $\Pi \cup \{\#\}$ such that all l_j are $\leq k+1$ check if $e(1, \ldots, 1)$ on x evaluates to FALSE, considering the #'s appropriately, and make in that case x a forbidden subword. This set of forbidden subwords suffices to determine the local language because the expression can not "reach further" in the n-dimensional word. **q.e.d.**

For each $n \geq 1$ let $\exists^{2,n}\forall^1[\min, \max, \mathrm{S}]$ be the set of 1-dimensional languages over alphabet Σ definable with signature $[\min, \max, \mathrm{S}, (\mathrm{P}_l)_{l\in\Sigma}]$ with existential second variables having arity at most n and with one universal first-order quantifier only (from now the predicates $\mathrm{s}(\mathrm{P}_l)_{l\in\Sigma}$ are no longer mentioned in the signature).

Lemma 5.2 $\mathrm{COL}^n = \exists^{2,n}\forall^1[\min, \max, \mathrm{S}]$.

Proof. For the direction $\subseteq$ translate k colors into k n-ary predicates and require via the universal first order quantifier that exactly one of them holds for each tupel and that these predicates do obey the restrictions of the set of forbidden subwords, see Lemma 5.1 above. For the other direction let every combination of predicates become a color of a new alphabet, and construct a set of forbidden subwords like in Lemma 5.1. **q.e.d.**

Lemma 5.2 can be seen as another charactization of the n-dimensionally recognizable languages. Note that for $n = 1$ this gives a logical characterization of the regular languages as the languages definable with signature $[\min, \max, \mathrm{S}]$ in monadic second order having only one first-order quantifier which is universal. For $n = 2$ this gives a logical characterization of the bounded grid context-sensitive languages BgridCS [Gi03]: The languages definable with signature $[\min, \max, \mathrm{S}]$ in duadic second order having only one first-order quantifier which is universal.

Let $\exists^2\forall^1$ be the union of all $\exists^{2,n}\forall^1$, and let $\exists^2\mathrm{FO}[\sigma]$ be the set of languages expressible in second order with signature σ with no restriction on the first order part. The first of the following equalities follows from Lemma 5.2 together with Theorem 4.4. The second follows from the fact that a first order part can be evaluated in polynomial time, and the last equality follows from the fact that $[<]$ is definable via $[\min, \max, S]$ in existential second order, and vice versa (even in first order).

Theorem 5.3 (Fagin [Fa74]) $NP = \exists^2\forall^1[\min, \max, S] = \exists^2 FO[\min, \max, S] = \exists^2 FO[<]$.

6 A Characterization of P

Reinhardt [Re98] introduced the following concept of *deterministic* recognizability. Instead of just asking for the existence of a coloring the coloring has to be constructed in a deterministic fashion, starting from the boundary extension of the 2-dimensional word on letters, and flipping letters successively into colors only when that color is the only one at the position locally not hurting the neighborhood requirements. Only if this iterated procedure results in a fully colored word the word is deterministically colorable.

For a formal treatment and the generalization from 2 to n dimensions the following definition is introduced. Let like in the definition of colorability two alphabets Σ (the *letters*) and Π (the *colors*) be given (let them be disjunct), together with an alphabet projection $\pi : \Pi \to \Sigma$ and a finite set of forbidden subwords Θ of n-dimensional words over the alphabet $\Pi \cup \{\#\}$. Like in [Re98] the set Θ is required to consist only of domino words, i.e. words of size $(2, 1, \ldots, 1), (1, 2, 1, \ldots, 1), \ldots, (1, \ldots, 1, 2)$.

Define the following relation $\stackrel{\Theta,\tau}{\longrightarrow}$ among n-dimensional words x, x' on the alphabet $\Sigma \cup \Pi \cup \{\#\}$ to hold if the following conditions (1)-(3) are met: (1) x, x' only differ at one position i at which $x(i)$ is a letter and $x'(i)$ is an appropriate color for $x(i)$, (2) both x and x' do not contain a forbidden subword, and (3) every coloring of i and its yet uncolored neighbors $j_1, \ldots, j_k$ with appropriate colors $c(i), c(j_1), \ldots, c(j_k)$, resp., which results, after replacing these letters in x by these colors $c(i), c(j_1), \ldots, c(j_k)$, resp., in a word not containing a forbidden subword from Θ, has $c(i) = x'(i)$. An n-dimensional language L over an alphabet Σ is *deterministically recognizable* if there are Π, π, and Θ like above such that an n-dimensional word x is in L if and only if there are words $x_1, \ldots, x_{f-1}$ over the alphabet $\Sigma \cup \Pi$ and a word x_f over the alphabet Π such that it holds

$$\widehat{x} \stackrel{\Theta,\tau}{\longrightarrow} \widehat{x}_1 \stackrel{\Theta,\tau}{\longrightarrow} \ldots \stackrel{\Theta,\tau}{\longrightarrow} \widehat{x}_{f-1} \stackrel{\Theta,\tau}{\longrightarrow} \widehat{x}_f.$$

Corollary 6.1 *For a language L it holds: $L \in P \iff n$-padded-cube(L) is deterministically recognizable for some n.*

The corollary is a proof corollary of Lemmata 4.1 and 4.3, note that the simulation of a determinstic Turing machine by the local language, see Lemma 4.2, is a simple case of the above notion of determinism: starting from the head position of a deterministic Turing machine one first colors the current line containing the start configuration and after that one moves, at the position of the head, one line up and continues there. Like in the previous section one can conclude that levels n and $2n$ of the deterministic colorability hierarchy are different (via the deterministic time hierarchy theorem, see [Pa94, Th. 7.1]).

7 Characterizations of Counting Classes

In this section colorability characterizations of some counting complexity classes are given. The idea is the following: Instead of asking whether a coloring exists one counts the number of valid colorings.

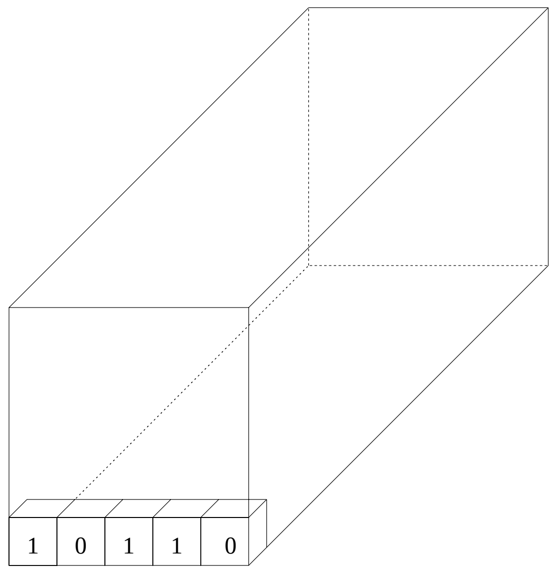

Figure 9: A 3-dimensional word which is a cube in the first 2 dimensions

An 1-dimensional language L over an alphabet Σ is *n-dimensionally complement (exactly-1, parity, majority, unambiguously) colorable* if there is an alphabet Π together with a alphabet projection $\pi : \Pi \rightarrow \Sigma^n$ of colors to n-tupels of letters and a set of forbidden subwords Θ of n-dimensional words over alphabet $\Pi \cup \{\#\}$ such that a 1-dimensional word x is in L if and only if the number of colorings of the n-tupels $i = (i_1, \ldots, i_n)$ of positions of x with an appropriate color $c(i)$ such that $\widehat{c}$ does not contain a forbidden subword from Θ is 0 (is exactly 1, is odd, is at least half as large as the total number of colorings with appropriate colors, is 1 and it is given that for every word x there is at most one such coloring). The recognizability version of the 2-dimensionally unambiguously colorable languages was defined as UREC in [GR92].

The following is a proof corollary of Lemmata 4.1 and 4.3. For the definition of the classes occurring refer for example to [Pa94]. More counting classes could be characterized in an analogous fashion.

Corollary 7.1 *Let L be a language in Σ^+. $L \in$ co-NP(1-NP, $\oplus$P, PP, UP) $\iff$ L is n-dimensionally complement (exactly-1, parity, majority, unambiguously) colorable for some n.*

8 A Characterization of PSPACE

The definition of colorabilty, according to characterization (b) of Lemma 3.2, is generalized to additional unbounded dimensions. Let in the following $n \geq 1$ and $m \geq 0$. Call an $(n+m)$-dimensional word of size $(k, \ldots, k, l_{n+1}, \ldots, l_m)$ a *cube in the first n dimensions*, and call k its *edge length*. See Figure 9 for a 3-dimensional word which is a cube in the first 2 dimensions. A 1-dimensional language L over an alphabet Σ is called *colorable in n bounded and m unbounded dimensions* if there exists an $(n+m)$-dimensional local (or recognizable) language L' such that L consists of the frontiers of the

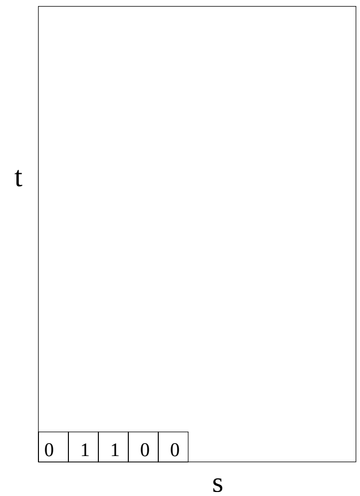

Figure 10: Turing computation, cf. Lemma 4.2

cubes in the first n dimensions in L'. Let COL^{n+mU} be the set of these languages. By Lemma 3.2(b), $\mathrm{COL}^{n} = \mathrm{COL}^{n+0U}$. One could have, equivalently, extended the original definition of colorability, or the equivalent characterization via circumferential frontiers.

The following more general version of Lemma 4.2 is needed, see Figure 10. The space constraint is obeyed by not allowing the Turing machine head to move right of the right border.

Lemma 8.1 (cf. Wang [Wa61, Wa62], Lewis [Le78]) *Let M be a nondeterministic Turing machine. Then the following 2-dimensional language is recognizable: The set of words $C(x, s, t)$ such that M accepts x within space s and time t.*

From this Lemma 8.1 one can conclude that with two or more unbounded dimensions one gets the recursively enumerable languages RE.

Lemma 8.2 *If $m \geq 2$ then it holds for every $n \geq 1$:* $\mathrm{COL}^{n+mU} = \mathrm{RE}$.

Proof. $\subseteq$: For an input word x put all possible $(n+m)$-dimensional words which are cubes in the first n dimensions and having x as its frontier into a sequence and accept the first time one of them does not contain a forbidden subword. $\supseteq$: Copy via a diagonal the input word x into the first frontier positions of the first unbounded dimension and accept if one of the $C(x, s, t)$ is accepted according to Lemma 8.1 applied to the first two unbounded dimensions and ignoring the other. **q.e.d.**

It remains to study the classes colorable in n bounded and 1 unbounded dimension.

Lemma 8.3 $\mathrm{COL}^{n+1U} = \mathrm{NSPACE}(|x|^n)$.

Proof. $\subseteq$: Let the local language for a language in COL^{n+1U} have k colors. Then suffices to check for an input x all cubes in the first n dimensions having x as a frontier with length of the last dimension being bounded by $k^{(|x|^n)}$ because all colorings of cubes in the first n dimensions with a longer last dimension would contain identical slices in the first n dimensions and could therefore be shortened in the last dimension by dropping the part between the identical slices (including one of the two identical slices) until the length becomes less than $k^{(|x|^n)}$. This gives an NSPACE($|x|^n$) algorithm which nondeterminstically guesses slice by slice all these cubes in the first n dimensions up to this length in the last dimension and looks if one of them does not contain a forbidden subword. $\supseteq$: Like in the proof of Lemma 4.3 one considers the local language for the 2-dimensional recognizable language from Lemma 8.1. But here only the horizontal lines (i.e. the configurations) will be folded, not the vertical (time) dimension. This gives a $n+1$-dimensional local language L such that an input x is accepted by a Turing machine M working with space $c|x|^n$ iff x is the frontier of a cube in the first n dimensions in L and one unbounded dimension which represents the time dimension. **q.e.d.**

Together with the result of Kuroda [Ku64], stating NSPACE($|x|$) = CSL, this implies as the special case $n = 1$ the following characterization of the context sensitive languages CSL.

Theorem 8.4 (Sperber 1985, Latteux & Simplot 1997) $\text{COL}^{1+1U} = \text{NSPACE}(|x|) = \text{CSL}$.

It remains to follow immediately from Lemma 8.3 that the union limit of the classes COL^{n+1U} is NPSPACE, which equals PSPACE according to Savitch's Theorem [Sa70] stating, for polynomial space bounds, $\text{NSPACE}(|x|^n) \subseteq \text{DSPACE}(|x|^{2n})$.

Theorem 8.5 $\text{PSPACE} = \text{NPSPACE} = \bigcup_{n\geq 1} \text{COL}^{n+1U}$.

Conclusions and Open Problems

The characterization of NP given in this paper is the first one in terms of Formal Language Theory, likewise the one for PSPACE (for P there exists already the characterization as the set of languages accepted by alternating two-way multihead finite automata [Ki88]). These characterizations demonstrate an even closer relation of Formal Language Theory and Complexity Theory.

A problem not solved is the separation of level n from level $n-1$ (and not only from level $n/2$) in the colorability hierarchy, likewise in the deterministic colorability hierarchy. As another open problem it would be interesting to see whether for some $k \geq 3$ the complement of the k-slice of the CLIQUE problem (or similar problems) could be shown not to be in the second level COL^2: It is easy to see that for every k the problem k-CLIQUE (i.e. the set of graphs, given via the adjacency matrix, which contain a clique of size k) is in COL^2 but for $k \geq 3$ its complement co-k-CLIQUE does not seem to be contained in COL^2, only COL^k is an obvious upper bound for co-k-CLIQUE. (Proving for every m that COL^m does not contain all co-k-CLIQUE problems would imply co-NP $\not\subseteq$ NP: Assume co-NP $\subseteq$ NP, then the complement of CLIQUE would be in COL^m for some m, and for every k the problem co-k-CLIQUE would be in COL^{m+1}, contradicting the hypothesis). Another question: Can one interpret the characterization of PSPACE in Section 8 as a logical characterization of PSPACE?

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