BEYOND LOGIC

Proceedings of the Conference held in
Cerisy-la-Salle, 22–27 May 2017

Edited by

Jean Fichot
Thomas Piecha

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Contents

Preface . . . . . **vii**

Programme . . . . . **viii**

Abstracts

Pablo Arrighi, Alejandro Díaz-Caro and Benoît Valiron
Towards a quantum Curry-Howard correspondence 3

Jean-Louis Giavitto
Sharing (musical) time between machines and humans: simultaneity, succession and duration in real-time computer-human musical interaction 3

Maël Pégny
Useless old machines? On the interest of analog computation today 5

Myriam Quatrini
From proofs/programs to natural language dialogues modelling 5

Shahid Rahman and Muhammad Iqbal
Unfolding parallel reasoning in Islamic jurisprudence 6

Florian Steinberger
On the normativity of logic 6

Thomas Piecha
Popper's notion of duality and his theory of negations 7

Enrico Moriconi
From Popper's Decomposition of Logical Notion to Lakatos's Decomposition of the Notion of Proof 7

David Binder
Popper and the role of inference rules in logic 8

Jean-Baptiste Joinet
Dynamic of informational time and space, objects and propositions: an Husserlian reading of Church and Gentzen 10

René Gazzari
The Calculus of Natural Calculations 11

Gilles Dowek
Logiques décidables, règle axiome et (non) élimination des coupures 12

Peter Schroeder-Heister
The problem of semantic completeness in proof-theoretic semantics 12

Paolo Pistone
On propositional variables: the atomic and the parametric view 13

Michele Abrusci
Methodological remarks on completeness theorems and incompleteness theorems 13

Mitsuhiro Okada
Some hints to answer the question "What is logic?" 14

Thomas Seiller
Why complexity theorists should care about philosophy 14

Alberto Naibo
Representing inferences and proofs: the case of harmony and conservativity . . 15

Andrzej Indrzejczak
Hypothetical Reasoning in the setting of Sequent Calculi 15

Nils Kürbis
Intuitionist Bilateralism 16

Luiz Carlos Pereira and Ricardo O. Rodríguez
Ecumenism: a new perspective on the relation between logics 16

Mattia Petrolo
Normality beyond logic 17

Kosta Došen
Deductive Systems and Categories in Logic and Beyond 17

Presentations

Pablo Arrighi, Alejandro Díaz-Caro and Benoît Valiron
Towards a quantum Curry-Howard correspondence 21

Jean-Louis Giavitto
Sharing (musical) time between machines and humans: simultaneity, succession and duration in real-time computer-human musical interaction 41

Maël Pégny
Useless old machines? On the interest of analog computation today 75

Myriam Quatrini
From proofs/programs to natural language dialogues modelling 91

Shahid Rahman and Muhammad Iqbal
Unfolding parallel reasoning in Islamic jurisprudence [paper] 109

Shahid Rahman and Muhammad Iqbal
Unfolding parallel reasoning in Islamic jurisprudence [slides] 139

Florian Steinberger
On the normativity of logic 167

Thomas Piecha
Popper's notion of duality and his theory of negations 179

Enrico Moriconi
From Popper's Decomposition of Logical Notion to Lakatos's Decomposition of the Notion of Proof 195

David Binder
Popper and the Role of Inference Rules in Logic 233

Jean-Baptiste Joinet
From the dynamic of informational time and space to objects and propositions: an Husserlian reading of Church and Gentzen 259

René Gazzari
Calculus of Natural Calculations 271

Gilles Dowek
Decidable logics, the axiom rule, and (non) cut-elimination 297

Peter Schroeder-Heister
The problem of semantic completeness in proof-theoretic semantics 315

Paolo Pistone
On propositional variables: the atomic and the parametric view 327

Michele Abrusci
Methodological remarks on completeness theorems and incompleteness theorems 347

Mitsuhiro Okada
A hint to answer the question "What is logic?" — "Proof-formation precedes proposition-rule formation" 367

Thomas Seiller
Why complexity theorists should care about philosophy 381

Alberto Naibo
Representing inferences and proofs: the case of harmony and conservativity . . 395

Andrzej Indrzejczak
Hypothetical Reasoning in the setting of Sequent Calculi 413

Nils Kürbis
Intuitionist Bilateralism: Negations, Implications and some Observations and Problems about Hypotheses 429

Luiz Carlos Pereira and Ricardo O. Rodríguez
Ecumenism: a new perspective on the relation between logics 439

Mattia Petrolo
Normality beyond logic 469

Preface

The project BEYOND LOGIC is devoted to what hypothetical reasoning is all about when we go beyond the realm of "pure" logic into the world where logic is applied. As such extralogical areas we have chosen philosophy of science as an application within philosophy, informatics as an application within the formal sciences, and law as an application within the field of social interaction. The aim of the conference was to allow philosophers, logicians and computer scientists to present their work in connection with these three areas.

The conference took place 22–27 May, 2017 in Cerisy-la-Salle at the *Centre Culturel International de Cerisy*. The proceedings collect abstracts, slides and papers of the presentations given, as well as a contribution from a speaker who was unable to attend.

The conference and its proceedings were supported by the French-German ANR-DFG project "Beyond Logic: Hypothetical Reasoning in Philosophy of Science, Informatics, and Law", ANR-14-FRAL-0002-01, DFG grant Schr 275/17-1.

We would like to thank Jean-Baptiste Joinet and Alberto Naibo who co-organized the conference, and Edith Heurgon and her team of the *Cerisy Centre* who made our conference a very enjoyable event.

Jean Fichot
Thomas Piecha

Back row, left to right: Myriam Quatrini, Michele Abrusci, Paolo Pistone, Nils Kürbis, Mattia Petrolo, T.P., Andrzej Indrzejczak, Peter Schoeder-Heister, N.N., Alberto Naibo, Jean-Louis Giavitto, René Gazzari, Gilles Dowek, Jean-Baptiste Joinet, Thomas Seiller

Front row, left to right: Mitsuhiro Okada, Jean Fichot, Shahid Rahman, Edith Heurgon, Maël Pégny, Florian Steinberger, Luiz Carlos Pereira, Enrico Moriconi, David Binder

Programme

Monday, 22 May

15.30 *Presentation of the programme and of the project* Beyond Logic

Tuesday, 23 May

9.00–10.00 Pablo Arrighi: *Towards a quantum Curry-Howard correspondence*

10.00–11.00 Jean-Louis Giavitto: *Sharing (musical) time between machines and humans: simultaneity, succession and duration in real-time computer-human musical interaction*

11.15–12.15 Maël Pégny: *Useless old machines? On the interest of analog computation today*

14.15–15.15 Myriam Quatrini: *From proofs/programs to natural language dialogues modelling*

15.30–16.30 Shahid Rahman: *Unfolding parallel reasoning in Islamic jurisprudence*

16.30–17.30 Florian Steinberger: *On the normativity of logic*

Wednesday, 24 May

9.00–10.00 Thomas Piecha: *Popper's notion of duality and his theory of negations*

10.00–11.00 Enrico Moriconi: *From Popper's Decomposition of Logical Notion to Lakatos's Decomposition of the Notion of Proof*

11.15–12.15 David Binder: *Popper and the role of inference rules in logic*

14.15–15.15 Jean-Baptiste Joinet: *Dynamic of informational time and space, objects and propositions: an Husserlian reading of Church and Gentzen*

15.30–16.30 René Gazzari: *The Calculus of Natural Calculations*

Thursday, 25 May

9.00–10.00 Gilles Dowek: *Decidable logics, the axiom rule, and (non) cut-elimination*

10.00–11.00 Peter Schroeder-Heister: *The problem of semantic completeness in proof-theoretic semantics*

11.15–12.15 Paolo Pistone: *On propositional variables: the atomic and the parametric view*

14.15–15.15 Michele Abrusci: *Methodological remarks on completeness theorems and incompleteness theorems*

15.30–16.30 Mitsuhiro Okada: *Some hints to answer the question "What is logic?"*

16.30–17.30 Thomas Seiller: *Why complexity theorists should care about philosophy*

Friday, 26 May

9.00–10.00 Alberto Naibo: *Representing inferences and proofs: the case of harmony and conservativity*

10.00–11.00 Andrzej Indrzejczak: *Hypothetical Reasoning in the setting of Sequent Calculi*

14.15–15.15 Nils Kürbis: *Intuitionist Bilateralism*

15.30–16.30 Luiz Carlos Pereira: *Ecumenism: a new perspective on the relation between logics*

16.30–17.30 Mattia Petrolo: *Normality beyond logic*

Saturday, 27 May

9.00–11.30 *Final discussion*

Abstracts

Towards a quantum Curry-Howard correspondence

Pablo Arrighi, Alejandro Díaz-Caro and Benoît Valiron
Aix-Marseille Université · Universidad Nacional de Quilmes · Université Paris-Sud

The linear-algebraic λ-calculus extends the λ-calculus by allowing for arbitrary linear combinations of terms. We introduce a type system that accounts for this: it is able to statically describe the linear combinations of terms that will be obtained when reducing the programs. This gives rise to an original type theory where types, in the same way as terms, can be superposed into linear combinations. We prove that the resulting typed λ-calculus is strongly normalizing and features weak subject reduction. We show how to naturally encode matrices and vectors in this typed calculus. This is a step towards the program of having a fuzzy / quantum / quantified logic to arise from a Curry-Howard correspondence.

This presentation is based on https://arxiv.org/abs/1308.1138, published in *Information and Computation* 254(1): 105–139 (2017).

Sharing (musical) time between machines and humans: simultaneity, succession and duration in real-time computer-human musical interaction

Jean-Louis Giavitto
CNRS IRCAM UPMC

The functional approach to AI has focused on the ability to provide a computer the cognitive capabilities usually attributed to humans: translating a text, recognizing an object in an image, playing chess, planning a route, *etc*. Even if perception and actions have been largely considered, cognitive capabilities related to the apprehension and the organization of time have been less studied. Music is a prime area to address these issues. In all written forms of music, the act of music composition is a choreography of events and expectations in time to allow sophisticated continuous interactions among musicians. This powerful effect is the result of intrinsic combination of strong language formalisms (for authoring music) and performance mechanisms that allow synchrony, real-time coordination of actions and robustness of expected results in ensemble music.

Bringing such capabilities to computers and providing them with the ability to take part in musical interactions with human musicians is an excellent workbench to investigate and to test, from an experimental viewpoint, several temporal notions.

This presentation focuses on the various temporal notions put at work in the *Antescofo* system. This system couples a *listening module* and a *domain specific programming language*. It is used by music composers, and more generally by interactive multimedia designers, to specify and to implement *augmented scores*, *i.e.*, temporal scenarios where electronic musical processes are computed and scheduled in tight interaction with a live musician performance[1].

Interaction scenarios are expressed at a symbolic level through the specification of musical

[1] Used in the mixed music piece produced at IRCAM, *Antescofo* has gained a wide audience attracting composers in contemporary music such as Pierre Boulez, Philippe Manoury, Marco Stroppa or Emmanuel Nunes which have used the system for the creation of new musical mixed pieces and for their performances by the Los Angeles Philharmonic Orchestra, the Berliner Philharmoniker, the Orchestre de Radio France, *etc.* Videos of actual performances and additional informations are available on the project web site http://repmus.ircam.fr/antescofo.

time in the score (musical events like notes and beats in relative tempi) and the management of the physical time of the performance (with relationships like succession, delay, duration, velocity of the occurrence of the events on stage). During the performance, human performers "implement" the instrumental part of the augmented score, while the language runtime evaluates the electronic part with the help of the information provided by a listening module, to control and synchronize the electronic actions with the musical environment.

These two main phases of the usual workflow of written music, *composition* and *performance*, relies on two different notion of time reminiscent of the distinction pointed by John McTaggart [7]: the *B-series* of the "deferred time" specified in the score during the authoring phase and the *A-series* of the real-time relationships occurring on stage during the performance when a score is interpreted.

Succession and *simultaneity* are the two forms of the perception of the interval in music [9]. It is also two of the three relationships classically used to analyze time and the basis of *event-driven*, or reactive, systems in computer science. But *Antescofo* is also a *timed* system [4] tacking into account the relationships of delay and duration in the score. As acknowledged by many philosophers [8], duration is irreducible to succession and simultaneity: it cannot be abstracted by two instantaneous events starting and ending the duration. This leads to the notions of *striated* and *smooth time* exemplified by Pierre Boulez in music [3] but also to the notion of *inner tempo*, drawing near to the concept of duration developed by Henry Bergson [2, 9].

The timing relationships (duration, succession, simultaneity) between events denoted in the score are *relative* (to each others), *virtual* (the timing relationships expressed at the level of the score will be instantiated during the performance) and *undetermined* (several performances comply with the same score). During live performance of a music score, musicians instantiate the high-level processes denoted in the score by musical gestures. At this point, the durations and delays become physical time (measurable in second). Nevertheless, events with the same relative duration in a score (and in different positions) do not necessarily lead to the same duration during the execution and vice versa. Their value depends highly on the performance, individual performers and musical interpretation strategies such as stylistic features that are neither determined nor easily formalizable: the tempi, accelerations, rubato, *etc.* are personal choices that will vary in every performance. The notion of internal tempo may appear problematic for at least two reasons: first, it leads directly to the questionable notion of "speed of time" and secondly, it calls for an actual measurement in order to relate continuously the different and subjective time frames. We will explain how *Antescofo* faces these two problems in an effective way.

Time is a resource and programmers organize it in programs through the figures of succession, simultaneity and duration. This analytical grid remains however limited because computers increasingly interact with us and our experienced time. Our perception of the duration, our sense of the passing of time, must be taken into account to achieve a fluid and seamless interaction between human and machines. *Antescofo* achievements show that this goal is not out of reach. But then, the computer scientist has to confront other dimensions of time as movement, memory, expectation, passage, anticipation, emergence . . .

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Useless old machines? On the interest of analog computation today

Maël Pégny

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In this programmatic presentation, I explore the interest of the history of analog computation for an integrated History and Philosophy of Science approach. In particular, I examine the 1940–1970 debate on the comparative merits of analog and digital computers, with a particular question in mind: what role did computation theory play in this debate? By reviewing the current historical literature, I show how tricky it will be to actually solve that question, because computer scientists of that era were trying to formulate computability and complexity arguments without having all the necessary theoretical tools. Understanding the history of analog computation and its demise will demand an integration of considerations coming from the foundations of computability and computational complexity, and the use of analog computers in simulation.

From proofs/programs to natural language dialogues modelling

Myriam Quatrini

Université de la Méditerranée

The logical theory due to J.-Y. Girard, called Ludics, which is also presented as a theory of interaction, is a relevant frame for modelling natural language dialogues and especially argumentative dialogues. We will introduce this theory, focusing on the features which enable a proof theoretical modelling of controversies. We will discuss the potential applications by illustrating the modelling on an example of legal debate.

Unfolding parallel reasoning in Islamic jurisprudence

Shahid Rahman and Muhammad Iqbal
Université Lille, CNRS, UMR 8163

One of the epistemological results emerging from this initial study is that the different forms of *co-relational inference*, known in the Islamic jurisprudence as *qiyās*, represent an innovative and sophisticated form of reasoning that not only provides new epistemological insights into legal reasoning in general but also furnishes a fine-grained pattern for *parallel reasoning* which can be deployed in a wide range of problem-solving contexts and does not seem to reduce to the standard forms of analogical argumentation studied in contemporary philosophy of science. In the present paper we will only discuss the case of so-called *co-relational inferences of the occasioning factor*.

On the normativity of logic

Florian Steinberger
Birkbeck University of London

Logic, the tradition has it, is normative for reasoning. Famously, the tradition was challenged by Gilbert Harman who argued that there is no straightforward connection between logical consequence and norms of reasoning. A number of authors (including John MacFarlane and Hartry Field) have sought to rehabilitate the traditional view of the normative status of logic against Harman. In this paper, I argue that the debate as a whole is marred by a failure of the disputing parties to distinguish three different types of normative assessment, and hence three distinct ways in which the question of the normativity of logic might be understood. Logical principles might be thought to provide first-personal directives to the reasoning agent, they might be thought to serve as third-personal evaluative standards, or they might underwrite our third-personal appraisals of others whereby we attribute praise and blame. I characterize the three normative functions in general terms. I then show how a failure to appreciate this three-fold distinction has impeded progress since it has led the participants in the debate to talk past one another. Moreover, I show how the distinction paves the way for a more fruitful engagement with the issue.

Popper's notion of duality and his theory of negations

Thomas Piecha
University of Tübingen

We discuss Karl Popper's theory of deductive logic that he developed in the late 1940s, focusing on his treatment of different kinds of negation. In his approach, logic is a meta-linguistic theory of deducibility relations that are based on certain purely structural rules. Logical constants are then characterized in terms of deducibility relations. Characterizations of this kind are also called inferential definitions by Popper.

An explanation of Popper's treatment of negation needs to take his conception of duality into account. We analyze his conception and give a formal definition of duality that agrees with Popper's use of the notion in propositional logic, extending into the treatment of several kinds of negation, as well as into the domain of modal logic. This wide applicability is only possible because his notion of duality does not depend on truth functions but is based on deducibility, and it illustrates the importance of Popper's notion of duality as a structuring principle in various areas of logic.

In this talk we present some of his ideas and results, and we will show how they correspond to later developments in logic.

From Popper's Decomposition of Logical Notion to Lakatos's Decomposition of the Notion of Proof

Enrico Moriconi
University of Pisa

Popper's logical enquiries of the late 40s are experiencing a renewed interest from many scholars, focusing on different subjects. His intention was to reverse Tarski's order of priority – in his 1936 paper "On the notion of logically following" – taking the notion of "derivability", or "deducibility", or "logical consequence", as primitive, and trying to show that those signs are logical or formative which can be defined with the help of that primitive concept. This approach considers logic a metalinguistic enterprise, which can be applied to any language in which we can identify statements. Devising logical notions, consequently, becomes a matter strictly linked to discussing principles of rational discussion. This is the origin of (e.g.) the detection of different notions of negation, and of the study of their possible coexistence, on the one hand; on the other hand, of the necessity, stressed by Lakatos, to supplement Popper's approach by paying due attention to the fact that the validity of an intuitive inference depends also on the translation we adopt to translate inferences from ordinary language into the logical language.

Popper and the role of inference rules in logic

David Binder
University of Tübingen

K. R. Popper published in the 1940's a series of articles on deductive logic [4–9]. A discussion of his philosophical ideas, the construction of his logical theory and its problems can be found in [10, 11, 1]. A critical edition of his articles is currently being prepared by Peter Schroeder-Heister, Thomas Piecha and David Binder. This new edition will contain revised versions of the published articles, selected correspondence, contemporary reviews, unpublished material and individual introductions for the articles. This is a short outline of Popper's involvement in formal logic.

Vienna. Popper probably first got in contact with logic and the foundations of mathematics by enrolling in a course of Hans Hahn in 1922 in which *Principia Mathematica* was part of the curriculum. He soon got in contact with members of the Vienna circle, among them Gödel and Carnap, and in 1934 he met Tarski who had a profound influence on Popper's views on logic. We do know about the impression that Tarski's analysis of truth made on him, and in a letter written in 1943 he calls himself a "disciple of Tarski" and mentions that he helped Tarski in the translation of "Über den Begriff der logischen Folgerung" into German.[2] There is little written testimony about his views on formal logic during the Vienna years, due to the lack of publications on formal logic and the poor archival situation regarding the time before his departure to New Zealand.

Christchurch. In 1937 Popper had to flee from Austria and found employment as a lecturer at the University of Canterbury in Christchurch, New Zealand. Part of his teaching duties was a course in logic for philosophers. He was not content with the available logic textbooks suitable for philosophers and planned to write a logic textbook in ∼1937/38. The three people whom Popper discussed logical problems with during his time in Christchurch are, as far as we can see, John Findlay, Henry George Forder and Rudolf Carnap. The evidence for Findlay, who taught at the University of Otago at the time, is rather slim and rests on (1) handwritten remarks on a paper that is likely to be an early version of [2], and (2) the fact that Popper discussed that article with Paul Bernays in 1946. With Forder, a professor of mathematics at Auckland University College, on the other hand, the situation is clear since there is an extensive correspondence from February 1943 to July 1945 (23 letters in total). They discuss university politics but also problems in the philosophy of mathematics, logic and quantum physics. It is in these letters that Popper mentions for the fist time his conception of logic as a "meta-propositional calculus"; a particular interpretation of the inequations of boolean algebra. The contact with Carnap is through exchange of letters, averaging about three letters per year. Every time Carnap finishes another book, *Introduction to Semantics* in 1942 and *Formalization of Logic* in 1943, he sends a copy to Popper who replies with questions and sometimes long sheets of comments. Carnap is certainly, together with Tarski, the one person who inspired most of the logical investigations Popper undertook during that time. Remarks in letters and published and unpublished articles show that it is through reading Carnap that he found the problems that he tried to solve.

In 1943 Popper writes a series of articles on boolean algebra, at least one of which he intended to publish in the Journal of Symbolic Logic.[3] They are called "Extensionality in a Rudimentary Boolean Algebra", "An Elementary Problem of Boolean Algebra", "Completeness and Extensionality of a Rudimentary Boolean Algebra", "Postulates for Boolean Algebra" and "Simply Independent Postulates for Boolean Algebra". Forder supported Popper by proofreading his typoscripts and by lending him articles that were not available in Christchurch, most importantly Huntington's [3] on which much of the development in Popper's articles is based.

London. In 1946 Popper gets a position at the London School of Economics and moves back to Europe. For reasons that are still opaque, he met with Bernays in Zürich in December

[2] Letter from Popper to H. G. Forder, May 7th 1943. Karl-Popper-Sammlung (KPS) 296, 15.

[3] In the LaTeX-version that we work with, these articles take up about 100 pages. They are from KPS 12,3; 12,4; 12,5; 16,13.

1946. During discussions, Bernays proposed to publish an article together with Popper who eagerly accepted and set himself to work in the first months of 1947. He finished the article, entitled "On Systems of Rules of Inference" by March 3rd and sent a copy of the manuscript to Bernays[4]. The reason why the article never got published is unclear, but it seems that Bernays was not in full agreement with Popper regarding some of the arguments of the article. The content of this article is already quite close to the content of [4] and [5], but contains significant material that was omitted in those later articles. Among other things, it contains an explicit comparison with Tarski's system [12] and a criterion for the "purity" of inference rules.[5]

Popper wrote on the distinction between derivation and demonstration in three unpublished drafts, written some time between the completion of "On Systems of Rules of Inference" and the writing of [5]. One of them is untitled; the other two are called "Derivation and Demonstration in Propositional and Functional Logic" and "The Propositional and Functional Logic of Derivation and Demonstration"[6]. They contain material which would later be incorporated in section 8: "Derivation and Demonstration", of [5]. He draws the distinction between demonstrational logic, exemplified by the systems of Russell–Whitehead, Hilbert–Ackermann and Hilbert–Bernays, and derivational logic, to which only Gentzen has come close with his system of natural deduction. In these drafts Popper formulates an idea much more radically than in his published articles: the logic of derivation should be primary and the logic of demonstration should be introduced via a definition of demonstrability as a second step.

The reception of Popper's articles was rather negative, partly due to the fact that they contained errors, pointed out in the reviews, and partly due to the fact that some passages could easily lead to misunderstanding[7]. But not all reception was negative; William Kneale and Brouwer responded positively. Brouwer had presented three of Popper's articles to the Koninklijke Nederlandse Akademie van Wetenschappen and spoke very warmly about Popper's articles on logic. The overall cold reception greatly discouraged Popper from pursuing further publications in logic. Regrettably so, since they do contain interesting philosophical ideas that are not affected by the technical blunders. Even though Popper did not publish anything substantial on formal logic for the rest of his life, he continued to work on logical problems as diverse as the philosophy of logic of Boole, the quantum logic of von Neumann and, especially around 1950, on the different concepts of implication.

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[5] This definition of purity appears as definition D8.1 in [5], where it is discussed rather elliptically.

[6] KPS 36,20; 36,21.

[7] And they did confuse McKinsey, in his review, and Quine, in personal correspondence.

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Dynamic of informational time and space, objects and propositions: an Husserlian reading of Church and Gentzen

Jean-Baptiste Joinet

Université Jean Moulin Lyon 3

The thirties of last century are in one hand years in which the Proof Theory (Hilbert, Gentzen) re-centered Logic around the spatial and temporal inscription of reasoning (this with several respects: temporality and spatiality of the representation of argumentations, of heuristic, of analytisation of proofs), in the other hands years during which a new science emerged, the theory of the transformation of information a.k.a Informatics or Computing theory (Church, Turing). In my talk, I will evaluate in which sense one can read those scientific programmes (Computing Theory and Proof Theory) and their dialogue around a common question (Sense in time and space) as a relevant contribution to the Husserlian project to complete modern Formal Logic by a Transcendental Logic.

The Calculus of Natural Calculations

René Gazzari
University of Tübingen

Gentzen discusses two formal calculi in his renowned paper "Untersuchungen über das logische Schließen". Before defining the Sequent Calculus, a technically convenient tool for proving his Hauptsatz, Gentzen first introduces the so called Calculus of Natural Deduction. Gentzen's reason to introduce Natural Deduction is its close relationship to informal mathematical reasoning. In Gentzen's own words:

> We wish to set up a formalism that reflects as accurately as possible the actual logical reasoning involved in mathematical proofs. ([1], p. 74)
>
> The calculus lends itself in particular to the formalization of mathematical proofs. ([1], p. 80)

Such natural calculi minimise the inevitable gap between informal argumentations and their formalisations. This minimality may justify to carry over the notions defined with respect to formal derivations and the results obtained from their investigations to the realm of informal argumentations. This way, we may obtain formally justified answers to philosophical questions about the properties of informal proofs.

With respect to argumentations involving only statements, Gentzen's Calculus of Natural Deduction is, indeed, pretty close to the argumentations found in mathematics. But mathematicians do not only argue; they also calculate within their proofs. Usually, such calculations are formalised with the help of axioms and rules dealing with equality statements. From a technical point of view, this approach is perfect. But it is not a very natural approach.

In our talk, we present an alternative and more natural approach of formalising informal calculations: we extend the Calculus of Natural Deduction by some term inference rules (which may be understood as elimination and introduction rules for equality statements). These term inference rules allow the syntactical manipulation of the terms of a formal language (within a derivation) in the very same way as the mathematicians calculate with informal objects (as numbers and sets) in their informal proofs. After introducing the Calculus of Natural Calculations we provide some basic proof theoretic results about this calculus, in particular, we briefly discuss its completeness and the problem of normalisation. Finally, we consider some further extensions of this calculus aiming towards a uniform formal frame for the natural formalisation of informal mathematical reasoning in all of its aspects.

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Logiques décidables, règle axiome et (non) élimination des coupures

Gilles Dowek
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(travail commun avec Ying Jiang)

Nous proposons une modification de la notion de coupure, telle que la propriété d'élimination des coupures implique la décidabilité de la logique. Pour les logiques indécidables (telle la logique des prédicats), la volonté d'éliminer les coupures "autant que possible" nous mène de la déduction naturelle au calcul des séquents de Gentzen, du calcul des séquents de Gentzen à celui de Kleene et de celui de Kleene à celui de Vorob'ev–Hudelmaier–Dyckhoff–Negri.

The problem of semantic completeness in proof-theoretic semantics

Peter Schroeder-Heister
University of Tübingen

Various options for the definition of proof-theoretic validity and their respective problems are discussed. It is shown that intuitionistic logic is not complete with respect to any of these options. Only a weak version of completeness can be established: Every valid rule which has the form of an elimination inference, is derivable in intuitionistic logic. If Prawitz's completeness conjecture of 1971 is understood in the sense of this weak version of completeness, then it is correct, otherwise it cannot be upheld.

On propositional variables: the atomic and the parametric view

Paolo Pistone
Università degli Studi Roma Tre

What consequences are we entitled to draw from a proof that $p \to p$ (where p is a propositional variable)? The standard answer, coming from model-theory, is that every interpretation $[p]$ of p will obey the truth-table of implication. However, when considering proof-theoretic interpretations, quite different answers might be found in (quite different) literatures. In Prawitz's and Dummett's proof-theoretical interpretation a proof of $p \to p$ warrants that a canonical argument for the interpreted statement can be found for a special class of interpretations only, i.e. those associating with propositional variables so-called atomic bases, i.e. sets of atomic inferential rules. Indeed, in this approach, if every interpretation were admitted, a vicious circle would result in the definition. This apparent limitation is grounded in the view (we call it the atomic view) that proof conditions must be explained in a hierarchical way, with simple (atomic) propositions grounding complex ones. Such limitations do not appear in the proof-theoretic interpretations of polymorphism (i.e. of second order quantification): by defining the interpretation in a relational frame one can express the fact that a variable p figures as a parameter in the proof, and hence that it can be replaced by any interpretation, yielding a canonical argument in a uniform way. This parametric view does not demand for a hierarchical explanation of logical consequence, but takes propositional variables as free parameters in the proofs. I will argue that the latter view, far from concerning second order logic only, provides a perspicuous picture of proofs in propositional logic. Indeed, parametricity expresses a naturality condition (in the sense of category theory) for proofs which, on the one side, provides significant information concerning the identity of proofs and, on the other side, allows to characterize correct proofs by purely semantical means, yielding several completeness results for intuitionistic propositional logic, a problematic issue in the atomic view.

Methodological remarks on completeness theorems and incompleteness theorems

Michele Abrusci
Università degli Studi Roma Tre

Completeness and incompleteness theorem as answers to a general philosophical question applied to specific formats.

Some hints to answer the question "What is logic?"

Mitsuhiro Okada
Keio University

In my opinion, "What is logic?" is an oldest and newest question in logic, and it would be also an important question, among many others, for us to go 'beyond logic'. In this talk, I would challenge to clarify why this question is not so easy to answer. In the course of the discussion we take a slight look at some of our recent works and results on some topics of logical inferences, including those on diagrammatic logical reasoning, behavioral experimental study of logical inferences, forcing-Fitting model construction for informatics applications, consideration of the relationship between the traditional logics and linear logic, consideration of relationship between decision making and logical inferences, then, we also touch some of our recent works on re-consideration of logic with some classical works in the early 20th century, such as those of Husserl, Peirce, Hilbert, Wittgenstein.

Why complexity theorists should care about philosophy

Thomas Seiller
University of Copenhagen

Theoretical computer science was somehow born almost a hundred years ago when logicians asked themselves the question: "What is a computable function?". This question, purely theoretical, was answered before the first computer was designed, in the form of the Church-Turing thesis: a computable function is one that can be defined in one of the following equivalent models: recursive functions, Turing machines, or lambda-calculus. The apparition of actual computing devices however made it clear from the start that another question made more sense for practical purposes, namely: "What is an *efficiently* computable function?". This question was tackled by three different works in the span of a single year, marking the birth of computational complexity. Nowadays, computational complexity is an established field: many methods and results have been obtained, and the number of complexity classes grows every year. However, a number of basic open problems remain unanswered, in particular concerning classification of complexity classes. Even worse than that, a number of results – called barriers – show that no known method will succeed in producing a new separation result, i.e. show that two classes (e.g. P and NP, or L and P) are disjoint. From a purely theoretical point of view, this lack of methods might be explained by a historic tradition of viewing programs as functions. Once this misconception is identified, it points to a lack of adequate foundations for the theory of computation. Fortunately, some recent technical developments may provide a solution to this problem.

Representing inferences and proofs: the case of harmony and conservativity

Alberto Naibo
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Traditionally, proof-theoretic semantics focuses on the study of logical theories from a general point of view, rather than on specific mathematical theories. Yet, when mathematical theories are analyzed, they seem to behave quite differently from purely logical theories. A well-known example has been given by Prawitz (1994): adding a set of inferentially harmonious rules to arithmetic does not always guarantee to obtain a theory which is a conservative extension of arithmetic itself. This means that outside logic the nice correspondence between harmony and conservativity (advocated for example by Dummett 1991) seems to be broken. However, as it has been pointed out by Sundholm (1998), this is not necessarily a consequence due to the passage from a logical setting to a mathematical one. It could depend also on the way in which proofs are represented. In particular, if proofs are seen as composed by rules which act on judgments involving proof-objects, rather than on rules which act on propositions, then the aforementioned correspondence can in fact be re-established. An analysis of this phenomenon is proposed. In particular, two different ways of representing proof-objects are taken into consideration: the Church-style presentation and the Curry-style presentation. It is then shown that a crucial difference can be obtained by choosing the first rather than the second.

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Hypothetical Reasoning in the setting of Sequent Calculi

Andrzej Indrzejczak
University of Łódź

We consider different ways of representing hypothetical reasoning in the framework of sequent calculi (SC). The basic approaches, not necessarily dependent on the features of specific proof systems, were already considered by Schroeder-Heister. It seems that a deeper analysis of special features of SC leads to further distinctions. We are going to focus on various factors that have an impact on the ways in which hypothetical reasoning may be formally conducted in SC. The most important ones are connected with different interpretations of the notion of sequent and with different kinds of deducibility relation induced by SC. In particular, we will show that hypotheses may be expressed not only by special sequents but also by special rules. This issue is strongly connected with the problem of possible ways of formalizing theories in the framework of SC and with expressing definitions by means of specific rules. We will state a general theorem on possible ways of transforming sequents into rules and focus on the problem of their good proof-theoretical behaviour.

Intuitionist Bilateralism

Nils Kürbis
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There is widespread agreement that in a proof-theoretic approach to the meanings of the logical constants, where the rules governing them are required to be in harmony, intuitionist logic is the correct logic if the meanings of sentences are specified in terms of the conditions for their correct assertibility, while classical logic is the correct logic on a bilaterlist account, where meanings are in addition specified in terms of deniability conditions. There are two arguments to this effect, a formal one, due to Ian Rumfitt, and an informal one, due to Huw Price. I'll present an intuitionist bilateral logic that shows Rumfitt's claim that on a bilateralist account only the rules for classical logic are harmonious. The rules of the intuitionist system satisfy all of Rumfitt's requirements on bilateral logics, in particular, they are in harmony. I will then show how this system of intuitionist bilateral logic matches with the principles Price adduces for his argument. This exhibits clearly where Price's argument fails and that his conclusion that a bilateralist account must validate double negation elimination does not follow. It even looks as if intuitionist logic is preferable for Price's account.

Ecumenism: a new perspective on the relation between logics

Luiz Carlos Pereira
PUC-Rio
(joint work with Ricardo Oscar Rodríguez, Universidad de Buenos Aires)

Eclecticism is not a position available to an intuitionist mathematician/logician of "faith". The classical mathematician/logician may even consider the intuitionist position quite interesting, since constructive proofs, although usually longer, are more informative than indirect classical proofs, since they have an algorithmic nature and satisfy interesting informative properties such as the disjunction property and the property of the existential quantifier. To the intuitionist mathematician/logician however, there seems to be no alternative but to revise and revoke the universal validity of certain classical principles of reasoning; for the intuitionist, mathematics must be constructed exclusively on constructively valid forms of argument. From the point of view of the classical mathematician, the intuitionist proposition, if taken seriously, would imply a mutilation of the mathematical corpus; for the intuitionist it is simply the only correct way of doing mathematics. (We cannot lose what we do not have!) In 2015 Dag Prawitz proposed the idea of an ecumenical system, a codification where the classical and the intuitionist could coexist "in peace". The main idea behind this codification is that the classical and the intuitionist share the constants for conjunction and negation, but each have their own disjunction and implication. Similar ideas were present in Dowek (2015) and Krauss (1992), but without Prawitz' philosophical motivations. The aims of the present paper are: (1) to investigate the proof theory for Prawitz' Ecumenical system, (2) to propose a truth-theoretical semantics for which Prawitz' system is sound and complete, (3) to compare Prawitz' system with other ecumenical approaches, and (4) to propose a generalization of the ecumenical idea.

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Normality beyond logic

Mattia Petrolo
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(joint work with Paolo Pistone)

How should one characterize reductions and their associated normal forms once one steps outside the realm of "standard" systems of natural deduction? On which criteria it is possible to rely in order to provide such a characterization? We address these questions in a framework in which they become particularly pressing, namely the proof-theoretic analysis of paradoxes.

Deductive Systems and Categories in Logic and Beyond

Kosta Došen
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[unable to attend]

In the investigation of deductions, conceived as hypothetical proofs, one reaches naturally the notion of deductive system, which is a special kind of directed graph, with loops and possibly multiple arrows between objects, i.e. vertices. Deductive systems have identity arrows and the arrows are closed under composition. Arrows correspond to deductions and objects to propositions. Categories are deductive systems that satisfy the associative law for composition and the unit laws concerning composition with identity arrows. These notions are very general and are not rectricted to logic. They occur throughout mathematics and mathematizable areas of thought.

The introduction of the notions of deductive system and category can be motivated by identifying an object with the set of arrows having it as source or, alternatively, as target. Proof-theoretically, this means identifying a proposition with the set of deductions having it as conclusion or, alternatively, as premise. In intuitionism on finds something similar when a proposition is identified with the set a proofs of it, but usually without taking care that these proofs be hypothetical. The introduction of the notion of deductive system is motivated by a generalization of an elementary aspect of Stone's Representation of distributive lattices in sets, involving preorders, while the introduction of the notion of category is motivated by a generalization of the representation of monoids involved in Cayley's representation of groups (see [1] and [2], Section 1.9).

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Presentations

Towards a Quantum Curry-Howard correspondence

Pablo Arrighi
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BeyondLogic
Cerisy-la-Salle, May 2017
* Slides' and principal author

A proof-as-programs approach to quantum logic

Motivation

Curry-Howard correspondence

Intuitionistic logics $\Longleftrightarrow$ Typed λ-calculus

hypotheses	free variables
implication elimination (modus ponens)	application
implication introduction	abstraction

A proof is a program
(the formula it proves is a type for the program)

Goal: To find a **quantum** Curry-Howard correspondence

Between what?

- A *quantum* λ-calculus (quantum control/quantum data)
- Any logic, even if we need to define it!

A *physics free* introduction to quantum computing

Quantum vs. Classic, side by side

- **Classic computing Bit**: $0, 1$
  Quantum computing Qubit: Normalised vector from

$$\mathbb{C}^2 = \mathrm{Span}\{|0\rangle, |1\rangle\} = \{\alpha|0\rangle + \beta|1\rangle, \alpha, \beta \in \mathbb{C}\}$$

$$\text{where } |0\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \text{ and } |1\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

- **CC 2-bits system**: one of $00, 01, 10, 00$
  QC 2-qubits system: $\mathbb{C}^2 \otimes \mathbb{C}^2 = \mathrm{Span}\{|00\rangle, |01\rangle, |10\rangle, |11\rangle\}$
  where $|xy\rangle = |x\rangle \otimes |y\rangle$
- **CC Reading data**: No problem
  QC Measuring the system: Measurement of $\alpha|0\rangle + \beta|1\rangle$ returns a **bit**, and the system collapses to
  - $|0\rangle$ if 0 was measured, with probability $p_0 = |\alpha|^2$
  - $|1\rangle$ if 1 was measured, with probability $p_1 = |\beta|^2$.

A *physics free* introduction to quantum computing

Quantum vs. Classic, side by side (cont.)

- **Classic computing computation**: Logic gates {NOT, AND, etc...}
  Quantum computing operations: Unitary matrices ($U^\dagger U = I$)

Example:
$$H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

$$H|0\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$$

$$H|1\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$$

We can also combine them:
$(H \otimes I)$ = Apply H to the first qubit, and identity to the second

- **No-cloning theorem** "There is no universal cloning machine"
  i.e. $\nexists U$ s.t. $U|\phi\psi\rangle = |\psi\psi\rangle$ for an arbitrary qubit $|\psi\rangle$
- **Entanglement** n-qubit $\neq \bigotimes_i |\psi_i\rangle$ e.g. $\frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{2}}|11\rangle$
  Consequence: Measuring the first qubit... both collapse!

Untyped algebraic extensions to λ-calculus

Two origins:

- *Alg* [Vaux'09] (from Linear Logic)
- *Lineal* [Arrighi,Dowek'08] (for Quantum computing)

Equivalent formalisms [Díaz-Caro,Perdrix,Tasson,Valiron'10]

$$\mathbf{t},\mathbf{r} ::= \mathbf{v} \mid \mathbf{tr} \mid \mathbf{t}+\mathbf{r} \mid \alpha.\mathbf{t} \mid 0 \quad \alpha \in (\mathcal{S},+,\times), \text{ a ring}$$
$$\mathbf{v} ::= x \mid \lambda x.\mathbf{t}$$

$$\beta\text{-reduction: } (\lambda x.\mathbf{t})\mathbf{v} \to \mathbf{t}[x := \mathbf{v}]$$

"Algebraic" reductions:

$$\alpha.\mathbf{t} + \beta.\mathbf{t} \to (\alpha+\beta).\mathbf{t},$$
$$\alpha.\beta.\mathbf{t} \to (\alpha\times\beta).\mathbf{t},$$
$$\mathbf{t}(\mathbf{r}_1+\mathbf{r}_2) \to \mathbf{tr}_1 + \mathbf{tr}_2,$$
$$(\mathbf{t}_1+\mathbf{t}_2)\mathbf{r} \to \mathbf{t}_1\mathbf{r} + \mathbf{t}_2\mathbf{r},$$
$$\ldots$$

(oriented version of the axioms of vectorial spaces)

Vectorial space of values

$\mathcal{B} = \{$ vars. and abs. $\}$

Space of values $::= \mathrm{Span}(\mathcal{B})$

A minimal language. . . and its semantics.

Higher-order computation

$\mathbf{t} ::= \quad x \mid \lambda x.\mathbf{t} \mid (\mathbf{t}\,\mathbf{t}) \quad \mid$

$\lambda x.\mathbf{t}\,\mathbf{b} \longrightarrow \mathbf{t}[\mathbf{b}/x]\,(*) \qquad (B)$

(*) $\mathbf{b}$ an abstraction or a variable.

Linear algebra.

$\mathbf{t}+\mathbf{t} \mid \alpha.\mathbf{t} \mid \mathbf{0}$

$1.\mathbf{u} \longrightarrow \mathbf{u},\ 0.\mathbf{u} \longrightarrow \mathbf{0},\ \alpha.\mathbf{0} \longrightarrow \mathbf{0},$
$\mathbf{u}+\mathbf{0} \longrightarrow \mathbf{u},\ \alpha.(\beta.\mathbf{u}) \longrightarrow \alpha\times\beta.\mathbf{u},$
$\alpha.(\mathbf{u}+\mathbf{v}) \longrightarrow \alpha.\mathbf{u} + \alpha.\mathbf{v} \qquad (E)$

$\mathbf{u}+\mathbf{u} \longrightarrow (1+1).\mathbf{u}$
$\alpha.\mathbf{u}+\mathbf{u} \longrightarrow (\alpha+1).\mathbf{u}$
$\alpha.\mathbf{u}+\beta.\mathbf{u} \longrightarrow (\alpha+\beta).\mathbf{u} \qquad (F)$

$\mathbf{t}\,(\mathbf{u}+\mathbf{v}) \longrightarrow (\mathbf{t}\,\mathbf{u}) + (\mathbf{t}\,\mathbf{v})$
$(\mathbf{u}+\mathbf{v})\,\mathbf{t} \longrightarrow (\mathbf{u}\,\mathbf{t}) + (\mathbf{v}\,\mathbf{t})$
$\mathbf{t}\,\alpha.\mathbf{u} \longrightarrow \alpha.(\mathbf{t}\,\mathbf{u})$
$\alpha.\mathbf{u}\,\mathbf{t} \longrightarrow \alpha.(\mathbf{u}\,\mathbf{t})$

Linearity

. . . is a matter for confluence!

$$\begin{array}{l} \lambda x.(x\,x)(u+v) \not\longrightarrow^* u\,u+u\,v+v\,u+v\,v \\ \quad\downarrow \\ \lambda x.(x\,x)u+\lambda x.(x\,x)v \longrightarrow u\,u+v\,v \end{array}$$

The restriction forbids the above banch.
Is it now confluent?

Higher-order

Fixed points, quantum control, black-box algos. . .
Higher-order usually means

$$\lambda x.t\,\lambda y.u \longrightarrow t[\lambda y.u/x]$$

Hence base vectors are
- abstractions (i.e. terms of the form $\lambda y.u$);
- variables (by some kind of recurrence).

Machine description interpretation, LISP quotes. . .

Encoding booleans

$$\textbf{true} \equiv \lambda x.\lambda y.x$$
$$\textbf{false} \equiv \lambda x.\lambda y.y$$

Encoding the Not gate

$$\textbf{Not} \equiv \lambda y.\Big((y * \textbf{false}) * \textbf{true}\Big)$$

Encoding the Phase gate

$$\textbf{Phase} \equiv \lambda y.\Big(\big((y * \lambda x.(e^{i\frac{\pi}{4}}.\textbf{true})) * \lambda x.\textbf{false}\big) * \_\Big)$$

where _ stands for dead code.

Running the Phase gate

Phase $*$ **true** yields

$\lambda y.\Big(\big((y * \lambda x.(e^{i\frac{\pi}{4}}.\textbf{true})) * \lambda x.\textbf{false}\big) * \_\Big) * \textbf{true}$

$\big((\textbf{true} * \lambda x.(e^{i\frac{\pi}{4}}.\textbf{true})) * \lambda x.\textbf{false}\big) * \_$

$\big((\lambda x.\lambda y.x) * \lambda x.(e^{i\frac{\pi}{4}}.\textbf{true})) * \lambda x.\textbf{false}\big) * \_$

$\big(\lambda x.\lambda x.(e^{i\frac{\pi}{4}}.\textbf{true}) * \lambda x.\textbf{false}\big) * \_$

$\lambda x.(e^{i\frac{\pi}{4}}.\textbf{true}) * \_$

$e^{i\frac{\pi}{4}}.\textbf{true}$

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$\Big(((\lambda x.\lambda y.x) * \lambda x.(e^{i\frac{\pi}{4}}.\mathbf{true})) * \lambda x.\mathbf{false}\Big) * \_$

$\Big(\lambda x.\lambda x.(e^{i\frac{\pi}{4}}.\mathbf{true}) * \lambda x.\mathbf{false}\Big) * \_$

$\lambda x.(e^{i\frac{\pi}{4}}.\mathbf{true}) * \_$

$e^{i\frac{\pi}{4}}.\mathbf{true}$

Encoding the Hadamard gate

$$\mathbf{Hadamard} \equiv \lambda y.\Big(\big((y * \lambda x.(\mathbf{false} - \mathbf{true})) * \lambda x.(\mathbf{false} + \mathbf{true})\big) * \_\Big)$$

where _ stands for dead code.

Encoding tensors

$$\otimes \equiv \lambda x.\lambda y.\lambda f.\big((f * x) * y\big)$$
$$\pi_{\mathbf{1}} \equiv \lambda x.\lambda y.x$$
$$\pi_{\mathbf{2}} \equiv \lambda x.\lambda y.y$$
$$\bigotimes \equiv \lambda f.\lambda g.\lambda x.\Big(\big(\otimes * (f * (\pi_1 * x))\big) * \big(g * (\pi_2 * x)\big)\Big)$$
$$\text{E.g. } \mathbf{H}^{\otimes \mathbf{2}} \equiv \big((\bigotimes \mathbf{Hadamard}) * \mathbf{Hadamard}\big)$$

Encoding the CNOT gate

$$\mathbf{Cnot} \equiv$$
$$\lambda x.\Big(\big(\otimes * (\pi_{\mathbf{1}} * x)\big) * \Big(\big((\pi_{\mathbf{1}} * x) * (\mathbf{Not} * (\pi_{\mathbf{2}} * x))\big) * (\pi_{\mathbf{2}} * x)\Big)\Big)$$

Expressing Deutsch's algorithm parametrically

$$\textbf{Deutsch} \equiv$$
$$\lambda f.\left(\mathbf{H}^{\otimes \mathbf{2}} * \left(f * \left(\mathbf{H}^{\otimes \mathbf{2}} * \left((\otimes \mathbf{false}) * \mathbf{true}\right)\right)\right)\right)$$

The *Scalar* Type System [Arrighi,Díaz-Caro'09]

A polymorphic type system *tracking scalars*:

$$\frac{\Gamma \vdash M : T}{\Gamma \vdash \alpha.M : \alpha.T}$$

- Barycentric restrictions
- Characterises the "amount" of terms

The *Additive* Type System [Díaz-Caro,Petit'10]

A polymorphic type system *with sums*:

$$\frac{\Gamma \vdash M : T \qquad \Gamma \vdash N : R}{\Gamma \vdash M + N : T + R}$$

- Sums $\sim$ Assoc., comm. pairs
- distributive w.r.t. application

Can we combine them?

Typed *Lineal*: λ^{vec} (or the *Vectorial* System)

[Arrighi,Díaz-Caro,Valiron'12-13]

$$\begin{array}{rcl} T,R & ::= & U \mid \mathbb{X} \mid \alpha.T \mid T+R \\ U & ::= & \mathrm{X} \mid U \to T \mid \forall \mathrm{X}.U \mid \forall \mathbb{X}.U \end{array}$$

$$\begin{array}{c} T+R \equiv R+T \\ T+(R+S) \equiv (T+R)+S \\ 1.T \equiv T \\ \alpha.(\beta.T) \equiv (\alpha \times \beta).T \\ \alpha.T+\alpha.R \equiv \alpha.(T+R) \\ \alpha.T+\beta.T \equiv (\alpha+\beta).T \end{array}$$

Typing rules

$$\frac{}{\Gamma, x : U \vdash x : U}\, ax \qquad \frac{\Gamma \vdash M : T}{\Gamma \vdash 0 : 0.T}\, 0_I \qquad \frac{\Gamma \vdash M : T}{\Gamma \vdash \alpha.M : \alpha.T}\, \alpha_I$$

$$\frac{\Gamma \vdash M : \sum_{i=1}^{n} \alpha_i.\forall \vec{X}.(U \to T_i) \qquad \Gamma \vdash N : \sum_{j=1}^{m} \beta_j.V_j \qquad \forall V_j, \exists \vec{W}_j / U[\vec{W}_j/\vec{X}]=V_j}{\Gamma \vdash (M)N : \sum_{i=1}^{n}\sum_{j=1}^{m} \alpha_i \times \beta_i.T_i[\vec{W}_j/\vec{X}]}\, \to_E$$

$$\frac{\Gamma, x : U \vdash M : T}{\Gamma \vdash \lambda x.M : U \to T}\, \to_I \qquad \frac{\Gamma \vdash M : T \qquad \Gamma \vdash N : R}{\Gamma \vdash M+N : T+R}\, +_I$$

$$\frac{\Gamma \vdash M : U \qquad X \notin FV(\Gamma)}{\Gamma \vdash M : \forall X.U}\, \forall_I \qquad \frac{\Gamma \vdash M : \forall X.U}{\Gamma \vdash M : U[V/X]}\, \forall_E$$

Infinity

Untyped λ-calculus + linear algebra $\Rightarrow$ ∞

$$\mathbf{Yb} \equiv \lambda \mathbf{x}.(\mathbf{b} + (\mathbf{x}\,\mathbf{x}))\,\lambda \mathbf{x}.(\mathbf{b} + (\mathbf{x}\,\mathbf{x}))$$
$$\mathbf{Yb} \longrightarrow \mathbf{b} + \mathbf{Yb}$$

But whoever says infinity says trouble says... indefinite forms.
These are again a matter for confluence!

$$\begin{array}{l} \mathbf{Yb} - \mathbf{Yb} \longrightarrow \mathbf{b} + \mathbf{Yb} - \mathbf{Yb} \longrightarrow \mathbf{b} \\ \downarrow_* \\ \mathbf{0} \end{array}$$

High school teacher says we must restrict (F) to **finite vectors**.

Hiding indefinite forms

We restrict (F) to $\mathbf{u}$ normal.
Now

$$\mathbf{Yb} - \mathbf{Yb} \not\longrightarrow \mathbf{0}$$

But

$$\begin{array}{l} \lambda \mathbf{x}.(\mathbf{x}\,\mathbf{y} - \mathbf{x}\,\mathbf{y})\,\lambda \mathbf{y}.\mathbf{Yb} \longrightarrow^* \mathbf{0} \\ \quad \downarrow_* \\ \mathbf{Yb} - \mathbf{Yb} \end{array}$$

Confluence and Strong normalisation

In the original untyped setting: "confluence by restrictions":

$$Y_B = (\lambda x.(B + (x)x))\lambda x.(B + (x)x)$$

$$Y_B \to B + Y_B \to B + B + Y_B \to \dots$$

$$\begin{array}{l} Y_B + (-1).Y_B \longrightarrow (1-1).Y_B \longrightarrow^* 0 \\ \quad\downarrow \\ B + Y_B + (-1).Y_B \\ \quad\downarrow_* \\ \quad B \end{array}$$

Solution in the untyped setting:
$\alpha.M + \beta.M \to (\alpha + \beta).M$
only if M is closed-normal

In the typed setting: **Strong normalisation solves the problem**

Theorem (Strong normalisation)

$\Gamma \vdash M : T \Rightarrow M$ *strongly normalising.*

Proof.

Reducibility candidates method.

Main difficulty: Show that

$$\{M_i\}_i \text{ strongly normalizing} \Rightarrow \sum_i \alpha_i.M_i \text{ strongly normalizing}$$

Done by using a measurement on terms decreasing on algebraic rewrites. □

Theorem (Confluence)

$$\forall M \;/\; \Gamma \vdash M : T \qquad \begin{matrix} M \to^* N_1 \\ M \to^* N_2 \end{matrix} \Rightarrow \exists L \; such\ that \; \begin{matrix} N_1 \to^* L \\ N_2 \to^* L \end{matrix}$$

Proof.

1) **local confluence**: $\begin{matrix} M \to N_1 \\ M \to N_2 \end{matrix} \Rightarrow \exists L \text{ such that } \begin{matrix} N_1 \to^* L \\ N_2 \to^* L \end{matrix}$

- Algebraic fragment: Coq proof
- Beta-reduction: Straightforward extension
- Commutation: Induction

2) Local confluence + Strong normalisation ⇒ Confluence [TeReSe'03] □

$$(\alpha + \beta).T \sqsubseteq \alpha.T + \beta.T' \qquad \text{if } \exists M \;/\; \Gamma \vdash M : T \text{ and } \Gamma \vdash M : T'$$
(and its contextual closure)

Theorem (A weak subject reduction)

If $\Gamma \vdash M : T$ *and* $M \to_R N$, *then*

- *if* R *is not a factorisation rule:* $\Gamma \vdash N : T$
- *if* R *is a factorisation rule:* $\exists S \sqsubseteq T \;/\; \Gamma \vdash N : S$

How weak?

Let $M \to N$,

Subject reduction
$\Gamma \vdash M : T \Rightarrow \Gamma \vdash N : T$

Subtyping
$\Gamma \vdash M : T \Rightarrow \Gamma \vdash N : S$, but $S \leq T$, so $\Gamma \vdash N : T$

Our theorem
$\Gamma \vdash M : T \Rightarrow \Gamma \vdash N : S$, and $S \sqsubseteq T$

The factorisation rule problem

$$\frac{\Gamma \vdash M : T \qquad \Gamma \vdash M : T'}{\Gamma \vdash \alpha.M + \beta.M : \alpha.T + \beta.T'}$$

- However, $\alpha.M + \beta.M \to (\alpha + \beta).M$
- In general $\alpha.T + \beta.T' \neq (\alpha + \beta).T \neq (\alpha + \beta).T'$

(and since we are working in System F, there is no principal types neither)

Typed *Lineal*: $\mathcal{X}^{\mathrm{vec}}$ (or the *Vectorial* System)

[Arrighi,Díaz-Caro,Valiron'12-13]

$$\begin{array}{rcl} T, R & ::= & U \mid \mathbb{X} \mid \alpha.T \mid T + R \\ U & ::= & \mathsf{X} \mid U \to T \mid \forall \mathsf{X}.U \mid \forall \mathbb{X}.U \end{array}$$

$$\begin{array}{c} T + R \equiv R + T \\ T + (R + S) \equiv (T + R) + S \\ 1.T \equiv T \\ \alpha.(\beta.T) \equiv (\alpha \times \beta).T \\ \alpha.T + \alpha.R \equiv \alpha.(T + R) \\ \alpha.T + \beta.T \equiv (\alpha + \beta).T \end{array}$$

Most important property of $\mathcal{X}^{\mathrm{vec}}$

$$\begin{array}{lcll} \vdash \mathbf{t} : \sum_i \alpha_i.T_i & \Rightarrow & \mathbf{t} \to^* \sum_i \alpha_i.\mathbf{r}_i & \text{where } \vdash \mathbf{r}_i : T_i \\ \mathbf{t} \to^* \sum_i \alpha_i.\mathbf{r}_i & \Rightarrow & \vdash \mathbf{t} : \sum_i \alpha_i.T_i + 0.R & \end{array}$$

A type system capturing the "vectorial" structure of terms
... able to type matrices and vectors
... able to check for probability distributions
... or whatever application needing the structure of the vector

Example: Typing vectors and matrices

$$\begin{array}{l} \vdash \overbrace{\lambda x.\lambda y.x}^{|0\rangle} : \overbrace{\forall XY.X \to Y \to X}^{\mathbf{T}} \\ \vdash \underbrace{\lambda x.\lambda y.y}_{|1\rangle} : \underbrace{\forall XY.X \to Y \to Y}_{\mathbf{F}} \end{array} \quad \left| \quad \begin{array}{l} H|0\rangle \to \overbrace{\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)}^{|+\rangle} : \overbrace{\frac{1}{\sqrt{2}}(\mathbf{T} + \mathbf{F})}^{\boxplus} \\ H|1\rangle \to \underbrace{\frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)}_{|-\rangle} : \underbrace{\frac{1}{\sqrt{2}}(\mathbf{T} - \mathbf{F})}_{\boxminus} \end{array} \right.$$

Example: Typing vectors and matrices

$$\begin{array}{l} \vdash \overbrace{\lambda x.\lambda y.x}^{|0\rangle} : \overbrace{\forall XY.X \to Y \to X}^{\mathbf{T}} \\ \vdash \underbrace{\lambda x.\lambda y.y}_{|1\rangle} : \underbrace{\forall XY.X \to Y \to Y}_{\mathbf{F}} \end{array} \quad \left| \quad \begin{array}{l} H|0\rangle \to \overbrace{\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)}^{|+\rangle} : \overbrace{\frac{1}{\sqrt{2}}(\mathbf{T} + \mathbf{F})}^{\boxplus} \\ H|1\rangle \to \underbrace{\frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)}_{|-\rangle} : \underbrace{\frac{1}{\sqrt{2}}(\mathbf{T} - \mathbf{F})}_{\boxminus} \end{array} \right.$$

$$\vdash \underbrace{\lambda x.\{x\ \ [|+\rangle]\ \ [|-\rangle]\}}_{H} : \forall \mathbb{X}.([\boxplus] \to [\boxminus] \to [\mathbb{X}]) \to \mathbb{X}$$

Example: Typing vectors and matrices

$$\begin{array}{l} \vdash \overbrace{\lambda x.\lambda y.x}^{|0\rangle} : \overbrace{\forall XY.X \to Y \to X}^{\mathbf{T}} \\ \vdash \underbrace{\lambda x.\lambda y.y}_{|1\rangle} : \underbrace{\forall XY.X \to Y \to Y}_{\mathbf{F}} \end{array} \quad \left| \quad \begin{array}{l} H|0\rangle \to \overbrace{\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)}^{|+\rangle} : \overbrace{\frac{1}{\sqrt{2}}(\mathbf{T} + \mathbf{F})}^{\boxplus} \\ H|1\rangle \to \underbrace{\frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)}_{|-\rangle} : \underbrace{\frac{1}{\sqrt{2}}(\mathbf{T} - \mathbf{F})}_{\boxminus} \end{array} \right.$$

$$\vdash \underbrace{\lambda x.\{x\ \ [|+\rangle]\ \ [|-\rangle]\}}_{H} : \forall \mathbb{X}.([\boxplus] \to [\boxminus] \to [\mathbb{X}]) \to \mathbb{X}$$

$$\vdash H(\tfrac{1}{\sqrt{2}}(|0\rangle + |1\rangle)) : \tfrac{1}{\sqrt{2}}(\boxplus + \boxminus) \equiv \mathbf{T}$$

Example: Typing vectors and matrices

$$\begin{array}{l} \vdash \overbrace{\lambda x.\lambda y.x}^{|0\rangle} : \overbrace{\forall XY.X \to Y \to X}^{\mathbf{T}} \\ \vdash \underbrace{\lambda x.\lambda y.y}_{|1\rangle} : \underbrace{\forall XY.X \to Y \to Y}_{\mathbf{F}} \end{array} \quad \left| \quad \begin{array}{l} H|0\rangle \to \overbrace{\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)}^{|+\rangle} : \overbrace{\frac{1}{\sqrt{2}}(\mathbf{T} + \mathbf{F})}^{\boxplus} \\ H|1\rangle \to \underbrace{\frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)}_{|-\rangle} : \underbrace{\frac{1}{\sqrt{2}}(\mathbf{T} - \mathbf{F})}_{\boxminus} \end{array} \right.$$

$$\vdash \underbrace{\lambda x.\{x\ \ [|+\rangle]\ \ [|-\rangle]\}}_{H} : \forall \mathbb{X}.([\boxplus] \to [\boxminus] \to [\mathbb{X}]) \to \mathbb{X}$$

$$\vdash H(\tfrac{1}{\sqrt{2}}(|0\rangle + |1\rangle)) : \tfrac{1}{\sqrt{2}}(\boxplus + \boxminus) \equiv \mathbf{T}$$

In general

$$\begin{pmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{pmatrix} := \lambda x_1 \ldots \lambda x_n.x_i \qquad \left(\boxed{c_1}\ \boxed{c_2} \ldots \boxed{c_n} \right) = \lambda x.\{x[c_1][c_2]\ldots[c_n]\}$$

$$\forall \mathbb{X}.([C_1] \to [C_2] \to \cdots \to [C_n] \to [\mathbb{X}]) \to \mathbb{X}$$

$$\forall X_1 \cdots X_n.X_1 \to \cdots \to X_n \to X_i$$

Contributions

- Scalar ∪ Additive ("AC, distributive pairs")
  ⇒ linear-combination of types
- Strong normalisation
  ⇒ Confluence without restrictions
- Weak SR, types tell you
  ⇒ "how much weight of terms of a given type" there will be in the normal form
- Types as matrices and vectors
  ⇒ "finite-dimensional linear algebra abstraction interpretation of computable linear operators"

Open

- Simplify
  ⇒ Church-style polymorphism, or just intersection types instead
- Interprete, understand as a proof theory
  ⇒ Superpositions of hypotheses: $\longrightarrow_I$ for $\mathbb{X}$

1 / 2

Annex

Not *one* zero... a lot of them

$$\dfrac{\dfrac{\Gamma \vdash \lambda x.x : U \to U \qquad \dfrac{\Gamma \vdash M : R \qquad \Gamma \vdash -M : -R}{\Gamma \vdash M - M : \overline{0}}}{\Gamma \vdash \lambda x.x + M - M : U \to U} \qquad \dfrac{}{\Gamma \vdash B : U}}{\Gamma \vdash (\lambda x.x + M - M)B : U}$$

However, $(\lambda x.x + M - M)B \to (\lambda x.x)B + (M)B - (M)B$

We need to keep track of the zeros... where they came from!

Instead of a type $\overline{0}$, we have $0.R$ and $T + 0.R \not\equiv T$

Extra: Example of arrow elimination

$$\begin{array}{l}\Gamma \vdash B_1 : V_1 \\ \Gamma \vdash B_2 : V_2\end{array} \Rightarrow \Gamma \vdash \beta_1.B_1 + \beta_2.B_2 : \beta_1.V_1 + \beta_2.V_2 \qquad \boxed{\begin{array}{l}U[W_1/X] = V_1 \\ U[W_2/X] = V_2\end{array}}$$

$$\begin{array}{l}\Gamma \vdash \lambda x_1.M_1 : \forall X.(U \to T_1) \\ \Gamma \vdash \lambda x_2.M_2 : \forall X.(U \to T_2)\end{array}$$

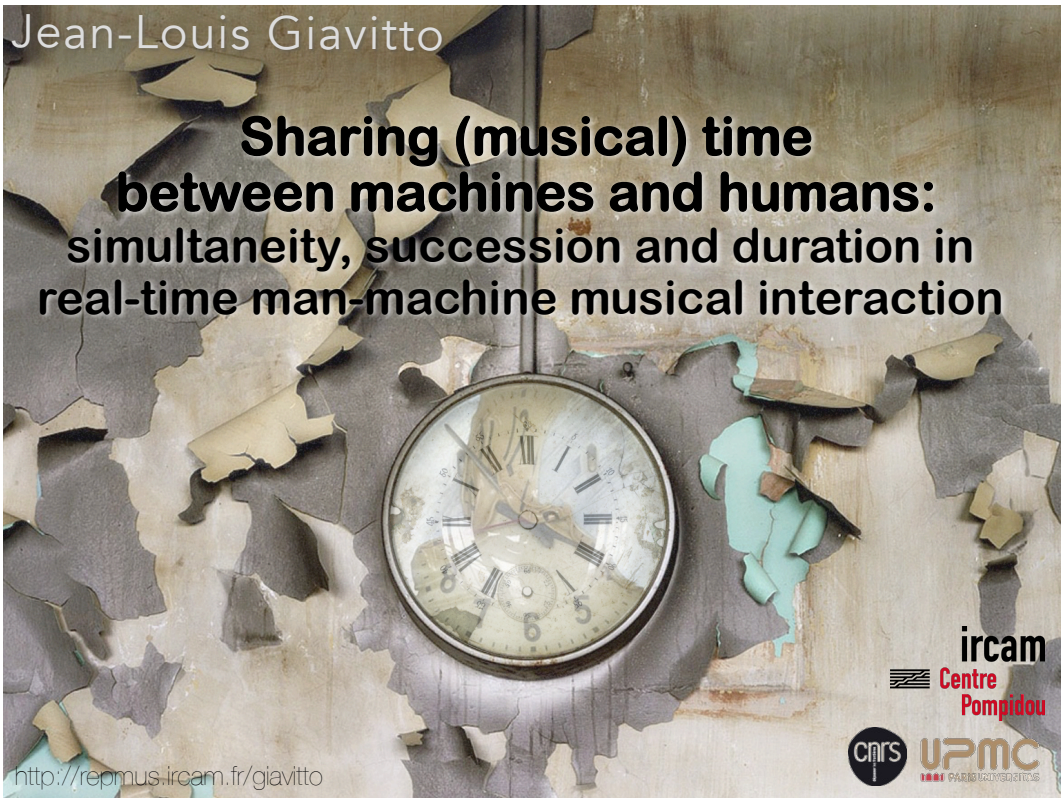
$$\Rightarrow$$

$$\Gamma \vdash (\alpha_1.\lambda x_1.M_1) + (\alpha_2.\lambda x_2.M_2) : (\alpha_1.\forall X.(U \to T_1)) + (\alpha_2.\forall X.(U \to T_2))$$

$$\dfrac{\begin{array}{l}\Gamma \vdash \blacktriangle + \blacktriangledown : \alpha_1.\forall X.(U \to T_1) + \alpha_2.\forall X.(U \to T_2) \\ \Gamma \vdash \triangle + \triangledown : \beta_1.V_1 + \beta_2.V_2\end{array}}{\Gamma \vdash (\blacktriangle + \blacktriangledown)(\triangle + \triangledown) : \sum_{i=1}^{2}\sum_{j=1}^{2} \alpha_i \times \beta_j.T_i[W_j/X]} \to_E$$

$$(\blacktriangle + \blacktriangledown)(\triangle + \triangledown) \to^* (\blacktriangle)\triangle + (\blacktriangle)\triangledown + (\blacktriangledown)\triangle + (\blacktriangledown)\triangledown$$

e.g.
$(\blacktriangledown)\triangle = (\alpha_2.\lambda x_2.M_2)\beta_1.B_1 \to^*$
$(\alpha_2 \times \beta_1).(\lambda x_2.M_2)B_1$



Computer music as an experimental enquiry in temporal concepts

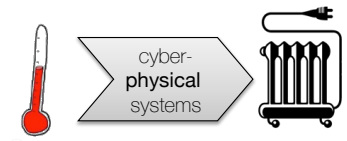
- Music makes time audible, and its form and continuity sensible (S. Langer)
- But, if music is the paradigmatic "art of time", for which kind of time is music an art?
- *Real musical time is only a place of exchange and coincidence between an infinite number of different times.* (Gérard Grisey, Tempus Ex Machina: A Composer's reflections on musical time. *Contemporary Music Review*, 1987)
- Test the relevance and the effectiveness of temporal notions in musical applications

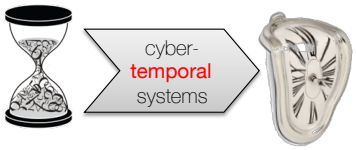
Is a computer able to…

- play chess
- check logical reasoning
- find a path from A to B in a city
- …
- recognize a smiling face
- walk on two legs
- …
- *play music* together *with human performers ?*

Cyber-temporal systems: computing time in real-time

- *from:* physical entities monitored by algorithms
- *to:* temporal relationships sensed and organized by algorithms
- *example:* interactive music systems
  Antescofo





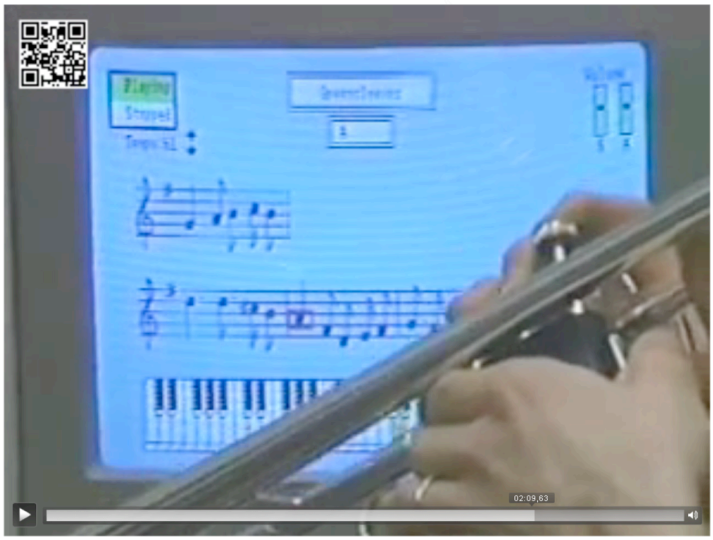
- notionS of TIME:
  - multiple times: deferred time, real-time
  - multiple models of time: event-driven, time-driven
  - multiple scales: from audio (0.02 ms) to control (hours)
  - time programmability: time is a denotable entity

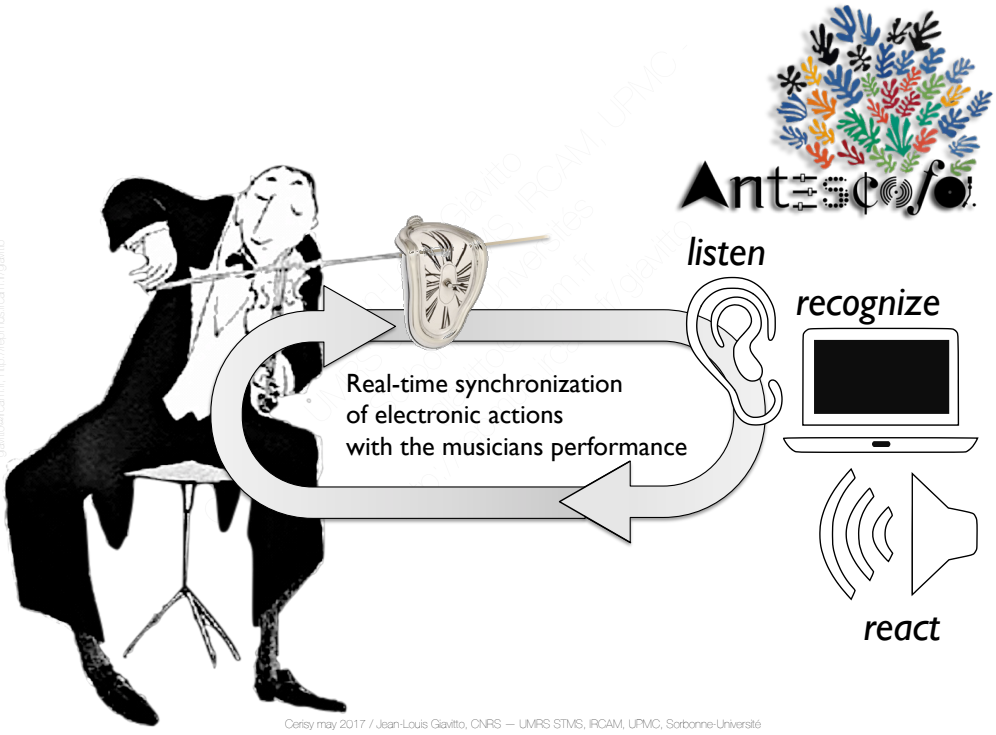
An example of cyber-temporal systems : Automatic Accompaniment in *Antescofo*

Concerto pour main gauche, Ravel.
Performer: Jacques Comby
Orchestra: recording Orchestre de Paris *synchronized with* Antescofo *in real-time*

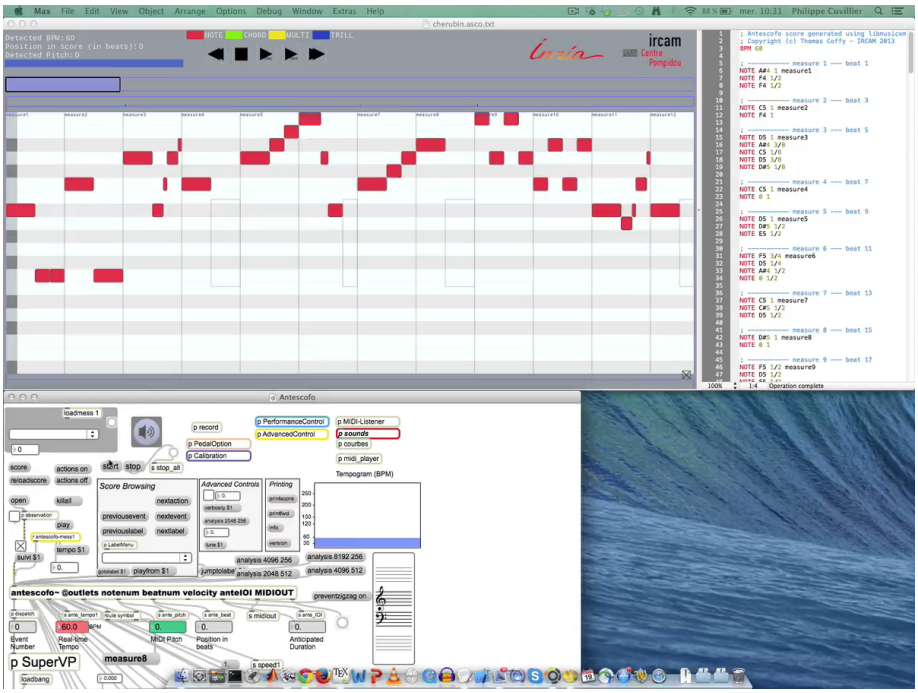
A long way from 1983 (~35th anniversary for score following)

http://repmus.ircam.fr/antescofo/videos

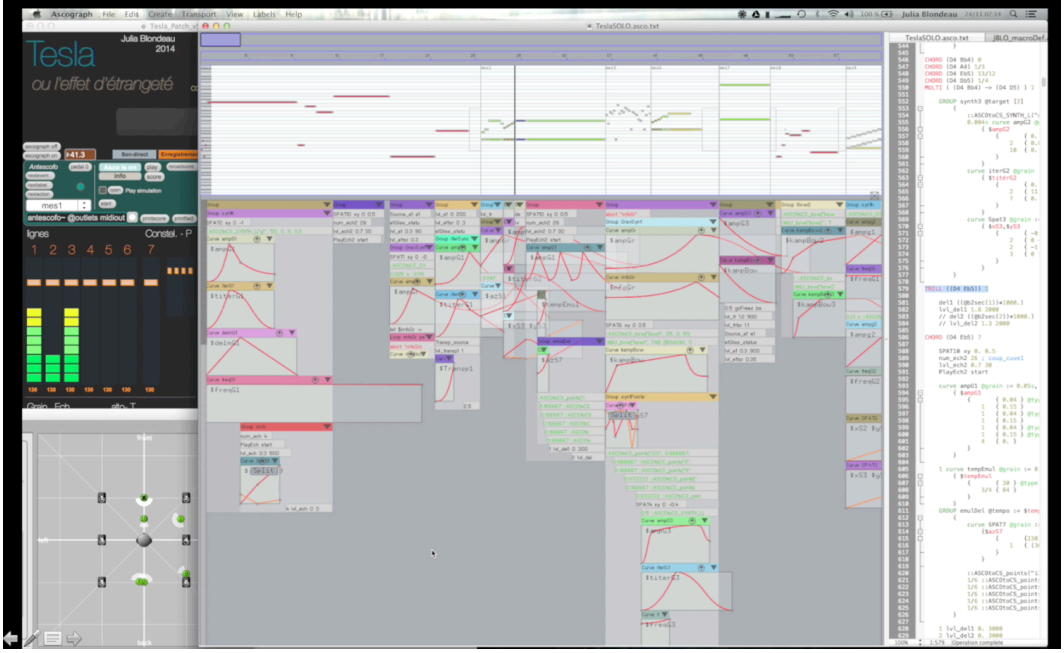




Dealing with wide ranges of interpretation and errors



Tesla ou l'effet d'étrangeté
Julia Blondeau (2014)
alto: Christophe Desjardins, real-time electronic: Antescofo



ODEI
(performance Les Nuits Sonores, Lyon, 2014)

Outlines

Interactive Music Systems as interpreters (computer meaning of « interpretation »)

- time in a score
- score as program, performance as execution !?
- the interpretation problem

Times in Antescofo

- time in computer programing languages
- events and duration
- from triggers to synchronization strategy
- time-time diagrams
- tempo extraction
- temporal scope

- Beyond chronometric time
  - why multiple times ?
  - fungible or incomparable times ?
  - time and causality ?

Some other artistic applications

- the Polyrhytmic Machine
- Marco Stroppa's Totem
- open score by Jason Freeman
- gesture-following by José-Miguel Fernandez

Final remarks

- from functions to interactions through processes
- sharing our time with machines

11

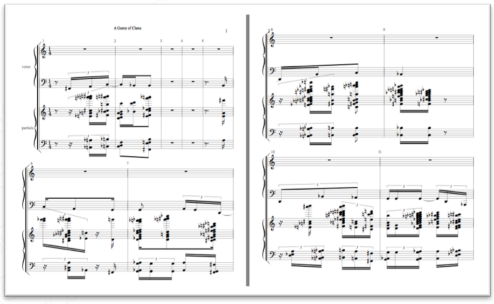
- event and duration
- continuous and discrete time
- building time together
- deferred and real time

MULTIPLE TIMES

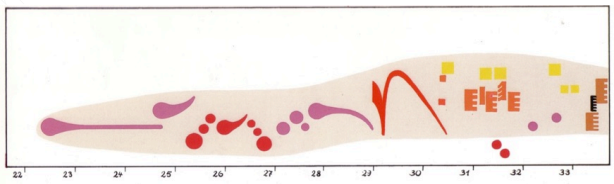
12

A Score

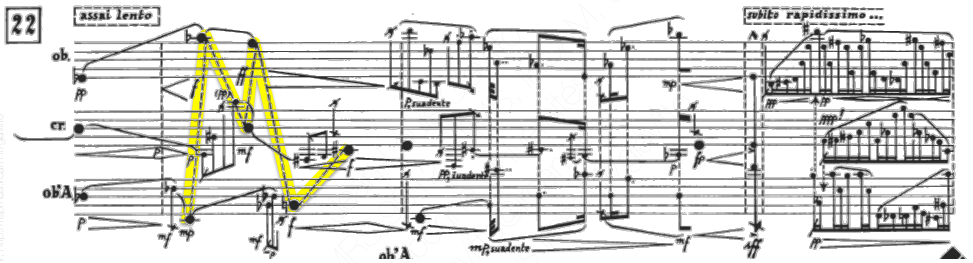
- instantaneous events
  (e.g. the onset of a note)
- events that last
  (duration of a note)



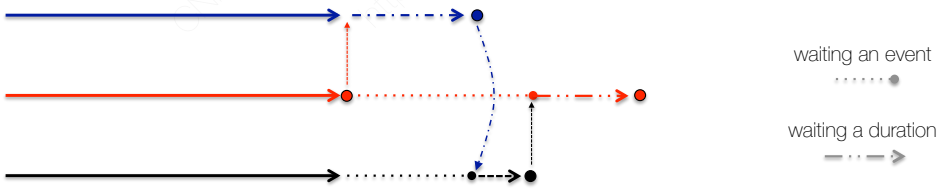
- continuous change of parameter (*movement*, *gesture*)
  - frequency
  - ambitus
  - sound localization
  - etc.



Organizing time together (and in a distributed way)



Stockhausen, *Gruppen*



“hors temps” (deferred time)

When a composer write a score, usually:

- his « flow of time » is not the « flow of time » which is written
- The « written time » is “spatialized”: every instants are accessible on the same level

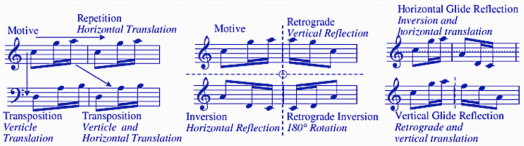
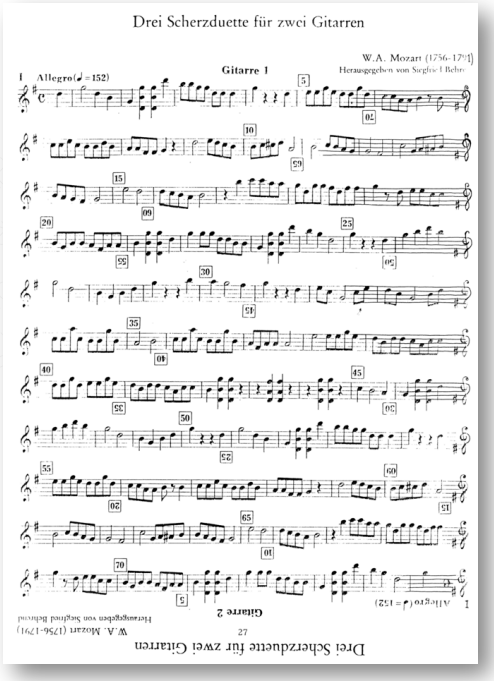
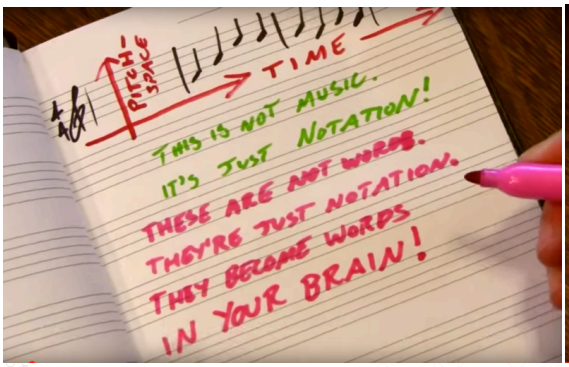
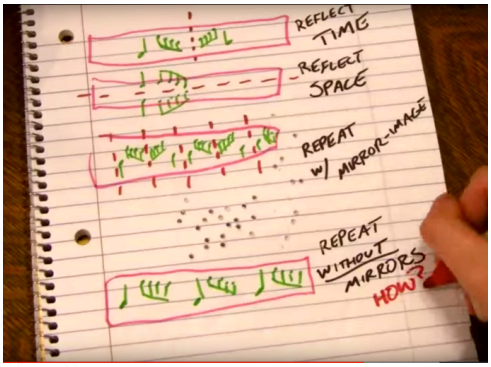


Figure 1: *All Eight Isometric Transformations in Musical Space*

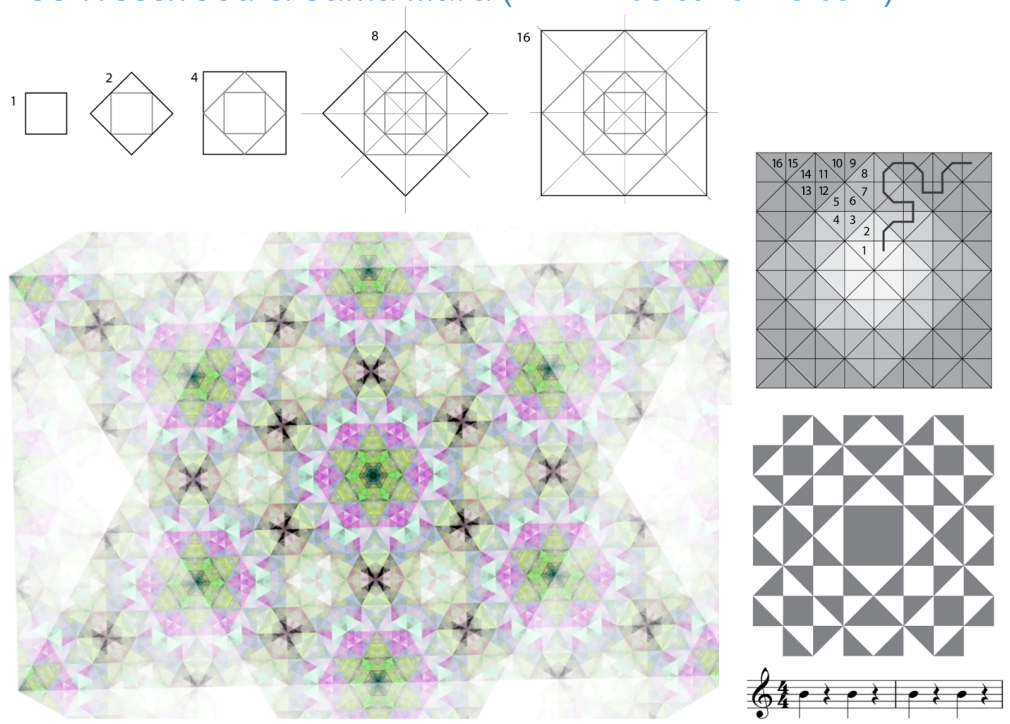


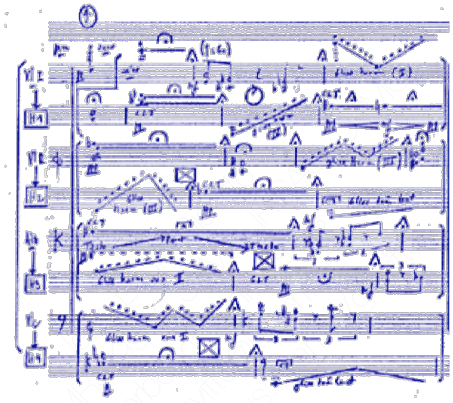




Vi Hart
http://vihart.com/

Lee Westwood & Sama Mara (www.musicalforms.com)





THE SCORE AS A PROGRAM AND THE PERFORMANCE AS PROGRAM EXECUTION?

Confronting scores and programs ?

Score	Program
music composition	program spécification
interpretation	execution
a score / its denotation	an expression / its evaluation
composer	programmer
one (multiple) performer(s)	one (many distributed) computer(s)
…	…

- Score ≠ Program
  - G. Mazzola : a « frozen gesture », « unfrozen is not heating »
  - T. Adorno: solid state of the score vs. the liquid state of the performance

Scoring with Code: Composing with algorithmic notation

THOR MAGNUSSON

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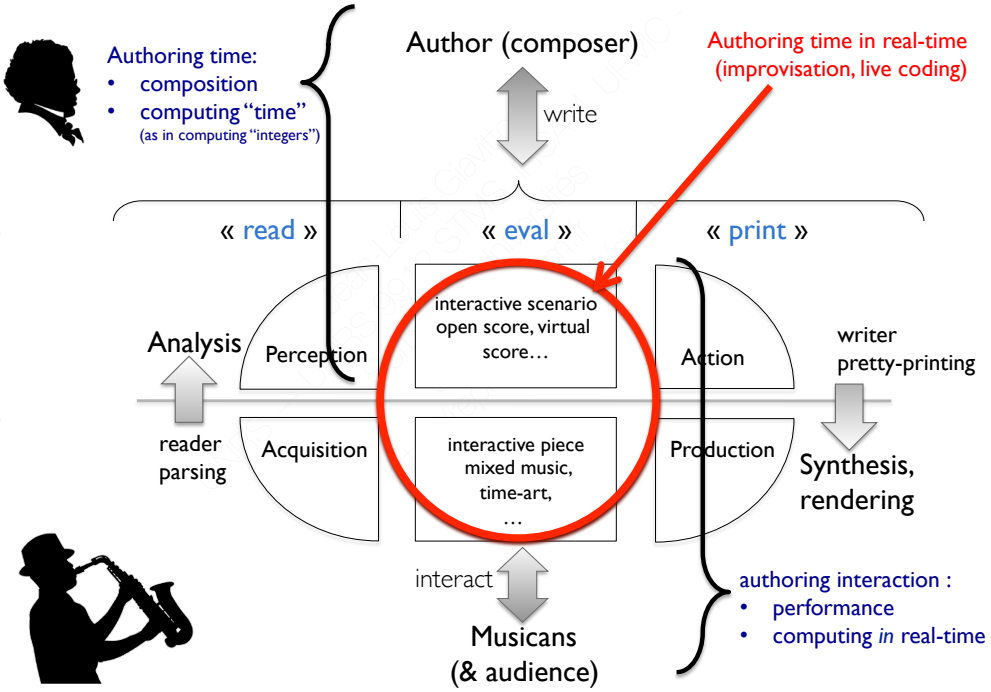
Email: t.magnusson@sussex.ac.uk

Computer code is a form of notational language. It prescribes actions to be carried out by the computer, often by systems called interpreters. When code is used to write music, we are therefore operating with programming language as a relatively new form of musical notation. Music is a time-based art form and the traditional musical score is a linear chronograph with instructions for an interpreter. Here code and traditional notation are somewhat at odds, since code is written as text, without any representational timeline. This can pose problems, for example for a composer who is working on a section in the middle of a long piece, but has to repeatedly run the code from the beginning or make temporary arrangements to solve this difficulty in the compositional process. In short: code does not come with a timeline but is rather the material used for building timelines. This article explores the context of creating linear 'code scores' in the area of musical notation. It presents the *Threnoscope* as an example of a system that implements both representational notation and a prescriptive code score.

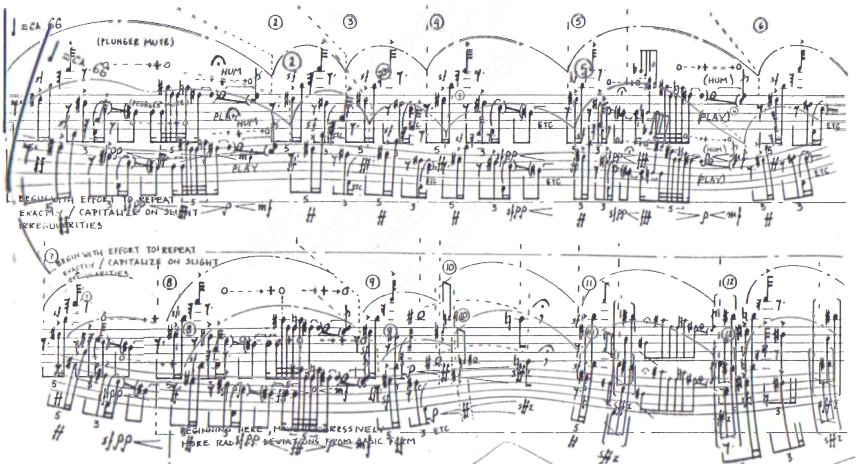
A "Language Approach" to IMS



The Antescofo aproach (to mixed music)

- An *augmented score* is a real-time program
  - the composer is a programmer that specifies bith human and electronic parts
  - program evaluation is done jointly by <musician | machine>
  - the composer specifies the synchronization between human and electronic parts
- The listening machine provides the inputs to the machine
  <musician | listening & recognition | strongly timed program>
- **BUT music interpretation is not program evaluation**
  the gap between the score and its implementation is intentional
- **Time is a first class entity in the DSL**
  - time is not an operational property (*e.g.*, a quality of service or a performance metric)
  - handling of events and duration
  - chronometric and relational time
  - computing dynamic timelines

Shrinking and stretching the score into a performance

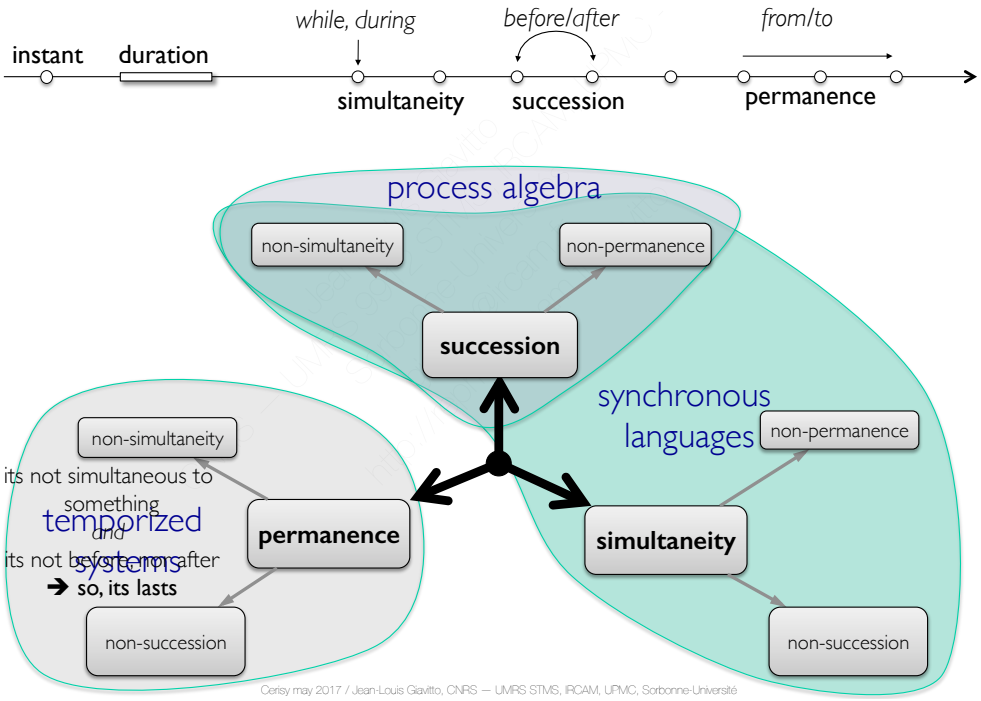


Times in Antescofo

Daniel Firmin, 2011

- multiple time(s)
- tempo extraction
- from triggers to synchronization
- the interpretation problem
- time-time diagrams

Instant and Duration: Simultaneity, Succession & Permanence



Can we deal with *instants* only?

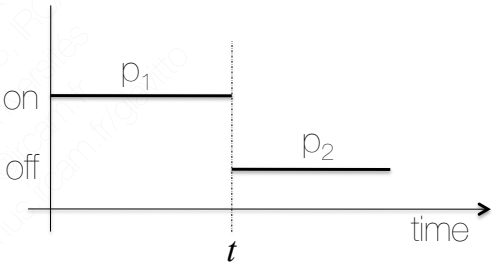
➔ duration as a set of contiguous instants

- evenemential-time

versus

- the fluxion: continuous passage of time
  - going twice faster
  - finishing together
  - accelerando
  - rubato
  - tempo
  - etc…

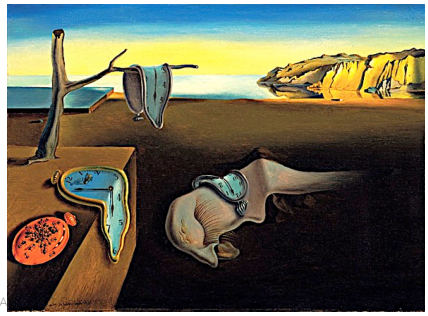
Can duration be reduced to instant? (in temporal logic)



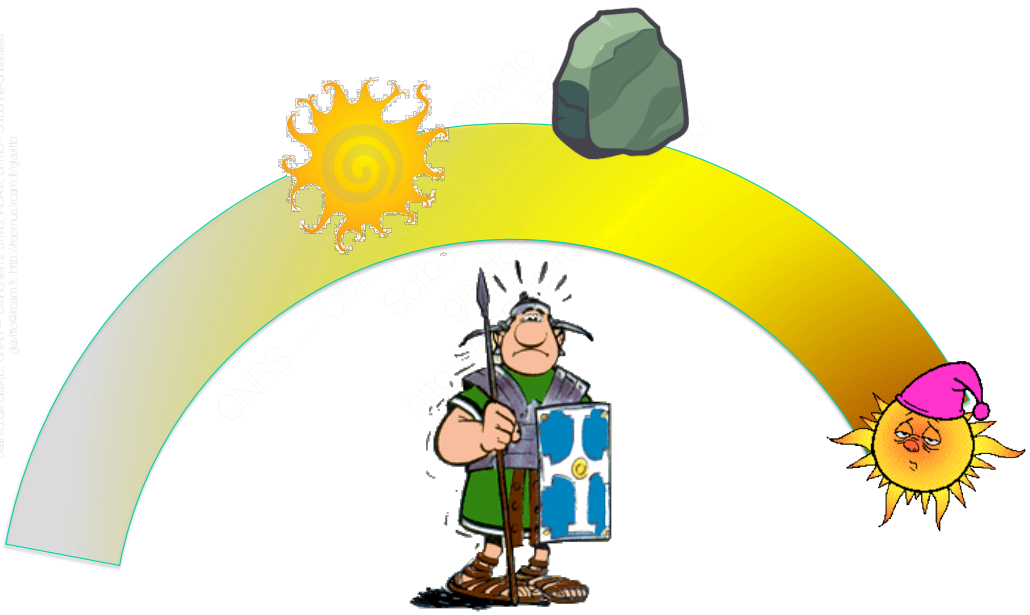
doing real analysis and topology
or
making instant and duration primitive notions

From one global time to multiple, relational, distributed times

- a shared global time: *one external objective master clock*
  - events inhabit time
  - newtonian time, a priori fungible unit
  - a shared prescriptiion (which can be only partially known)
- multiple times: *co-dependant clocks*
  - events build time (Bluedorn: epochal time is defined by events)
  - leibnizian, relational time
  - Examples :
    - score: multiple temporal layers
    - relationships score / performance
    - co-construction during the performance



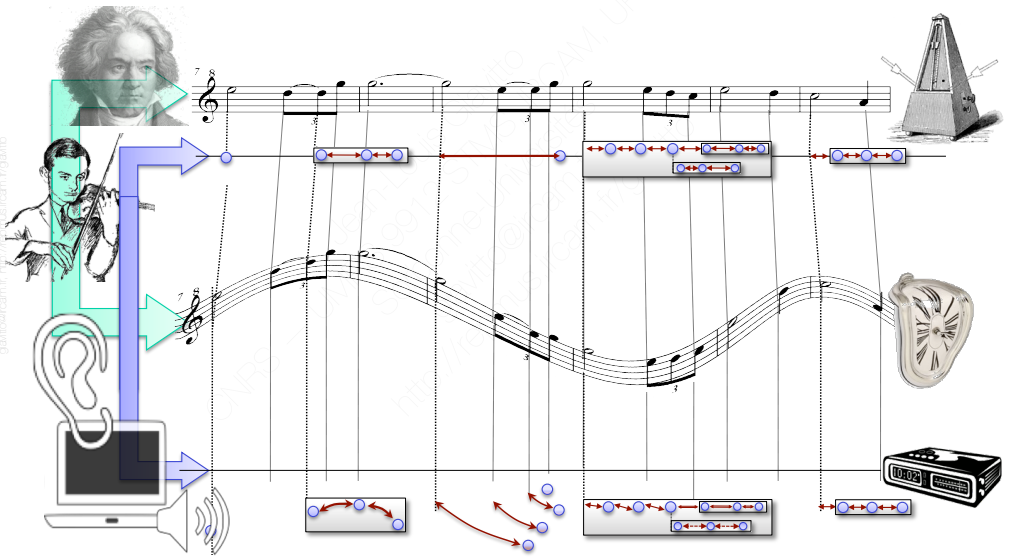
Example of relational (event-specified) time
Roman notions of summer hours and winter hours



times in Antescofo

FROM TRIGGER TO SYNCHRONIZATION

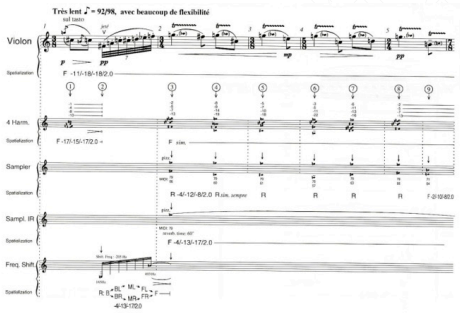
The Multiples Times of Temporal Scenarios



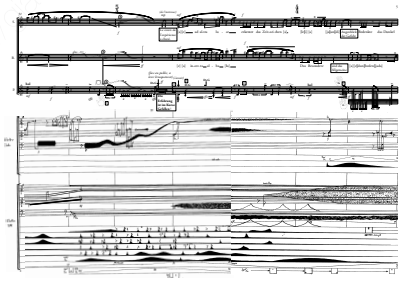
Notice the difference…
from trigger (event alignment)
to synchronization (full timeline event + duration alignment)

electronic + acoustic

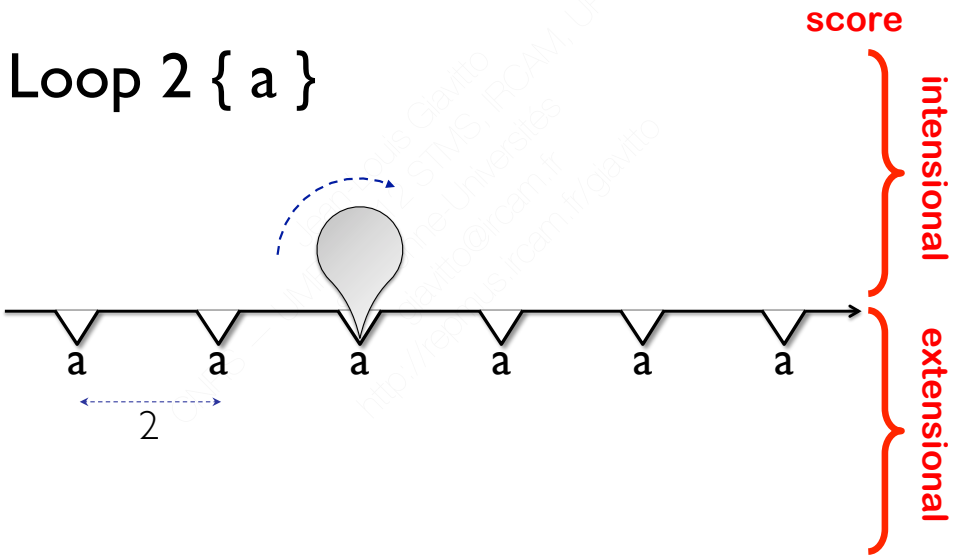
Antheme2
Pierre Boulez



Nachleben
Julia Blondeau



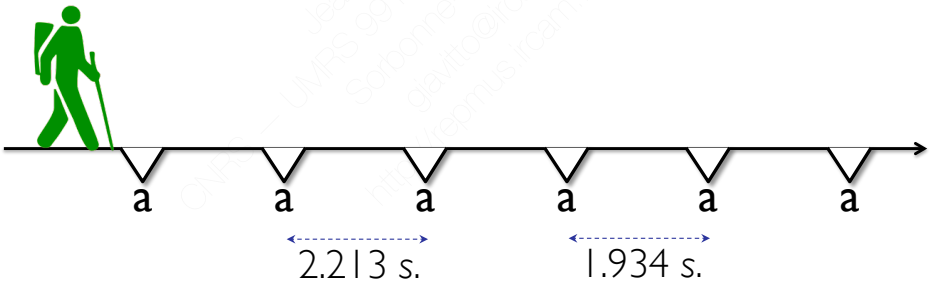
Writing music = filling a timeline



-Louis Giavitto, IRCAM – CNRS – INRIA MuTAnt 35

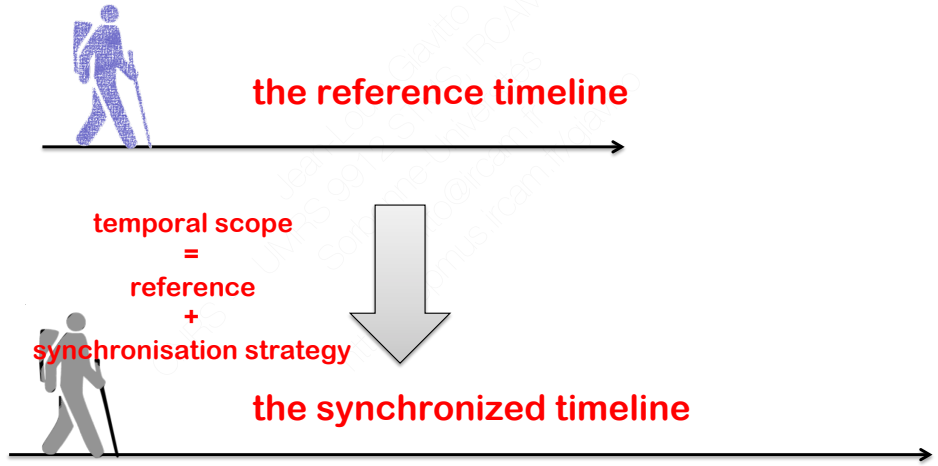
performing = progressing on the timeline

"variable speed" = deformation (stretching and shrinking) of the temporal relationships between score and performance



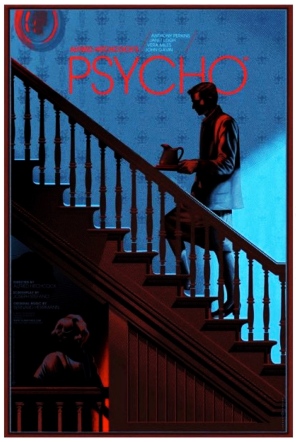
Cerisy may 2017 / Jean-Louis Giavitto, CNRS — UMRS STMS, IRCAM, UPMC, Sorbonne-Université 36

performing = progression on the timeline relatively to another timeline



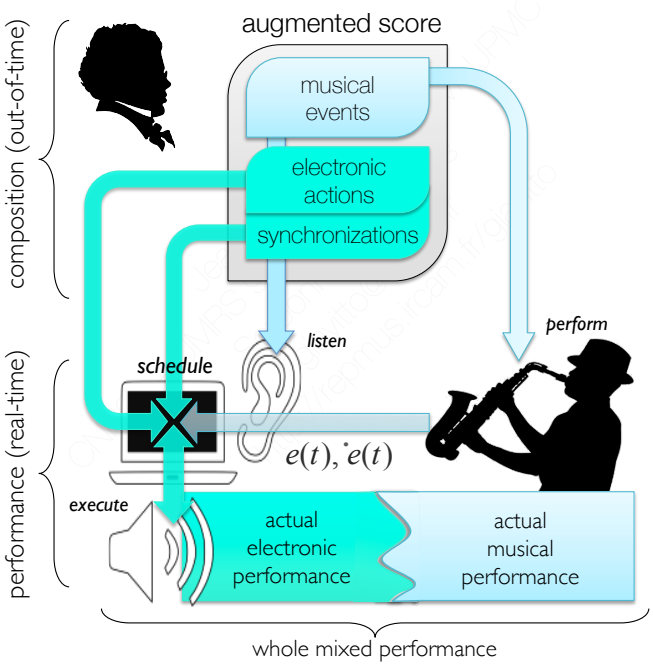
Striated time and smooth time

Bernard Hermann, *Psycho (1960)*



György Ligeti *Lontano (1967)*

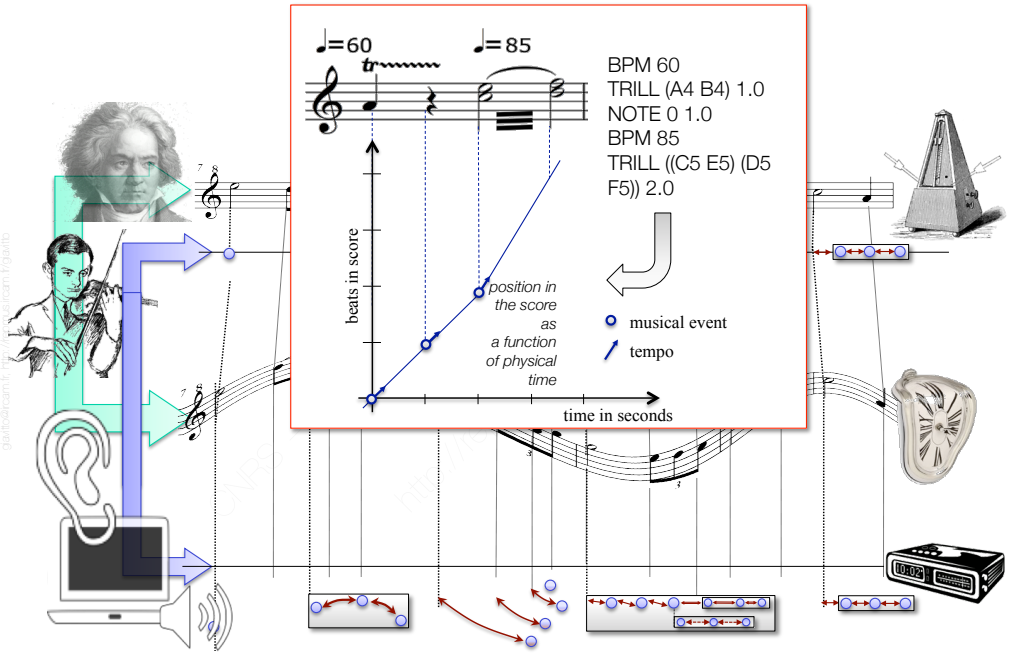
Tackling the interpretation problem in mixed music



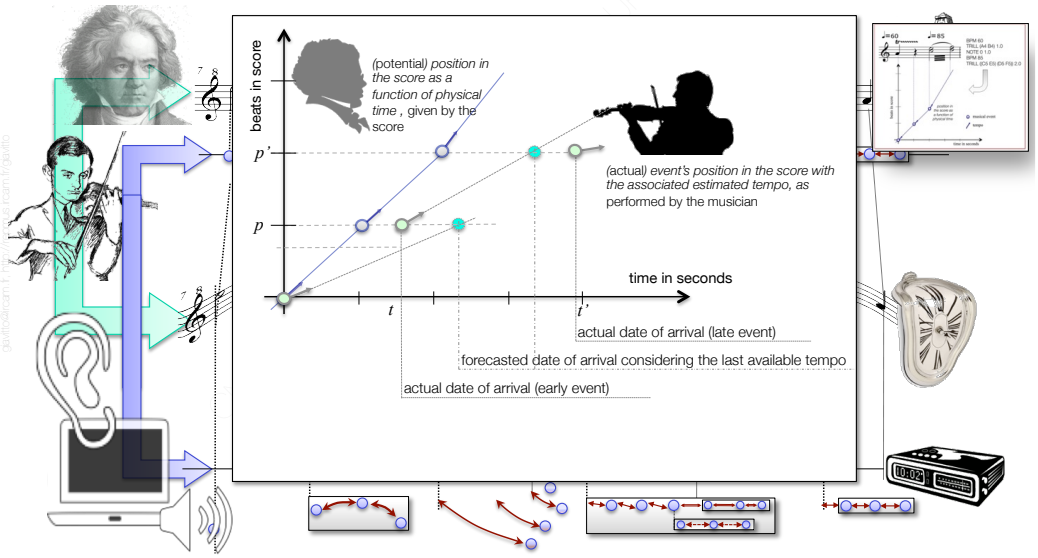
synchronizing timeline in Antescofo

TIME-TIME DIAGRAM

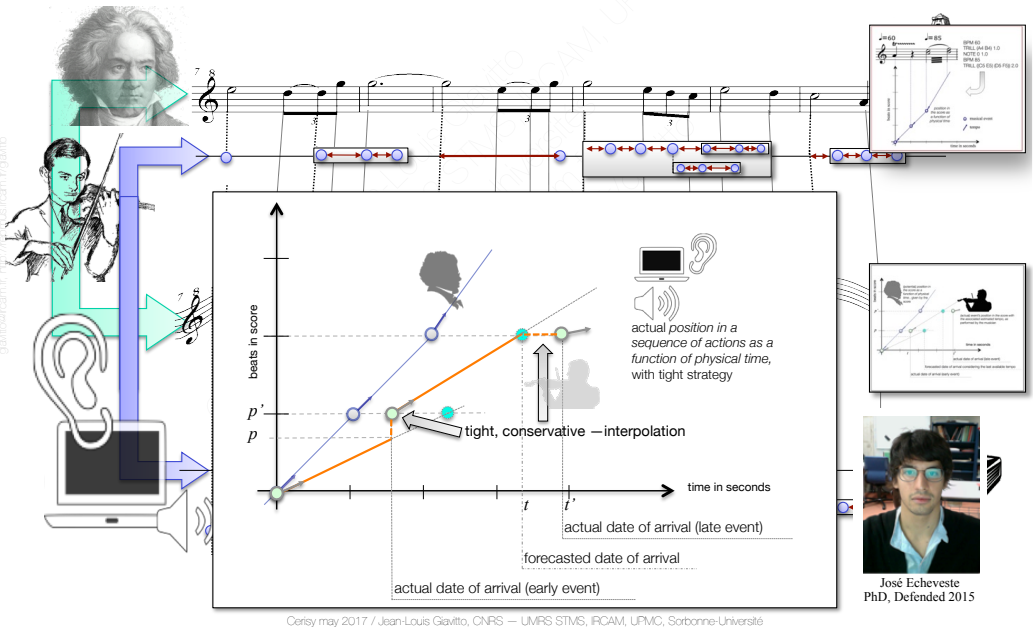
Time-time diagrams



Time-time diagrams



Time-time diagrams



Dynamic Target

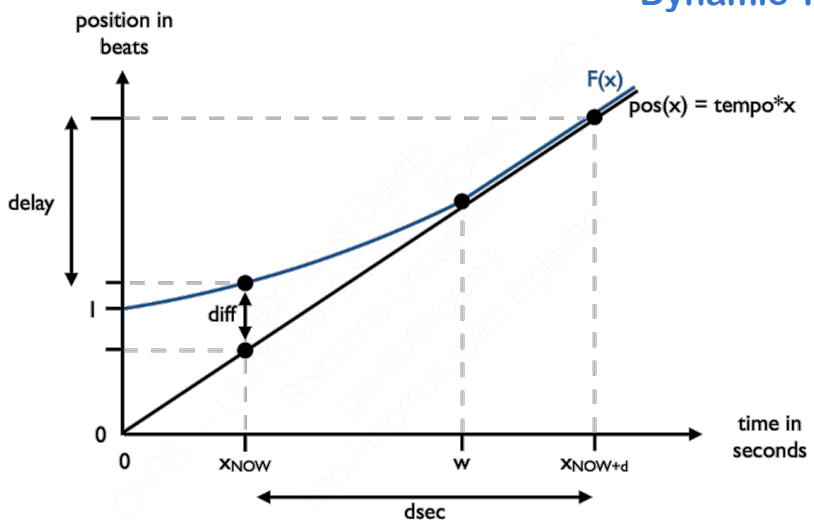
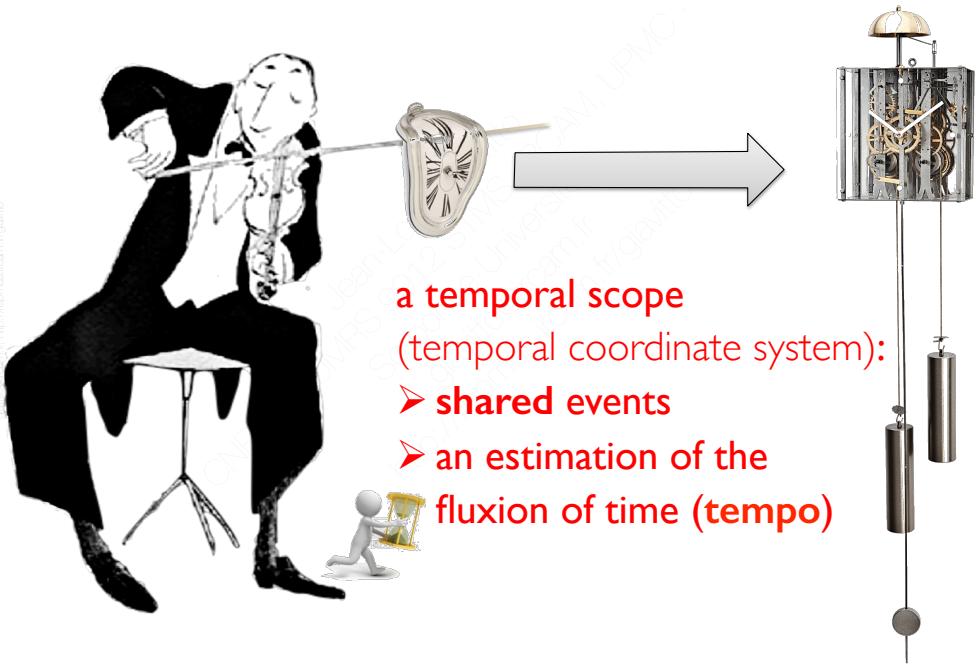


Fig. 10. Method used to compute a duration *dsec* in seconds corresponding to a delay *delay* in beats with a dynamic target $[w]$ and with a initial difference of position $diff = \tau.beatPos - position$. Function F represents the position in τ as a function of time x. It is made of two parts: a part G where the $\tau.tempo$ changes linearly until it becomes equal to *tempo*. From this time, F evolve as *pos*, with $\tau.tempo = tempo$ (a constant). Function G is the part of the parabola that goes from $x = 0$ to $x = w$. Because the origin is translated w.r.t. the origin of the physical time, the date x_{NOW} of the current instant is localized on the X axis by looking at the point which achieve the current difference *diff*.

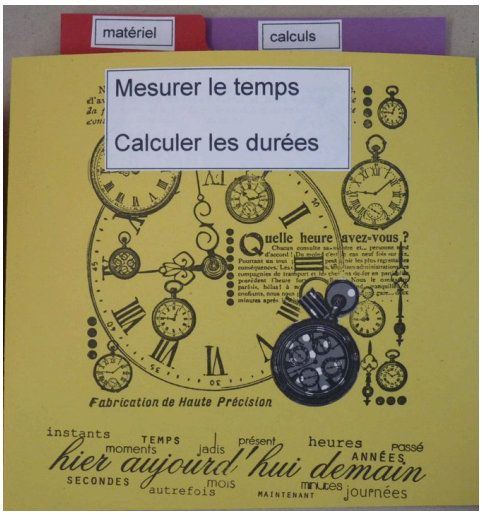
aligning (*aka* synchronizing) timelines



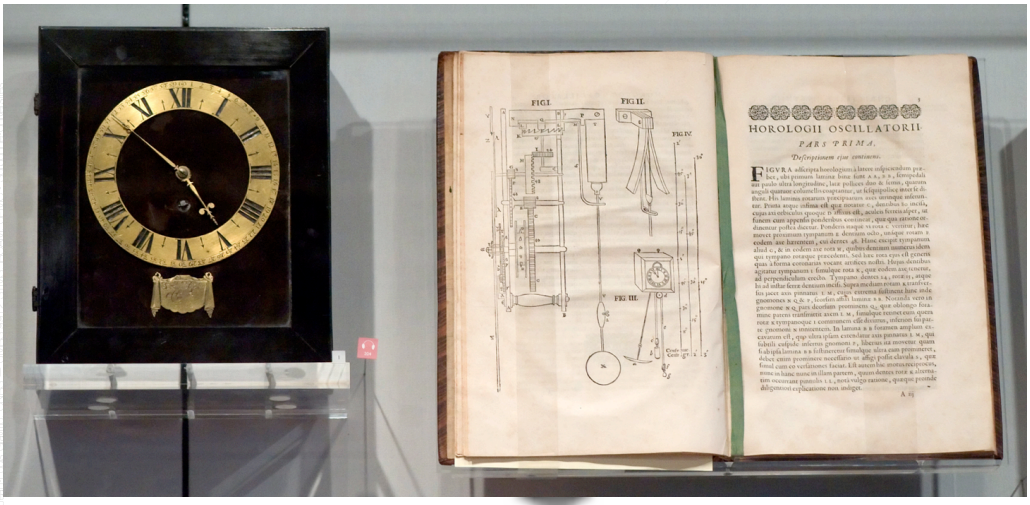
a temporal scope
(temporal coordinate system):
- **shared** events
- an estimation of the fluxion of time (**tempo**)

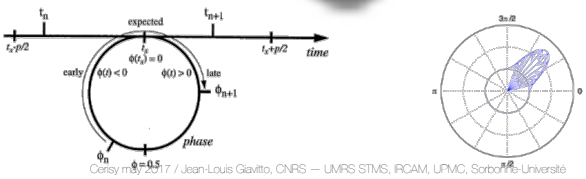
times in Antescofo

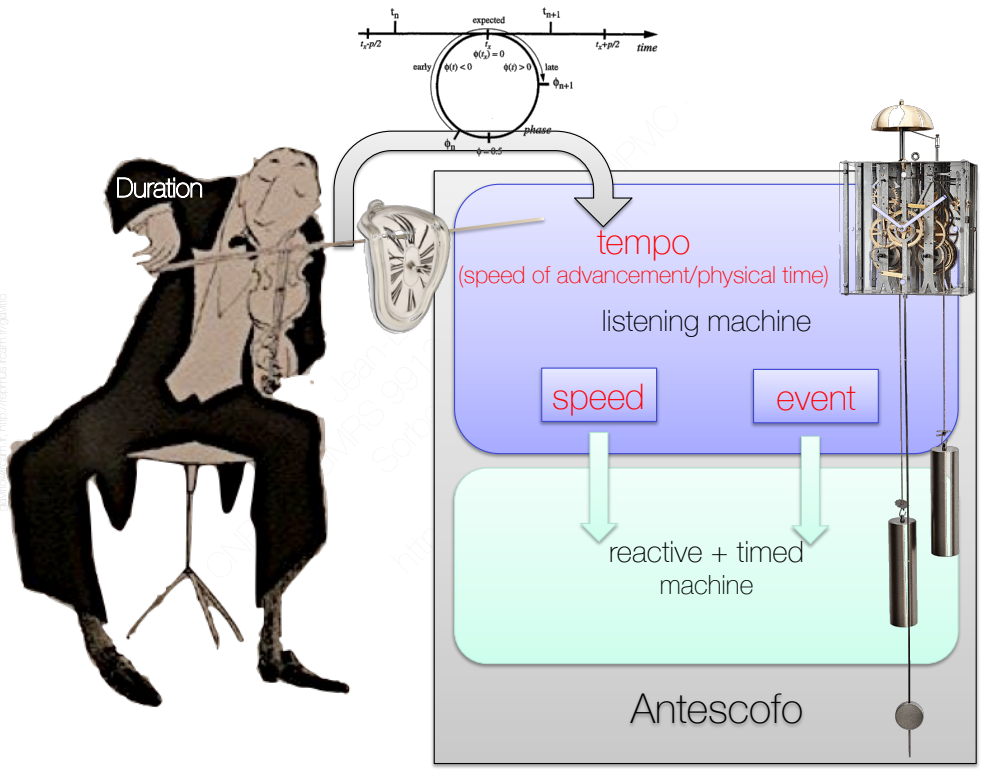
TEMPO EXTRACTION
(Large's algorithm)



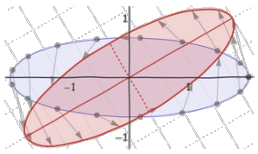
Tempo inference and odd sympathy





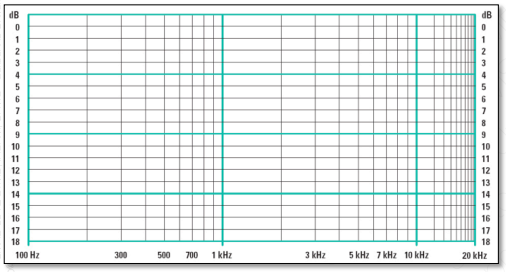


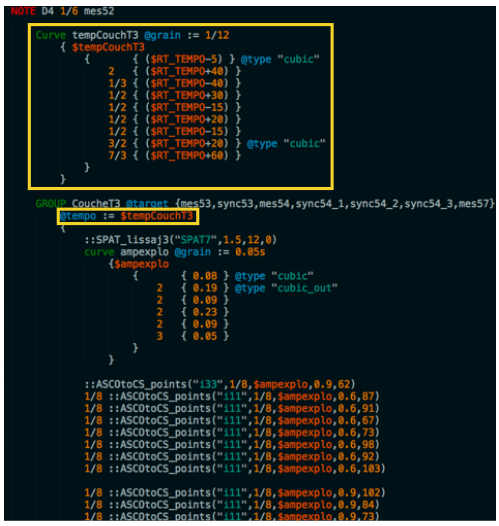
Temporal Scope

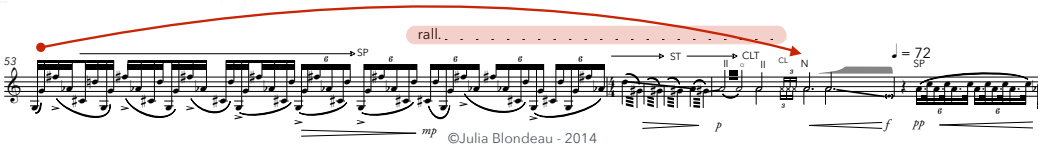


- time transformations are for *Antescofo* what changes of coordinates are for postscript…
- **BUT**
  - time is only spent in real-time
  - time is evenmential and durative
  - time is causal
    (I don't know the transformation in the future)
  - the transformation comes from the environment
    (synchronization)
  - transformations are not necessarily *newtonian*
    (when human is in-the-loop position ≠ ∫ tempo)

Julia Blondeau
Phrasé







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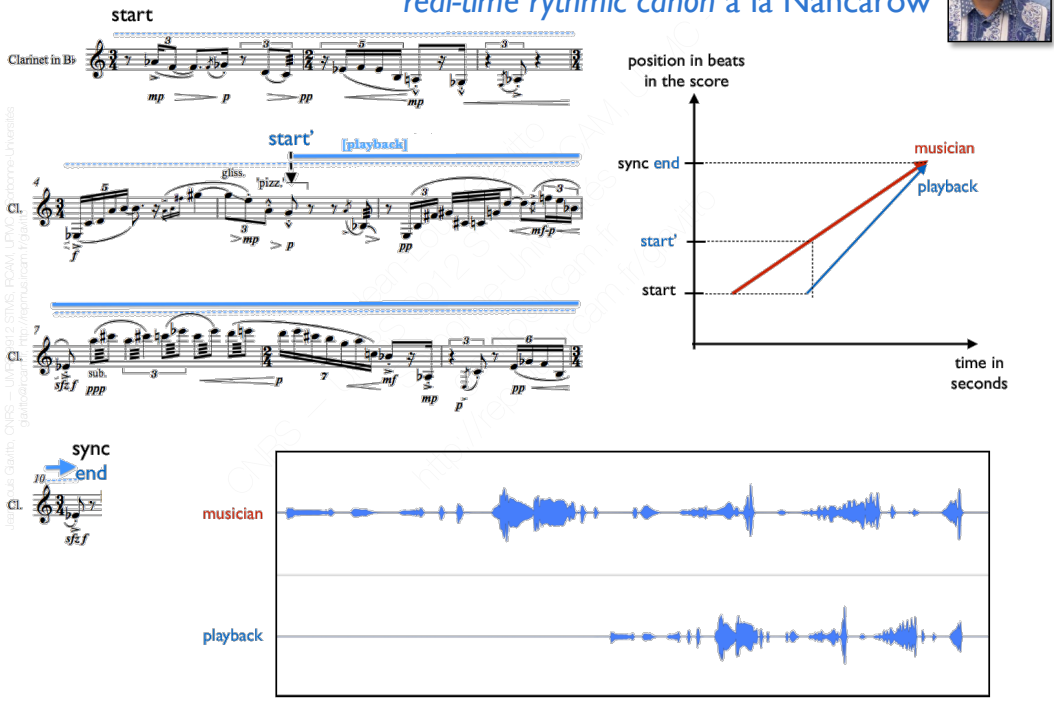
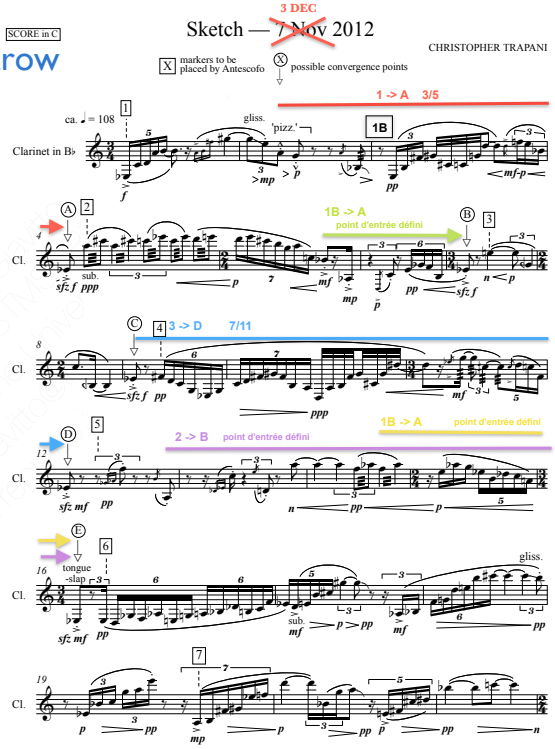
Christopher Trapani
real-time rythmic canon à la Nancarow



Christopher Trapani
real-time rythmic canon à la Nancarow

SCORE in C



José-Miguel Fernandez
gesture-driven synthesis

OSC *or* setvar

GeKiPe (Gest Kinect Percussion),
Philippe Spiesser (percu),
Alexander Vert (composition),
Jose Miguel Fernandez (RIM)

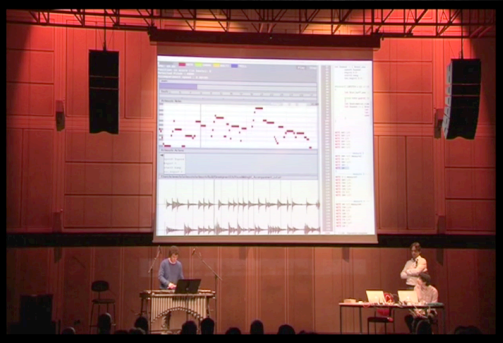
Haute école de musique Genève

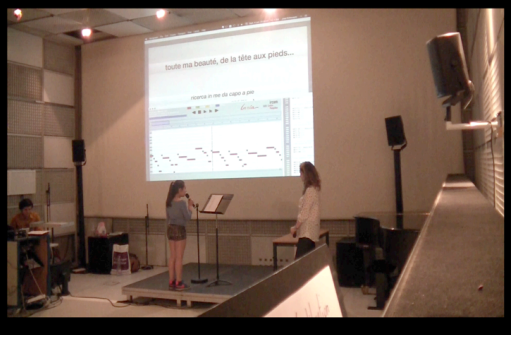
Hypersphère, Jose Miguel Fernandez,
séance de travail IRCAM 26/2/16

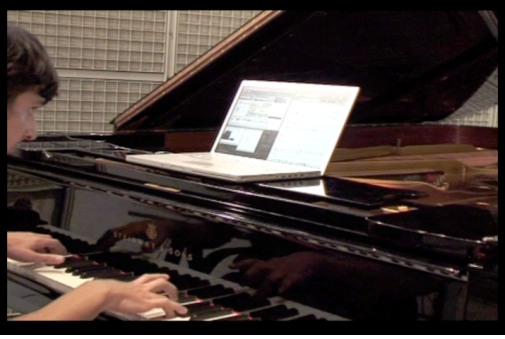
Towards Greater Public

Videos at http://repmus.ircam.fr/antescofo/videos

Antescofo

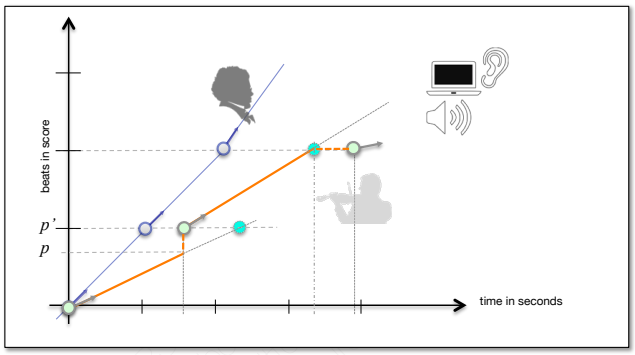






FINAL REMARKS

BEYOND CHRONOMETRIC TIME (IN MUSIC)



FUNGIBLE TIMES
OR
INCOMPARABLE TIMES ?

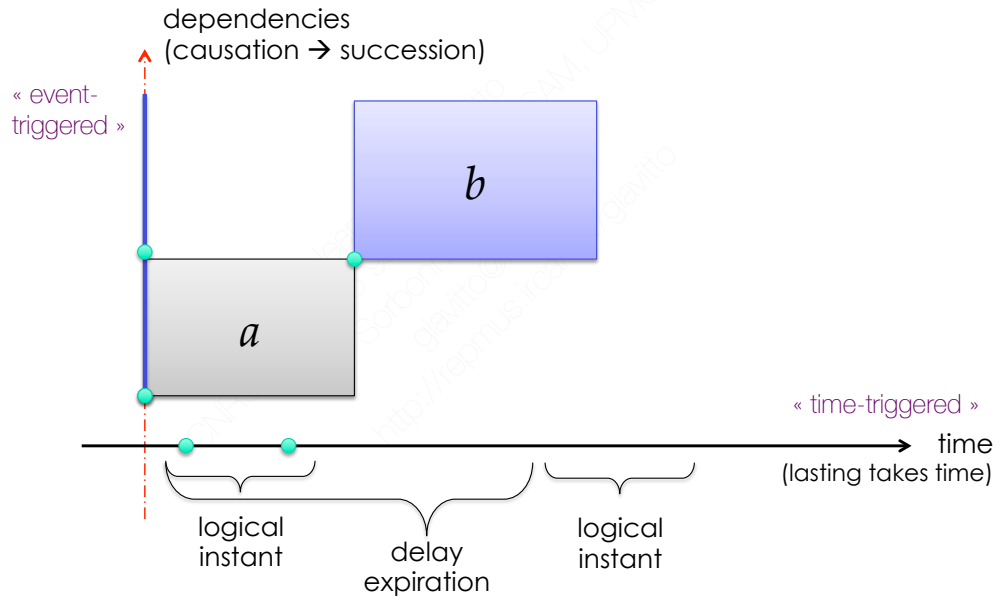
One second per second

Subordination of the objective time to the subjective ones and not the reverse !

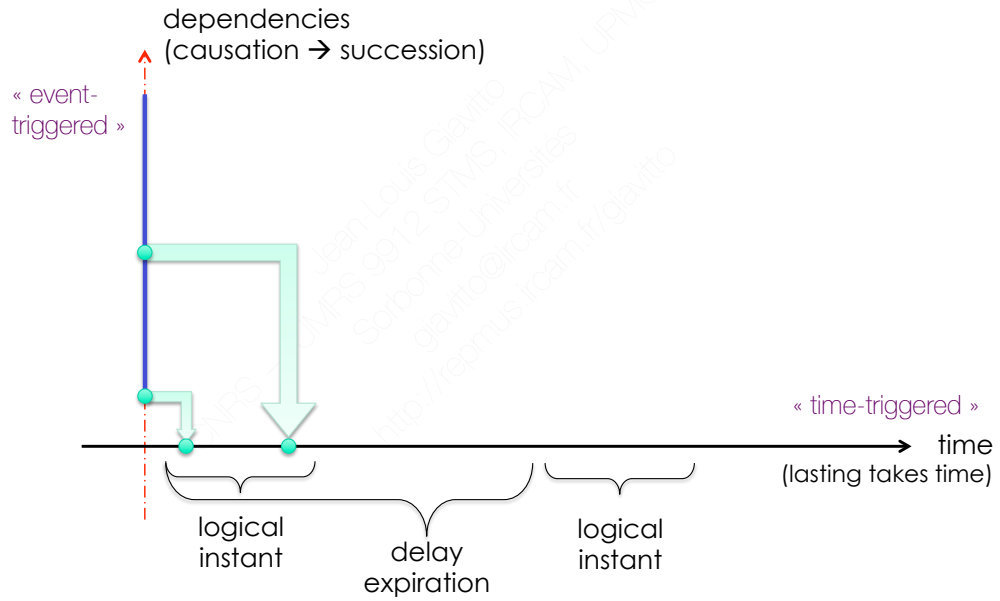
- shared events are not enough:
  duration is not reducible to instants
  - halving a duration
  - accelerando
  - *phrasé* (ex. rubato)
- the "conversion rate" changes in time and is known "after". The conversion rate is *established with the weaving of time itself.*
  - A-series et B-series,
  - "out of time" ("deferred time") of the composition *versus* the real-time of the performance
- Subjective time is useful: the score refers to this subjective time, not to physical time in second
- *In fine*, it enables a effective musical interaction between the performer and the computer

THE PASSING OF TIME AND CAUSALITY

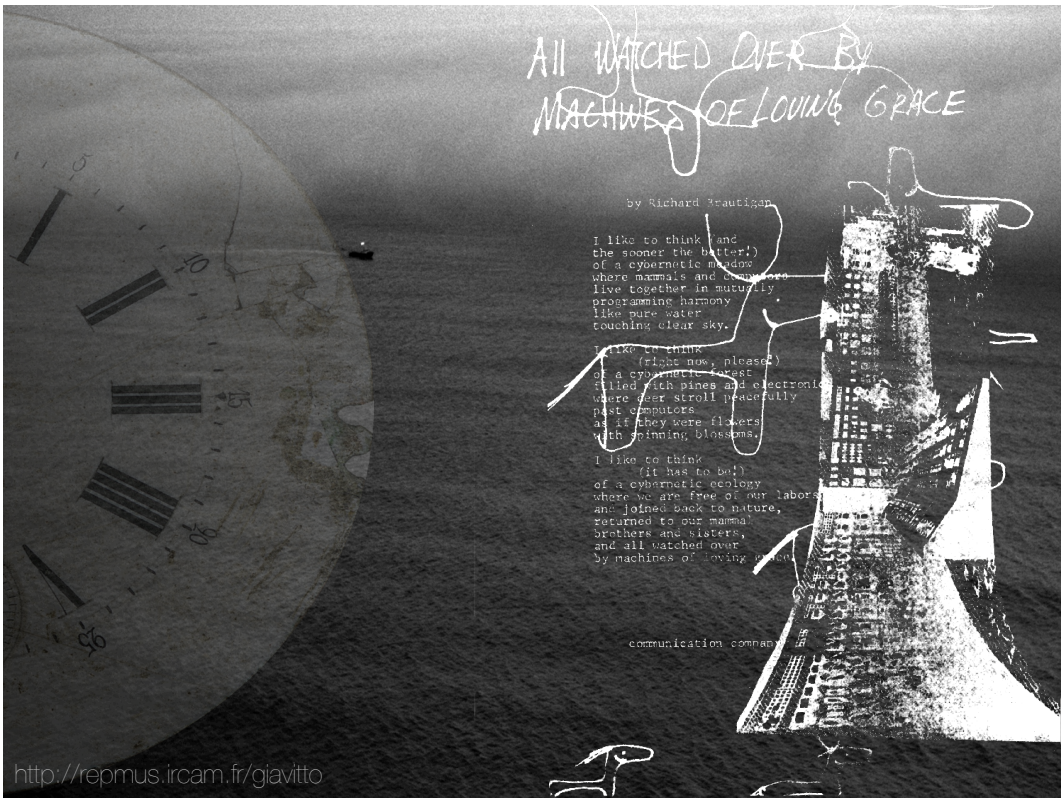
The passing of time: causality & duration



The passing of time: causality & duration



Arshia Cont, Philippe Cuvillier,
José Echeveste, Jean-Louis Giavitto



Credits

Arshia Cont

PhD students

J. Blondeau, P. Cuvillier, J. Echeveste, C. Poncelet

Scientific Collaborations

- SIERRA & PARKAS (ENS), FLOWERS & POSET (Bordeaux), Inria Chile,
- GRAME (Lyon) ...
- and many more: UC Berkeley, UCSD, Salzburg U., Twente U., …

Composers (and their assistants !)

P. Manoury, M. Stroppa, J. Freeman, C. Trapani, J.-M. Fernandez, J. Blondeau, G. Nouno, Y. Maresz, O. Neuwirth, L. Morciano, … T. Goepfer, G. Beller, G. Lorieux... and many more


http://www.antescofo.com/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PhD, Defended 2015


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Useless old machines? On the interest of analog computation today

Maël Pégny. (CNRS, Paris 1, IHPST).
ANR/DFG Project Hypo 3 *Beyond Logic*

Beyond Logic Congress. Cerisy.
Tuesday May 23rd 2017

CAUTION

THIS IS PRELIMINARY WORK: PLEASE DO NOT QUOTE WITHOUT PERMISSION!

CONTACT ME AT: maelpegny@gmail.com

The Digital Age and Its History

We hear *ad nauseam* that we live in a "digital age".
The claim that we live in a digital age is based on the following technical aspects (among others):

1. Digital computers replaced analog computation.
2. Numerical simulations replace experiment and modeling.
3. Analog signal is replaced and/or converted into numerical signal.

Analog computing: contemporary approach

Before they became completely dominant, digital computers co-existed with another model of computation, analog computation (1940-1970).

In contemporary terms, analog computation is usually defined as "computation with continuous parameters" (time, space, or the state of the processor).

In more intuitive terms, analog computation is a computational model with a different data representation: data is encoded into the continuous variables of a physical system, processed by the system dynamics, and retrieved by measurement.

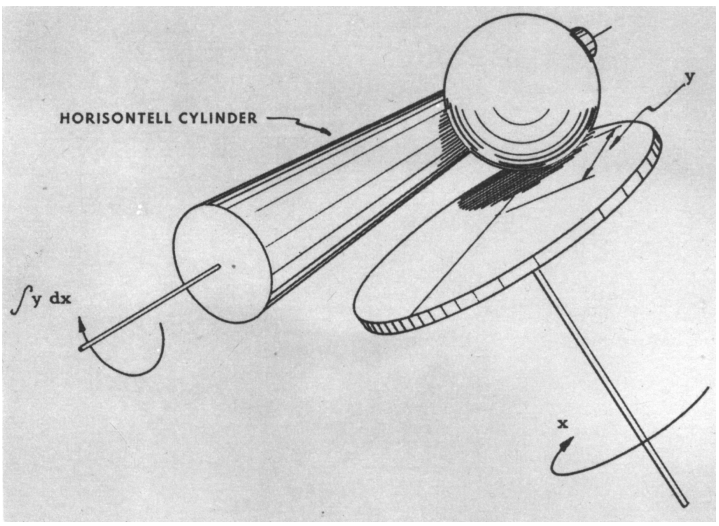
Analog computation as a non-effective model

Why do analog computers belong to a different class than the many variants of digital computation?

Effective computation is an old-fashioned term for an idealized view of pen-and-paper computation, the type of computation practiced by a human computer following the instructions of an algorithm. Digital computers are, at a certain level of abstraction, an automatization of effective computation.

Analog computing was a **non-effective** model of computation. For instance, an analog integrator can perform integration in one step, an utterly impossible operation for effective computation.

Illustration: a ball-and-cylinder mechanical integrator



Historical Relevance of Analog Computing

Analog computing was used in major computing applications, for instance :

- Hydrology and oil reservoirs simulation.
- Nuclear reactors design.
- Power networks simulation.
- Car design.
- Aircraft and guided missiles design and simulation (Ceruzzi, 1989, "the midwife of computer science").

Philosophical points of interest

Numerical simulation has become the topic of a virtual industry in philosophy of science (Hartmann, Humphreys, Morrisson, Frigg...).

The digital vs analog debate has been relatively ignored in philosophy, and has been mainly a topic for historians (machines and models of computation).

An Integrated HPS approach should show that analog computation is actually a topic of philosophical relevance.

On the interest of old useless machines

Interest in alternatives, non-effective models of computation has experienced a regain in interest, mainly because of quantum computing.

What is the point of studying outdated computational models?

- Theoretical (theory of computation): foundations enriched by the study of non-standard models of computation.
- Analog computing is important for the understanding of simulation and modeling.
- Theory and practice: **what was the role of computation theory in the analog vs digital debate?**

Particular point of interest: computation theory and practice

Question: what is the relation between theory and practice in the theory of computation (computability and complexity)?

The analog vs digital debate is the main exemple of a genuine historical debate between two deeply different computational models: quantum computers are not running yet!

It is obvious that theoretical arguments cannot be the only arguments: money, training, and organizational constraints always talk in concrete applications.

A priori, computation theory should have played a role in that debate: interesting for both history and foundations of the discipline.

Outline

1. Meaning and use of analog computing
2. The digital vs analog debate

Two definitions of analog computation

To understand the analog vs digital debate, we have to understand how analog computers were used.

Basic problem raised by the recent historical and philosophical literature: analog computing was **not** primarily understood as computation with continuous parameters.

Charles Care, James Small, Cameron Beebe, Bernd Uldmann all defend that the practice of analog computing is not about a different data representation, but also about **models based on analogy**: discrete models can also be analog models.

Basic principles of analog simulation

Analog "computing" was primarily used by engineers and experimental scientists for modeling and simulation.

Basic idea: In order to facilitate the study of (behavior of) a target-system S, or "target", use (the behavior of) an analog system S_a, also known as "the source".

The nature of the analogy depends on the **nature of the desired knowledge** (behavior of interest, desired accuracy of predictions...) and the **optimized costs** (energy, time, storage space, money, amount of material, organizational planning, exhausting or fastidious work, safety and/or environmental hazards...).

Two types of analogy

- **Material analogy**: target system S and analog system S_a abide to the same (relevant) physical laws.
  Ex: scale models. The system bridge + wind and wood bridge model + wind obeys the same relevant laws of hydrodynamics.
- **Formal analogy**: S and S_a obey similar equations, even if the physical laws are different.
  Ex: harmonic oscillator in electrodynamics and mechanics.
  Spring-mass oscillator and RLC circuit.

Rise in generalization: the use of mathematical analogy allows to treat a class of systems ruled by analog equations, not just systems ruled by the same physical laws. You can switch from a problem to another by a modification of parameters.
As C. Care has demonstrated, "analog models" in this sense were extremely common in the 20-30s.

From formal analogy to computation

Instead of using the formal analogy to study a target-system, let's use a system to compute the solution of a class of equations.

We are not aiming anymore at physical knowledge on the target-system S, but at **mathematical knowledge** on a class of equations: the analog system S_a has become an analog computer (ex. of that evolution: MIT Network Analyser).

"**General purpose analog computer**" (vs digital general purpose computer): a computer which can solve any differential equation that an analog computer can solve.
The notion of an "analog computer" = late 30s-40s. Work on general purpose analog computers like the famous Differential Analyzer, and their comparison with the new digital computer. But the original meaning of "analog models" remained in use.

Continuity and modeling

From a historical point of view, "analog model" denotes both continuous models of computation **and** experimental devices set up for the study of some system through a theoretical analogy.

There is no competition between "two definitions" of analog model: these are two orthogonal dimensions necessary to describe both historical and contemporary practice.

While general arithmetical computation and equation solving was quickly digitalized, the more lasting analog models were used by scientists and engineers for **modelization** purposes.

Advantages of analog computation

The advantages of analog computation was often not conceived in terms of computational performances, but in terms of modeling use:

- Efficient tool for vizualisation;
- Interactive work with the source allowing to develop an intuitive "feeling" and "insight" for the behavior of the target;
- Analogy between the dynamics of the source and the behavior of the target.

Ex: based on the formal analogy between "electric current" and "hydrodynamic flow", electric analog models were intensively used in hydrology, oil reservoirs simulation, and aircraft wing design.

Moniac as a vizualisation tool for money flow in Keynesian Economics

Contemporary relevance of analog modeling

Analogy modeling is far more being eliminated from contemporary research practice:

- (material analogy) Engineering: Millau Viaduct (2004), Olympiastadion, München (1972);
- (formal analogy) Analog hydrodynamical model for black holes (2010s).

Whatever its epistemological value, numerical simulation has not been substituted for all other forms of modeling: analog modeling is alive and well.

Millau Viaduct: use of a wood model for study of wind resistance

Olympiastadion: use of soap bubbles to minimize roof surface

Historical context

From the early 50s on, the claim that the digital model would ultimately replace analog computers became pervasive.

Analog computers were still built and used for about 30 years, sometimes in an hybrid setting.

Proponents of the analog argued for the **complementarity** of the two models.

Suming up the main arguments

Main arguments:

- Quantitative: speed (real-time analog computation) against precision (numerical precision vs measurements limits);
- Universality: digital computation is a **genuine** general purpose model: it can compute every computable function;
- Qualitative:
  - Analog model: interactivity, development of a "feeling" for the target-system;
  - Digital model: ease of programming with the new levels of abstraction.

Lack of a theoretical framework: computability

Proponents of digital computation said that they had a truly general purpose computer (CT thesis):

- TM machines were understood as a model of digital computers only in the late 50s-early 60s.
- Having a general purpose computer is not necessarily a master argument on its own. Some special purpose applications do not need universality.
- The argument is false (Bournez and alii, 2010): one can have a **Turing universal general purpose computer**.

Lack of a theoretical framework: computational complexity II

It was difficult to make a theoretical comparison of complexity performances between the two models for several reasons:

- Computational Complexity theory was still in its infancy when most of the competition took place (NP-complete problems: early 1970s).
- Rigorous definition of complexity measures for analog computation is extremely recent (Amaury Pauly's dissertation, 2015).
- The comparison involved the comparison of different resources on different computational models: real-time computation does not grow with the size of the input but the number of elements does.

Difficulty with practical computational arguments

Definition of comparative speed was difficult. One operation was not enough, but the combinaison of operations necessary to carry out different problems could be complex and vary from device to device: task-to-task comparison.

Growth in computing elements made the computational set-up more and more difficult, and the ability to draw an intuition out of it was weakened. That's why a digital interface to automatically set-up the computer was sometimes created, defeating the initial mottos of analog computers: **interaction between quantitative and qualitative aspects**.

Other arguments in favor of analog computers

Arguments rooted in the context of use were very important in the actual historical debate:

- Analog computers were powerful tools of vizualisation and modelization. Good understanding could trump numerical precision (Small, 18).
- Artificiality introduced in your simulation by the discretization process. A purely computational process does not give us a peak into the dynamics of your target system.
- Aircraft design general-purpose vs special-purpose (UK): general-purpose will be cheaper to buy and maintain vs a special purpose computer neednot be shared.
- Weighting of the cost of training against the actual advantages brought by numerical simulation.
- Ease of programming (conception and modularization of the simulation).

Decisive arguments

Many partisans of analog computing were trained in the culture of engineering design, and they were not so sensitive to general theoretical arguments (Small, 248). They explicitly argued that the debate could not be solved by a general, theoretical argument, but by a case-by-case examination: decision based on the context of use.

According to Care's account, they were partly right! Context-sensitive arguments were decisive in the final victory of the digital model:

- Digitalization of data;
- Development of good vizualisation tools;
- Sound numerical simulations algorithms (DFT) and programming languages.

The development of a good digital environment for numerical simulation was another factor in the ultimate triumph of digital computing.

A complete story?

The story we told shows a debate where computation theory did not, and could not play a decisive role.

Is the story complete? Care and Small's accounts are based mainly on the British and American cases, and arguments should be studied in all details.

Theory and practice: a long road ahead

Computability and complexity type arguments played a decisive role in that debate, but alongside more context-sensitive arguments.

Future direction: rewrite the history of that debate from our current theoretical perspective.

From proofs/programs to natural language dialogues modelling

Myriam Quatrini

23 May 2017

Sommaire

1. Ludics, a logical theory arising in the proofs/programs context
   - Recent developments of Proof Theory in a nutshell
   - A few words on Ludics
2. A modelling of natural language dialogue in Ludics
   - Dialogue surface
   - Cognitive Bases
   - Tools for argumentation

Curry-Howard Isomorphism

Programming	Specifications	Programs	Evaluation program/ entries	Execution
λ**-calculus**	Types	λ-terms	Application $(\lambda x M)N$	Normalisation $M[N/x]$
Logic	Formulas	Proofs	Cut	Cut Elimination

A perfect correspondance

- Not only between objects.
- But also with respect to the dynamics.

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Logic as a model of calculus

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The main object of Proof Theory moved

- From *formulas*
- To *proofs*
- Even more, to the *cut elimination* and to the properties by means of which cut elimination may provide a *"good" model of calculus*.

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Myriam Quatrini From proofs/programs to natural language dialogues modelling

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A crucial step in this evolution: Linear Logic (Girard 1985)

A step further towards new objects in Proof Theory

- Refinement of Formulas:
  $A \wedge B$ becomes either $A \otimes B$, or $A \& B$,
  According to the identification/juxtaposition of contexts.
- A new expression of Proofs:
  - Proof nets,
  - Hypersequentialised sequent calculus.
- The formulation of a program: (*GoI*)
  the expression of proofs giving an account of the dynamics of cut elimination.

Ludics (Girard 2001)

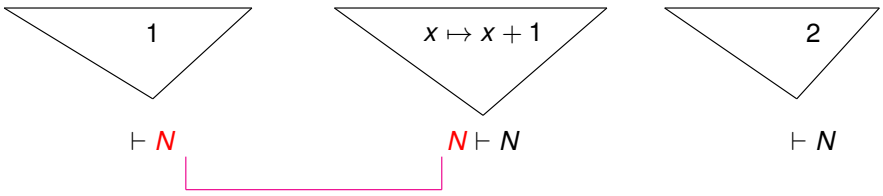
- *A logical theory* :
  a result/step towards the expression of proofs giving an account of the cut elimination dynamics
- *Obtained via an "ontological" reverse*:

Usually Formulas $\rightarrow$ Proofs $\rightarrow$ Cut and cut elimination

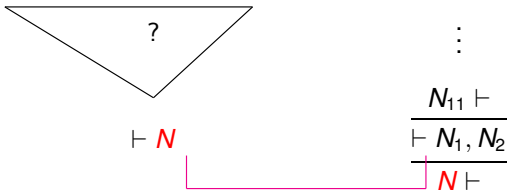
In Ludics Behaviours $\leftarrow$ Designs $\leftarrow$ Interaction Interaction process

Two aspects of cut elimination as interaction

- Calculus

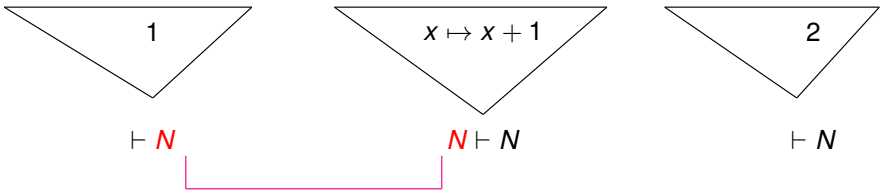


- Proof search



Two aspects of cut elimination as interaction

- Calculus



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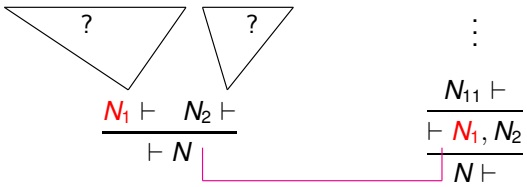

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Two aspects of cut elimination as interaction

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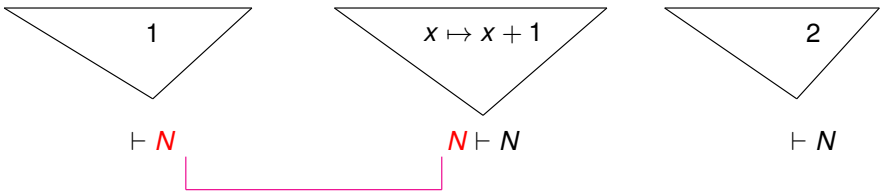

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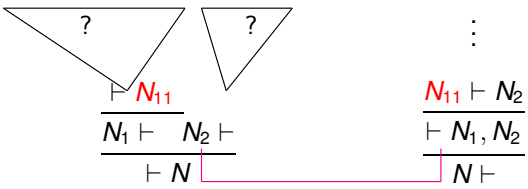

Towards an abstraction of Proofs

By means of the properties that Linear Logic makes explicit:

- Polarity
  To prove $\vdash \Gamma, A\&B$, you have to prove $\vdash \Gamma, A$ and to prove $\vdash \Gamma, B$.
  There is no choice, no commitment. The rule and the connective are said negative.
  To prove $\vdash \Gamma, A \otimes B$, you have to prove $\vdash \Delta, A$ and to prove $\vdash \Sigma, B$.
  You have to separate the context in two parts. The rule and the connective are said positive.

- Focalisation
  Roughly speaking: if a formula is provable, it has a focalized proof, i.e. alternated sequence of positive and negative steps, decomposing in only one big step a formula until its subformas on opposite polarity.

Towards an abstraction of Proofs as **Designs**

Then, only **cut-free, focalized** proofs,

And a radical addition:

- **Daimon** rule (†).

$$\dfrac{\dfrac{\dfrac{}{\vdash N_{11}}\dagger}{N_1 \vdash}\quad\dfrac{\dfrac{}{\vdash N_{22}}\emptyset\quad\dfrac{}{\vdash N_{23}}\emptyset}{N_2 \vdash}}{\vdash N}\qquad\dfrac{N_{11} \vdash N_2}{\vdash N_1, N_2}\over{N \vdash}$$

Therefore: **cut-free, focalized** proof-like objects

Instead of Proofs: **Designs** as supports of interaction

We keep only what is relevant

for cut elimination/proof search.

$$\dfrac{\dfrac{\dfrac{}{\vdash N_{11}}\dagger}{\neg N_{11}\vdash}\quad\dfrac{\dfrac{}{\vdash N_{22}}\emptyset\quad\dfrac{}{\vdash N_{23}}\emptyset}{\neg(N_{22}\&N_{23})\vdash}}{\vdash(\neg\neg N_{11})\otimes(N_{22}\&N_{23})}\quad\rightarrow\quad\dfrac{\dfrac{}{\vdash L_{11}}\dagger}{L_1\vdash}\quad\dfrac{\dfrac{}{\vdash L_{22}}\emptyset\quad\dfrac{}{\vdash L_{23}}\emptyset}{L_2\vdash}}{\vdash L}$$

Instead of formulas, their addresses:

L instead of $N = N_1 \otimes N_2$
L_1 instead of $N_1 = \neg N_{11}, \ldots$

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Instead of rules, actions:

$(+, L, \{1, 2\})$ means that you decompose the formula $N_1 \otimes N_2$ in its two subformulas

$(-, L_1, \{1\})$ means that you decompose the formula N_1 in its subformula N_{11}

Instead of Proofs: **Designs** as supports of interaction

We keep only what is relevant

for cut elimination/proof search.

$$\dfrac{\dfrac{\dfrac{}{\vdash N_{11}}\dagger}{\neg N_{11}\vdash}\quad\dfrac{\dfrac{}{\vdash N_{22}}\emptyset\quad\dfrac{}{\vdash N_{23}}\emptyset}{\neg(N_{22}\&N_{23})\vdash}}{\vdash(\neg\neg N_{11})\otimes(N_{22}\&N_{23})}\quad\rightarrow\quad\dfrac{\dfrac{}{\vdash L_{11}}\dagger}{L_1\vdash}\quad\dfrac{\dfrac{}{\vdash L_{22}}\emptyset\quad\dfrac{}{\vdash L_{23}}\emptyset}{L_2\vdash}}{\vdash L}$$

A special action † to stop the interaction/to give up a proof search

Two modes of interaction

- **The open case:** (the usual one) a process towards a result obtained once a calculus is done.

$$\dfrac{\begin{array}{cc}\vdots & \vdots\\ \vdash L & L\vdash R\end{array}}{\vdash R}$$

- **The closed case:** a process going through two sequences of actions, as long as they are dual each other.

$$\dfrac{\begin{array}{cc}\vdots & \vdots\\ \vdash L & L\vdash\end{array}}{\vdash}$$

Interaction between two designs is said **convergent** when it ends on †. Then, the designs are said **orthogonal**.

Features of Ludics

- Ludics is a theory of interaction (according to two modes) :

  - The **closed mode** enables to retrieve the concepts of Game Theory: designs are strategies

  - The **open mode** enables to retrieve the Computation Theory: designs are λ-terms, closed sets of designs are types.

- Ludics is a logical theory :
  Designs are proofs, closed sets of designs are formulas.

One retrieves logical concepts as non primitive ones

- One retrieves formulas amongs sets of designs: the set which are closed with respect to the biorthogonality., called **behaviours**. :
  $\mathbf{A} = \mathbf{A}^{\perp\perp}$.
- A **good design** belongs to the interpretation of a formula iff it is the interpretation of a proof.
- One retrieves the notion of valid formula: a formula is true when its associated behaviour **contains a good design**.

$\rightarrow$ *It is possible to define new connectives.*
$\rightarrow$ *Amongs the justifications of an assertion, some are proofs, some are not.*
$\rightarrow$ *The logical form may be refined by new justification.*

The dialogue modelling in Ludics

is done in two steps (corresponding to the two modes of interaction).

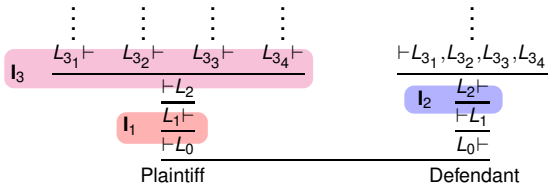
- **Dialogues surface**: the flow of interventions between two protagonists,
  One design representing the dialogue for each protagonist
- **Cognitive Bases** Objects enabling to describe how interventions are built, how they are received, how they are recorded, how calculi are done
  One set of designs for each protagonist

Dialogue as an interaction between two strategies

A dialogue is the interaction between two designs, one for each protagonist. Where:

- The **actions**(dialogue acts) are the primitive elements of interventions,relying interventions to previous ones, opening discussion threads.
- A dialogue act is positive for the locutor who produces it and negative for the locutor who receives it.

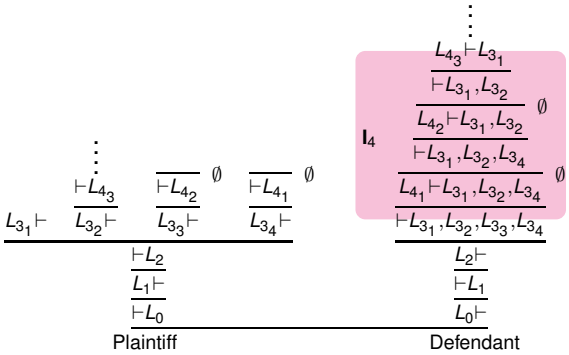
An example: the one developt by H. Prakken, a juridic debate between *plaintiff* and *defendant*



I_1 : I claim that defendant owes me 500 euro.
I_2: I dispute plaintiff's claim.
I_3 : Defendant owes me 500 euro by r_1 since we conclude a valid sales contract, I delivered but defendant did not pay.

The fourth intervention

I_4: I concede that plaintiff delivered and I did not pay, but I dispute that we have valid contract.



Principles for accounting for a juridic debate

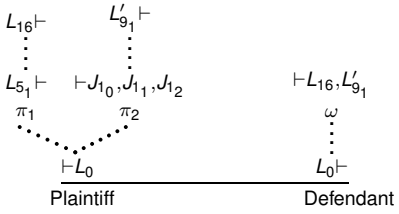
Principles

1. *An account of the juridic debate as controverse between only two parties.*
   The interventions due to the Judge occur as intervention of one of the two parties, according to the turn of speech.
2. *The sentence: both interacting designs are completed,*
   - still open branches are closed
   - the last concession has to be followed by a daimon, making explicit who is the looser.

By this way, we account for:
- the distribution of the burden to prove,
- the sentence.

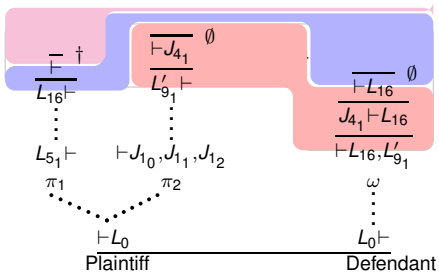
La sentence

Before



It is the defendant's turn. Two loci are available:
- L'_{9_1} (*his signature is not valid*)
- L_{16} (*his illness*)

After



I am convinced by plaintiff's evidence that defendant's signature under the contract is authentic.
Yet I cannot grant plaintiff's claim
since the fact that defendant looked normal during the negotiations is insufficient to conclude that defendant's insanity could not be known to plaintiff: he might have known if he had checked the court's register.
Therefore I deny plaintiff his claim.

Second part of the modelling

For accounting for the context of the dialogue:

A cognitive base for each locutor

in which **designs** are:
- associated with his knowledges, his commitments, his inferential calculi ...
- updated after each intervention.

Focus on intervention I_3

Example

I_3 : Defendant owes me 500 euro by r1 since we conclude a valid sales contract, I delivered but defendant did not pay.

- **In the unfolding dialogue:**

$$\frac{\begin{array}{cccc}\mathcal{D}_{r_1} & \mathcal{D}_{val-contrat} & \mathcal{D}_{del} & \mathcal{D}_{not-pay}\\ \vdots & \vdots & \vdots & \vdots \\ L_{3_1} \vdash & L_{3_2} \vdash & L_{3_3} \vdash & L_{3_4} \vdash\end{array}}{\vdash L_2}$$

- **In plaintiff's base**, some designs

$$\mathcal{D}_{r_1}, \mathcal{D}_{val-contrat}, \mathcal{D}_{del} \text{ and } \mathcal{D}_{not-pay},$$

on which rests his argumentation.

Designs in Plaintiff's cognitive base

$$\dfrac{\dfrac{\dfrac{}{\vdash L_{sign}}\emptyset}{L_{?sign} \vdash}}{\vdash L_{val-contrat}}$$

making explicit the contract validity which rests on a signature

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\vdash L_{Pay}}\emptyset}{L_{Del} \vdash, L_{Pay}}\emptyset}{\vdash L_{?Del}, L_{Pay}}}{L_{sign} \vdash L_{?Del}, L_{Pay}}\emptyset}{\vdash L_{?sign}, L_{?Del}, L_{Pay}}}{L_{val-contrat} \vdash L_{?Del}, L_{Pay}}}{\vdash L_{?val-cont}, L_{?Del}, L_{Pay}}}{L_{Cont-and-Del} \vdash L_{Pay}}$$

the expression of the law r_1 conditions entailing the obligation to pay

Interactions in Plaintiff's base

$$\dfrac{\dfrac{\dfrac{}{\vdash L_{sign}}\emptyset}{L_{?sign} \vdash}}{\vdash L_{val-contrat}} \qquad \dfrac{\dfrac{}{\vdash L_{Del}}\emptyset}{L_{?Det} \vdash}$$

$$\dfrac{\vdash L_{val-contrat} \qquad L_{?Det} \vdash}{L_{?Cont} \vdash}$$

$$\vdash L_{Cont-and-Del}$$

and

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\vdash L_{Pay}}\emptyset}{L_{Del} \vdash, L_{Pay}}\emptyset}{\vdash L_{?Del}, L_{Pay}}}{L_{sign} \vdash L_{?Del}, L_{Pay}}\emptyset}{\vdash L_{?sign}, L_{?Del}, L_{Pay}}}{L_{val-contrat} \vdash L_{?Del}, L_{Pay}}}{\vdash L_{?Cont}, L_{?Del}, L_{Pay}}}{L_{Cont-and-Del} \vdash L_{Pay}}$$

The interaction gives as result: $\dfrac{}{\vdash L_{Pay}}\emptyset$, Defendant has to pay.

And not in Defendant's base

- The contract validity is negated, by means of a first argument: the signature, then a second one: defendant insanity.
- Even if the validity of the signature is finally conceded, the calculus of plaintiff can not be done
- The contract validity has not only a sub-formula but two sub-formulas: the validity of signature **and** the sanity of contractants.

Correction of Plaintiff's design

The contract validity has not only a sub-formulas but two sub-formulas: the validity of signature **and** the sanity of contractants.

$$\dfrac{\dfrac{\dfrac{}{\vdash L_{sign}}\emptyset}{L_{?sign}\vdash} \quad \dfrac{\dfrac{}{\vdash L_{sanity}}\emptyset}{L_{?san}\vdash}}{\vdash L_{val-contrat}} \quad \text{instead of} \quad \dfrac{\dfrac{}{\vdash L_{sign}}\emptyset}{L_{?sign}\vdash}\ \vdash L_{val-contrat}$$

Summary

In this formal frame **we may retrieve several manifestations of the logical dimension of natural language.**

- Via the closed mode:
  - the notion of proposition: a sentence which may be true or false;
  - the notion of argument;
  - a BHK semantics of utterances as set of their justifications
- Via the open mode;
  - the notion of modus ponens, logical inference,
  - updating (algorithmic dimension of Logic).

Ludics, a logical theory arising in the proofs/programs context
A modelling of natural language dialogue in Ludics
Dialogue surface
Cognitive Bases
Tools for argumentation

Summary

By separating dialogue surface and cognitive bases, we succed in departing several levels: logical, argumentative, rhetorical.

- The utterances occur at the dialogue surface via addresses. The explicitation as proposition is not necessarily **the same** in both cognitive bases.
- The inferences are done in cognitive bases. Their implications are imposed to all, only once the designs, supports of interaction are shared.
- Therefore, it is possible to unfold controversies and to observe which branches **either are closed or are still open**, and therefore to account precisely for the **nature of divergences**.

Thank you for your attention !

UNFOLDING PARALLEL REASONING IN ISLAMIC JURISPRUDENCE (I)

Epistemic and Dialectical Meaning in

Abū Isḥāq al-Shīrāzī's System of Co-Relational Inferences of the Occasioning Factor

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To the memory of my late Great-Uncle ***Alama Khalil Bin Mohammad Arab*** and to his son ***Yahya Ansari***

S. Rahman

Abstract: One of the epistemological results emerging from this initial study is that the different forms of *co-relational inference*, known in the Islamic jurisprudence as *qiyās*, represent an innovative and sophisticated form of reasoning that not only provides new epistemological insights into legal reasoning in general but also furnishes a fine-grained pattern for *parallel reasoning* which can be deployed in a wide range of problem-solving contexts and does not seem to reduce to the standard forms of analogical argumentation studied in contemporary philosophy of science. In the present paper we will only discuss the case of so-called *co-relational inferences of the occasioning factor*.

I Introduction

Uṣūl al-fiqh (أصول الفقه), that is, Islamic Legal Theory, is deeply rooted in the notion of rational knowledge and understanding. Indeed, *uṣūl al-fiqh* constitutes the body of knowledge and methods of reasoning that Islamic jurists – led by the aim of delving into God's intended norms for human conduct – deploy in order to provide solutions to legal problems based on the juridical understanding of the sources. According to *uṣūl al-fiqh*, legal knowledge is achieved by rational endeavour, the intellectual effort of human being: this is what is meant when the term *ijtihād* (اجتهاد), endeavour of the intellect, is attached to *fiqh*. Let us quote the beautiful paragraph on *ijtihād* by Wael B. Hallaq in his landmark work *A History of Islamic Legal Theories* (1997, p. 117).

> *In his Mustaṣfā Ghazali depicts the science of legal theory in terms of a tree cultivated by man. The fruits of the tree represent the legal rules that constitute the purpose behind planting the tree; the stem and the branches are the textual materials that enable the tree to bear the fruits and to sustain them. But in order for the tree to be cultivated, and to bring it to bear fruits, human agency must play a role.* […]. *We shall now turn to the "cultivator," the human agent whose creative legal reasoning is directed toward producing the fruit, the legal norm. The jurist* ***(faqīh)*** *or jurisconsult* ***(muftī)*** *who is capable of practising such legal reasoning is known as the* ***mujtahid****, he who exercises his utmost effort in extracting a rule from the subject matter of revelation while following the principles and procedures established in legal theory. The process of this reasoning is known as* ***ijtihād****, the effort itself.*

One of the most remarkable features of the practice of *ijtihād* is that it presupposes that *fiqh* is dynamic in nature. Indeed, since the ultimate purpose of such a kind of rational endeavour is to achieve decisions for new circumstances or cases not already established by the juridical sources, the diverse processes conceived within Islamic jurisprudence were aimed at providing tools able to deal with the evolution of the practice of *fiqh*. This dynamic feature animates Walter Edward Young's (2017) main thesis as developed in his book *The Dialectical Forge: Juridical Disputation and the Evolution of Islamic Law*.[1] In fact the main claim underlying the work of Young is that the dynamic nature of *fiqh* is put into action by both the dialectical understanding and the dialectical practice of legal reasoning. The following lines of Young (2017, p.1) set out the motivations for the development of a dialectical framework such as the one we are aiming at in the present paper. [2]

> *The primary title of this monograph is "The Dialectical Forge," and its individual terms provide a suitable launching point for discussing the current project as a whole. As for the first, the most common Arabic terms for "dialectic" are* jadal *and* munāẓara, *both denoting formal disputation between scholars in a given domain, with regard to a specific thesis. When one encounters the term "dialectical" in the present work, one should think foremost of procedure-guided debate and the logic inherent to this species of discourse. A dialectical confrontation occurs between two scholars, in question and answer format, with the ultimate aims of either proving a thesis, or destroying it and supplanting it with another. A proponent-respondent introduces and attempts to defend a thesis; a questioner-objector seeks (destructively) to test and undermine that thesis, and (constructively) to supplant it with a counter-thesis. Through progressive rounds of question and response the questioner endeavours to gain concession to premises which invalidate the proponent's thesis, justify its dismantling, and provide the logical basis from which a counter-thesis necessarily flows.*
>
> *Ultimately, and most importantly, a truly dialectical exchange—though drawing energy from a sober spirit of competition—must nevertheless be guided by a cooperative ethic wherein truth is paramount and forever trumps the emotional motivations of disputants to "win" the debate. This truth-seeking code demands sincere avoidance of fallacies; it views with abhorrence contrariness and self-contradiction. This alone distinguishes dialectic from sophistical or eristic argument, and, in conjunction with its dialogical format, from persuasive argument and rhetoric. And to repeat: dialectic is formal—it is an ordered enterprise, with norms and rules, and with a mutually-committed aim of advancing knowledge.*

According to this perspective, the practice of *ijtihād* takes the form of an interrogative enquiry where the intertwining of giving and asking for reasons features the notion of meaning that grounds legal rationality.[3] More precisely, the conception of legal reasoning developed by Islamic jurisprudence is that it is a combination of deductive moves with hermeneutic and heuristic ones deployed in an epistemic frame. Let us once more quote Hallaq (1997, p. 82):

> *Armed with the knowledge of hermeneutical principles, legal epistemology and the governing rules of consensus, the mujtahid is ready to undertake the task of inferring rules. Inferring rules presupposes expert knowledge in hermeneutics because the language of the texts requires what may be called verification; namely, establishing, to the best of one's ability, the meaning of a particular text as well as its relationship to other texts that bear upon a particular case in the law. For this relationship, as we have seen, may be one of particularization, corroboration or abrogation. Before embarking on inferential reasoning, the mujtahid must thus verify the meaning of the text he employs, and must ascertain that it was not abrogated by another text. Knowledge of the principles of consensus as well as of cases subject to the sanctioning authority of this instrument is required to ensure that the mujtahid's reasoning does not lead him to results contrary to the established consensus in his school. This knowledge is also required in order to ensure that no case that has already been sanctioned by consensus is reopened for an alternative rule.*

[1] Young (2017, pp. 21-32) acknowledges and discusses his debt to the work of Hallaq in many sections of the book.

[2] Also relevant are the following lines of Hallaq (1997, pp. 136-137), quoted by Young (2017, p. 25):
In one sense, dialectic constituted the final stage in the process of legal reasoning, in which two conflicting opinions on a case of law were set against each other in the course of a disciplined session of argumentation with the purpose of establishing the truthfulness of one of them. The aim of this exercise, among other things, was to reduce disagreement (ikhtilāf) among legists by demonstrating that one opinion was more acceptable or more valid than another. Minimizing differences of opinion on a particular legal question was of the utmost importance, the implication being that truth is one, and for each case there exists only one true solution.

[3] See too Hallaq (1987a,b, 2004, 2009a,b). Another early study that stressed this point is Larry Miller's (1984) PHD thesis of 1984 on the development of dialectic in Islam. Hassan Tahiri (2008, pp. 183-225) discusses the crucial role of dialectical reasoning for astronomy and for the development of sciences in general – Tahiri (2014, 2015, 2016).

In fact, the dissatisfaction with the efficiency of the standard post-Aristotelian notion of syllogism in jurisprudence led to an ambitious dialectical frame for argumentation by parallelisms (including exemplification, symmetry and analogy) which should offer a new unifying approach to epistemology and logic for the practice of *ijtihād*.[4] The finest outcome of this approach to legal reasoning within *fiqh* is the notion of *qiyās* (قياس), known as *co-relational inference* (Young, 2017).[5]

The aim of co-relational inferences is to provide a rational ground for the application of a juridical ruling to a given case not yet considered by the original juridical sources. It proceeds by combining heuristic (and/or hermeneutic) moves with logical inferences. The simplest form follows the following pattern:

> In order to establish if a given juridical ruling applies or not to a given case, we look for a case we already know that falls under that ruling – the so-called source-case. Then we search for the property or set of properties upon which the application of the ruling to the source-case is grounded. If that grounding property (or set of them) is known, we ponder if it can also be asserted of the new case under consideration. In the case of an affirmative answer, it is inferred that the new case also falls under the juridical ruling at stake, and so the range of its application is extended.

Complications arrive when the grounds behind a given juridical ruling are not explicitly known or even not known at all. In such a case, other devices are put into action. The latter situation, as discussed in the next sections, yields a system of different forms of *qiyās* that are hierarchically organized in relation to their epistemic strength.

More generally, one interesting way to look at the contribution of the inception of the juridical notion of *qiyās* is to compare it with the emergence of European Civil-Law (not Common Law). Indeed, European Civil Law emerged as a system of general norms or rules that were thought to generalize the repertory of cases recorded mainly by Roman-Law. The idea of *qiyās* can be seen as providing an epistemological instrument to stablish those general norms behind the cases recorded by the sources and the tradition. The dynamics triggered by implementing such instrument "forges" the general norms that structure Islamic Law.

According to our view, the dialogical conception of Per Martin-Löf's *Constructive Type Theory* provides both a natural understanding and a fine-grained instrument to stress three of the hallmarks of this form of reasoning: [6]

> (a) the interaction of heuristic and epistemological processes with logical steps,
> (b) the dialectical dynamics underlying the meaning-explanation of the terms involved,[7]

[4] Cf. *Ibn Taymiyya against the Greek Logician*s, edited and translated by Hallaq (1993).
[5] Cf. Young (2017, pp. 10). The term has quite often a broader meaning encompassing legal reasoning in general. However, Young's choice for its translation renders a narrower sense that stems from al-Shīrāzī's approach. It seems that Young's translation is based on the one by David Weiss (1998).
[6] In fact there is ongoing work on deploying the dialogical setting in order to reconstruct logical traditions in ancient philosophy (see Castelnérac/Marion (2009), Marion/Rückert (2015) and medieval logical theories (C. Dutilh Novaes (2007), Popek (2012)).
[7] The term *meaning-explanation* stems from Martin-Löf's CTT (see Appendix I). It refers to a way of providing meaning to an expression by setting out rules that determine what needs to be knownn in order to make and assertion involving that expression.

(c) the unfolding of parallel reasoning as similarity in action.

Our study is focused on Abū Isḥāq al-Shīrāzī[8]'s classification of *qiyās* as discussed in his *Mulakhkhaṣ fi'l-Jadal* (*Epitome on Dialectical Disputation*). Let us point out that, though our paper is grounded on confrontation with the original textual sources, we deploy the thorough studies of these texts (and others) by Hallaq (1987a,b, 1997, 2004, 2009a,b) and Young (2017).
Furthermore, we are not claiming (yet) that the framework we propose in the present paper is either a literal description or a complete formalization of the *jadal*-disputation-form in which the *qiyās* is carried out. Our study provides a *dialectical meaning-explanation* of the main notion of co-relational inference relevant for the development of al-Shīrāzī's system of *qiyās*. In other words, what we are aiming at is to set out a kind of interactive language game that makes apparent the dialectical meaning of the main notions involved in these forms of reasoning.

Actually,

- since all of the steps prescribed by our dialogical framework are based on moves involved in al-Shīrāzī's dialectical conception of *qiyās al-'illa* , we think that our proposal can be further developed into a system for actual juridical disputation that provides a full reconstruction of *jadal* (جدل) as deployed in *uṣūl al-fiqh*. [9].

Thus, on the one hand our reconstruction might provide researchers on the Arabic tradition with some instruments for epistemological analysis, and on the other, we hope to motivate epistemologists and researchers in argumentation theory to explore the rich and thought-provoking texts produced by this tradition. Indeed, one of the main epistemological results emerging from this initial study is that the different forms of *qiyās* as developed in the context of *fiqh* represent an innovative approach that not only provides new epistemological insights into legal reasoning in general but also furnishes a fine-grained pattern for *parallel reasoning*[10] that can be deployed in a wide range of problem-solving contexts where degrees of evidence and inferences by drawing parallelisms are relevant.

II. A Dialectical Genealogy of Abū Isḥāq al-Shīrāzī's System of *Qiyās*

In the classical studies on juridical argumentation or *jadal* by Abū al-Ḥusayn al-Baṣrī (436H/1044 CE) in his *Kitāb al-Qiyās al-Shar'ī* (*Book of Correlational Inference Consonant to God's Law,* edited 1964) and by Abū Isḥāq al-Shīrāzī (393-476 H/1003-1083 CE) in his *Mulakhkhaṣ fi'l-Jadal* (*Epitome on Dialectical Disputation*), recorded, commented and worked out by Young (2017, chapter 4.3), we can find the following description of the *qiyās*:

- The aim of a *qiyās*, in its more general form, is to provide a rational ground to the ascription of some *juridical ruling* or *ḥukm* (حكم) such as (forbidden, allowed,

[8] Actually, al-Shīrāzī, who was a follower of the *Shāfi'ī* school of jurisprudence, endorsed the mistrust of the *Shāfi'ī-s* in relation to what they considered *subjective features* of *istiḥsān* and *maṣlaḥa*. Indeed, although he accepted that the extension of the scope of a juridical ruling is necessary, he was convinced that extensions should result from a rational process such as the one deployed by a *qiyās*.
[9] It is also worth mentioning that, to the best of our knowledge, there is no systematic study yet comparing the theory of juridical argumentation as developed within the Islamic tradition with the dialectical form of medieval disputations known as *Obligationes*. Such a study, that will fill up some flagrant gaps in the history of the development of rational argumentation, is certainly due.
[10] We have borrowed the term *parallel reasoning* from Bartha (2010).

obligatory) to a given case not yet considered by the sources acknowledged by *uṣūl al-fiqh* (for short, *juridical sources*).[11]

In fact, in this context, a *qiyās* involves bringing forward a case to which, according to the claim of the thesis, a particular *ḥukm* applies. The point is to ground this claim by relating it to an already juristically acknowledged application of such a ruling. Accordingly, the grounding is carried out in two main steps (involving two alternative developments):

1. It starts by bringing forward a case, known as *al-aṣl* or *the root-case* (الأصل), which the juridical sources have already established falls under the scope of the same juridical ruling as the one claimed to apply to the new case, called *al-far'* (الفرع), *the branch-case*.[12]

2.1 (First alternative). It proceeds by the assumptions that the property (*waṣf*) determining the *ground* or *occasioning factor* (*'illa*) for the ruling of the root-case can be found,[13] and this property also applies to the branch-case. Moreover, the proceeding assumes that the relevant property is to be found either by inspecting the sources or by epistemological considerations.

2.2 (Second alternative). It proceeds by finding some way to relate the branch-case to the root-case *in absence of knowledge of the occasioning factor* by developing a parallel reasoning based on some kind of similarity and it includes three cases:
- 2.2.1 both the root-case and the branch-case share some other juridical ruling,
- 2.2.2 in the absence of the similarities between the root-case and the branch case, it can nevertheless be established that there is some parallelism between a pair of source-cases and a pair of branch-cases such that if some particular juridical ruling applies to the pair of source-cases, it also applies to the pair of branch-cases,
- 2.2.3 both the root-case and the branch-case share some properties.

The second of the alternatives to step two is called *qiyās al-dalāla* (قياس الدلالة) or *correlational inference of indication,* also known as *qiyās al-shabah* (قياس الشبه), and also as *correlational inference of resemblance* – though it might be perhaps useful to restrict the term *qiyās al-*

[11] In general the term *ḥukm* refers to norm or ruling. In the context of the *qiyās* it indicates the ruling of the *aṣl* which the proponent seeks to transfer to the *far'* - see Young (2016, p. 610).

[12] The Arabic terminology makes use of the botanic metaphor of, respectively, *root* and *branch* in order to express the relation between the case established by the juridical sources, *al-aṣl*, and the case under consideration, *al-far'*. The idea is not that the *far'* is a subcase of the *aṣl*, but that the ruling claimed to apply to the *far'* is rooted on that of the *aṣl*.

[13] The term *'illa* is also translated into as *effective cause*, *operative cause*, *ratio legis and ratio decidenci*. Some of these translations do not seem to bear the causal significance of the term. The term *'illa* is derived from ancient Syriac, where it means a "fault" or "blame" constituting the cause for returning articles or property. The term penetrated from Syriac into the lexicon of rational thought even before Aristotelianism penetrated Arabic culture (we owe the remark on the etymology of term *'illa* to David Joseph (2010, 2014).

In a general context, a distinction is drawn between providing a *ground* (*'illa*) and providing a *factual cause* or *reason* (*sabab*): while grounding is a rational endeavour, providing a *sabab* might be limited to an empirical task. It seems to be related to St. Thomas' (*Summa Theologiae* I.2.2c:) distinction between *propter quid* and *quia* that stems from Aristotle's distinction in *Posterior Analytics* I.13) (for a discussion in the context of CTT see Granström (2011), p. 157). In the context of the *qiyās* the notion of *sabab* seems to allude to the justification underlying the choice of one specific occasioning factor. This use is witnessed by al-Shīrāzī's denomination of the second subtype of *qiyās al-'illa* as *qiyās plainly evident by reported reason* (*al-wāḍiḥ bi'l-sabab*).That is, those *qiyās* where the *'illa* is not found in the *naṣṣ* but specified on the basis of some reason stemming from a specific historical background of *naṣṣ* reported by the Companion of the Prophet. In fact we should also mention the notion *ḥikma* that stands for the underlying higher purpose of the *'illa*. Moreover, the notion of *ḥikma* underlies the *doctrine of rational juridical preference* or *istiḥsān*, and *the theory of public welfare* or *maṣlaḥa* mentioned before. However, this notion does not seem to play a role in the inferential processes deployed by the use of a *qiyās*.

shabah for the last form of *qiyās al-dalāla*.[14] *Qiyās al-dalāla* based on the resemblance of the branch-case to the root-case in relation to a *set of properties* is considered to be the weakest, epistemically speaking, and is very close to what is known in other traditions as analogical argumentation by similarity or agreement. By contrast, the *qiyās* based on the resemblance of the branch-case to the root-case in relation to a *set of juridical rulings* is considered to be epistemically the strongest form of inference of the type *al-dalāla.* The form of inference-form of *qiyās al-dalāla* based on double parallelisms constitutes a generalization and a deeply innovative approach to what is known as *proportionality-based analogical reasoning.*[15] In relation to its epistemic strength it is placed between the former two.

> وأما قياس الدلالة فهو أن يحمل الفرع على الأصل بضرب من الشبه غير العلة التي علق الحكم عليها في الشرع .وهذا ضرب من القياس لا تعرف صحته إلا بالاستدلال بالأصول وهو على ثلاثة أضرب[16].

> "As for *Qiyās al-Dalāla*, it is that one that link the branch-case with the source-case by way of a type of resemblance other than the occasioning factor upon which the ruling is made contingent in God's Law. The validity of this type of correlational inference is not known except by way of drawing indication from the authoritative source-cases; and it is [also] of three types[17]."

Al-Shīrāzī calls the first alternative to the second step *qiyās al-'illa* (*co-relational inference of the occasioning factor*) – that provides the subject of our paper – and distinguishes three main cases classified by the strength of the evidence for the *'illa*:

> (i): the evidence for the determination of the *'illa* stems from the authoritative texts (*naṣṣ*) of the Qur'ān and prophetic tradition (*al-jalī bi'l-naṣṣ*), or from a consensus of the jurists (*al-jalī bi'l-ijmā'*)
> (ii): it stems from some hermeneutical process of the texts (*al-wāḍiḥ bi'l-nuṭq*) or it is based upon some historical background reported by the Companion of the Prophet (*al-wāḍiḥ bi'l-sabab*)
> (iii) the *'illa* is specified by positing some suitable hypothesis (*al-khafī*).[18]. The latter has some relation to Aristotle's *argument from example* (*paradeigma*) described in the *Rhetoric* (1402b15) and the *Prior Analytics* (*Pr. An.* 69a1).

> فأما قياس العلة فهو أن يحمل الفرع على الأصل بالعلة التي علق الحكم عليها في الشرع وذلك على ثلاثة أضرب جلي وواضح وخفي[19].

> "As for *Qiyās al-ʿIlla*, it is that one link the branch-case with the source-case by way of the occasioning factor upon which the ruling is made contingent in God's Law; and that is according to three types: *al-jalī* (clearly-disclosed), *al-wāḍiḥ* (plainly-evident), and *al-khafī* (latent)." [20]

[14] See al-Shīrāzī, *Mulakhkhaṣ, fi'l-Jadal*, p. 5.
[15] Cf. Cellucci (2013, pp. 340-41). Moreover, it seems to be very close to Bartha's (2010) own model.
[16] See al-Shīrāzī, *Mulakhkhaṣ fi'l-Jadal*, pp. 5
[17] Cf. Young (2017, pp. 115).
[18] See al-Shīrāzī, *Mulakhkhaṣ, fi'l-Jadal*, p. 5, cf. Young (2017, 113-114). Al-Baṣrī distinguishes a positive inferential process (*Qiyās al-Ṭard, correlational inference of co-presence*), covered by the description above, from a negative one (*Qiyās al-ʿAks, correlational inference of the opposite*). The result of the negative one is to deny that some designated juridical ruling that applies to the root case also applies to the branch-case, on the grounds that the occasioning factor does not apply to the branch-case – see al-Baṣrī, *Muʿtamad*, Ḥamīd Allāh ed., vol. 2, pp. 697-699.; and *K. al-Qiyās al-Sharʿī*, pp. 1031-3 (trans. of the latter in Hallaq (1987a), quoted by Young (2017, p. 109)).
[19] See al-Shīrāzī, *Mulakhkhaṣ fi'l-Jadal*, p. 5.
[20] Cf. Young (2017, p. 109).

Remarks:

- One way to express the rationale behind AL-Shīrāzī's typology (not shared by all of the other authors) is that he conceives *qiyās* as a system of parallel reasoning that deploys arguments by
  a) exemplification (of a general law): *qiyās al-'illa*.
  b) symmetry between structures (established by either chains of rulings or pairs of parallel rulings) (the two first forms of *qiyās al-dalāla*).
  c) resemblance between the root-case and the branch-case (*qiyās al-shabah*)

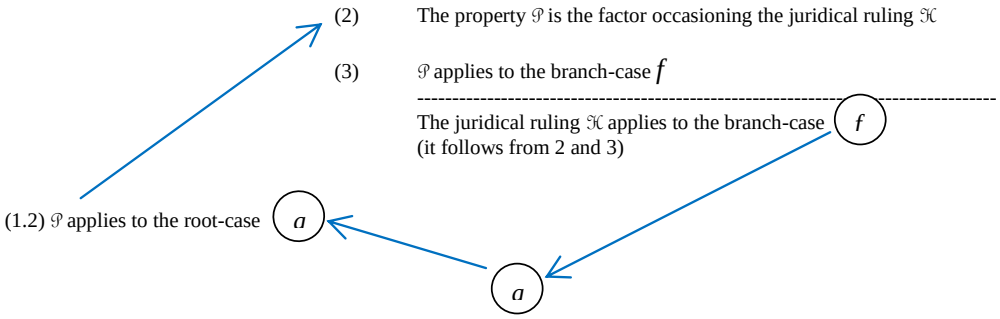
- Some paragraphs of AL-Shīrāzī's *Al-Lumaʿ fī Uṣūl al-Fiqh* seem to support a three-fold rather than a two-fold classification – the three-fold classification comes close to the triad a, b, c.[21] However the *Mulakhkhaṣṣ* and the *Ma'ūna* provide solid textual evidence of a two –fold classification, where b and c are both included in a general category of *qiyās* where the occasioning factor is not present. [22]

- *Qiyās* constitutes a system of juridical reasoning that is in the middle of two other (sometimes contested) forms of rational juridical change deployed in *fiqh* called, respectively, *the doctrine of rational juridical preference* or *istiḥsān* (استحسان), that might produce the withdrawal of a conclusion achieved by a *qiyās*-procedure, and *the theory of public welfare* or *maṣlaḥa* (مصلحة), that can trigger the production of a new juridical ruling. Indeed, while the use of a *qiyās* might extend the scope of application of a particular juridical ruling, it does not actually refute the ruling or the occasioning factor that the juridical source explicitly declares as the ground for that ruling. The changes possible by the use of *qiyās* are, in some sense, of a more logical and semantic nature.

Before delving into the structure of *qiyās al-'illa*, let us motivate the underlying dialectical processes with the help of an informal diagram. The diagram presents the most general form of the *qiyās al-'illa*, without (for the moment) drawing a distinction between subdivisions inside each type of co-relational inference.

Qiyās al-'Illa[23]

(schema 1)



[21] See al-Shīrāzī (2003, pp. 99-101) and al-Shīrāzī (1995, pp. 204-210).
[22] See Al-Shīrāzī (1987, pp. 36-38)
[23] The diagram has been adapted from Bartha's (2010, p. 36) figure for Aristotle's reasoning by *paradeigma.*

(1.1) The juridical ruling ℜ applies to the root-case

The point of the *al-'illa*-form of co-relational inference is to find a general law and a property, shared by both the branch and the source-case, which allows the inference of the ruling we are looking to ground. It is not really a case of analogy by resemblance, but a kind of what is nowadays called *deductive parallel reasoning*, since it combines some kind of *symmetric* reasoning with inferential moves. Notice that neither 1.1. nor 1.2 are premises for the last inferential step. Indeed, steps 1.1 and 1.2 have the heuristic role of leading to the required general rule, and these steps are moves carried out within a dialectical structure. In order to extract from the diagram the underlying *jadal*-structure, we need to read the arrows as dialectical actions or argumentative moves, whereby the first action (the arrow right of the diagram) amounts to the heuristic move of finding a suitable root-case, then the short arrow from 1.1 to 1.2 indicates the result of finding out the property that provides the occasioning factor specific to the ruling of the root-case, and the last arrow stresses the core of the process, namely: *to learn from the ruling of the root-case that it instantiates a general juridical norm*. Once this has been achieved, a simple logical mechanism leads us to the conclusion sought.

Now, before delving into the dialectical structure, let us motivate the use of a notation inspired by Constructive Type Theory. In fact, we only deploy very basic features of the CTT-framework; a deep and thorough development is still due.

III. Motivating the Deployment of a CTT-Framework

The expressive power of Per Martin Löf's Constructive Type Theory[24] allows the following features underlying the *qiyās* to be expressed at the object language level:

- The stress on assertions (or judgements) rather than on propositional sentences. The dialectical process underlying co-relational inferences is triggered by both an assertion concerning the identification of the factor occasioning the relevant ruling and the process of the justification of such an assertion. In the specialized literature these assertions are called *ʿa'līl* (affirmation of the relevance of a particular property for the determination of the *'illa*), or more generally *ithbāt* (affirmation).
- The intensional rather than extensional understanding of the sets underlying the semantics of the *qiyās*.
- The deployment of hypothetical judgements. This dovetails with the *qiyās*-notion of dependence of a given juridical ruling on a particular occasioning factor.
- The restrictive form of the substitution rules.

In the present paper the last point will be left out since it relates to co-relational inferences by indication, which will not be discussed here.

Certainly, other formal reconstructions are possible, and in particular, we might not need an intensional framework in order to deal with changing extensions. However,

1. the deployment of intensional frameworks seems to be a natural approach in historical contexts, [25]

[24] See Appendix I. For a systematic presentation of CTT see Martin-Löf (1984, 1996), Nordström/ Petersson/Smith (1990, 2000), Ranta (1994), Granström (2011). For philosophical and historical insights into CTT see Ranta (1988), Primiero (2008), Sundholm (2009, 2012).
[25] See for example, Marion/Rückert (2015) and Martin-Löf (2012).

2. CTT provides a solid theory for the deployment of intensionally grounded sets:
3. CTT seems to match well with dialectical approaches to meaning and normative approaches to logic, such as the dialogical one. This is particularly so in a CTT-framework where non-mathematical propositions are understood as language-games, as suggested by Ranta (1994, pp. 55-57).

The main idea to be developed in sections III.1 and III.2 is that the relevance of a given property $\mathcal{P}$ (conceived as a set) for the correspondent juridical ruling $\mathcal{H}(x)$ is displayed by explaining the meaning of the latter as being defined over that set. In this context the factor occasioning the ruling of some particular case under scrutiny obtains as the application to this case of a method that provides the justification of applying the ruling to every instance of $\mathcal{P}$ (and dually, the justification of applying $\sim\mathcal{H}(x)$, given instances of $\sim\mathcal{P}$).

III. The Dialectical Framework

In order to provide meaning explanations to the basic notions al-Shīrāzī's System of *qiyās* we deployed CTT, but al-Shīrāzī's approach is a dialectical framework. Thus, we need now to motivate the interface of CTT with a dialectical framework. We will develop this motivation in two main steps, namely

(1) by a (brief) discussion of the interface *epistemic-assumption, formal rule* and the notion of *epistemic strength*
(2) by the distinction of play and strategic level and the notion of winning and losing within the dialectical framework underlying the system of *qiyās al-'illa*

….

A Dialogical Framework for *Co-Relational Inferences of the Occasioning Factor*

We will not be able to present here the full-formalization of the dialogical framework for *qiyās al-'Illa.* However, the following presentation should provide the reader the means to follow how to develop a dialogue for this kind of *qiyās*.

The dialogical approach to logic is not a specific logical system but rather a framework rooted on a rule-based approach to meaning in which different logics can be developed, combined and compared. More precisely, in a dialogue two parties argue about a thesis respecting certain fixed rules. The player that states the thesis is called Proponent (**P**), his rival, who contests the thesis is called Opponent (**O**). Dialogues are designed in such a way that each of the plays end after a finite number of moves with one player winning, while the other loses. Actions or moves in a dialogue are often understood as speech-acts involving *declarative utterances or posits and interrogative utterances or requests*. The point is that the rules of the dialogue do not operate on expressions or sentences isolated from the act of uttering them. The rules are divided into particle rules or rules for logical constants (*Partikelregeln*) and structural rules (*Rahmenregeln*). Particle rules provide an abstract description of how the game can proceed locally: they specify the way a formula can be challenged and defended according to its main logical constant. In this way the particle rules govern the local level of meaning. Strictly speaking, the expressions occurring in the table above are not actual moves because they feature formula schemata and the players are not specified. Moreover, these rules are indifferent to any particular situations that might occur during the game. For these reasons we say that the description provided by the particle rules is abstract. The structural rules determine the development of a dialogue game.[26]

[26] The main original papers are collected in Lorenzen/Lorenz (1978) – see too Lorenz (2010a,b), Felscher (1985), Krabbe (2006). For an account of recent developments see Rahman/Keiff (2005), Keiff (2009), Rahman/Tulenheimo (2009), Rückert (2011), Clerbout (2014a, b). The most recent work links dialogical logic and Constructive Type Theory (see, Clerbout/Rahman (2015)), Rahman/Clerbout/Redmond (2017).

AII.1 Local meaning of the logical constants

It is presupposed in standard dialogical systems that the players use well-formed formulas. The well formation can be checked at will, but only with the usual meta reasoning by which the formula is checked to indeed observe the definition of a wff. We want to enrich the system by first allowing players to enquire on the status of expressions and in particular to ask if a certain expression is a proposition. We thus start with dialogical rules explaining the formation of propositions. These rules are local rules which we are added to the particle rules giving the local meaning of logical constants.

Moreover, we extend the first-order language assumed in standard dialogical logic with two labels **O** and **P**, standing for the players of the game, and the two symbols '!' and '?'. When the identity of the player does not matter, we use variables **X** or **Y** (with **X**≠**Y**).

A move **M** is an expression of the form '**X**-*e*', where *e* is one of the forms specified by the particle rules.

Local meaning: Formation

Posit	Challenge	Defence
X $A \vee B$: **prop**	**Y** $?_{F\vee 1}$ Or **Y** $?_{F\vee 2}$	**X** A: **pro** **X** B : **prop**
X $A \wedge B$: **prop**	**Y** $?_{F\wedge 1}$ Or **Y** $?_{F\wedge 2}$	**X** A : **prop** **X** B : **prop**
X $A \supset B$: **prop**	**Y** $?_{F\supset 1}$ Or **Y** $?_{F\supset 2}$	**X** A : **prop** **X** B : **prop**
X $\sim A$: **prop**	**Y** $?_{F\sim}$	**X** A : **prop**
X $(\forall x : A)\ B(x)$: **prop**	**Y** $?_{F\forall 1}$ Or **Y** $?_{F\forall 2}$	**X** A : **set** **X** $B(x)$: **prop** $(x : A)$
X $(\exists x : A)\ B(x)$: **prop**	**Y** $?_{\exists 1}$ Or **Y** $?_{F\exists 2}$	**X** A : **set** **X** $B(x)$: **prop** $(x : A)$

Because of our deployment expressions coming from Constructive-Type Theory the language contains expressions such as the following (further expressions are provided in the section on terminology in the main text)

X ! $a : A$ Player **X** claims that *a* instantiates *B* / Player **X** claims that *a* provides a *local reason* for *B.*

X $b.\mathbf{Y}^a : B(a)$ Player **X** claims that *b* provides a *local reason* for *a* being *B* given that the antagonist **Y** claims that *a* provides a *local reason* for *A*, and given that $\boldsymbol{B}(x)$: **prop** $(x : A)$.

X $b.\mathbf{Y}^a : B$ Player **X** claims that *b* provides a *local reason* for *B* given that the antagonist **Y** claims that *a* provides a *local reason* for *A*, and given that $A \supset B$.

Similarly

X $b : B(\mathbf{X}^a)$ Player **X** claims that *b* provides a *local reason* for *a* being *B* given that it is himself (**X**) who claims that *a* provides a *local reason* for *A*, and given that $\boldsymbol{B}(x)$: **prop** $(x : A)$.

The canonical argumentation form of a local reason as determined by the rules of synthesis is given by the triple

Posit by **X** Challenge by **Y** Defence by **X**

This yields the following table

Canonical argumentation form: Rules of Synthesis of the Local Reasoning

Posit	Challenge	Defence
X ! $(\exists x : A)B(x)$	**Y** $?_L$ Or **Y** $?_R$	**X** $p_1 : A$ Respectively **X** $p_2 : B(\mathbf{X}^{p1})$
X ! $\{x : A \mid B(x)\}$	**Y** $?_L$ Or **Y** $?_R$	**X** $p_1 : A$ Respectively **X** $p_2 : B(\mathbf{X}^{p1})$
X ! $A \wedge B$	**Y** $?_L$ Or **Y** $?_R$	**X** ! $p_1 : A$ respectively **X** $p_2 : B$
X ! $(\forall x : A)\ B(x)$	**Y** $p_1 : A$	**X** $p_2.\mathbf{Y}^{p1} : B(\mathbf{Y}^{p1})$
X ! $A \supset B$	**Y** $p_1 : A$	**X** $p_2.\mathbf{Y}^{p1} : B$
X ! $\neg A$	**Y** $p_1 : A$	–
X ! $A \vee B$	**Y** $? \vee$	**X** $p_1 : A$ Or **X** $p_2 : B$

We add too rules for the operators **F** and **V** adapted to the purposes of our present paper.

The operator F[27]

In uttering the formula **F***A* the argumentation partner **X** claims that he can find a counterexample during a play where the antagonist **Y** asserts *A*.
The antagonist **Y** challenges **F***A* by asserting that *A* can be challenged successfully. Thus, the challenge of **Y** compels **Y** to open a *sub-play* where he (**Y**) utters *A*.

	Challenge	Defence
X ! **F***A*	**Y** ?**F**	
	Sub-play $\mathfrak{D}_1$	*Sub-play*$\mathfrak{D}_1$
	Y ! *A* **Y** must play under the restriction of the *Socratic-Rule* in the sub-play	**X** $?_A$(he challenges *A*)

In uttering the formula **V***A* the argumentation partner **X** claims that he can win a play where he (**X**) asserts *A*.
The antagonist **Y** responds by challenging **X** to open a *sub-play* where he (**X**) defends *A*.

[27] Cf. Rahman/Rückert (2001, pp. 113-116). The main difference of the present formulation of F is that here it is the defender of the operator and not the challenger who must play under the *copy-cat* rule. The changes is due to the fact that in the context of the present paper the assertion of F*A* occurrs only as a *challenge* to a previous move of the Proponent.

	Challenge	Defence
X: V*A*	Y: ?V	
	Sub-play $\mathfrak{D}_1$	*Sub-play* $\mathfrak{D}_1$
	Y $?_A$(he challenges *A*) **Y** must play under the restriction of the *Socratic Rule*	**X** ! *A*

Special Local Rules for *Qiyās al-'Illa*

Expressions "*p*" in "*p* : *A* " stand for either some branch-case *far'* or some root-case *aṣl*

Posit	Challenge	Defence
X $ṭard^{\mathfrak{K}(\mathfrak{P})}$: $(\forall x : \mathfrak{P})\mathfrak{K}(x)$ Notation for a posit without specified reason **X** ! $(\forall x : \mathfrak{P})\mathfrak{K}(x)$	**Y** $p : \mathfrak{P}$	**X** $'illa^{\mathfrak{K}(\mathfrak{P})+}.p : \mathfrak{K}(p)$
X $'aks^{\mathfrak{K}(\mathfrak{P})}$: $(\forall x : \sim\mathfrak{P})\sim\mathfrak{K}(x)$ Notation for a posit without specified reason **X** ! $(\forall x : \sim\mathfrak{P})\sim\mathfrak{K}(x)$	**Y** $p : \sim\mathfrak{P}$	**X** $'illa^{\mathfrak{K}(\mathfrak{P})-}.p : \sim\mathfrak{K}(p)$
X $ta'thīr^{\mathfrak{K}(\mathfrak{P})}$: $(\forall x : \mathfrak{P})\mathfrak{K}(x) \wedge (\forall x : \sim\mathfrak{P})\sim\mathfrak{K}(x)$ Notation for a posit without specified reason **X** ! $(\forall x : \mathfrak{P})\mathfrak{K}(x) \wedge (\forall x : \sim\mathfrak{P})\sim\mathfrak{K}(x)$	**Y** $p : \mathfrak{P}$ **or** **Y** $p : \sim\mathfrak{P}$	**X** $'illa^{\mathfrak{K}(\mathfrak{P})+}.p : \mathfrak{K}(p)$ respectively **X** $'illa^{\mathfrak{K}(\mathfrak{P})-}.p : \sim\mathfrak{K}(p)$
X ! *A* (or *p* : *A*) ... **X** ~*A* (or *p'* : ~*A*) **(**it can also be the case that one explicitly displays the local reason but the other not)	**Y** ! *tanāquḍ* ϕ **The antagonist Indicates the contradiction**	**X** ! I concede

Qiyās al-'Illa also require the following moves prescribed by the **development rules** specific to the dialectical framework underlying this form of *qiyās*

Requests
Our framework for qiyās al-'Illa includes moves by the means of which players can request the contender to endorse some particular assertion. the general form of a request and the positive response is the following

X *A*?
Y ! *A*

If the request has a form that indicates sources, **must** be endorsed by the respondent

X $p^{\mathfrak{S}} : A?$ **X !** $A^{\mathfrak{S}}?$
Y $p^{\mathfrak{S}} : A$ **Y !** $A^{\mathfrak{S}}$

This general form of the request might trigger a different form of answer if it involves the endorsement of a particular occasioning factor. In such a case, the following responses are possible

X ! $illa^{\mathfrak{K}(\mathfrak{P})+}.aṣl : \mathfrak{K}^{\mathfrak{S}}(aṣl)?$

Y Cooperative criticism	**Y** Destructive Criticism	**Y** ! *muṭālaba* Asking for Justification	**Y** ! $(\forall x : \mathcal{P})\mathcal{K}(x) \wedge (\forall x : \sim\mathcal{P})\sim\mathcal{K}(x)$ Endorsing the request by asserting the efficiency of the property $\mathcal{P}$

Which of the options are available is determined by the rules prescribing the overall development of a play for *qiyās al-'Illa.* We proceed to describe the development of the first three responses, the development of the fourth one (the conjunction of universals) has been already described above

Muṭālaba
This move presupposes that player X requested the contender to endorse that the property $\mathcal{P}$ occasions the ruling of the root-case
That is, it presupposes the following request

X ! $illa^{\mathcal{K}(\mathcal{P})+}.aṣl : \mathcal{K}^{\mathcal{S}}(aṣl)$?
Y ! *muṭālaba*
X must be able to bring forward arguments showing that the property satisfies *ṭard* (! $(\forall x : \mathcal{P})\mathcal{K}(x)$), *'aks* ((! $(\forall x : \sim\mathcal{P})\sim\mathcal{K}(x)$), and *ta'thīr* (! $(\forall x : \mathcal{P})\mathcal{K}(x) \wedge (\forall x : \sim\mathcal{P})\sim\mathcal{K}(x)$)

***Mu'āraḍa* or cooperative criticism**

This move presupposes that the Proponent requested the Opponent to endorses that the property $\mathcal{P}$ occasions the ruling of the root-case
That is, the deployment of cooperative criticism presupposes the following request
P ! $illa^{\mathcal{K}(\mathcal{P})+}.aṣl : \mathcal{K}^{\mathcal{S}}(aṣl)$?

1. The Opponent refuses to endorse the requested assertion and starts by asserting that that the relevant factor for the root-case at stake is the property $\mathcal{P}^*$ rather than $\mathcal{P}$ – however, the Opponent believes that the main thesis is correct though it was poorly defended
   O ! $\forall illa^{\mathcal{K}(\mathcal{P}^*)+}.aṣl : \mathcal{K}^{\mathcal{S}}(aṣl)$.
2. If the assertion of the Opponent is rooted in the sources, the Proponent must accept it and the play will continue from step 5. If it is not based on the sources the Proponent responds by challenging the Opponent to open a *sub-play* where the latter must defend his thesis.
   P ! *muṭālaba*
3. In the sub-play, before providing the required justification, the Opponent might first choose to force the Proponent to accept that there is there is a root-case that contradicts the Proponent's choice of $\mathcal{P}$ as relevant for the juridical ruling at stake. Driving the Proponent to contradiction is carried out by means of the following steps:
   Start of a sub-play
   O searches for a new root-case to which $\mathcal{P}$. applies.
   O *aṣl** : $\mathcal{P}$**?**
   P **!** *aṣl** : $\mathcal{P}$
   O forces **P** to agree that according to the presupposition $\mathcal{P}$ has the efficiency required for producing the ruling
   O ! $(\forall x : \mathcal{P})\mathcal{K}(x) \wedge (\forall x : \sim\mathcal{P})\sim\mathcal{K}(x)$?
   P ! $(\forall x : \mathcal{P})\mathcal{K}(x) \wedge (\forall x : \sim\mathcal{P})\sim\mathcal{K}(x)$
   O forces then **P** to contradict himself in relation to the applicability of the ruling to the new-root case
   O **!** *aṣl** : $\mathcal{P}$ (The Opponent challenges the *ṭard*- component of **P**'s last assertion.
   P **!** : $illa^{\mathcal{K}(\mathcal{P}^*)+}.aṣl^* : \mathcal{K}^{\mathcal{S}}(aṣl^*)$ (The Opponent responds by conceding that the ruling applies to the new root-case)
   O **!** $\sim\mathcal{K}^{\mathcal{S}}(aṣl^*)$?
   P **!** $\sim\mathcal{K}^{\mathcal{S}}(aṣl^*)$
   O ! *tanāquḍ* $\mathcal{K}^{\mathcal{S}}(aṣl^*)$ (the Opponent indicates that **P** just contradicted himself by asserting both that the ruling applies and not to the new root-case)
   P ! I concede
   The Opponent starts now his constructive contribution by displaying the efficiency of a new property. Herewith he answers to the request of justification

   P concedes and this ends the sub-play
4. The Proponent accepts the suggestion and making use of the fact that the new property applies to the branch-case he will proceed that this will lead to the justification of the thesis.
5. The tree displaying the winning strategy will delete the unsuccessful attempts and also the justification of the sub-play.

Destructive Criticisms

This move also presupposes that the Proponent requested the Opponent to endorses that the property $\mathcal{P}$ occasions the ruling of the root-case
That is, the deployment of cooperative criticisms presupposes the following request
P ! $illa^{\mathcal{K}(\mathcal{P})+}.aṣl : \mathcal{K}^{\mathcal{S}}(aṣl)$?
However; different to cooperative criticism the Opponent aims to refute the main thesis. We will be more succinct in the description since after the description of the cooperative criticism and after the examples in the main text, the development is quite straightforward.
O ! **F** $(\forall x : \mathcal{P})\ \mathcal{K}(x)$ (***qalb'***)
The Opponent is committed to a sub-play where he brings forward a root-case of which it is recorded that an opposite ruling to the claimed ruling applies. Hence the root-case is presented as a counterexample to the

Proponent's claim that every $\mathcal{P}$ falls under the ruling $\mathcal{K}$ and in particular to the claim that this ruling applies to the branch-case.

O ! **F** ($\forall x : \mathcal{P}$) $\mathcal{K}(x)$, given $aṣl^* : \mathcal{P}$, $\mathcal{K}^{\mathcal{C}*}(aṣl^*)$, and ~ ($\mathcal{K}(aṣl^*) \wedge \mathcal{K}^*(aṣl^*)$) (***naqḍ***)

The Opponent is committed to a sub-play where he brings forward a root-case of which it is recorded that a different ruling to the claimed ruling applies and both rulings are incompatible. Hence the root-case is presented as a counterexample to the Proponent's assertion that every $\mathcal{P}$ falls under the ruling $\mathcal{K}$ and in particular to the claim that this ruling applies to the branch-case.

O ! **F** ($\forall x : \{ x : \mathcal{P} \mid B(x)\}$) $\mathcal{K}(x)$ (***kasr***)

The Opponent is committed to a sub-play where he brings forward a root-case which instantiates a subset of $\mathcal{P}$ and of which it is recorded that the claimed ruling does not apply. Hence the root-case is presented as a counterexample to the Proponent's assertion that every $\mathcal{P}$ falls under the ruling $\mathcal{K}$ and in particular to the claim that this ruling applies to the branch-case.

O ! **F** ($\forall x : \mathcal{P}$) $\mathcal{K}(x)$ (***fasād al-waḍ'***)

The Opponent is committed to a sub-play where he brings forward a root-case of which it is recorded in the sources that a property assumed in the thesis to apply to the branch-case occasions in fact, the opposite ruling to the one posited by the Proponent. In other words, the Opponent brings forward an *'illa* that destroys the thesis.

O ! **F** ! ($\forall x : \mathcal{P}$)$\mathcal{K}(x) \wedge (\forall x : \sim\mathcal{P})\sim\mathcal{K}(x)$ (***'adam al-ta'thīr***)

The Opponent is committed to a sub-play where he brings forward a root-case which constitutes a counterexample to the efficiency of the proposed property asserted by the Proponent.

AI.2.2 Global meaning

As mentioned above global meaning is defined by means of *structural rules* that determine the general development of the plays, by specifying who starts, what are the allowed moves and in which order, when does a play end and who wins. The structural rules include the following rule on elementary expressions, i.e., expressions of one of the forms a: B, $a : B(c)$, A, B:

- **P** may not utter an elementary expression unless **O** uttered it first. Elementary expressions cannot be challenged.

This, rule is one of the most salient characteristics of dialogical logic. As discussed by Marion / Rückert (2016), it can be traced back to Aristotle's reconstruction of the Platonic Dialectics: the main idea is that, when an elementary expression is challenged then, from the purely argumentative point of view – that is, without making use of an authority beyond the moves brought forward during an argumentative interaction-, the only possible response is to appeal to the concessions of the challenger:

> *my grounds for the proposition you are asking for are exactly the same as the ones you bring forward when you conceded the same proposition*.[28]

In previous literature on dialogical logic this rule has been called the *copy-cat rule or Socratic rule*. Now, if the ultimate grounds of a dialogical thesis are elementary propositions and if this is implemented by the use of the copy-cat rule, then the development of a dialogue is in this sense necessarily asymmetric. Indeed, if both contenders were restricted by the copy-cat rule no elementary proposition can ever be uttered. Thus, we implement the copy-cat rule by designing one player, called the *Proponent*, whose utterances of elementary propositions are, restricted by this rule. It is the win of the Proponent that provides the dialogical notion of validity. More precisely, in the dialogical approach validity is defined via the notion of *winning strategy*, where winning strategy for X means that for any choice of moves by Y, X has at least one possible move at his disposal such that he (X) wins:

> *Validity (definition):* A proposition is valid in a certain dialogical system iff **P** has a winning strategy for this formula.

In present context we will deploy a variant of the formal-rules. Before providing the structural rules let us precise the following notions:

> ***Play***: A *play* is a legal sequence of moves, i.e., a sequence of moves which observes the game rules. Particle rules are not the only rules which must be observed in this respect. In fact, it can be said that the second kind of rules, namely, the *structural rules* are the ones giving the precise conditions under which a given sequence is a play.
>
> ***Dialogical game***: The *dialogical game* for φ, written $\mathcal{D}(\varphi)$, is the set of all plays with φ being the *thesis* (see the Starting rule below).
>
> The ***structural rules*** are the following:[29]

[28] Cf. Rahman/Clerbout/Keiff (2009) and in Rahman/Keiff (2010).
[29] For a formal formulation see Clerbout (2014a,b)

SR0 (Starting rule). Any dialogue starts with the Opponent positing initial concessions, if any, and the Proponent positing the thesis. After that the players each choose a positive integer called *repetition rank*.

- The *repetition rank* of a player bounds the number of challenges he can play in reaction to a same move.

SR1i (Classical game-playing rule). Players move alternately. After the repetition ranks have been chosen, each move is a challenge or a defence in reaction to a previous move and in accordance with the particle rules.

SR1ii (Intuitionistic game-playing rule). Players move alternately. After the repetition ranks have been chosen, each move is a challenge or a defence in reaction to a previous move and in accordance with the particle rules.
Players can answer only against the ***last non-answered*** challenge by the adversary.[30]

SR2 (Socratic- rule).
The following rule only applies to elementary posits (of the form $a : A$, or $! A$) covered neither the rules for requests stemming from the sources described above nor by the prescriptions involving the development rule for ***qiyās al-'illa***

Modified Copy-cat rule 1. **O**'s elementary posits (of the form $a : A$, or $! A$) . However, **O** can challenge a **P**-elementary move iff he (**O**) did not posit the same elementary posit before. The challenge and correspondent defence is ruled by the following table. Once **P** answered the challenge on this posit is not any more available.

Posit	**Challenge**	**Defence**
P $! a : A$ (for elementary A)	**O** ?	**P** *sic (n)* (**P** indicates that **O** posited $a : A$ at move n)

SR3 (The overall development of a dialogue for *qiyās al-'illa*). We describe this rule below

The following structural rule requires some additional terminology:

- ***Terminal play***: A play is called *terminal* when it cannot be extended by further moves in compliance with the rules.

- ***X-terminal***: We say it is **X**-*termina*l when the last move in the play is an **X**-move.

SR4 (Winning rule). Player **X** wins the play ζ only if it is **X**-terminal.

Strategy: A *strategy* for player **X** in $\mathbf{D}(\varphi)$ is a function which assigns an **X**-move M to every non terminal play ζ having a **Y**-move as last member such that extending ζ with M results in a play.

X-winning-strategy: An ***X***-*strategy* is *winning* if playing according to it leads to **X**-terminal play no matter how **Y** moves.

Winning-strategy resulting from a cooperative move: Winning strategies constituted by plays where cooperative moves took place, will disregard the unsuccessful attempts and also the justification of the sub-play. More precisely it will proceed as if the Proponent has chosen the property resulting from the sub-play. Accordingly the winning strategy will include moves where the Proponent rather than the Opponent asserted the efficiency of the right property

The overall development of a dialogue for *qiyās al-'illa*

1. A dialogical play starts with the Proponent claiming that some specific legal ruling applies to a certain branch-case.

[30] This last clause is known as the *Last Duty First* condition, and is the clause making dialogical games suitable for Intuitionistic Logic, hence the name of this rule.

P ! $\mathcal{K}$(*far'*)

2. After agreement on the finiteness of the argument to be developed, the Opponent will launch a challenge to the assertion by asking for justification.

O Why?

The Proponent's aim is to develop an argument in such a way that if forces the Opponent to concede the justification of the challenged assertion. In other words **P** will try to obtain (see step 13)

! *'illa* $^{\mathcal{K}(\mathcal{G}^{\#})+}$(*far'*) : $\mathcal{K}$(*far'*)

3. In order to develop his argument, the Proponent will start by choosing (to the best of his juridical knowledge) a suitable root-case from the sources for which the ruling at stake has been applied. The move consists in the Proponent forcing the Opponent to acknowledge this fact.

4. Since the evidence comes from the sources the Opponent is forced to concede it.

Steps 4 and 5 yield :
P $\mathcal{K}^{\copyright}$(*aṣl*)?
O ! $\mathcal{K}^{\copyright}$(*aṣl*)

5. Once conceded, the Proponent will start by choosing (to the best of his juridical and epistemological knowledge) a suitable property (that should lead to the relevant occasioning factor). The move consists in the Proponent forcing the Opponent to acknowledge that the root-case instantiates that property. – recall (section III.2.1) that we adopt here al-Baṣrī's and al-Shīrāzī's practice of keeping only those plays where the Opponent responds positively to this form of request.

P *aṣl* : $\mathcal{G}$?
O ! *aṣl* : $\mathcal{G}$

Once the Opponent concedes both that the ruling and the selected property apply to the root-case, the Proponent will ask the Opponent to concede that the property just selected is the one that constitutes the relevant occasioning factor.[31] The request can carry out indicating to the sources or not.

P *'illa* $^{\mathcal{S}\text{-}\mathcal{K}(\mathcal{G})+}$*.aṣl* : $\mathcal{K}$ (*aṣl*)?
P *'illa* $^{\mathcal{K}(\mathcal{G})+}$*.aṣl* : $\mathcal{K}$(*aṣl*)?

6. If the *'illa* has been determined by the sources the Opponent must accept by endorsing the efficiency of the property (thus, the Opponent must assert the universal ! $(\forall x : \mathcal{G})\mathcal{K}(x) \wedge (\forall x : \sim\mathcal{G})\sim\mathcal{K}(x)$). Otherwise he might ask for justification (*muṭālaba*), cooperate in the justification or *strongly* reject it.

7. If the Opponent asks for a justification, the Proponent will switch to the development of a dialogue of the form *qiyās al-'illa al-khafī*) and will develop an argument towards establishing its efficiency. In other words, the Proponent must be able to bring forward arguments showing that the property satisfies *ṭard* (! $(\forall x : \mathcal{G})\mathcal{K}(x)$), *'aks* ($(\forall x : \sim\mathcal{G})\sim\mathcal{K}(x)$), and *ta'thīr* (! $(\forall x : \mathcal{G})\mathcal{K}(x) \wedge (\forall x : \sim\mathcal{G})\sim\mathcal{K}(x)$)

8. If he does not succeed, the play stops unless the Opponent decides to cooperate as described in the next step.

9. The Opponent might react by deciding to cooperate by first proposing a more precise formulation of the property advanced or by proposing a new property for the constitution of the occasioning factor.[32] This will trigger a sub-play where the Opponent will defend the choice of an alternative property following the procedure prescribed for a *mu'āraḍa*-move or constructive criticism. Once the sub-play ended, the play proceeds to step 12. A *mu'āraḍa*-move assumes (1) that the choice of the root-case and the choice of ruling are relevant for the thesis , despite the fact that the Proponent chooses the wrong property for determining the occasioning factor (2) that the branch-case instantiates the "right" (newly proposed property).

The launching of a constructive criticism will be indicated with the following notation
! V *illa* $^{\mathcal{K}(\mathcal{G}^{\#})+}$ *.aṣl* : $\mathcal{K}$(*aṣl*)

10. The Opponent might also react by *strongly* rejecting the Proponent's proposal. We distinguish two cases that we call (1) *Destruction of the thesis*. The main target of this form objection is the thesis rather than only objecting against to the Proponent proposal for determining the *'illa*. In such a case it is he, the Opponent, who has to bring forward a counterexample from the sources. This will trigger a sub-play where the Opponent develops his counter argumentation, following the prescriptions for one of the forms of destructive criticism, namely: *qalb'* (reversal), *naqḍ* (*inconsistency*), or *kasr* (*breaking apart*). (2) *Destruction of the 'illa*. The counter-argument involves bringing forward objections against the proposed *waṣf* proposed as determining the *'illa,* following the prescriptions for attacks of the forms *fasād al-waḍ'* (invalidity of occasioned status) or *'adam al-ta'thīr* (lack of efficiency). If the Opponent succeeds the play stops.

[31] In the context of *jadal* this move is called "*ta'līl*" by the means of which the Proponent asserts that a given property determines the factor occasioning the relevant ruling – see Young (pp. 568, nn. 24-25, p. 624).
[32] This counterattack of the Opponent is a *mu'āraḍa* move, extensively discussed by Miller (1985, pp. 33-39) and by Young (2017, pp. 151), who calls it *constructive criticism*. It is opposed to the *destructive criticism* or *naqḍ* displayed in the following step.

11. If the Opponent concedes that the property determines the occasioning factor for the ruling of the root-case, the Proponent will ask the Opponent to acknowledge that this exemplifies a general law binding the ruling with the relevant property.[33]

12. If the Opponent concedes that the property does determine the occasioning factor for the ruling of the root-case, the Proponent will ask the Opponent to acknowledge that the property also applies to the branch-case . – recall (section III.2.1) that we adopt here al-Baṣrī's and al-Shīrāzī's practice of keeping only those plays where the Opponent responds positively to this form of request. If the property does not apply, though it determines the occasioning factor, then it is the main thesis that should be rejected. In other words, if the Opponent refuses to concede that the branch-case instantiates the relevant property a kind of strong rejection results.

The request and answer will be expressed by means of the following notation:

P *far'* : $\mathcal{P}$? (or $\mathcal{P}^*$)
O *far'* : $\mathcal{P}$ (or $\mathcal{P}^*$)

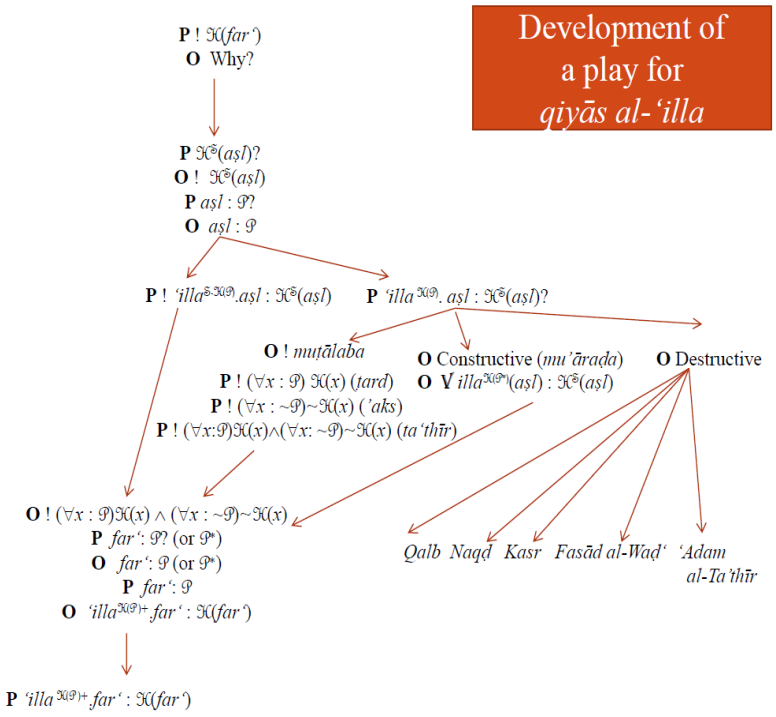
13. After the Opponent concedes that the property does apply to the branch case, and since the Opponent also concedes that the property is the one that characterizes the relevant occasioning factor, the Proponent will ask the Opponent to acknowledge that the branch-case falls under the ruling at stake. This move forces the Opponent to concede the challenged thesis. In fact the play will end (if successful) by the Proponent indicating that the Opponent has finished by conceding the thesis under scrutiny. For short, a play that ends by bringing the Opponent to silence (*iflḥām*), is a play a won by the Proponent. Otherwise it is a play won by the antagonist – i.e. the Proponent accepts the objections of the Opponent (*ilzām*). A play ends if there are no more moves allowed.

The final moves of a successful play have the following form

P *far'* : $\mathcal{P}$ (challenging the universal that expresses the *ṭard* -condition)
O *'illa*$^{\mathcal{H}(\mathcal{P})+}$(*far'*) : $\mathcal{H}$(*far'*)
P *'illa*$^{\mathcal{H}(\mathcal{P})+}$(*far'*) : $\mathcal{H}$(*far'*) (answer to the request for justification of the thesis)

(or involving the alternative property $\mathcal{P}^*$)

[33] Recall our remark in section III.1.1 concerning the fact that identifying an occasioning factor amounts characterizing it as a general law.



Global Reasons, Applications and the Constitution of Strategies:
While building the core of a *winning* **P**-strategy play objects are linked not only to the local meaning of expressions, but also to their justification. This cannot be achieved while considering single plays—nor non-winning strategies. Consider for example the case of a **P**-conjunction such that the Proponent claims that it has a (winning) strategic object for it. Single plays cannot provide a way to check if a conjunction is justified: this would require **P** to win the play for the two conjuncts. However, if the repetition rank chosen by the Opponent is 1, then in no single play can **P** bring forward the strategic object for the whole conjunction. It is only within the tree that displays the winning-strategy that both plays can be brought together as two branches with a common root. . Indeed, if we think of the tree as developed through the plays, the root of the tree will not explicitly display the information gathered while developing the plays. When a play starts it is just a *posit.* Only at the end of the construction-process of the relevant plays **P** will be able to have the knowledge required to *assert* the thesis. Similarly, in the case of a disjunction, we will able to display the strategic object correspondent to the choice that yielded the canonical argumentation form of the strategic object, only after the choices involving the defence have been made. More generally:

- The *assertion* of the thesis that makes explicit the reaason resulting from the plays is a *recapitulation* of the result achieved after running the relevant plays, after **P**'s initial posit of that thesis. This is, what the canonical argumentation form of a reason is at the strategic level, and this is what renders the dialogical formulation of a canonical proof-object. We call those reasons that constitute a winning strategy *global reasons.*

In the case of material implication (and universal quantification) a winning **P**-strategy literally displays the procedure by which the Proponent chooses the local reason for the consequent *depending* on the local reason chosen by the Opponent for the antecedent. What the canonical argumentation form of a global reason does is to make explicit the relevant *choice-dependence* by means of a *recapitulation* of the thesis.
This corresponds to the general description of proof-objects for material implications and universally quantified formulas in CTT: a method which, given a proof-object for the antecedent, yields a proof-object for the consequent. The dialogical interpretation of this functional dependence amounts rendering the canonical argumentation form of a global reason for **P** ! $A \supset B$ as **P** $p(x)[\![\mathbf{O}^{x:A}]\!] : A \supset B$ that expresses that if **P** is looking to make his claim legitimate he must be able to assert the consequent for any reason that the Opponent brings forward for backing his (the Opponent's) own assertion of the antecedent. Thus, the global reason for the material implication $A \supset B$ is the "strategic-object" $p(x)[\![\mathbf{O}^{x:A}]\!]$ – in CTT it corresponds to the lambda-abstract of the local reason for the consequent, namely the lambda-abstract of the function $p(x) : B$.

Let us express all this in the form of a table:

Canonical argumentation form and global reasons within winning strategies for P
Strategic objects as *Recapitulation*

Posit	Challenge	Defence	Recapitulation
P ! $(\exists x : A)\, B(x)$	**O** $?_L$ Or **O** $?_R$	**P** $p_1 : A$ Respectively **P** $p_2 : B(p_1)$	**P** $< p_1, p_2 > : (\exists x : A)B(x)$
P ! $(\forall x : A)B(x)$	**O** $p_1 : A$	**P** $p_2.\mathbf{O}^{p1} : B(\mathbf{O}^{p1})$	**P** $p(x)[\![\mathbf{O}^{x:A}]\!] : (\forall x : A)B(x)$ Notice that the CTT-notation $(\lambda x)p(x) : (\forall x : A)B(x)$ does not make explicit that the the arguments of $p(x)$ are chosen by **O**.
P ! $(\forall x : \mathcal{P})\mathcal{H}(x)$	**O** $p_1 : \mathcal{P}$	**P** $\textit{ţard}^{\mathcal{H}(\mathcal{P})}.\mathbf{O}^{p1} : \mathcal{H}(\mathbf{O}^{p1})$	**P** $\textit{ţard}^{\mathcal{H}(\mathcal{P})}(x)[\![\mathbf{O}^{x:\mathcal{P}}]\!] : (\forall x : \mathcal{P})\mathcal{H}(x)$
P ! $(\forall x : \sim\mathcal{P})\sim\mathcal{H}(x)$	**O** $p_1 : \sim\mathcal{P}$	**P** $\textit{'aks}^{\mathcal{H}(\mathcal{P})}.\mathbf{O}^{p1} : \sim\mathcal{H}(\mathbf{O}^{p1})$	**P** $\textit{'aks}^{\mathcal{H}(\mathcal{P})}(x)[\![\mathbf{O}^{x:\sim\mathcal{P}}]\!] : (\forall x : \sim\mathcal{P})\sim\mathcal{H}(x)$

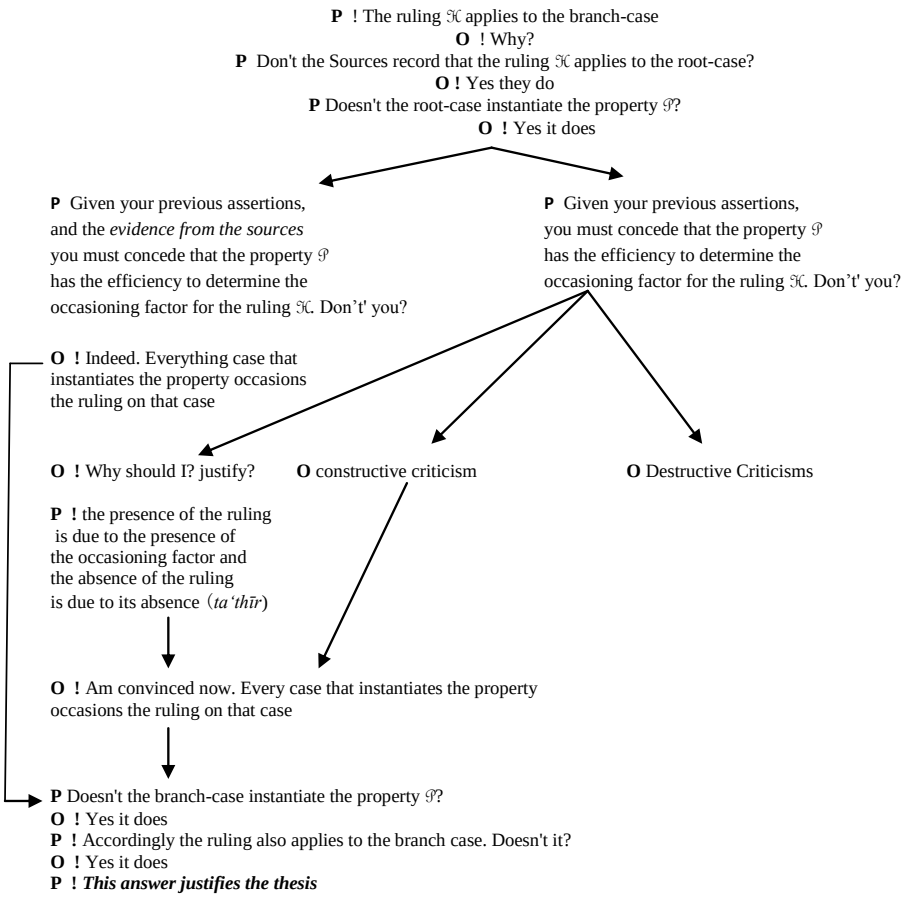
Notice that the canonical form of a global reasoning has been defined only for **P**. There is not general reason to do so; however we proceeded in this way since we are after a notion of winning strategy that corresponds to that of a CTT-demonstration, and these strategies have being identified as those where **P** wins. In fact the table above is the dialogical analogue to the *introduction rules* in CTT. Dialogically speaking those rules display the *duties* required by *P*'s own assertions – we will come back to this issue later on.

Now, we also need to specify the global-reason that provides the legitimation of the (Proponent's) thesis, when it is the Opponent who made the choice: a winning-strategy for P should also include those cases where it is the contender who brought forward some assertion. In our context, the dialectical meaning of the notion of occasioning factor, is that the Proponent justifies his thesis relying on the endorsements of the Opponent. In particular, if the Opponent endorses the efficiency of the property $\mathcal{P}$ in relation to the ruling $\mathcal{H}$, and also concedes that the branch-case instantiates $\mathcal{P}$; then the Proponent can legitimate his thesis by claiming that the reason endorsed by the Opponent provides the occasioning factor that justifies his thesis.

Posit	Challenge	Defence	Application
O $\textit{ţard}^{\mathcal{H}(\mathcal{P})} : (\forall x : \mathcal{P})\mathcal{H}(x)$ Notation for a posit without specified reason **O** ! $(\forall x : \mathcal{P})\mathcal{H}(x)$	**P** $p : \mathcal{P}$ (provided O endorses this assertion)	**O** $\textit{'illa}^{\mathcal{H}(\mathcal{P})+}.p : \mathcal{H}(p)$	**P** $\textit{'illa}^{\mathcal{H}(\mathcal{P})+}[\![\mathbf{O}^{x:\mathcal{P}}]\!].\ \mathbf{O}^{p:\mathcal{P}} : \mathcal{H}(p)$ Where "$\textit{'illa}^{\mathcal{H}(\mathcal{P})+}[\![\mathbf{O}^{x:\mathcal{P}}]\!].\ \mathbf{O}^{p:\mathcal{P}}$" stands for the application "$\textit{'illa}^{\mathcal{H}(\mathcal{P})}(\textit{ţard}^{\mathcal{H}(\mathcal{P})}(x)[\![\mathbf{O}^{x:\mathcal{P}}]\!]).\ \mathbf{O}^{p:\mathcal{P}}$"
O $\textit{'aks}^{\mathcal{H}(\mathcal{P})} : (\forall x : \sim\mathcal{P})\sim\mathcal{H}(x)$ Notation for a posit without specified reason **O** ! $(\forall x : \sim\mathcal{P})\sim\mathcal{H}(x)$	**P** $p : \sim\mathcal{P}$ (provided O endorses this assertion)	**O** $\textit{'illa}^{\mathcal{H}(\mathcal{P})}.p : \sim\mathcal{H}(p)$	**P** $\textit{'illa}^{\mathcal{H}(\mathcal{P})}[\![\mathbf{O}^{x:\sim\mathcal{P}}]\!].\mathbf{O}^{p:\sim\mathcal{P}} : \sim\mathcal{H}(p)$ Where "$\textit{'illa}^{\mathcal{H}(\mathcal{P})}[\![\mathbf{O}^{x:\sim\mathcal{P}}]\!].\mathbf{O}^{p:\sim\mathcal{P}}$" stands for the application "$\textit{'illa}^{\mathcal{H}(\mathcal{P})}(\textit{'aks}^{\mathcal{H}(\mathcal{P})}(x)[\![\mathbf{O}^{x:\sim\mathcal{P}}]\!]).\ \mathbf{O}^{p:\mathcal{P}}$"

EXAMPLES

Development of a play for *quiyas-al-'illa l-'lll*



Terminological Conventions and Strategy Procedure:

We slightly changed the usual notation of the dialogical framework and added some further indications specific to the *qiyās.* More precisely

- **General Notational Devices**:
  1. Proponent's moves are numbered with even numbers starting from **0**. Those, moves are recorded at the outmost right column.
  2. Opponent's moves are numbered with odd numbers starting from **0**. Those, moves are recorded at the outmost left column.
  3. The inner columns record the form (challenge or defence) of response and the line to which the move responds. So, while " ? 0 " indicates that the corresponding move is a challenge (by the Opponent) to line 0 of the Proponent; " ! 3 " indicates that corresponding move is a defence of a challenge launched by the Opponent in move 3.
  4. Formal expression **with preceding exclamation mark** such as ! $\mathfrak{K}^{\mathfrak{s}}(aṣl)$ indicates the assertion that there is some (not yet specified) occasioning factor for the fact that, according to the sources the ruling ℜ, applies to the root-case. Similar applies to expression such as ! $\mathfrak{K}(far')$.
  5. Formal expressions **without preceding exclamation mark** such as $illa^{\mathfrak{K}(\mathfrak{P})+}.\ far' : \mathfrak{K}(far')$ by the Proponent indicate that the justification for the application of the ruling to the branch-case follows from applying that branch-case to the universal $(\forall x : \mathfrak{P})\mathfrak{K}(x)$ conceded by the Opponent. The point of the Proponent is that he will try during the play to force the Opponent to provide the missing justification for the thesis. In other words, the Proponent will try to motivate the passage from ! $\mathfrak{K}(far')$ to $illa^{\mathfrak{K}(\mathfrak{P})+}.\ far' : \mathfrak{K}(far')$.

6. For the sake of notational simplicity we did not include the moves related to the repetition rank..

- **Specific addenda to *qiyās***:
  The dialectical framework for *qiyās al-'illa* deploys not only the usual challenges and defences but also requests. With a request a player brings forward an assertion and asks the contender to endorse it.
  The notation deployed for a request has the form "¿ 1, ! 2" (that reads: the Proponent responds to move 1 of the Opponent by requesting him to endorse assertion brought forward in move 2.)
  Sometimes a request formulated in move *k* responds to move *n* of the antagonist X, given a previous move *m* of X, this request will be indicated with the notation "¿ *n* (*m*), ! *k*"

  Before endorsing the requested assertion brought forward with move *m* the requested contender might ask for justification of this request. This response will be indicated with the notation "? *m* ¿".

Example of *a qiyās al-'illa (al-jalī bi'l-naṣṣ)*

The importance of this form of this *qiyās al-'illa,* despite its simplicity, is that it has the canonical form of a *qiyās al-'illa*. Moreover, it is related to Aristotle's *reasoning by exemplification* or *paradigmatic inference* (cf. Aristotle, *Pr. An*. 69a1, Bartha (2010), pp. 36-40), though, as pointed out before (III.1.1), it is not to be understood as involving one-step induction.[34]

O			**P**		
		responses	responses	Reading (without permission) the mail of someone else is forbidden ! $\mathcal{H}(far')$	0
1	Why?	? 0 (challenges move 0)	¿ 1, ! 2 (responds to 1 with the request of endorsing 2)	Entering (without permission) into a house of someone else is forbidden by the Quran, isn't it? [35] $\mathcal{H}^{\mathcal{S}}(aṣl)$?	2
3	Yes ! $\mathcal{H}^{\mathcal{S}}(aṣl)$	! 2 (responds to the request of move 2)	¿ 3, ! 4	Does entering (without permission) into a house of someone else violate privacy. Don't you agree? $aṣl : \mathcal{P}$?	4
5	I do. ! $aṣl : \mathcal{P}$	! 4	¿ 5 (3), ! 6	Given 3 and 5, and the evidence from the sources you must concede that violation of privacy has the efficiency to determine the *'illa* of that *ḥukm.* Don't' you? $'illa^{\mathcal{S}.\mathcal{H}(\mathcal{P})+}.aṣl : \mathcal{H}(aṣl)$?	6
7	I see. I endorse it since it comes from the sources the assertion ! $(\forall x : \mathcal{P})\mathcal{H}(x) \wedge (\forall x : \sim\mathcal{P})\sim\mathcal{H}(x)$	! 6	¿ 7, ! 8	Does reading (without permission) the mail of someone else violate the privacy of that person? $far' : \mathcal{P}$?	8
9	Yes, it does	! 8	? 7	So, since reading (without permission) the mail of someone else violates the privacy of that person, it instantiates the antecedent of the *ṭard* -component of your assertion linking privacy-violation and interdiction.	10

[34] It might be argued that Aristotle's notion does not involve one-step induction either.

[35] In fact this interdiction is explicitly sanctioned in the Quran:
يَاأَيُّهَا الَّذِينَ ءامَنُوا لاتَدْخُلُوا بُيُوتًا غَيْرَ بُيُوتِكُمْ حَتَّى تَسْتَأْنِسُوا وَتُسَلِّمُوا عَلَى أَهْلِهَا
(*O believers! Do not enter houses other than your own until you have sought permission and said greetings of peace to the occupants*) [Q.S. 24: 27].

	far' : $\mathcal{P}$			You should now assert the consequent. Right? *far'* : $\mathcal{P}$	
11	Indeed, I endorse this interdiction to the branch-case too *'illa*$^{\mathcal{S}:\exists(\mathcal{P})+}$. *far'* : $\mathcal{K}$(*far'*)	! 12	! 1	So, this provides the justification for the thesis you were asking for with your first move: the branch-case falls under the ruling because it instantiates the property you just endorsed as constituting the occasioning factor. *'illa*$^{\mathcal{S}:\exists(\mathcal{P})+}$. *far'* : $\mathcal{K}$(*far'*)	12
	ifḥām				

Examples o*f qiyās al-'illa al-khafī*

The following example is a reconstruction that follows closely al-Shīrāzī's (1987, p. 112) refutation of Ḥanafī's analysis of the argument on the purity status of beasts of prey. As pointed out by Young (2017, p. 159) al-Shīrāzī himself thought that the argument should be developed following a *fasād al-waḍ'* (invalidity of the occasioned status) –move.[36] Indeed, al-Shīrāzī sees the argument as indicating that the main thesis is fundamentally false since it assumes that beasts of prey are impure, but there is direct evidence from the sources contradicting this. Thus, according to al-Shīrāzī we do not need to be involved in a discussion about the suitability or not of the property chosen by the Proponent. Our take on the example corresponds rather to Miller's (1984, p. 119) presentation of *qalb* or *destructive criticism by reversal*. Moreover, it corresponds to a particular form of *qalb* called *reversal* and *oppositeness* (*al-qalb' wa'l-'aks*) – see Young (2017, pp. 166-167) . We made the choice to reconstruct the *qalb*-version of this argument since it provides the chance to display the deployment of a sub-play while developing a *destructive criticism.*

On beasts of prey, impure saliva and the deployment of *qalb'*

O			**P**		
				The saliva of the beast of prey qualify as impure (*najāsa*) ! $\mathcal{K}$(*far'*)	0
1	Why?	? 0	¿ 1, ! 2	Does the saliva of pigs qualify as impure (*najāsa*)? $\mathcal{K}^{\mathcal{S}}$ (*aṣl*)?	2
3	Yes I do ! $\mathcal{K}^{\mathcal{S}}$ (*aṣl*)	! 2	¿ 3, ! 2	Does the saliva of pigs come from an animal that has canine teeth (*dhū nābin*)? *aṣl** : $\mathcal{P}$?[37]	4
5	Yes they do ! *aṣl** : $\mathcal{P}$	! 4	¿ 5 (3), ! 6	Given 3 and 5 it seems plausible to conclude that the saliva of animals with canines has the required efficiency for determining the relevant *'illa* for its impurity. Don't you agree? *'illa*$^{\exists(\mathcal{P})+}$.*aṣl* : $\mathcal{K}^{\mathcal{S}}$(*aṣl*) ?	6
7	*qalb'* ! Do not agree! I have a counterexample ! $\mathbb{F}$ ($\forall x$: $\mathcal{P}$)$\mathcal{K}$(x) START OF THE SUB-PLAY ----------	? 6]		START OF THE SUB-PLAY ----------	

[36] Different to Young's (2017, p. 159) analysis, Miller (1984, p. 119) concludes that al-Shīrāzī' presentation suggests that both forms of destructive criticism, namely *qalb* and *fasād al-waḍ'*, are indistinguishable.

[37] For the sake of simplicity we do not reflect in our formalization the mereological relation between animals and their saliva.

			? 7	I rather endorse the following: It is true that impurity applies to any saliva of an animal possessing canines ! $(\forall x : \mathcal{P})\ \mathcal{H}x$	8
9	Cats possess canine teeth. Thus, according to your characterization of $\mathcal{P}$ (*saliva of animals possessing canines*), their saliva is impure. ! *cat-saliva* : $\mathcal{P}$	? 8	! 9	Indeed I have to concede this *'illa*$^{\mathcal{H}\mathcal{P}+}$. *cat-saliva* : $\mathcal{H}$(*cat-saliva*)	10
11	We know (from the sources) that the saliva of cats is not impure. Do you see this? ~$\mathcal{H}^{\mathcal{S}}$(*cat-saliva*)?.	¿ 10, ! 11	! 11	I must. It comes from the sources ! ~$\mathcal{H}^{\mathcal{S}}$(*cat-saliva*)	12
13	*tanāquḍ* ! You asserted before that according to your view on the occasioning factor, it follows that the saliva of cats is impure. You contradict yourself![38] Therefore possessing canine teeth is not the occasioning factor of saliva's impurity. ! $\mathcal{H}$(*cat-saliva*)	? 12		I concede.	14
15	Moreover, cats are beasts of prey. So, their saliva is the saliva of a beast of prey. Furthermore, the saliva of a beast of prey is a case of the saliva of animals with canines. Right?[39] *far'* : $\mathcal{P}$?	¿ 14, ! 15		Yes, it is *far'* : $\mathcal{P}$	16
17	So you must also concede that their saliva is impure neither? ~$\mathcal{H}^{\mathcal{S}}$(*far'*) ?	¿ 16, ! 17		Indeed. *'illa*$^{\mathcal{H}\mathcal{P}+}$. *far'*: ~$\mathcal{H}^{\mathcal{S}}$(*far'*)	18
19	*tanāquḍ* ! This contradicts your main thesis. ! $\mathcal{H}^{\mathcal{S}}$(*far'*)	? 18		I give up *ilzām.*	

What the Opponent is doing is displaying a winning strategy for a claim that denies that $\mathcal{P}$ determines the relevant occasioning factor. Notice that it is stronger than the rejection of endorsing a claim. The opponent is changing the roles and defending that he has a winning strategy in order to reject $\mathcal{P}$ as determining occasioning factor. This move *a switch of roles* pointed out by scholars as Hallaq (1985) and Young (2017).

The following example is one that has received very much attention in the specialized literature.

The Wine-example

O	P

[38] The player that brings up the expression *tanāquḍ*, accuses the antagonist of self-contradiction – for a thorough discussion on this notion see Young (2017, pp. 537-543).

39 In order to focus on the main argumentation thread we did not include (formally) the moves that lead from *saliva of animals of prey* to *saliva of the cats*.

				(Consuming) Date-wine (*nabīdh*) is forbidden (*ḥarām*)[40]. ! $\mathcal{K}(far')$	0
1	Why?	? 0	¿ 1, ! 2	Isn't drinking grape-wine (*khamr*) forbidden by the Quran? [41] $\mathcal{K}^{\mathcal{C}}(aṣl)$?	2
3	Yes, it is forbidden. ! $\mathcal{K}^{\mathcal{C}}(aṣl)$	! 2	¿ 3, ! 4	Isn't grape-wine a drink made of fruit-juice which contains *euphoric intensity* (*shiddat muṭriba*)? $aṣl : \mathcal{P}$?	4
5	Yes ! $aṣl : \mathcal{P}$	! 4	¿ 3, (5), ! 6	So, according to your moves 3 and 5, the presence of euphoric intensity occasions the proscription of consuming grape-wine. Right? $'illa^{\mathcal{KP}+}.aṣl : \mathcal{K}^{\mathcal{C}}(aṣl)$?	6
7	*muṭālaba* ! Justify !	? 6	! 7	'*aks*: Before the occurrence of the euphoric intensity, the lawfulness of consuming a drink made of fruit-juice is the object of consensus. ! $(\forall x : \sim\mathcal{P})\sim\mathcal{K}(x)$ '*ṭard*: After the euphoric intensity occurs [i.e., when it becomes wine] and nothing else occurs the proscription of consuming a drink made of fruit-juice is object of consensus. (ratification of) '*aks*: When the euphoric intensity of a drink made of fruit-juice falls away [i.e., when it becomes vinegar] and nothing else falls away it is object of consensus that it should not be forbidden. ! $(\forall x : \mathcal{P})\mathcal{K}(x)$ *ta'thīr*: Therefore, the presence of the *ḥukm* is due to the presence of the *'illa*, and the absence of the *ḥukm* is due to its absence ! $(\forall x : \mathcal{P})\mathcal{K}(x) \wedge (\forall x : \sim\mathcal{P})\sim\mathcal{K}(x)$	8
9	Given these arguments I concede your previous request ! $(\forall x : \mathcal{P})\mathcal{K}(x) \wedge (\forall x : \sim\mathcal{P})\sim\mathcal{K}(x)$	! 6 (8)	¿ 9, ! 10	Isn't *nabīdh* a drink made of fruit-juice which contains 'euphoric intensity'? far': $\mathcal{P}$?	10
11	Yes, I agree far': $\mathcal{P}$	! 10	? 9	If it is the case that date-wine contains euphoric intensity, and, given 9, should not this lead to its interdiction? far': $\mathcal{P}$	12
13	Indeed, the presence of euphoric intensity should occasion its interdiction.	! 10	! 1	So, this provides the justification for the thesis you were asking for with your first move: the branch-case falls under the ruling because it instantiates the property you just endorsed as constituting the	14

[40] The original text deploys the word *ḥarām.* This notion, the opposite of *ḥalāl*, refers (in this context) to the interdiction of consuming certain food.

[41] It is sanctioned in the Quran that wine is *ḥarām* (forbidden [to be consumed]) :
يَا أَيُّهَا الَّذِينَ آمَنُوا إِنَّمَا الْخَمْرُ وَالْمَيْسِرُ وَالْأَنصَابُ وَالْأَزْلَامُ رِجْسٌ مِّنْ عَمَلِ الشَّيْطَانِ فَاجْتَنِبُوهُ لَعَلَّكُمْ تُفْلِحُونَ
(*O you believe! Wine, gambling, altars and divining arrows are filth, made up by Satan. Therefore, refrain from it, so that you may be succesful*). [Q.S: 5: 90]

	'illa$^{\mathfrak{K}(\mathcal{P})+}$. *far'*: $\mathfrak{K}^{\mathfrak{S}}$(*far'*)			occasioning factor. *'illa*$^{\mathfrak{K}(\mathcal{P})+}$. *far'*: $\mathfrak{K}^{\mathfrak{S}}$(*far'*)	
	iḥām				

The Wine-example and the deployment of *mu'āraḍa*

As already mentioned *mu'āraḍa*-moves assume a cooperative attitude of the challenger. In the following example, we assume that the original argument in favour of choosing the property of being a drink made of pressed fruit-juice as relevant for the determining the relevant example misses one of those conditions, namely co-presence (the counterexample is vinegar).
At the end we will sketch the winning strategy, which, as discussed in section III.2.2, only keeps the result of the cooperation.

O			**P**		
				(Consuming) Date-wine is forbidden. [42]. ! $\mathfrak{K}$(*far'*)	0
1	Why?	? 0	¿ 1, ! 2	Isn't drinking grape-wine forbidden by the Quran? [43] $\mathfrak{K}^{\mathfrak{S}}$(*aṣl*)?	2
3	Yes, it is *ḥarām*. ! $\mathfrak{K}^{\mathfrak{S}}$(*aṣl*)	! 2	¿ 3, ! 4	Isn't grape-wine made of pressed fruit-juice *aṣl* : $\mathcal{P}$?	4
5	Yes *aṣl* : $\mathcal{P}$	! 4	¿ 3, (5), ! 6	So, according to your moves 3 and 5, the proscription of consuming grape-wine is caused by the fact that it is made of pressed fruit-juice. Right? *'illa*$^{\mathfrak{K}(\mathcal{P})+}$.*aṣl* : $\mathfrak{K}^{\mathfrak{S}}$(*aṣl*) ?	6
7	I am not convinced. I rather think that the relevant property is containing euphoric intensity ($\mathcal{P}^*$) ! $\forall$ *illa* $^{\mathfrak{K}(\mathcal{P}^*)+}$.*aṣl* : $\mathfrak{K}^{\mathfrak{S}}$(*aṣl*).	? 6	? 7	*muṭālaba* ! Justify !	8
	START OF THE SUB-PLAY ----------			**START OF THE SUB-PLAY** ----------	
9	Vinegar is made of pressed juice-fruit. Isn't it? *aṣl** : $\mathcal{P}$?	¿ 8, ! 9		Indeed. *aṣl** : $\mathcal{P}$	10
11	Given 6 you must agree that being a pressed-juice is efficient property for sanctioning them as *ḥarām*. Right? $(\forall x : \mathcal{P})\mathfrak{K}(x) \wedge (\forall x : \sim\mathcal{P})\sim\mathfrak{K}(x)$?	¿ 6, ! 10		Yes ! $(\forall x : \mathcal{P})\mathfrak{K}(x) \wedge (\forall x : \sim\mathcal{P})\sim\mathfrak{K}(x)$	12
13	But, given that you just agreed that vinegar is made of pressed-juice,	12		Indeed	14

[42] The original text deploys the word *ḥarām.* This notion, the opposite of *ḥalāl*, refers (in this context) to the interdiction of consuming certain food.
[43] It is sanctioned in the Quran that wine is *ḥarām* (forbidden [to be consumed]) :
يَا أَيُّهَا الَّذِينَ آمَنُوا إِنَّمَا الْخَمْرُ وَالْمَيْسِرُ وَالْأَنصَابُ وَالْأَزْلَامُ رِجْسٌ مِّنْ عَمَلِ الشَّيْطَانِ فَاجْتَنِبُوهُ لَعَلَّكُمْ تُفْلِحُونَ
(*O you believe! Wine, gambling, altars and divining arrows are filth, made up by Satan. Therefore, refrain from it, so that you may be successful*). [Q.S: 5: 90]

	(according to the *ṭard* -component of your assertion) it should be *ḥarām* ! $aṣl^* : \mathcal{P}$			*'illa*$^{\mathfrak{K}(\mathcal{P})+}$*.aṣl** : $\mathfrak{K}(aṣl^*)$	
15	But its consumption is not forbidden. Isn't it? ! $\sim\mathfrak{K}^{\mathfrak{S}}(aṣl^*)$?	¿ 14, ! 15		Yes it is not *ḥarām* ! $\sim\mathfrak{K}^{\mathfrak{S}}(aṣl^*)$	16
17	*tanāquḍ* ! You contradict yourself ! $\mathfrak{K}(aṣl^*)$	? 16		I concede !	18
19	Herewith my argument for the relevance of $\mathcal{P}^*$ '*aks*: Before the occurrence of the euphoric intensity, the lawfulness of consuming a drink made of fruit-juice is the object of consensus. ! $(\forall x : \sim\mathcal{P}^*)\sim\mathfrak{K}(x)$ '*ṭard*: After the euphoric intensity occurs [i.e., when it becomes wine] and nothing else occurs the proscription of consuming a drink made of fruit-juice is object of consensus. (ratification of) '*aks*: When the euphoric intensity of a drink made of fruit-juice falls away [i.e., when it becomes vinegar] and nothing else falls away it is object of consensus that it should not be forbidden. ! $(\forall x : \mathcal{P}^*)\mathfrak{K}(x)$ *ta'thīr*: Therefore, the presence of the *ḥukm* is due to the presence of the *'illa*, and the absence of the *ḥukm* is due to its absence *'illa* $^{\mathfrak{K}(\mathcal{P}^*)}$: $(\forall x : \mathcal{P}^*)\mathfrak{K}(x) \wedge (\forall x : \sim\mathcal{P}^*)\sim\mathfrak{K}(x)$	! 8			
	END OF THE SUB-PLAY ----------			**END OF THE SUB-PLAY** ----------	
21	Yes, it does. *far'*: $\mathcal{P}^*$	! 20	¿ 19, ! 20	Doesn't *nabīdh* contains *euphoric intensity*? *far'*: $\mathcal{P}^*$?	20
23	Indeed ! *'illa*$^{\mathfrak{K}(\mathcal{P}^*)+}$*.far'* : $\mathfrak{K}$(*far'*)	! 23	? 19	If it is the case that date-wine contains euphoric intensity, and, given 19, should not this lead to its interdiction? *far'*: $\mathcal{P}^*$	22
			! 1	So, this provides the justification for the thesis you were asking for with your first move: the branch-case falls under the ruling because it instantiates the property you just helped to identify as the one determining the occasioning factor. *'illa*$^{\mathfrak{K}(\mathcal{P}^*)+}$*.far'* : $\mathfrak{K}$(*far'*)	24
	iſḥām				

This yields the following tree displaying the winning-strategy, which is (essentially) the same as the one of the previous example – the only difference with the strategy for the previous example is that the tree of the latter will include the moves justifying the determination of the occasioning factor. The point is that the strategy deletes the unsuccessful attempts:

0. **P** ! $\mathcal{H}(far')$
1. **O** ! Why [?0]
2. **P** ! $\mathcal{H}^{\mathcal{S}}(aṣl)$?
3. **O** ! $\mathcal{H}^{\mathcal{S}}(aṣl)$
4. **P** $aṣl : \mathcal{P}^*$?
5. **O** $aṣl : \mathcal{P}^*$
6. **P** $'illa^{\mathcal{H}(\mathcal{P}^*)+}.aṣl : \mathcal{H}^{\mathcal{S}}(aṣl)$?
7. **O** ! $(\forall x : \mathcal{P}^*)\mathcal{H}(x) \wedge (\forall x : \sim\mathcal{P}^*)\sim\mathcal{H}(x)$
8. **P** $far' : \mathcal{P}^*$?
9. **O** $far' : \mathcal{P}^*$
10. **P** $far' : \mathcal{P}^*$ [?7]
11. **O** $'illa^{\mathcal{H}(\mathcal{P}^*)+}.far' : \mathcal{H}(far')$
12. **P** $'illa^{\mathcal{H}(\mathcal{P}^*)+}.far' : \mathcal{H}(far')$ (!1. answer to the request of justification in the second move)

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UNFOLDING PARALLEL REASONING IN ISLAMIC JURISPRUDENCE

Initial Remarks to the Epistemic Features and Dialectical Meaning within Abū Isḥāq al-Shīrāzī's System of Co-Relational Inferences

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1

- Those different forms of *co-relational inference*, known in the Islamic jurisprudence as *Qiyās* or co-relational inferences represent an innovative and sophisticated form of parallel reasoning developed in Islamic jurisprudence

- They claimed to be proud of the inception of an approach different to the Greek Tradition: a system of reasoning with content, (proper of the juridical reasoning?) that combines heuristical; epistemological and logical steps.

2

Young (2017, p.1) set out the motivations for the development of a dialectical framework such as the one we are aiming at in the present paper.

> *The primary title of this monograph is "The Dialectical Forge," and its individual terms provide a suitable launching point for discussing the current project as a whole. As for the first, the most common Arabic terms for "dialectic" are* jadal*and* munāẓara*, both denoting formal disputation between scholars in a given domain, with regard to a specific thesis. When one encounters the term "dialectical" in the present work, one should think foremost of procedure-guided debate and the logic inherent to this species of discourse. A dialectical confrontation occurs between two scholars, in question and answer format, with the ultimate aims of either proving a thesis, or destroying it and supplanting it with another. A proponent-respondent introduces and attempts to defend a thesis; a questioner-objector seeks (destructively) to test and undermine that thesis, and (constructively) to supplant it with a counter-thesis. Through progressive rounds of question and response the questioner endeavours to gain concession to premises which invalidate the proponent's thesis, justify its dismantling, and provide the logical basis from which a counter-thesis necessarily flows.*
>
> *Ultimately, and most importantly, a truly dialectical exchange—though drawing energy from a sober spirit of competition—must nevertheless be guided by a cooperative ethic wherein truth is paramount and forever trumps the emotional motivations of disputants to "win" the debate. This truth-seeking code demands sincere avoidance of fallacies; it views with abhorrence contrariness and self-contradiction. This alone distinguishes dialectic from sophistical or eristic argument, and, in conjunction with its dialogical format, from persuasive argument and rhetoric. And to repeat: dialectic is formal—it is an ordered enterprise, with norms and rules, and with a mutually-committed aim of advancing knowledge.*

3

A dialectical framework + a CTT-inferential system arethe right instrument to stress three of the most salient features of this form of inference:

1) the interaction of heuristic with logical steps
2) the dynamics underlying the meaning-explanation of the terms involved.
3) the unfolding of parallel reasoning as similarity in action.

4

The Motivations behind the inception of Co-relational inferences

- The universality of Law and the needs of a new contextualization instrument
- Co-relational inferences and the origins of Civil-Law: Ratio Legis within the « dialectical forge ».
- The difference between the Ratio Legis, Concomitance of pairs of Rulings, Resemblance
- Different to Common Law! It is not about induction

5

Open texture of the meaning of normative statements.

- The notion of co-relational inference suggests that every form of parallel reasoning that shares the formal structure of the *qiyās* presupposes that the concept of meaning involved is open to contextual changes.
- This strongly suggests that the whole process deployed is intrinsically dialectic.

6

The Play Level of Dialogica Logic

- Epistemic assumptions for formal and Material Dialogues: specific copy-cat rules: definitional equalities

These notions offer a good way to understand what the dialectical meaning in correlational inferences is about

- The play-level and cooperative moves

  The notion of « Geltung » and strategy as recapitulation of the optimal moves

  The possibility to make the process of the play-level explicit

7

The simplest form follows the following pattern:

In order to establish if a given juridical ruling applies or not to a given case, we look for a case we already know that falls under that ruling – the so-called source-case.

Then we search for the property or set of properties– called the occasioning factor -upon which the application of the ruling to the source-case is grounded.

If that grounding properties are known we ponder if they can also be asserted of the new case under consideration.

In the case of affirmative answer it is inferred that the new case also falls under the specific juridical ruling at stake and so the range of its application is extended.

8

- Complications arrive when the grounds behind a given juridical ruling are not explicitly known or even not known at all.

- The we appeal to parallel reasoning and/or similarity

While finding the Ration Legis is close to the Eurpean Law reasoning, similarity is closer to Comon-Law reasoning by precedent cases.

9

- Our study is focused on AbūIsḥāqal-Shīrāzī's (393–476/1003–1083) classification of *qiyās* as discussed in his *Mulakhkhaṣfi'l-Jadal* (*Epitome on Dialectical Disputation*).

- Walter Young (2017): *The Dialectical Forge*. Dordrecht: Springer

10

- Exemplification,
- Symmetry, and
- Resemblance.

11

Qiyās al-'Illa

(2) The property $\mathcal{P}$ is the factor occasioning the juridical ruling $\mathcal{H}$

(3) $\mathcal{P}$ applies to the branch-case *f*

The juridical ruling $\mathcal{H}$ applies to the branch-case (*f*)
(it follows from 2 and 3)

(1.2) $\mathcal{P}$ applies to the root-case (*a*)

(1.1) The juridical ruling $\mathcal{H}$ applies to the root-case (*a*)

12

- Notice that the root-case and the branch-case do not need

  to be similar.
- The point is that the root-case is brought forward as exemplifying a general law under which the branch-case will be subsumed

13

Example of *a qiyās al-'illa (al-jalī bi'l-naṣṣ)*

O			P		
		responses	responses	Reading (without permission) the mail of someone else is forbidden ! ℋ(*far'*)	0
1	Why?	? 0 (challenges move 0)	¿ 1, ! 2 (responds to 1 with the request of endorsing 2)	Entering (without permission) into a house of someone else is forbidden by the Quran, isn't it? ℋ$^{\mathfrak{S}}$(*aṣl*)?	2
3	Yes ! ℋ$^{\mathfrak{S}}$(*aṣl*)	! 2 (responds to the request of move 2)	¿ 3, ! 4	Does entering (without permission) into a house of someone else violate privacy. Don't you agree? *aṣl* : ℘?	4
5	I do. ! *aṣl* : ℘	! 4	¿ 5 (3), ! 6	Given 3 and 5, and the evidence from the sources you must concede that violation of privacy has the efficiency to determine the *'illa* of that *ḥukm*. Don't' you? *'illa*$^{\mathfrak{S}.\wp*}$.*aṣl* : ℋ (*aṣl*)?	6

14

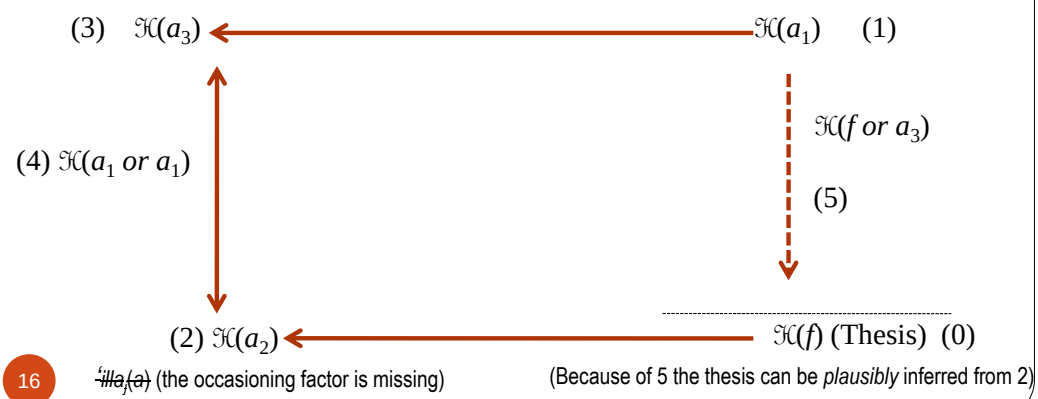
------Continue on the next page------

O			P		
7	I see. I endorse it since it comes from the sources the assertion ! $(\forall x : \mathcal{P})\mathcal{K}(x) \wedge (\forall x : \sim\mathcal{P})\sim\mathcal{K}(x)$	! 6	¿ 7, ! 8	Does reading (without permission) the mail of someone else violate the privacy of that person? *far'* : $\mathcal{P}$?	8
9	Yes, it does *far'* : $\mathcal{P}$	! 8	? 7	So, since reading (without permission) the mail of someone else violates the privacy of that person, it instantiates the antecedent of the *ṭard* -component of your assertion linking privacy-violation and interdiction. You should now assert the consequent. Right? *far'* : $\mathcal{P}$	10
11	Indeed, I endorse this interdiction to the branch-case too *'illa*$^{\mathcal{S}\mathcal{K}(\mathcal{P})+}$. *far'* : $\mathcal{K}$(*far'*)	! 12	! 1	So, this provides the justification for the thesis you were asking for with your first move: the branch-case falls under the ruling because it instantiates the property you just endorsed as constituting the occasioning factor. *'illa*$^{\mathcal{S}\mathcal{K}(\mathcal{P})+}$. *far'* : $\mathcal{K}$(*far'*)	12
	ifḥām				

15

Notation:
f =: branch-case: *modern form of divorce for foreigners*
a_1 =: simile to the branch-case: *old form of divorce for foreigners*
a_2 =: root-case: *ancient form of divorce for nationals*
a_3 = : root-case: *modern form of divorce for nationals*
$\mathcal{K}(x)$ =: *x is legally valid*

Qiyās al-dalāla II



16

~~*'illa*$_f$~~(*a*) (the occasioning factor is missing)

(Because of 5 the thesis can be *plausibly* inferred from 2)

Qiyās al-dalāla II

O			P		
				Thesis: the modern form of divorce as applied to foreigners is legally valid	0
1	Why?	0	1	Is the modern form as applied to foreigners legally valid?	2
3	Yes it does		3	Does the ancient form of divorce apply to nationals?	4
5	Yes it does		5	Is the modern form of divorce as applied to nationals legally valid?	6
7	Yes it does		1, 5	So , if a forms of divorce is either ancient or modern it does apply to nationals. Isn't it ?	8
9	Indeed		7	Can't we assert by parallel reasoning that similar applies for foreigners?	10
11	It seems plausibile		11	Thus, there seem to be reasonable grounds for assertin that the modern form of divorce as applied to foreigner is also legally valid.	12

17

Motivating the use of a CTT-framework

Per Martin Löf's Constructive Type Theory allows the following features underlying the *qiyās*t o be expressed at the object language level:

- The stress on assertions (or judgements) rather than on propositional sentences. The dialectical process underlying co-relational inferences is triggered by both **an assertion concerning the identification of the factor occasioning the relevant ruling** and **the process of the justification of such an assertion**. In the specialized literature these assertions are called *ta'līl* (affirmation of the relevance of a particular property for the determination of the *'illa*), or more generally *ithbāt* (affirmation).
- The intensional rather than extensional understanding of the sets underlying the semantics of the *qiyās*.
- The deployment of hypothetical judgements. This dovetails with the *qiyās*-notion of dependence of a given juridical ruling on a particular occasioning factor.
- The restrictive form of the substitution rules.

In the present paper the last point will be left out since it relates to co-lational inferences by indication, which will not be discussed here.

18

$b : A$ $\qquad\qquad$ *A true*

can be read as

b is an element of the set *A*	*A* has an element
b is a proof of the proposition *A*	*A* is true
b fulfils the expectation *A*	*A* is fulfilled
b is a solution to the problem *A*	*A* has a solution

The four basic forms of *categorical* judgements in CTT are

$A : set$
$A = B : set$
$A : prop$
$A = B : prop$

19

Four basic forms of hypothetical judgements with one assumption:

$B(x)$: **set** $(x : A)$
$B(x) = C(x)$: **set** $(x : A)$
$b(x) : B(x)$ $(x : A)$
$b(x) = c(x) : B(x)$ $(x : A)$

We read the first as "$B(x)$ is a set under the assumption $x : A$". Similar remarks apply to the other three forms of hypothetical judgement. Let us consider the more precise meaning explanations of these forms of judgement.

A judgement of the form $B(x)$: **set** $(x : A)$ means that

$B(a/x)$: **set** whenever $a : A$, and
20 $B(a/x) = B(a'/x)$: **set** whenever $a = a' : A$.

Let us consider that

$$B(x) : prop\ (\ x : A)$$

underlies the formation of our example on emails. This yields

"*B*(*x*) (*being forbidden*), renders a proposition once the free-variable *x* is substituted
by
some element *a* of the set *A* of cases of *violating privacy*."

If "*a*" stands for "*entering in someone else's house without permission*" we obtain
B(*a*)
That is, *"entering in someone else's house without permission is forbidden*"

21

Identifying the relevant set

In relation to the main step, let us start by pointing out that Islamic jurisprudents identified three general conditions to be met by the *waṣf* occasioning a ruling:

- Efficiency (*ta'thīr*).
- Co-extensiveness (*ṭard*) – the presence of the propertywhen the judgment is present.
- Co-exclusiveness (*'aks*) – the absence of the property when the judgment is absent.

22

Over the set *Privacy-Violation* we can then define the juridical ruling $\mathcal{H}ukm(x)$ (or for short $\mathcal{H}(x)$), that expresses a juridical ruling relevant to cases of *Privacy-Violation*:

$$\mathcal{H}(x) :prop\ (x : Privacy\text{-}Violation),$$

This displays the relations of content linking ruling and property, but we need to be more explicit. What we need is to make it apparent that *Privacy-Violation* has the efficiency (*ta'thīr*) required to occasion the relevant juridical ruling.

23

Let us then analyze

> *Privacy-Violation occasions the juridical ruling sanctioning its proscription (given the efficiency of Privacy-Violation in relation to that proscription)*

as the construction

> *Cases of Privacy-Violation ($\mathcal{P}$) occasion the interdiction $\mathcal{H}$ (given the efficiency of $\mathcal{P}$ in relation to $\mathcal{H}$)*

Furthermore, if the property $\mathcal{P}$ is efficient in relation to the ruling $\mathcal{H}$, then there a method that provides the justification of applying the ruling to every instance of $\mathcal{P}$ (and dually, the justification of applying $\sim\mathcal{H}(x)$, given instances of $\sim\mathcal{P}$).

24

In such a context the occasioning factor ***'illa***$^{\mathcal{P}}$ is conceived as the application of the method to a particular case: each particular instance of *Privacy-Violation* provides the factor occasioning the proscription of that instance. E.g. *entering into the house of someone else without permission*, an instance of *Privacy-Violation*, provides the *'illa* occasioning the proscription of such an action. In other words

- the occasioning factor (*'illa*$^{\mathcal{P}}$) in relation to a juridical ruling $\mathcal{H}(x)$ defined over the set $\mathcal{P}$ is the application of the function from all instances of $\mathcal{P}$ into the set of instances of $\mathcal{H}(x)$.

25

Establishing that a given ruling applies to the branch-case of the thesis involves two main steps

1) recognizing that the root-case is an application of the function that takes us from every instance of $\mathcal{P}$ to a suitable instance of $\mathcal{H}(x)$ (and dually, the application takes every instance of $\sim\mathcal{P}$ to the negation of the ruling) – that is, the function that verifies the universal norm *Every* $\mathcal{P}$ *falls under the ruling* $\mathcal{H}$(and its dual),
2) recognizing that this general norm also applies to the branch-case.

26

- The point is that the construction underlying the meaning of application of the ruling to the root-case is, to put it in Bartha's terms (2010, p. 109), *precursor to a generalization*. However, the idea is quite different from what is nowadays called *one-step induction* – see e.g. Bartha (2010, pp. 36-40). Indeed, identifying the occasioning factor for the root-case under consideration amounts grasping it as exemplifying (the application of) a general law: this is what the notion of causality in *uṣūl al-fiqh* comes down to.

27

We could deploy

$\mathcal{H}(x)$:*prop* ({ x : *Drinks* | *Toxic*(x)})

(subset-separation: the set of those elements of the set of drinks that are toxic)

instead of the simpler

$\mathcal{H}(x)$:*prop* (x : *ToxicDrinks*)

(the set of toxic drinks)

However, for the sake of perspicuity, and despite the fact that this will lead us to the somewhat awkward formulation *instantiating the property*, we will deploy the second, simpler notation.

28

Recall

1) Efficiency (*ta'thīr*).
2) Co-extensiveness (*ṭard*) – the presence of the property when the judgment is present.
3) Co-exclusiveness (*'aks*) – the absence of the property when the judgment is absent.

29

Let the expression $x : \mathcal{P}$, stand for a set of drinks x that are toxic, and likewise $\sim\mathcal{P}$for non-toxic drinks.

Let the expression $\mathcal{H}(x)$ stand for the juridical ruling that the consumption of x is forbidden. Similar paraphrase admits the negation $\sim\mathcal{H}(x)$.

If we spell out the precise formulation of the property as determined by *ṭard* and *'aks*, the point is that

ṭard: If x is a toxic drink then its consumption is forbidden.

'aks: If x is not a toxic drink then its consumption is not forbidden.

We deploy here the expression *set toxic drinks* for simplicity. As discussed in the last sections, the set at stake is rather the *set of all those substances of which the property of having euphoric intensity sapplies*.

30

This yield the general norm:

$$(\forall x :\mathcal{P})\mathcal{H}(x) \wedge (\forall x :{\sim}\mathcal{P}){\sim}\mathcal{H}(x)$$

That reads: *The consumption of toxic drinks is forbidden and the consumption of non-toxic drinks is not*. Notice that the formation of each side of the conjunction still presupposes the dependence of the ruling upon the property:

$$\mathcal{H}(x) :prop \;\; (x : \mathcal{P})$$
$${\sim}\mathcal{H}(x) :prop \;\; (x : {\sim}\mathcal{P})$$

31

Accordingly, the formation of the conjunction underlying the efficiency of the property $\mathcal{P}$ in relation to the ruling $\mathcal{H}$ is structured as follows:

$ta'thīr^{\mathcal{P}}$

- $ṭard(x) : \mathcal{H}(x) \;\; (x : \mathcal{P})$
- $'aks(x) : {\sim}\mathcal{H}(x) \;\; (x : {\sim}\mathcal{P})$

32

Furthermore, the efficiency of the property P for our example is the pair

$$ta'thīr^{\mathcal{P}} =_{df} \langle ṭard^{P}, \ 'aks^{P}\rangle : (\forall x :\mathcal{P})\mathcal{H}(x) \wedge (\forall x :\sim\mathcal{P})\sim\mathcal{H}(x)$$

According to this analysis the **occasioning factor** is the following pair of applications:

Given $ta'thīr^{P} : (\forall x :\mathcal{P})\mathcal{H}(x) \wedge (\forall x :\sim\mathcal{P})\sim\mathcal{H}(x)$
And $a :\mathcal{P}$, $a^* :\sim\mathcal{P}$, we obtain

the left component of the evidence for the co-extensiveness of $\mathcal{P}$ provides the factor that occasions the application of the ruling $\mathcal{H}(x)$ to $a :\mathcal{P}$ – we express this with the abbreviated notation $illa^{\mathcal{H}(\mathcal{P})+}.a : \mathcal{H}(a)$;and

the right component of the evidence for the co-exclusiveness of $\sim\mathcal{P}$ provides the factor that occasions the application of the ruling $\sim\mathcal{H}(x)$ to $a^* :\sim\mathcal{P}$, we express this with the abbreviated notation $illa^{\mathcal{H}(\mathcal{P})-}.a^*: \sim\mathcal{H}(a)$).

33

Martin-Löf's lecture on the truth of empirical propositions

- In the standard, or pure, formulation a member of a set S is a program which when executed yields a canonical member of S.
- In the new definition a member of a set is an `empirical quantity', i.e. something that is determined by *experiment* rather than calculation to belong to S. Canonical objects are defined as before. Thus we have for instance the set bool and the set N.
- An empirical quantity X : bool is a new kind of non-canonical element. In general the value of an element a : A is some canonical element of A. For instance, the value of 2+2 is ssss(0), say. We are used to thinking of the values of elements of A as being determined by calculation; an empirical quantity is a non-canonical element whose value is determined by "experiment".
-

34

- *From Formal to Material Dialogues*
- The Formal rule (**Copy-cat**) is not formal:

  The **implicit standard view** on the formal rule

  P ! *A*

  O why?

  P *ipse dixisti* (you already asserted it: Copy-Cat!)

  The explicit view on the formal rule: **Local Reasons (the interactive corrspondent to a « cause » or « truth-maker »)**

  P *a* : *A*

  O why?

  Given O ! *c* : *A&B*

  P **a = *Left*$^{\&}$(*c*) : *A***

35

Material Dialogues

- Material Dialogues
- The copy-cat strategy is defined explicity for each elementary proposition . Analagous the constitution of a type

Example

P ! 1 *:* $\mathbb{N}$ **P** ! *n :* $\mathbb{N}$

O ? $\equiv_{df}$ 1 **O** ! **s**(0) : $\mathbb{N}$ **O** ? $\equiv_{df}$ *n* **O** ! **s**(….(**s**(**s**(0)) : $\mathbb{N}$

P ! 1 $\equiv_{df}$ **s**(0) : $\mathbb{N}$ **P !** *n* $\equiv_{df}$ **s**(….(**s**(**s**(0)) : $\mathbb{N}$

36

- Back to our cases: Define X “empirical quantity of containing some substance”, X : bool. Then if 1 then “Euphoric Intensity”, if 0 , then it does not be considered to be element of Euphoric Intensity.

- Material Dialogue: {yes_e, no_e}: Bool
- Euphoric Intensity(x_e) (x_e : Bool)
- Socratic Rule: **P** is allowed to assert date-wine has Euphoric Intensity if **O** asserted Euphoric Intensity(yes_e) (, given yes_e : Bool)

37

Posit	Challenge	Defence	Strategic object
X ! Bool Synthesis of Boolean	Y ? Bool	X ! : yes : Bool X ! : no : Bool	P ! : yes : Bool P ! : no : Bool (dial version of introduction rules)
X ! p : C(c) [c : Bool] Analysis of Boolean	Y ? c^{Bool} Y ? …/L^{Bool} Y ? c^{Bool} Y ? …/R^{Bool}	X ! p_1 : C(L^{Bool}) X ! p_1 : C(yes) X ! p_2 : C(R^{Bool}) X ! p_2 : C(no)	P ! (c, p_1 | p_2) : C(c / yes | c / no) (dial version of elimination rules) With equality P ! (yes / L^{Bool}, p_1 | p_2) = p_1 : C(yes) P ! (no / R^{Bool}, p_1 |

Special Local Rules for *Qiyās al-'Illa*

Expressions "p" in "$p : \mathcal{P}$" stand for either some branch-case *far'* or some root-case *aṣl*

Posit	Challenge	Defence
X $tard^{\mathfrak{I}(\mathcal{P})} : (\forall x : \mathcal{P})\mathfrak{C}(x)$ Notation for a posit without specified reason **X** ! $(\forall x : \mathcal{P})\mathfrak{C}(x)$	**Y** $p : \mathcal{P}$	**X** $'illa^{\mathfrak{I}(\mathcal{P})+}.p : \mathfrak{C}(p)$
X $'aks^{\mathfrak{I}(\mathcal{P})} : (\forall x : \sim\mathcal{P})\sim\mathfrak{C}(x)$ Notation for a posit without specified reason **X** ! $(\forall x : \sim\mathcal{P})\sim\mathfrak{C}(x)$	**Y** $p : \sim\mathcal{P}$	**X** $'illa^{\mathfrak{I}(\mathcal{P})-}.p : \sim\mathfrak{C}(p)$
X $ta'thīr^{\mathfrak{I}(\mathcal{P})} : (\forall x : \mathcal{P})\mathfrak{C}(x) \wedge (\forall x : \sim\mathcal{P})\sim\mathfrak{C}(x)$ Notation for a posit without specified reason **X** ! $(\forall x : \mathcal{P})\mathfrak{C}(x) \wedge (\forall x : \sim\mathcal{P})\sim\mathfrak{C}(x)$	**Y** $p : \mathcal{P}$ **or** **Y** $p : \sim\mathcal{P}$	**X** $'illa^{\mathfrak{I}(\mathcal{P})+}.p : \mathfrak{C}(p)$ respectively **X** $'illa^{\mathfrak{I}(\mathcal{P})-}.p : \sim\mathfrak{C}(p)$
X ϕ ... **X** $\sim\phi$ (one of them might not explicitly display local reason)	**Y** ! *tanāquḍ* ϕ **The antagonist** **Indicates the contradiction**	**X** ! I concede

39

In uttering the formula $\mathbf{F}\varphi$ the argumentation partner **X** claims that he can find a counterexample during a play where the antagonist **Y** asserts φ.

The antagonist **Y** challenges $\mathbf{F}\varphi$ by asserting that φ can be challenged successfully. Thus, the challenge of **Y** compels **Y** to open a *sub-play* where he (**Y**) utters φ.

	Challenge	Defence
X ! Fφ	**Y ? F**	
	Sub-play $\mathfrak{D}_1$	*Sub-play* $\mathfrak{D}_1$
	Y ! φ **Y** must play under the restriction of the *Socratic-Rule* in the sub-play	**X** $?_\varphi$ (he challenges φ)

40

In uttering the formula $\mathbb{V}\varphi$ the argumentation partner **X** claims that he can win a play where he (**X**) asserts φ.

The antagonist **Y** responds by challenging **X** to open a *sub-play* where he (**X**) defends φ.

	Challenge	Defence
X ! $\mathbb{V}\varphi$	Y ?$\mathbb{V}$	
	Sub-play $\mathfrak{D}_1$	*Sub-play* $\mathfrak{D}_1$
	Y ?$_\varphi$ (he challenges φ) **Y** must play under the restriction of the *Socratic Rule*	**X** ! φ

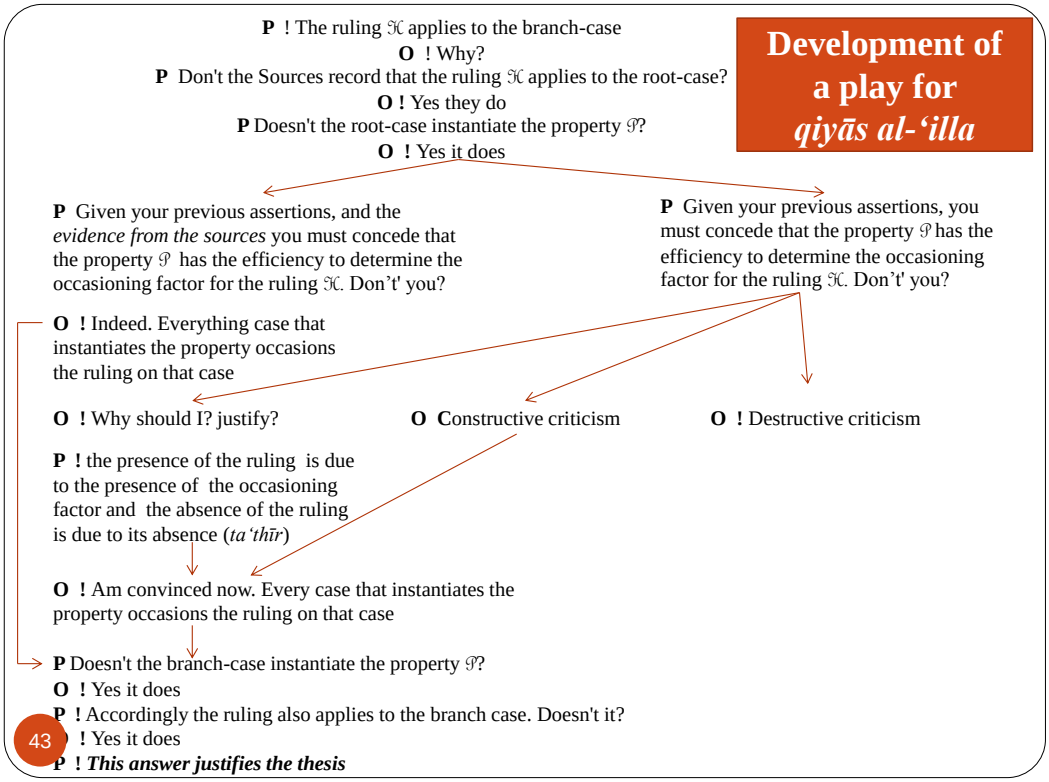
41

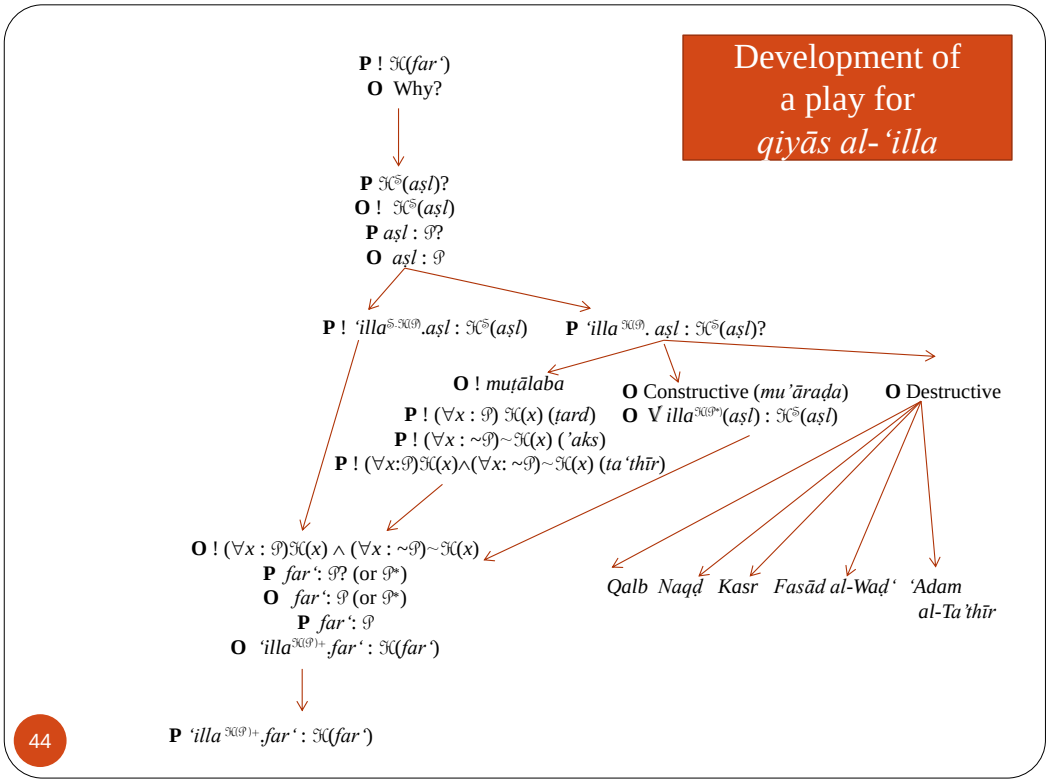
- These operators require “global reasons” as justification.

What justifies these operators is winning the subplay

Reason in the more usual way

42


Development of a play for qiyās al-'illa
P ! The ruling ℋ applies to the branch-case
O ! Why?
P Don't the Sources record that the ruling ℋ applies to the root-case?
O ! Yes they do
P Doesn't the root-case instantiate the property 𝒫?
O ! Yes it does
P Given your previous assertions, and the evidence from the sources you must concede that the property 𝒫 has the efficiency to determine the occasioning factor for the ruling ℋ. Don't' you?
P Given your previous assertions, you must concede that the property 𝒫 has the efficiency to determine the occasioning factor for the ruling ℋ. Don't' you?
O ! Indeed. Everything case that instantiates the property occasions the ruling on that case
O ! Why should I? justify?
O Constructive criticism
O ! Destructive criticism
P ! the presence of the ruling is due to the presence of the occasioning factor and the absence of the ruling is due to its absence (ta'thīr)
O ! Am convinced now. Every case that instantiates the property occasions the ruling on that case
P Doesn't the branch-case instantiate the property 𝒫?
O ! Yes it does
P ! Accordingly the ruling also applies to the branch case. Doesn't it?
O ! Yes it does
P ! This answer justifies the thesis
43


Development of a play for qiyās al-'illa
P ! ℋ(far')
O Why?
P ℋ𝒮(aṣl)?
O ! ℋ𝒮(aṣl)
P aṣl : 𝒫?
O aṣl : 𝒫
P ! 'illa𝒮-ℋ(𝒫).aṣl : ℋ𝒮(aṣl)
P 'illa ℋ(𝒫). aṣl : ℋ𝒮(aṣl)?
O ! muṭālaba
P ! (∀x : 𝒫) ℋ(x) (ṭard)
P ! (∀x : ~𝒫)~ℋ(x) ('aks)
P ! (∀x:𝒫)ℋ(x)∧(∀x: ~𝒫)~ℋ(x) (ta'thīr)
O Constructive (mu'āraḍa)
O V illaℋ(𝒫*)(aṣl) : ℋ𝒮(aṣl)
O Destructive
O ! (∀x : 𝒫)ℋ(x) ∧ (∀x : ~𝒫)~ℋ(x)
P far': 𝒫? (or 𝒫*)
O far': 𝒫 (or 𝒫*)
P far': 𝒫
O 'illaℋ(𝒫)+.far' : ℋ(far')
Qalb Naqḍ Kasr Fasād al-Waḍ' 'Adam al-Ta'thīr
P 'illa ℋ(𝒫)+.far' : ℋ(far')
44

The Dialague Example of *Mu'āraḍa*

O			P		
			(Consuming) Date-wine is forbidden. ! $\mathfrak{K}(far')$		0
1	Why?	? 0	¿ 1, ! 2	Isn't drinking grape-wine forbidden by the Quran? $\mathfrak{K}^{\mathfrak{S}}(aṣl)$?	2
3	Yes, it is *ḥarām*. ! $\mathfrak{K}^{\mathfrak{S}}(aṣl)$	! 2	¿ 3, ! 4	Isn't grape-wine made of pressed fruit-juice $aṣl : \mathfrak{P}$?	4
5	Yes $aṣl : \mathfrak{P}$	! 4	¿ 3, (5), ! 6	So, according to your moves 3 and 5, the proscription of consuming grape-wine is caused by the fact that it is made of pressed fruit-juice. Right? $'illa^{\mathfrak{K}(\mathfrak{P})+}.aṣl : \mathfrak{K}^{\mathfrak{S}}(aṣl)$?	6
7	I am not convinced. I rather think that the relevant property is containing euphoric intensity ($\mathfrak{P}^*$) ! V $illa^{\mathfrak{K}(\mathfrak{P}^*)+}.aṣl : \mathfrak{K}^{\mathfrak{S}}(aṣl)$.	? 6	? 7	*muṭālaba* ! Justify !	8
	START OF THE SUB-PLAY			**START OF THE SUB-PLAY**	
9	Vinegar is made of pressed juice-fruit. Isn't it? $aṣl^* : \mathfrak{P}$?	¿ 8, ! 9		Indeed. $aṣl^* : \mathfrak{P}$	10
11	Given 6 you must agree that being a pressed-juice is efficient property for sanctioning them as *ḥarām*. Right? $(\forall x : \mathfrak{P})\mathfrak{K}(x) \wedge (\forall x : \sim\mathfrak{P})\sim\mathfrak{K}(x)$?	¿ 6, ! 10		Yes ! $(\forall x : \mathfrak{P})\mathfrak{K}(x) \wedge (\forall x : \sim\mathfrak{P})\sim\mathfrak{K}(x)$	12

45

------Continue on the next page------

O			P		
13	But, given that you just agreed that vinegar is made of pressed-juice, (according to the *ṭard*-component of your assertion) it should be *ḥarām* ! $aṣl^* : \mathfrak{P}$	12		Indeed $'illa^{\mathfrak{P}+}.aṣl^* : \mathfrak{K}(aṣl^*)$	14
15	But its consumption is not forbidden. Isn't it? ! $\sim\mathfrak{K}^{\mathfrak{S}}(aṣl^*)$?	¿ 14, ! 15		Yes it is not *ḥarām* ! $\sim\mathfrak{K}^{\mathfrak{S}}(aṣl^*)$	16
17	*tanāquḍ* ! You contradict yourself ! $\mathfrak{K}(aṣl^*)$	? 16		I concede !	18
19	Herewith my argument for the relevance of $\mathfrak{P}^*$ '*aks*: Before the occurrence of the euphoric intensity, the lawfulness of consuming a drink made of fruit-juice is the object of consensus. ! $(\forall x : \sim\mathfrak{P}^*)\sim\mathfrak{K}(x)$ '*ṭard*: After the euphoric intensity occurs [i.e., when it becomes wine] and nothing else occurs the proscription of consuming a drink made of fruit-juice is object of consensus. (ratification of) '*aks*: When the euphoric intensity of a drink made of fruit-juice falls away [i.e., when it becomes vinegar] and nothing else falls away it is object of consensus that it should not be forbidden. ! $(\forall x : \mathfrak{P}^*)\mathfrak{K}(x)$ *ta'thīr*: Therefore, the presence of the *ḥukm* is due to the presence of the *'illa*, and the absence of the *ḥukm* is due to its absence $'illa^{\mathfrak{P}^*} : (\forall x : \mathfrak{P}^*)\mathfrak{K}(x) \wedge (\forall x : \sim\mathfrak{P}^*)\sim\mathfrak{K}(x)$	! 8			
	END OF THE SUB-PLAY			**END OF THE SUB-PLAY**	

46

------Continue on the next page------

O			P		
21	Yes, it does. *far'*: $\mathcal{P}^*$	! 20	¿ 19, ! 20	Doesn't *nabīdh* contains *euphoric intensity*? *far'*: $\mathcal{P}^*$?	20
23	Indeed ! *'illa*$^{\mathcal{K}(\mathcal{P}^*)+}$. *far'* : $\mathcal{K}$(*far'*)	! 23	? 19	If it is the case that date-wine contains euphoric intensity, and, given 19, should not this lead to its interdiction? *far'*: $\mathcal{P}^*$	22
			! 1	So, this provides the justification for the thesis you were asking for with your first move: the branch-case falls under the ruling because it instantiates the property you just helped to identify as the one determining the occasioning factor. *'illa*$^{\mathcal{K}(\mathcal{P}^*)+}$. *far'* : $\mathcal{K}$(*far'*)	24
	iṭṭirād				

47

Mu'āraḍa

P ! $\mathcal{K}$(*far'*)
O ! Why [?0]
P ! $\mathcal{K}^{\mathcal{S}}$(*aṣl*)?
O ! $\mathcal{K}^{\mathcal{S}}$(*aṣl*)
P *aṣl* : $\mathcal{P}^*$?
O *aṣl* : $\mathcal{P}^*$
P *'illa*$^{\mathcal{K}(\mathcal{P}^*)+}$.*aṣl* : $\mathcal{K}^{\mathcal{S}}$(*aṣl*)?
O ! $(\forall x : \mathcal{P}^*)\mathcal{K}(x) \wedge (\forall x : \sim\mathcal{P}^*)\sim\mathcal{K}(x)$
P *far'* : $\mathcal{P}^*$?
O *far'* : $\mathcal{P}^*$
P *far'* : $\mathcal{P}^*$ **[?7]**
O *'illa*$^{\mathcal{K}(\mathcal{P}^*)+}$.*far'* : $\mathcal{K}$(*far'*)
P *'illa*$^{\mathcal{K}(\mathcal{P}^*)+}$.*far'* : $\mathcal{K}$(*far'*)
(!1. answer to the request of justification
in the second move)

48

Qalb

O ! $\mathbb{F}$ $(\forall x : \mathcal{P})$ $\mathcal{H}(x)$ (***qalb, reversal***)

The Opponent is committed to a sub-play where he brings forward a root-case of which it is recorded that an opposite ruling to the claimed ruling applies. Hence the root-case is presented as a counterexample to the Proponent's claim that every $\mathcal{P}$ falls under the ruling $\mathcal{H}$ and in particular to the claim that this ruling applies to the branch-case.

THESIS: Saliva of beasts of prey (*far'*) is impure ($\mathcal{H}$).
CLAIM: *Having canine teeth* determines the *'illa*.

COUNTEREXAMPLE: The saliva of cats which are beasts of prey with canine teeth is not impure.

49

Naqḍ

O ! $\mathbb{F}$ $(\forall x : \mathcal{P})$ $\mathcal{H}(x)$, given $aṣl^* : \mathcal{P}$, $\mathcal{H}^S{}^*(aṣl^*)$, and ~ ($\mathcal{H}(aṣl^*) \wedge \mathcal{H}^*(aṣl^*)$) (***naqḍ, inconsistency***)

The Opponent is committed to a sub-play where he brings forward a root-case of which it is recorded that a different ruling to the claimed ruling applies and both rulings are incompatible. Hence the root-case is presented as a counterexample to the Proponent's assertion that every $\mathcal{P}$ falls under the ruling $\mathcal{H}$ and in particular to the claim that this ruling applies to the branch-case.

THESIS: Killing (*far'*) should be punished with jail ($\mathcal{H}$). **CLAIM**: *Having commited homicide* determines the *'illa*.

COUNTEREXAMPLE: Some forms of homicide do neither lead to jail nor to be set free but to the obligation of carrying out certain specific social services.

50

Kasr

O ! 𝔽 $(\forall x : \{ x : \mathcal{P} \mid B(x)\})\ \mathcal{H}(x)$ (***kasr, breaking appart***)

The Opponent is committed to a sub-play where he brings forward a root-case which instantiates a subset of $\mathcal{P}$ and of which it is recorded that the claimed ruling does not apply. Hence the root-case is presented as a counterexample to the Proponent's assertion that every $\mathcal{P}$ falls under the ruling $\mathcal{H}$ and in particular to the claim that this ruling applies to the branch-case.

Thesis: Interdiction ($\mathcal{H}$) of transaction of goods that the buyer did not have direct access to the goods before the contract was closed (*farʿ*). **Claim**: *Establishing a contract with someone in such a way that the beneficiary has no access to the object of the contract*, determines the *ʿilla*.

Counterexample: Contract-on houses closed by internet are not (always) forbidden.

51

Fasād al-Waḍʿ

O ! 𝔽 $(\forall x : \mathcal{P})\ \mathcal{H}(x)$ (***fasād al-waḍʿ, invalidation of the ʿilla***)

The Opponent is committed to a sub-play where he brings forward a root-case of which it is recorded in the sources that a property assumed in the thesis to apply to the branch-case occasions in fact, the opposite ruling to the one posited by the Proponent. In other words, the Opponent brings forward an *ʿilla* that destroys the thesis.

Thesis: Modern forms of divorce (*farʿ*) do not apply to foreigners ($\mathcal{H}$). **Claim**: *Being foreigner* determines the *ʿilla*.

Counterexample: Modern forms of divorce apply to foreigners or nationals..

52

'Adam al-Ta'thīr

O ! $\mathbb{F}$! $(\forall x : \mathcal{P})\mathcal{H}(x) \wedge (\forall x : \sim\mathcal{P})\sim\mathcal{H}(x)$ (***'adam al-ta'thīr, lack of efficiency***)

The Opponent is committed to a sub-play where he brings forward a root-case which constitutes a counterexample to the efficiency of the proposed property asserted by the Proponent.

Thesis: Interdiction of the consumption (of wine (*far'*).
Claim: *Presence of euphoric intensity and having red-colour*, determines the *'illa*

Counterexample: White wine is forbidden, despite the fact that it is not red.

53

Conclusion

- Parallel Argumentation, which shapes meaning by symmetry, highlights one particular form of that plural act of collective understanding we call reasoning.
- All in all **argumentation is nothing-more and nothing-less than a collaborative enquiry into the ways of building up those symmetries that ground rationality and harmony within inquisitive interaction.**
- By building these symmetries we provide meaning to our actions, meaning which is deployed in our actions' internal coordination with the actions of others**.**

On the normativity of logic

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May 23, 2017

- Is logic normative for reasoning?
- Three-way ambiguity.

Background

- Harman: there is no 'significant way in which *logic* is specially [normatively] relevant to reasoning' (*Change in view*, MIT Press,1986, p. 20)

The implication principle

- Logical implication principle (IMP):
  if S's beliefs logically imply A, then S ought to believe A.

Harman's objections

- Objection from Belief Revision
- Objection from Clutter Avoidance
- Objection from Excessive Demands
- Objection from Preface Paradox

- Task: Formulate a *bridge principle*
- If $A_1, \ldots, A_n \models C$, then $N(\alpha(A_1), \ldots, \alpha(A_n), \beta(C))$.
- If $\gamma(A_1, \ldots, A_n \models C)$, then $N(\alpha(A_1), \ldots, \alpha(A_n), \beta(C))$.

MacFarlane's taxonomy

- Bridge principles may differ in the deontic operator they deploy: Does the normative constraint take the form of an *ought* (o), a permission (p) or merely of having (defeasible) reasons (r)?
- What is the *polarity* of the normative claim? Is it a *positive* obligation/permission/reason to *believe* a certain proposition given one's belief in a number of premises (+)? Or rather is it a *negative* obligation/permission/reason *not to disbelieve* (-)?

MacFarlane's taxonomy

- Different bridge principles result from giving the deontic operator different scope. Let O stand generically for one of the above deontic operators. Given that the consequent of a bridge principle will typically itself take the form of a conditional, the operator can take
  - narrow scope with respect to the consequent (C) ($P \supset O(Q)$);
  - wide scope (W) $O(P \supset Q)$;
  - or it can govern both the antecedent and the consequent of the conditional (B) ($O(P) \supset O(Q)$).

Examples

- (Co+) If $A_1, \ldots, A_n \models C$, then S ought to believe C, if S believes the A_i.
- (Wo-) If $A_1, \ldots, A_n \models C$, then S ought to (not disbelieve C, if S believes the A_i).
- (Co+K) If S knows that ($A_1, \ldots, A_n \models C$), then S ought to believe C, if S believes the A_i.

Evaluating bridge principles

- We now evaluate bridge principles in terms of how well they perform against our criteria.
- Our criteria: Immunity to the objections + Strictness and Priority...

Strictness Test

- **The Strictness Test:** At least when it comes to ordinary, readily recognizable logical implications leading to conclusions that the agent has reason to consider, there is something amiss about an agent who endorses the premises but fails to believe the conclusion.
- Seems to tell against defeasible principles.

Priority Question

- **The Priority Question**: 'we seek logical knowledge so that we will know how we ought to revise our beliefs: not just how we will be obligated to revise them when we acquire this logical knowledge, but how we are obligated to revise them even now, in our state of ignorance' (MacFarlane, 'In what sense (if any) is logic normative for thought?', Manuscript, 2004, p. 12).
- Seems to tell against attitudinally restricted principles.
- Set of desiderata is inconsistent.

Positions

- Harman denies that any BP is up to the job.
- MacFarlane argues for Wo- and Wr+.
- Streumer ('Reasons and entailment,' *Erkenntnis*, 66, 2007) goes in for Wr-.
- Field ('What is the normative role of logic?,' *Proceedings of the Aristotelian Society*, 83, 2009) proposes a quantitative principle:
- (DB) If $A_1, \ldots, A_n \models C$, then S's degrees of belief ought to be such that: $cr(C) \geq cr(A_1) + cr(A_2) + \cdots + cr(A_n) - (n-1)$.
- If $u(A) = 1 - cr(A)$
- (DB') If $A_1, \ldots, A_n \models C$, then S's degrees of belief ought to be such that: $u(C) \leq u(A_1) + u(A_2) + \cdots + u(A_n)$.
- If the uncertainty of each premise is at most ε, then if the argument has at least $1/\varepsilon$ premises, the conclusion will be maximally uncertain.

- Distinguishing normative roles...

Norms

(N) If C, then S ought to/may/has reason to Φ.

(AU) If Φing is the action (among the actions available to S) that maximizes net happiness, then S ought to Φ.

Misgivings

- Unhelpful
- Provides *no* guidance.
- Unfair
- Does *not* support attributions of praise or blame.

Evaluations

- Provides a standard against which to evaluate a state or act.
- Evaluates an agent's doxastic state (or part of it) or act from a third-person perspective.
- The *state* or *act* (not the agent) is being evaluated.
- Does *not* support attributions of praise or blame.
- 'Ought'; not 'can'-implying.

Directives

- Provides guidance in first-personal doxastic deliberation.
- Guiding principles must be 'followable'.
- Deontic modals are 'deliberative'.
- 'Can'-implying

Appraisals

- Third-personal assessments of the agent.
- *Do* deal in attributions of praise and blame, etc.
- Hold agents accountable; hence, presupposes sensitivity to the agent's epistemic situation.
- Varying degrees of idealization: the sliding scale.
- Deliberative deontic modals. Subjective/objective deontic modals.

- (Directive) If in S's best estimation $A_1, \ldots, A_n \models C$, then $N_1(\alpha(A_1), \ldots, \alpha(A_n), \beta(C))$.
- (Evaluation) If $A_1, \ldots, A_n \models C$, then $N_2(\alpha(A_1), \ldots, \alpha(A_n), \beta(C))$.
- (Appraisal) If S can reasonably be expected to believe that $A_1, \ldots, A_n \models C$, then $N_3(\alpha(A_1), \ldots, \alpha(A_n), \beta(C))$

Upshot

- Is logic normative for reasoning? $\Rightarrow$ Does logic provide guiding/evaluative/appraising norms for reasoning?
  - Does logic provide evaluative norms reasoning?
  - Does logic provide directive norms for reasoning?
  - Does logic provide appraising norms for reasoning?

Talking past one another

- Harman: Directives
- Field: Evaluations
- MacFarlane: Appraisals?

- Field: is explicit that his principle plays an *evaluative* role.
- MacFarlane: Logical implications have normative force even when the agent is incapable of being aware of them (Priority).
- Weight he gives to the Excessive Demands worry suggests that appraisals play a role.
- Attitudinal principles are all factive.

Upshot

	Directives	Evaluations	Appraisals
Objection from Belief Revision	✓	✓	✓
Clutter Avoidance	✓		✓*
Excessive Demands	✓		✓*
Preface Paradox	✓	✓	✓
Strictness Test	✓!	✓	✓!
Priority Question		✓	✓*

Popper's notion of duality and his theory of negations

Thomas Piecha
(University of Tübingen)

Joint work with David Binder

Beyond Logic, Cerisy-la-Salle, 22–27 May 2017

Karl Popper's articles on logic, 1940s

- 1943. 'Are contradictions embracing?', *Mind*, **52**, 47–50.
- 1947a. 'Logic without assumptions', *Proc. of the Arist. Soc.*, **47**, 251–292.
- 1947b. 'New foundations for logic', *Mind*, **56**, 193–235.
- 1947c. 'Functional logic without axioms or primitive rules of inference', *Koninklijke Nederlandse Akademie van Wetenschappen*, **50**, 1214–1224.
- 1948a. 'On the theory of deduction, part I. Derivation and its generalizations', *K. Nederlandse Akad. van Wetenschappen*, **51**, 173–183.
- 1948b. 'On the theory of deduction, part II. The definitions of classical and intuitionist negation', *K. Nederlandse Akad. v. W.*, **51**, 322–331.
- 1949. 'The trivialization of mathematical logic', in: E. W. Beth, H. J. Pos and J. H. A. Hollak, eds., *Proceedings of the Tenth International Congress of Philosophy*, vol. 1, Amsterdam: North-Holland, pp. 722–727.

Literature

Contemporary (1948, 1949) reviews by:

- Ackermann
- Beth
- Curry
- Hasenjaeger
- Kleene
- McKinsey

Detailed investigations:

- Lejewski (1974)
- Schroeder-Heister (1984, 2006)

Unpublished theses:

- K. J. Cohen (1953)
- B. Brooke-Wavell (1958)
- J. M. Dunn (1963)

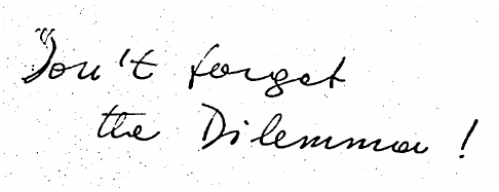
Critical edition of Popper's works on logic

Currently prepared by David Binder, Peter Schroeder-Heister and T. P.

To be ready for publication this year (Springer Trends in Logic)

Comprises:

- Popper's published works on logic
- unpublished material: talks, lecture notes etc.



- correspondence:
  Bernays, Brouwer, Carnap, Church, Kleene, Kneale, …

Interesting from historical and systematic points of view

For example:

- *tonk*-like connective
- dual-intuitionistic logic
- non-conservative language extensions
- Peirce's rule
- different kinds of negation
- bi-intuitionistic logic
- analysis of logicality; minimal and subminimal negation not logical
- inferentialist approach to logic

> *"Ultimately, we shall propose a definition [of validity of an inference] which makes use only of the ideas of absolute validity and of an inferential definition, and which no longer refers to truth."*
>
> (Popper; talk 5. 5. 1947 Aristotelian Society, London)

In this talk

Popper's

- framework
- general theory of derivation
- special theory of derivation
- notion of duality
- theory of negations

See
David Binder & T. P., 'Popper's Notion of Duality and His Theory of Negations', *History and Philosophy of Logic* **38(2)**, 2017, 154–189

- exposition of Popper's theory
- elaboration of some results that were only sketched by Popper

Popper's framework

Pairs of object language $\mathcal{L}$ and deducibility relation on $\mathcal{L}$

Deducibility relation characterized by axioms: basis

No specific object language

Elements $a, b, c, \ldots \in \mathcal{L}$ are statements

Deducibility: $a_1, \ldots, a_n/b$ "b can be deduced from $\{a_1, \ldots, a_n\}$"

Symbolic metalanguage:

Symbol	$\rightarrow$	$\leftrightarrow$	$\&$	$\bigvee$	(a)	$(\exists a)$
Meaning	if-then	if and only if	and	or	for all a	there is an a

Absolute and conditional rules of derivation

Atomic metalinguistic formulas

$$a_1, \ldots, a_n/b$$

are also called absolute rules of derivation

Metalinguistic formulas

$$a_1, \ldots, a_n/b \rightarrow c_1, \ldots, c_m/d$$

(and iterations thereof) are also called conditional rules of derivation or just rules of derivation

Not rules of a proof system

But metalinguistic statements about deducibility

Characterization of deducibility "/" by a basis

A basis is a complete axiomatization of deducibility:

- Generalized reflexivity principle (Rg)
- Generalized transitivity principle (Tg)

$$a_1, \ldots, a_n / a_i \qquad (1 \leq i \leq n) \qquad \text{(Rg)}$$

$$\left\{ \begin{array}{ll} & a_1, \ldots, a_n / b_1 \\ \& & a_1, \ldots, a_n / b_2 \\ \vdots & \vdots \\ \& & a_1, \ldots, a_n / b_m \end{array} \right\} \rightarrow (b_1, \ldots, b_m / c \rightarrow a_1, \ldots, a_n / c) \qquad \text{(Tg)}$$

(Tg) is schematic: instances depending on m

Basis is formulated in the metalanguage

Complete w.r.t. absolute validity (valid independent of logical form; structural inferences only)

Popper also considered alternative bases

The general theory of derivation

No reference to logical signs of object language

Studies

- properties of statements and
- relations between statements

definable by deducibility alone

Examples:

- mutual deducibility
- demonstrability
- refutability
- relative demonstrability

Mutual deducibility "$/\!/$"

Definition:

$$a/\!/b \leftrightarrow (a/b \,\&\, b/a) \qquad (\mathrm{D}/\!/)$$

Two mutually deducible statements a and b have the same logical force

Equivalence classes induced by $/\!/$ are logical forces

Mutually deducible statements are also called logically equivalent

(D$/\!/$): substitutivity principle for logical equivalence

Alternative definition:

$$a/\!/b \leftrightarrow (c)(a/c \leftrightarrow b/c) \qquad (\mathrm{D}/\!/')$$

Lemma

In the presence of (Tg) *and* (Rg), (D$/\!/$) *is equivalent to* (D$/\!/'$).

Complementarity and demonstrability

Complementarity:

$$\vdash a_1, \ldots, a_n \leftrightarrow (b)(c)((a_1/c \,\&\, \ldots \,\&\, a_n/c) \rightarrow b/c) \qquad (\mathrm{D}\vdash 1)$$

Intuitively: At least one of the statements $a_1, \ldots, a_n$ has to be true, i.e. taken together they exhaust all possible states of affairs

For $n = 1$ we get demonstrability:

$$\vdash a \leftrightarrow (b)(c)(a/c \rightarrow b/c) \qquad (\mathrm{D}\vdash 1')$$

From (D$\vdash 1'$):

$$\vdash a \leftrightarrow (b)(b/a)$$

by instantiating c by a and by using basic rules

A demonstrable statement follows from any statement

Contradictoriness and refutability

Contradictoriness:

$$\not\prec a_1, \ldots, a_n \leftrightarrow (b)(c)((b/a_1 \;\&\; \ldots \;\&\; b/a_n) \rightarrow b/c) \qquad (\mathrm{D}\not\prec)$$

Intuitively: A sequence of statements $a_1, \ldots, a_n$ is contradictory if its members cannot be true together

For $n = 1$ we get refutability:

$$\not\prec a \leftrightarrow (b)(c)(b/a \rightarrow b/c) \qquad (\mathrm{D}\not\prec')$$

Intuitively: A statement is refutable if it is false no matter the state of affairs

From (D$\not\prec'$):

$$\not\prec a \leftrightarrow (c)(a/c)$$

by substituting a for b and by using basic rules

Definition of a self-contradictory statement, i.e. of a refutable statement

From such a statement any other statement follows

Relative demonstrability

Definition:

$$a_1, \ldots, a_n \vdash b_1, \ldots, b_m \leftrightarrow (c)((b_1/c \;\&\; \ldots \;\&\; b_m/c) \rightarrow a_1, \ldots, a_n/c) \qquad (\mathrm{D}\vdash 2)$$

For technical reasons, we use:

Definition

Relative demonstrability $a_1, \ldots, a_n \vdash b_1, \ldots, b_m$ is defined by

$$a_1, \ldots, a_n \vdash b_1, \ldots, b_m \leftrightarrow (c)(d_1)\ldots(d_k)((b_1, d_1, \ldots, d_k/c \;\&\; \ldots \;\&\; b_m, d_1, \ldots, d_k/c) \rightarrow a_1, \ldots, a_n, d_1, \ldots, d_k/c) \qquad (\mathrm{D}\vdash 3)$$

Additional context statements $d_1, \ldots, d_k$ for $0 \leq l \leq k$

Definition (D$\vdash$3) is more general than (D$\vdash$2)

Relative demonstrability, properties

Lemma

The concept of relative demonstrability contains, as special cases, the concepts of complementarity, demonstrability, contradictoriness and refutability.

Lemma

For all $a_1, \ldots, a_n, b$: $\quad a_1, \ldots, a_n / b \;\leftrightarrow\; a_1, \ldots, a_n \vdash b$.

Lemma

The following structural rules hold for $a_1, \ldots, a_n \vdash b_1, \ldots, b_m$:

(i) *Weakening left/right, exchange left/right, contraction left/right*

(ii) *If there are* i, j *(for* $1 \leq i \leq n$ *and* $1 \leq j \leq m$*) such that* $a_i = b_j$*, then* $a_1, \ldots, a_n \vdash b_1, \ldots, b_m$.

(iii) $a_1, \ldots, a_n \vdash b_1, \ldots, b_m, c \rightarrow$
$(c, a_1, \ldots, a_n \vdash b_1, \ldots, b_m \rightarrow a_1, \ldots, a_n \vdash b_1, \ldots, b_m)$ (Cut)

Relative demonstrability and Gentzen's sequents

For object languages containing conjunction $\wedge$ and disjunction $\vee$ one can show:

$$a_1, \ldots, a_n \vdash b_1, \ldots, b_m \;\leftrightarrow\; a_1 \wedge \ldots \wedge a_n \vdash b_1 \vee \ldots \vee b_m$$

Concept of relative demonstrability gives an interpretation of Gentzen's sequents

However: object languages need not have conjunction or disjunction

The special theory of derivation

Special theory: definitions of logical constants

Relations between logically complex statements and their components

Object languages $\mathcal{L}$ not specified syntactically:
Logical connectives cannot be introduced by saying that for any two statements a and b in $\mathcal{L}$ there exists a statement having some specific syntactic form, say $a \wedge b$

Instead:
Logical constants have to be characterized in terms of deducibility "$/$"

Popper calls such definitions inferential definitions

A sign of an object language $\mathcal{L}$ is a logical constant iff it can be defined by an inferential definition

Inferential definitions of logical constants

Definitions of logical constants $\circ$ have the form:

$$a // a_1 \circ a_2 \;\leftrightarrow\; \mathcal{R}(a, a_1, a_2) \qquad (\mathrm{D}\circ)$$

$\mathcal{R}(a, a_1, a_2)$: metalinguistic formula containing

- the statements a, a_1, a_2 (among others)
- the deducibility relation $/$
- or the defined relations $\vdash$ and $\not\vdash$

Inferential, since $\circ$ defined in terms of deducibility

To simplify, we consider

$$\mathcal{R}(a_1 \circ a_2, a_1, a_2) \qquad (\mathrm{C}\circ)$$

This is the characterizing rule (C$\circ$), which corresponds to the definition (D$\circ$)

Problem: E.g. $\wedge \notin \mathcal{L}$.
We read $a \wedge b$ as $\varepsilon c\, \mathcal{R}_\wedge(c, a, b)$: $a \wedge b$ just picks one element, if it exists, of the equivalence class defined by the inferential definition of $\wedge$

Popper's definitional criterion of logicality

Does any rule $\mathcal{R}$ characterize a logical constant?

On the one hand: no restrictions on $\mathcal{R}$

Popper (1947) considers the following definition for opponent (*opp*):

$$a /\!/ opp(b) \leftrightarrow (c)(b/a \;\&\; a/c) \qquad \text{(D}opp\text{)}$$
$$(c)(b/opp(b) \;\&\; opp(b)/c) \qquad \text{(C}opp\text{)}$$

This obviously trivializes any system, since it implies $(c)(b/c)$

But this does not lead Popper to reject (D*opp*) as a definition

Historical: *tonk*-like connective, considered before Prior (1960)

Popper's definitional criterion of logicality

On the other hand: uniqueness condition

Popper considers fully characterizing rules:

Definition

A rule $\mathcal{R}(c, a_1, \ldots, a_n)$ is fully characterizing iff

$$(\mathcal{R}(a, a_1, \ldots, a_n) \;\&\; \mathcal{R}(b, a_1, \ldots, a_n)) \rightarrow a /\!/ b.$$

In other words:
$\mathcal{R}$ is fully characterizing a statement *c* iff $\mathcal{R}$ characterizes *c* up to mutual deducibility (i.e. if *c* is unique).

Fully characterizing rules are exactly those rules that satisfy uniqueness

Criterion: existence of fully characterizing rules distinguishes logical constants from non-logical constants

In 1948: definitions of logical constants that emphasize duality

Popper's notion of duality

Popper makes frequent use of duality, without making his notion of duality explicit

Duality applied also for non-classical logics, including modal logic

Our explanation:
An inferential definition is dual to another inferential definition, if it results from exchanging all statements on the left side of $\vdash$ with the statements on its right side, i.e. by transforming

$$a_1, \ldots, a_n \vdash b_1, \ldots, b_m$$

into

$$b_1, \ldots, b_m \vdash a_1, \ldots, a_n$$

For binary connectives: also swap the arguments to produce its dual

Popper's notion of duality

Definition

Let $\star$ be a unary connective and $\circ$ a binary connective.
The duality function δ is defined by the following clauses:

$$\begin{aligned} a^\delta &=_{\text{df}} a \\ (\star a)^\delta &=_{\text{df}} \star^\delta a^\delta \\ (a \circ b)^\delta &=_{\text{df}} b^\delta \circ^\delta a^\delta \\ (\Gamma \vdash \Pi)^\delta &=_{\text{df}} \Pi^\delta \vdash \Gamma^\delta. \end{aligned}$$

(Γ, Π lists of statements; Γ^δ means that δ is applied to each member of Γ)

No clauses for $/$ and $//$

But: no restriction of δ, since $/$ can always be replaced by $\vdash$

Example: conjunction and disjunction

Definition for conjunction ($\wedge$):

$$a//b \wedge c \leftrightarrow (d)(a \vdash d \leftrightarrow b, c \vdash d) \quad (\mathrm{D}\wedge)$$
$$b \wedge c \vdash d \leftrightarrow b, c \vdash d \quad (\mathrm{C}\wedge)$$

Apply δ to the characterizing rule (C$\wedge$):

$$d \vdash c \wedge^{\delta} b \leftrightarrow d \vdash b, c \quad (\mathrm{C}\wedge)^{\delta}$$

which is equivalent to Popper's definition of disjunction ($\vee$):

$$a//b \vee c \leftrightarrow (d)(d \vdash a \leftrightarrow d \vdash b, c) \quad (\mathrm{D}\vee)$$
$$d \vdash b \vee c \leftrightarrow d \vdash b, c \quad (\mathrm{C}\vee)$$

Lemma

The following rules for conjunction and disjunction hold:

(1) $a \wedge b/a$
(2) $a \wedge b/b$
(3) $a, b/a \wedge b$
(4) $a/a \vee b$
(5) $b/a \vee b$
(6) $(c)((a/c \,\&\, b/c) \rightarrow a \vee b/c)$

Example: logicality of conjunction and disjunction

Lemma

Conjunction and disjunction are logical constants, i.e. their rules are fully characterizing.

Proof.

For conjunction we have to show

$$((d)(a_1 \vdash d \leftrightarrow b, c \vdash d) \,\&\, (d)(a_2 \vdash d \leftrightarrow b, c \vdash d)) \rightarrow a_1//a_2$$

Assuming the antecedent, we substitute a_2 for d in both conjuncts:

$$a_1 \vdash a_2 \leftrightarrow b, c \vdash a_2$$
$$a_2 \vdash a_2 \leftrightarrow b, c \vdash a_2$$

By (Rg) we obtain $b, c \vdash a_2$ from the second, and with $b, c \vdash a_2$ we obtain $a_1 \vdash a_2$ from the first.
Likewise for $a_2 \vdash a_1$. The proof for disjunction is similar. □

Popper's theory of negations

Popper considered several different kinds of negation

Several definitions for classical negation ($\neg_k$), e.g.:

$$a /\!/ \neg_k b \leftrightarrow (a, b \vdash \ \& \ \vdash a, b) \qquad (\mathrm{D}\neg_k 1)$$
$$a /\!/ \neg_k b \leftrightarrow (c)(d)(d, a \vdash c \leftrightarrow d \vdash b, c) \qquad (\mathrm{D}\neg_k 2)$$

with the characterizing rules:

$$\neg_k b, b \vdash \ \& \ \vdash \neg_k b, b \qquad (\mathrm{C}\neg_k 1)$$
$$(c)(d)(d, \neg_k b \vdash c \leftrightarrow d \vdash b, c) \qquad (\mathrm{C}\neg_k 2)$$

(D$\neg_k$1): classical negation of b is a statement which is complementary and contradictory to b
(D$\neg_k$2) is similar to the rules in classical sequent calculus

Lemma

The definitions (D$\neg_k$1) *and* (D$\neg_k$2) *are equivalent.*

Intuitionistic and dual-intuitionistic logic

Intuitionistic negation ($\neg_i$):

$$a /\!/ \neg_i b \leftrightarrow (c)(c \vdash a \leftrightarrow c, b \vdash) \qquad (\mathrm{D}\neg_i)$$
$$c \vdash \neg_i b \leftrightarrow c, b \vdash \qquad (\mathrm{C}\neg_i)$$

If we dualize (D$\neg_i$), we get

$$a /\!/ \neg_i^{\delta} b \leftrightarrow (c)(a \vdash c \leftrightarrow \ \vdash c, b) \qquad (\mathrm{D}\neg_i)^{\delta}$$

Identical to Popper's definition for dual-intuitionistic negation ($\neg_m$):

$$a /\!/ \neg_m b \leftrightarrow (c)(a \vdash c \leftrightarrow \ \vdash c, b) \qquad (\mathrm{D}\neg_m)$$
$$\neg_m b \vdash c \leftrightarrow \ \vdash c, b \qquad (\mathrm{C}\neg_m)$$

Lemma

Intuitionistic negation $\neg_i$ and dual-intuitionistic negation $\neg_m$ are logical constants, i.e. their rules are fully characterizing.

Non-conservative language extensions

Theorem

In the presence of $\neg_k$ we have

$$\neg_k a /\!\!/ \neg_i a, \qquad \neg_k a /\!\!/ \neg_m a \qquad \neg_i a /\!\!/ \neg_m a$$

In other words, the three negations $\neg_k$, $\neg_i$ and $\neg_m$ collapse (i.e. they become synonymous).

Popper also considers the more general situation:

- two logical functions S_1 and S_2
- introduced by sets of primitive rules R_1 and R_2, respectively
- such that $R_2 \subset R_1$

If both S_1 and S_2 are definable, and S_1 is given, then one can show that S_1 and S_2 are equivalent

Popper's treatment of conservativeness

Throws some light on his logical approach in general

Schroeder-Heister (2006) argues:

- Popper does not use conservativeness as a criterion for accepting characterizing rules
- From a semantic theory we expect that the introduction of a new constant is always a conservative extension
- Thus Popper is not aiming at a semantic justification of logical theories
- Popper's theory is rather a means to metalinguistically describe logical theories

Bi-intuitionistic logic

Theorem

If a logic contains for any statement a also its intuitionistic negation $\neg_i a$ and its dual-intuitionistic negation $\neg_m a$, then these two negations do not (necessarily) collapse, i.e. we do not have $\neg_i a /\!/ \neg_m a$.

Proof.

By construction of logic $\mathcal{L}_1$ with this property.
(Proof sketched by Popper (1948), worked out by D. B. and T. P.) □

Popper (1948): there exists a bi-intuitionistic logic

This logic $\mathcal{L}_1$ may not be very interesting in itself

But: first example of a bi-intuitionistic logic to be found in the literature

Shows that already Popper had the idea of combining different logics

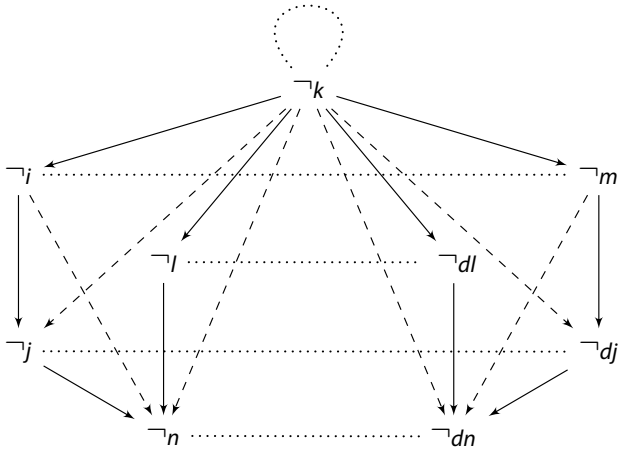
Six further kinds of negation

Negation	*Characterizing rule*	*Dual negation*
$\neg_j$	$a, b \vdash \neg_j c \to a, c \vdash \neg_j b$	$\neg_{dj}$
$\neg_{dj}$	$\neg_{dj} c \vdash a, b \to \neg_{dj} b \vdash a, c$	$\neg_j$
$\neg_l$	$a, \neg_l b \vdash c \to a, \neg_l c \vdash b$	$\neg_{dl}$
$\neg_{dl}$	$c \vdash a, \neg_{dl} b \to b \vdash a, \neg_{dl} c$	$\neg_l$
$\neg_n$	$a, b \vdash c \to a, \neg_n c \vdash \neg_n b$	$\neg_{dn}$
$\neg_{dn}$	$c \vdash a, b \to \neg_{dn} b \vdash a, \neg_{dn} c$	$\neg_n$
$\neg_k$	$a, \neg_k b \vdash c \leftrightarrow a \vdash b, c$	$\neg_k$
$\neg_i$	$a \vdash \neg_i b \leftrightarrow a, b \vdash$	$\neg_i$
$\neg_m$	$\neg_m a \vdash b \leftrightarrow \vdash a, b$	$\neg_m$

$\neg_n$: subminimal negation (cf. Dunn 1999)

Relations between negations

Solid arrows: negation satisfies rules of other negation
Dashed arrows: satisfaction by transitivity
Dotted lines: duality



(Several relations indicated by Popper, worked out by D. B. and T. P.)

Which negations are logical constants?

Theorem

None of the rules for $\neg_j$, $\neg_{dj}$, $\neg_l$, $\neg_{dl}$, $\neg_n$ and $\neg_{dn}$ are fully characterizing.

Interesting:

- minimal negation $\neg_j$ and
- subminimal negation $\neg_n$

are not logical constants

Overall, interesting approach

Abstraction from object languages that presuppose logical constants

Instead: discussion of logical constants in structural terms

From Popper's Decomposition of Logical Notions to Lakatos' Decomposition of the Notion of Proof

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INTRODUCTION THE LOGIC OF $\mathcal{P}\&\mathcal{R}$ POPPER COMES INTO PLAY REMARKS ON POPPER'S LOGICAL ENQUIRIES BA

OUTLINE

1. INTRODUCTION
2. THE LOGIC OF $\mathcal{P}\&\mathcal{R}$
3. POPPER COMES INTO PLAY
4. REMARKS ON POPPER'S LOGICAL ENQUIRIES
5. BACK TO LAKATOS
6. REFERENCES

Lakatos's masterpiece, *Proofs and Refutations*, say $\mathcal{P}\&\mathcal{R}$, constitutes a wonderful exploration of how mathematics evolves. The central theme of Lakatos' dissertation is a criticism of the concept of *formal proof*, which is an argument executed according to the rules of a precisely specified mechanism. His aim is to give some real sense to the claim that regimenting proofs in order to clarify their assumptions and the procedural rules involved –the process which formalization idealizes– is just *one* phase in the complex process that leads to the growth of mathematical knowledge.

INJECTING TRUTH AND MEANING

$\mathcal{P}\&\mathcal{R}$ can be seen as a renewal of the classical debate between analytical and axiomatic, or synthetic, procedures. In fact, it is framed within a broad context where the refusal of the latter kind of procedures is linked to a strong criticism of the *formalistic* school of mathematical philosophy.

- In the Euclidean Axiomatics injection of truth and meaning are given at the *outset* by means of definitions (which state which kind of entities we will dwell upon) and postulates (whose truth, guaranted by intuitive evidence, is propagated to any other assertion).

INJECTING TRUTH AND MEANING II

- In Hilbertian Axiomatic Formal Systems both questions are displaced at the very *end* of the construction.
- As regards the question of truth, it has been, so to speak, ignored and replaced by the requirement of consistency.
- As regards the question of meaning, it has been postponed to the formulation of the axioms: for instance, what it means to be an Euclidean point (or line or plane) is something which is (implicitly) fixed by the axioms.

INJECTING TRUTH AND MEANING III

Lakatos intends to occupy a *land* between the Euclidean and the Hilbertian perspectives: truth and meaning are injected in the course of the inquiry, without any hope to reach an ultimate point. Both notions have a tentative, conjectural nature.

- As regards *truth*: during the dialogue, after pupil Gamma has reminded that the very term "polyedron" has been stretched to the point that
  it does not figure in the theorem anymore, pupil Kappa adds

  > *because of concept-stretching, refutability means refutation. So you slide on to the infinite slope, refuting each theorem and replacing it by a more "rigorous" one, by one whose falsehood has not been "exposed" yet! But* you never get out of falsehood *(p. 99)*.

INJECTING TRUTH AND MEANING IV

- As regards *meaning*: if one tries to stop the previous "infinite slope" by exploiting monster-barring definitions, and so try to keep away counterexamples generated by concept-stretching, pupil Kappa warns that

  > *you will slide on to another infinite slope: you will be forced to admit of each "particular linguistic form" of your true theorem that it was not precise enough, and you will be forced to incorporate in it more and more "rigorous" definitions couched in terms whose vagueness has not been exposed yet! But* you never get out of vagueness *(p. 100).*

QUASI-EMPIRICAL THEORIES

In opposition to both kinds of perspectives, Lakatos proposed to assimilate mathematical theories to general scientific theories. In the *Introduction* to $\mathcal{P}\&\mathcal{R}$ Lakatos argues that he will challenge mathematical formalism elaborating

> *the point that informal, quasi-empirical, mathematics does not grow through a monotonous increase in the number of indubitably established theorems but through the incessant improvement of guesses by speculation and criticism, by the logic of proofs and refutations.*

(QUASI-)TECHNICAL NOTIONS

There are a few (quasi-)technical terms one has to get acquainted with in order to understand Lakatos's proposal.

1. Central for Lakatos' philosophy of mathematics is his characterization of the concept of mathematical **proof**, which occurs near the beginning of the text:

> *Teacher: I propose to retain the time-honoured technical term 'proof' for a thought-experiment –or 'quasi-experiment'– which suggests a* ***decomposition*** *of the original conjecture into subconjectures or lemmas, thus embedding it in a possibly quite distant body of knowledge (p. 9).*

QUASI-TECHNICAL NOTIONS II

2. After "proof", we remind the notion of **proof analysis**, which means the production of what we might normally call the "proof": the list of "lemmas" into which the proof (thought-experiment) was decomposed. We are doing proof analysis when we study the precise conditions under which the moves taken in the proof can be made, or are correct.
3. An important role is played by the notions of **local counterexample** and **global counterexample**. Global counterexamples show that some universal statement is false, but in a way that does not require any reference to the proof of that statement. A local counterexample, by contrast, is a property not of a statement but of a proof of the statement. Thus the definition of a local counterexample refers both to a statement and to a proof of it, regarded as a sequence of other statements.

QUASI-TECHNICAL NOTIONS III

4. The goal of the development of a proof, like that of Euler's formula, is a rigorous *theorem*, which Lakatos calls the **principle of retransmission of falsity**, meaning that all global counterexamples must become (also) local. Falsity must be retransmitted from naive conjecture to lemmata. That is, any counterexample to the theorem should be a counterexample to some step in the proof-analysis of the theorem:

   > *Lambda: A proof-analysis is 'rigorous' or 'valid' and the corresponding mathematical theorem true if, and only if, there is no 'third-type' counterexample to it (p. 47).*

   (We remind that the third-type counterexamples are those that are global – they refute the theorem at hand – but not local – they do not falsify any step of the proof.)

QUASI-TECHNICAL NOTIONS IV

5. The last notion to consider is the **principle of the retransmission of truth**, a notion which pertains to the case of counterexamples which are both local and global. The hollow cube, for instance, that is a cube with a cube shaped hole in it, is a counterexample which is both global (since $V - E + F = 16 - 24 + 12 = 4$), and local (since it cannot be stretched flat on the blackboard having had a face removed). To treat this type of counterexamples the faulty lemma is made up a condition of the original conjecture, restricting in this way its range of applicability. The proof is left unchanged, and just like with the question of the convexity, in this case too we have no assurance that even some polyhedron which does not satisfy the lemma is still an Eulerian polyhedron. .

COMMENTS

It is tempting to see the last two notions as very kin to, respectively, the *soundness* and the *semantic completeness* of a (formal) theory, the means to impede the overgeneration and undergeneration of mathematical truths.
However, because of the peculiarities of Lakatos's perspective, this is a temptation we must resist. The evolution of the initial "proof" sketch, or thought-experiment, results from interactions with various kinds of counterexamples. At each stage we examine the proposed counterexamples evaluating the reasons for the possible inadequacy of the proof, where such an examination may provide hints as to how modify both the steps and the notions involved in the proof.

AN OPEN FRAMEWORK

- Not necessarily this procedure produces a convergent sequence of proofs flowing into a definite, ultimate proof, least of all in a definite proof of the original conjecture. And the possibility to abandon the original conjecture cannot be excluded.
- $\mathcal{P}\&\mathcal{R}$ takes into account various ways of coping with counterexamples. It would be however a serious mistake to search for *the* correct method. The correct perspective is precisely given by the interplay of different methods to face different kinds of counterexamples; i.e., the interplay between generation and evaluation of counterexamples.

AN OPEN FRAMEWORK II

- What is worth to stress is that both generation and evaulation are driven by the proof.
- As student Beta admits, the logic of conjectures and refutations has no starting point (naive conjectures are preceded by many 'pre-naive' conjectures and refutations), but the logic of proofs and refutations has: it starts with the first naive conjecture to be followed by a thought-experiment (p. 74).

DISTANCING POPPER?

According to a largely shared opinion, Lakatos modified Popper's critical falsificationism with regard to two core aspects.

- He extends falsificationism also to mathematics (to which Popper himself did not venture to apply his ideas) proposing to consider also the latter as quasi-empirical. Mathematical theorems are not irrefutably true statements, but conjectures: one cannot know that a theorem will not be refuted.
- Refutation does not entail immediate rejection -as it was the case in Popper's Darwinistic account. He deploys instead a battery of strategic moves in order to cope with cropping out counterexamples.

DISTANCING POPPER? II

The previous characterization contains significant portions of truth, of course, but I don't feel satisfied that it offers a very balanced view of things, and I think that Popper's logical and epistemological papers of the late '40s can help us in better set the relationship between Popper's falsificationism and Lakatos's fallibilism.

POPPER [1946]

Interesting suggestions are already present in Popper [1946]:

- In this paper Popper wonders why ought we avoid those breaches of the rules of logic that we call "fallacies" if not because we are interested in formulating or deriving statements which are true, that is to say, which are true descriptions of facts?
- We undertake the "meta-linguistic" task of detecting the rules of inference of the language we are investigating aiming at *formalising all* those inferences which we intuitively *know how* to draw; much as we know that it is impossible to build a single calculus able to formalise *all* valid intuitive rules of inference.

POPPER [1946] II

- Turning to the question: Why are the logical calculi – which may contain arithmetic – applicable to reality? , he notes

> *In so far as a calculus is applied to reality, it loses the character of a logical calculus and becomes a descriptive theory which may be empirically refutable; and in so far it is treated as irrefutable, i.e., as a system of logically true formulae, rather than a descriptive scientific theory, is not applied to reality (p. 54).*

POPPER [1946] III

Commenting on this point, he discriminates "apparent" from "real" applications, considering proposition "2 + 2 = 4".

- If applied to apples, such proposition is considered irrefutable and logically true, but it does not describe any fact involving apples – any more than the statement "All apples are apples" does. Application is only apparent: by uttering that proposition we do not describe any reality, but only assert that a certain way of describing reality is equivalent to another way.
- Otherwise, the sentence "2 + 2 = 4" may be taken to mean that, if somebody has put two apples in a certain basket, and then again two, and has not taken any apples out of the basket, there will be four in it. In this interpretation the symbol "+" stands for a physical manipulation, and consequently we cannot be sure whether that sentence remains universally true.

POPPER [1946] III

- Very easily Popper provides examples of models in which "2 + 2 = 4" is not applicable (rabbits, drops,. . .), and to criticism based on the convinction that the equation "2 + 2 = 4" only applies to objects to which nothing happens, he replies that then that equation does not hold for "reality" (for in "reality" something happens all the time).
- And concludes maintaining that

> *whenever we are doubtful whether or not our statements deal with the real world, we can decide it by asking ourselves whether or not we are ready to accept an empirical refutation. If we are determined, on principle, to defend our statements in the face of refutations [. . .], then we are not speaking about reality. Only if we are ready to accept refutations do we speak about reality (p. 56).*

POPPER ON "PROOF"

Pertinent to the previous remarks is the following passage, quoted by Bar-Am in [2009], which reveals much about the character of the course in logic and scientific method Popper taught at LSE from 1946 to 1969:

> *The idea that science "proves" is wrong. The word "proves" is being misunderstood. In the sense of "prove" discussed above science has "proved" very little. Look at the changes in science in the last 2,000 years. If on important points science can change its teaching so much in the course of time, the proof, if it occurs at all in science must be comparatively rare . . . it marked a kind of false idea in science, an idea of science in which science cannot change, only grow.*

METHODOLOGICAL ISSUES

- Popper's idea is to consider a number of languages and *translations* from one of these languages into the others. The presupposition is that we master, or have competence, of the languages involved.
- We remind that for Popper Logic is a *metalinguistic* enterprise. A distinctive feature of his approach, compared with now usual approaches, is that no *assumption* is made about the form or syntactic structure of the (object-)language, say $\mathcal{L}$. $\mathcal{L}$ could also be a formally defined language, but nothing excludes its being a natural language.

METHODOLOGICAL ISSUES II

- The theory of inference that we are going to present will be able to be applied to any language in which we can identify statements, whatever their logical structure or lack of structure may be; that is to say, expressions of which we might reasonably say that they are true or that they are false.
- Popper starts by focusing on the *problem of giving a satisfactory definition of "valid deductive inference"*, where "deducibility" is the only undefined notion employed, as far as propositional and modal logic are concerned.

METHODOLOGICAL ISSUES III

- Popper's method, closely related to subsequent Lakatos's viewpoint, is the following:

  > *we shall propose a definition, criticize it, and replace it by a better one, and repeat this procedure ([1947a], p. 251).*

  (It is opportune to notice that [1947a] is Popper's paper Lakatos mainly refers to.)

METHODOLOGICAL ISSUES IV

- Popper presents a model consisting of an articulated structure given by inferential relations, which is to be laid on any $\mathcal{L}$, and aims at characterizing the meaning of logical compounds in the form of answers to questions like: "what does it mean for $\mathcal{L}$ to have an operation which has the inferential force of a negation, conjunction, . . . ?".
- This calls to mind the model of language which Quine was going to present shortly after (in 1951, with *Two Dogmas of Empiricism*): the model of an articulated structure, with some sentences lying at the periphery, where experience impinges, and others at varying levels within the interior. However, whereas Quine's proposal was driven by general meaning-theoretical issues, Popper just aimed at characterizing the meaning of logical notions.

METHODOLOGICAL ISSUES V

We collect here some notions which will be soon useful. Let's consider two languages, say $\mathcal{L}_1$ and $\mathcal{L}_2$:

1. A translation of $\mathcal{L}_1$ into $\mathcal{L}_2$ such that every complete statement of the former is co-ordinated with one complete and meaningful statement of the latter is called an *interpretation*.
2. If the interpretaion preserves re-occurrences of statements then it is called a *statement-preserving interpretation*.
3. In the case in which with every different statement of $\mathcal{L}_1$ a different statement of $\mathcal{L}_2$ is coordinated, Popper speaks of a *strictly* statement-preserving interpretation.
4. In case a translation preserves the meaning of the statements of $\mathcal{L}_1$, it is called a *proper translation*.

METHODOLOGICAL ISSUES VI

5. *Assuming* given the distinction between the *formative signs* and the *descriptive signs* of the languages we are considering, a *form-preserving interpretation* is now defined as an interpretation which
   - preserves the meaning of all the formative signs,
   - preserves recurrence of those groups of *descriptive* expressions which, in a proper translation, would fill the spaces between the translated formative signs.
6. Two statements a_1 and a_2, not necessarily belonging to the same language, have the same *logical form* if, and only if, there exist two form-preserving interpretations such that a_1 interprets a_2 and *vice versa*.
7. Then, the *logical form of the statement* a_1 is defined as the class of statements (of any number of languages) which have *the same logical form as* a_1.

METHODOLOGICAL ISSUES VII

8. The *logical skeleton* of a statement is obtained simply by eliminating all descriptive signs, and indicating, at the same time, recurrences of descriptive signs, by some method or other. Two statements a_1 and a_2, sharing the same "logical skeleton", belong to the *same* language.
9. *Assuming* given the distinction between the *formative signs* and the *descriptive signs* of the language, the notion of "logical skeleton" admits a direct definition, i.e. without passing through the idea of interpretation. On the other hand, and under the same *assumption*, the idea of a logical form is more general, and gives us the means of constructing a theory of language – or of languages – *without tying us down to any particular language*.

METHODOLOGICAL ISSUES VIII

- Tackling the notion of a valid (deductive) inference, Popper tries various proposals:

  (D1) An inference is valid iff every possible state of affairs which renders all the premises true also renders the conclusion true.

  A reformulation of (D1) is obtained by exploiting the notion of *counter-example*:

  (D1') An inference is valid iff no counter-example of it exists
- Problems: "state of affairs"? "possible state of affairs"? does "possible" mean "*logically* possible"?
  A second proposal uses the notion of "logical skeleton":

  (D2) An inference is valid iff every inference with the same logical skeleton whose premises are all true has a true conclusion.

METHODOLOGICAL ISSUES VI

- Another possibility is that we use the notion of "logical form" defined with the help of the term "form-preserving interpretation":

  (D3) An inference is valid iff every form-preserving interpretation of it whose premises are all true has a true conclusion.

- An intrinsic limit of (D2) is given by its referring to other arguments *of the same logical skeleton* as the argument in question and thereby confines its reference to other arguments *belonging to the same language*. (D2), moreover, is conditioned by the possible poverty in descriptive signs of the given $\mathcal{L}$: an invalid inference would appear as valid from the point of view of (D2), simply because no counterexample exists within $\mathcal{L}$.

METHODOLOGICAL ISSUES VII

- By its referring to all form-preserving interpretations, and therefore to *an unspecified number of different languages*, viz., to all those into which the formative signs can be properly translated, (D3) warrants that the validity or otherwise of an inference or rule of inference is independent of the language in which it is formulated.
- The problem with (D3) is that it is based on the distinction between formative and descriptive signs. A distinction that only "interpretation" and "statement-preserving interpretation" do not presuppose.

METHODOLOGICAL ISSUES VIII

- Popper proposes to start from a number of inferences which are valid *whatever the logical form of the statements involved*. Since they are valid indipendently of the distinction "formative/descriptive", they are called *absolutely valid*.
- Having a certain system of absolutely valid rules at our disposal, it is possible to define the logical force or import of the various formative signs in terms of deducibility: this is called an *inferential definition*. Formative signs are characterized as those signs which can be given an inferential definition.
- The following definition is proposed:

  (D4) An inference is absolutely valid if, and only if, every statement-preserving interpretation whose premises are true has a true conclusion.

INTRODUCTION THE LOGIC OF $\mathcal{P}\&\mathcal{R}$ POPPER COMES INTO PLAY REMARKS ON POPPER'S LOGICAL ENQUIRIES BA

METHODOLOGICAL ISSUES IX

- Absolute validity does no longer depend on the distinction between formative and descriptive signs. Popper admits that it depends on the distinction between statements and non-statements. But he emphasizes that whereas the former distinction affects the very central problem, validity, the latter can't affect the decision as to the validity or invalidity. It can only affect the question whether a certain sequence of expressions is an inference (valid or invalid) or no inference at all.
- We can not yet define, for instance, what we mean when we say: "*a* is the negation of *b*". But we do posses the means of defining what we mean when we say:

  "*a* has the same (logical) force as a negation of *b whatever the logical form of a and b may be*"

  And this is what Popper gets now ready to do.

LOGICAL ISSUES

- His program is to characterize the notion of deducibility through a certain system of absolutely valid rules, and then, *without any link to any particular language* ([1947], p. 260), he proceeds to provide definitions of logical compounds just in terms of the metalinguistically characterized deducibility relation.
- At the metalinguistic level, Popper adopts the following symbolic notations:

$$\rightarrow \qquad \leftrightarrow \qquad \& \qquad \vee \qquad (a) \qquad (\exists a)$$

LOGICAL ISSUES II

- To express the assertion: "From the statements $a_1, \ldots, a_n$, the statement b can be derived" Popper uses the notation

$$a_1, \ldots, a_n / b$$

noting that,

1. the symbols a, b, c, . . . , are *variables*, and their values are statements. Since we are at the metalinguistic level, *names of statements* (and not the statements themselves) may be substituted for the variables, which can be described as variable names of statements; and
2. although we may operate with as many premises as we like, we draw only *one* conclusion at a time.

LOGICAL ISSUES III

- Popper first attempts to determine the notion of deducibility by laying down a few very simple primitive rules for it, called a *Basis*. Basis I consists of
- a Generalised Principle of Reflexivity, referred to by (RG):

$$a_1, \ldots, a_n / a_i \qquad (1 \leq i \leq n)$$

- and a Generalized Principle of Transitivity, referred to by (TG):

$$\frac{\begin{array}{c}(a_1, \ldots, a_n / b_1) \\ \vdots \\ (a_1, \ldots, a_n / b_m)\end{array}}{(b_1, \ldots, b_m / c \rightarrow a_1, \ldots, a_n / c)}$$

LOGICAL ISSUES IV

- In a preliminary way, it is to be reminded that the introduction of compound statements starts by assuming postulates which assure, for every (pair of) statement(s), say a (and b), the existence of the corresponding compound statement. The function of postulates, which do not really form a part of Popper's theory of inference, is solely to indicate explicitly that the application of the theory is limited, if we wish to operate with certain compounds, to languages which contain these compounds.
- If a and b are mutually deducible, we write

$$a // b$$

We may also define in a obvious way "//" on the basis of "/":

(D//) $a // b$ if, and only if, $a/b \quad \& \quad b/a$.

NEGATION

- We focus on "negation": in [1947] the following definition is first given

$$(4.6) \qquad \neg a, b/\neg c \quad \leftrightarrow \quad c, b/a.$$

  It is interesting to note that (4.6) characterizes negation by means of the rule of contraposition in which the left to right direction $\neg a/\neg c \rightarrow c/a$ is an *intuitionistically* invalid form. In other words, (4.6) amounts in effect only to a principle underlying the *classical* theory of "indirect reduction".
- Popper notes that (4.6) is not completely satisfying since *two* negations occur on the left at the same time. Being the one somehow linked to the other, cannot always be eliminated alone.

NEGATION II

- Much more satisfying is considered the following definition:

$$\text{(D 5.6)} \qquad a//\neg b \quad \leftrightarrow \quad (a_1)(b_1)(a, a_1/b \rightarrow (a, a_1/b_1 \quad \& \quad a_1, b_1/b)).$$

  Popper notes that the last occurrence of "b_1" could be omitted. It is added only to make obvious the simmetry between the laws of contradiction and excluded middle.
- We think, however, that (5.6) combines in a somewhat cumbersome way a form of Peirce rule – if $\neg b, a_1/b$ then b follows from each a_1, b_1 – together with the law of contradiction: from $\neg b, a_1$ follows each b_1.

NEGATION III

- In an interesting way classical negation is compared with the intuitionistic one in the following two definitions, which contain one only occurrence of a quantifier, and are therefore not quite suitably related to (D 5.6):

$$a//\neg^c b \quad \leftrightarrow \quad (b_1)(a, b/b_1 \quad \& \quad (a, b_1/b \quad \rightarrow \quad b_1/b)) \qquad \text{(D 5.6c)}$$

$$a//\neg^i b \quad \leftrightarrow \quad (b_1)(a, b/b_1 \quad \& \quad (b, b_1/a \quad \rightarrow \quad b_1/a)) \qquad \text{(D 5.6i)}$$

NEGATION IV

- Seen from the point of view of sequent calculus, both rules contain an instance of "ex falso quodlibet", $(b_1)(\neg^{c,i} b, b/b_1)$, together with an application of the rules of negation (respectively: Peirce's rule and self-denial) and contraction:

$$\dfrac{\dfrac{\dfrac{\dfrac{\neg^c b, b_1 \vdash b}{\neg^c b, \neg^c b, b_1 \vdash}}{\neg^c b, b_1 \vdash}}{b_1 \vdash \neg^c \neg^c b} \qquad \dfrac{\dfrac{b \vdash b}{\vdash b, \neg^c b}}{\neg^c \neg^c b \vdash b}}{b_1 \vdash b} \qquad \dfrac{\dfrac{\dfrac{b, b_1 \vdash \neg^i b}{\neg^i \neg^i b, b, b_1 \vdash} \qquad \dfrac{\dfrac{b \vdash b}{\neg^i b, b \vdash}}{b \vdash \neg^i \neg^i b}}{b, b, b_1 \vdash}}{b, b_1 \vdash}}{b_1 \vdash \neg^i b}$$

NEGATION V

- This provides a first exemplification of the *decomposition* of logical notions we refer to in the title.
- In [1947a, p. 284] the same definition (5.6c) is taken as the starting point for further developing the analysis. In fact, Popper defines the "exclusiveness" (or "contradictoriness") of a couple of statements $[a \perp b]$, and their "exhaustiveness" (or "logical disjunctness") $[a \top b]$ (both notions can be extended to any number $n \leq 2$ of statements).

$$a \perp b \quad \leftrightarrow \quad (c)(d)(c/a \quad \rightarrow \quad (c/b \quad \rightarrow \quad c/d)) \tag{7.5}$$

$$a \top b \quad \leftrightarrow \quad (c)(d)(a/c \quad \rightarrow \quad (b/c \quad \rightarrow \quad d/c)) \tag{7.6}$$

NEGATION VI

- The *Exclusiveness* of a and b expresses the impossibility of their coexistence: if a statement c allows us to infer both a and b, then c allows us to infer any statement d; that is, any c capable to separately *infer* two exclusive sentences plays the role of "falsum": $c \equiv \perp$. A pair of exclusive sentences is the *weakest* "sentence" capable to infer any other sentence of the language.
- The *Exhaustiveness* of a and b expresses the fact that a and b fill any possibility: if a statement c can be inferred from both a and b, then c can be inferred from any statement d; that is, any statement which can be separately *obtained* from two exhaustive statements plays the role of "verum": $c \equiv \top$. A pair of exhaustive sentences is the *strongest* "sentence" capable to be inferred from any other sentence of the language.

NEGATION VII

- By means of definitions (7.5) and (7.6), Popper defines the notion of complementarity of statements a and b:

  (7.7) $\quad a//$ the complement of $b \quad \leftrightarrow \quad ((a \perp b)\&(a \top b))$

  If a and b are exhaustive as well as exclusive, then $a//$ the complement of b; that is, that a is the complement of b means that a and b cannot coexist whereas they cover any possibility. In a sense, "complement" and "negation" are equivalent notions:

  $$[a//\neg b \quad \& \quad c// \text{ the complement of } b] \rightarrow (a//c)$$

NEGATION VIII

- The two components of the notion of "complementarity" (and, through the equivalence, of the notion of "negation") focus on different features of the group of the "identity rules": exclusiveness looks at the "$\ldots \vdash$" perspective, whereas exhaustiveness looks at the "$\vdash \ldots$" direction.
- This analysis is already available in definition (7.2) of [1947a]

  (7.2) $\quad a//$the negation of b if, and only if,
  $(c)(a, b/c \quad \& \quad (a, c/b \quad \rightarrow \quad c/b))$.

  according to which "negation" has two components: the former exhibits (a variant of) the "$\ldots \vdash$" perspective, the latter expresses the content of the Peirce's rule, and is a variant of the "$\vdash \ldots$" perspective.

NEGATION IX

- Deepening the previous decomposition, in [1948a] Popper reminds that

  > *The intuitionistic negation of b is the* weakest *of those statements which are* strong *enough to contradict b (my emphasis)*

  meaning that, together with b, it is capable to infer any c.
- Thus, a is equivalent to the intuitionistic negation of b, $a//\neg^i b$, if and only if

$$(c)(c/a \quad \leftrightarrow \quad (d)(e)((d/c \quad \& \quad d/b) \quad \rightarrow \quad d/e).$$

NEGATION X

- Popper comments in [1948a] stressing that intuitionistic negation is characterized by contradictoriness, or exclusiveness, alone. This fact could induce the idea that an analogous link could exist between classical negation and complementarity, or exhaustiveness, *again* alone.
- This is an idea we must abandon: it would mean that a is equivalent to the classical negation of b, $a//\neg^c b$, if and only if

$$(c)(c/a \leftrightarrow (d)(e)((a/d \quad \& \quad b/d \quad \rightarrow \quad e/d))$$

  where we have dropped the part $(c)(a, b/c)$, the "principle of contradiction".

NEGATION XI

- Thus, the idea to be adandoned is that "excluded middle" *alone* could be enough to characterize classical negation, meaning that any c which can be inferred from both a statement and its classical negation, can be inferred from any statement. In other words, this is evidence for the mutual independence of the two components of (classical) negation.
- Pursuing this idea, Popper gets a different (from both classical and intuitionistic) notion of negation, say $\neg^m b$, which is called the "minimum definable negation of b":

$$\begin{array}{lll} \text{(D 4.2)} & a//\neg^m b & \leftrightarrow \quad (c)(a/c \quad \leftrightarrow \\ & (d)(e)((b/e \quad \& \quad c/e) \rightarrow d/e) & \end{array}$$

NEGATION XII

- Popper comments on (D 4.2) saying that $\neg^m b$ is the strongest of those statements which are weak enough to be complements of b. The right hand side of (D 4.2) means that a statement c can be inferred from $\neg^m b$ iff it is complementary to b.
- Deconstructing and reassembling logical notions, specifically *negation*, Popper has been able to draw attention to a "new" logic, which is now called *dual intuitionism*, and which is characterized by the "minimum definable negation", and which can have at most one formula on the left of the sequent arrow.

NEGATION XIII

- Continuing to dwell upon the various notions, Popper notes that besides taking care to use different names, there is no need to make sure that our system of definitions, and hence the language object whose inferential relations we are dealing with, is consistent.
- This is most probably a teaching which is part of A. Tarski's legacy. I mean, of his claim regarding the inconsistency of natural languages. Popper makes the example of the notion of "opponent" (we refer to [1947a]):

  (7.8) $a//$ opp (b) if, and only if, $(c)(b/a \quad \& \quad a/c)$.

NEGATION XV

- Popper emphasizes that, as a consequence of definition (7.8), every language which has a sign for "opponent of b" – analogous to the sign for "negation of b" – will be inconsistent. But this need not lead us to abandon (7.8); it only means that no consistent language will have a sign for "opponent of b".
- Popper's definition of "opponent of b" seems to contain already the idea of Prior's connective "tonk", and in fact Prior cites Popper's paper (even though just with regard to the clarification of the notion of "analytically valid inference" provided by Popper in [1947a]).

NEGATION XVI

- However, Popper doesn't feel obliged to raise barriers around the notion of "opponent". We can say that "opponent" is a(n *ante litteram*) generalization of "tonk"; in the sense that we get "tonk" when the (c) in the definition of "opp b" is particularized to b. In this way, in fact, we have that "$a \quad tonk \quad b$" is the opponent of a; in fact, by the same rules governing "tonk" it holds:

$$(a/a \quad tonk \quad b \quad \& \quad a \quad tonk \quad b/b).$$

NEGATION XVII

- Reflecting on $opp(b)$, and with respect to the *inferential definitions* provided by Popper for the logical compounds of a given object language $\mathcal{L}$, it is reasonable to wonder if any set of rules gives rise to a definition of a logical constant or not.
- The question is very near to the one people working in proof-theoretic semantics had to face after "tonk" came to the fore: is every "inferential definition" to be allowed? Popper's answer seems to be "yes", and this would teach us that his system seems not include any harmony requirement, since the transitivity rule used by Belnap to overcome "tonk" is part of Basis I.

NEGATION XVIII

- As D. Binder and T. Piecha say in [2017],

> *there is one condition that Popper seems to consider to be essential for any definition of a logical constant, namely uniqueness:[. . .] fully characterizing rules are exactly those rules that satisfy uniqueness.[. . .] It is the existence of fully characterizing rules that distinguishes logical constants from non-logical constants, and it is this criterion of logicality that leads Popper to reject, for example, minimal negation as a logical constant. (pp. 167-168)*

I think that attending Popper's courses and seminars was a true "training ground" for most of his students, Imre Lakatos included.

- It is difficult not to see a deep link between Popper's *playing* with the logical notions and Lakatos's perspective, in which the starting point is got by adapting a somehow devised "proof-sketch" to new problems waiting for a solution.
- The evolution of the initial proof results from interactions with various kinds of counterexamples which immediately start to crop up, leading to arguments over the meaning of terms involved in the definitions as they are put forward, so that various definitions of polyhedron, polygon, edge, area, vertex, . . . , are provided.

CONCEPT-STRETCHING

- At p. 93 pupil Pi notes that it is impossible to keep refutations and proofs on the one hand and changes in the conceptual, taxonomical, linguistic framework on the other. Facing a counterexample, one can choice to disregard it because involving notions not belonging to his language $\mathcal{L}_1$, or to accept the counterxample passing by concept-stretching to a new language $\mathcal{L}_2$, . . .

CONCEPT-STRETCHING II

- Commenting on this point, pupil Gamma foresees the possibility of *inconsistent* languages:

> *we may have two statements that are consistent in $\mathcal{L}_1$, but we switch to $\mathcal{L}_2$ in which they are inconsistent. Or, we may have two statements that are inconsistent in $\mathcal{L}_1$, but we switch in $\mathcal{L}_2$ in which they are consistent. [. . .] The growth of knowledge cannot be modeled in any* given *language. (My emphasis)*

CONCEPT-STRETCHING II

- The skeptic component of Lakatos's attitude surfaces in the claim that no single language can model the growth of knowledge, and that there is no hope that the mechanism of refutational success, i.e. "concept-stretching", could peter out. This same attitude is the source of some of his most brilliant insights.

INTRODUCTION THE LOGIC OF $\mathcal{P}\&\mathcal{R}$ POPPER COMES INTO PLAY REMARKS ON POPPER'S LOGICAL ENQUIRIES BA

CONCEPT-STRETCHING II

- As pupil Kappa explicitly spells it out:

  > *For any proposition there is always some sufficiently narrow interpretation of its terms, such that it turns out true, and some sufficiently wide interpretation such that it turns out false. Which interpretation is intended and which unintended depends of course on our intentions. [. . .] Concept-stretching will refute* any *statement, and will leave no true statement whatsoever (p. 99).*

I SHALL STRETCH "STRETCHING"

- The last notion to be submitted to "stretching" is that very same notion, and it is intruiguing to compare the last assertion with the following quotation from Wittgenstein (*Philosophical Investigations*, § 201):

  > *[. . .] no course of action could be determined by a rule, because every course of action can be made out to accord with a rule.*

- The thread of skepticism is then followed until touching the arithmetical (as for instance, addition) and logical (as for instance, the universal quantifier) notions.

I SHALL STRETCH "STRETCHING" II

- Answering to pupil Gamma, who hopes to reach a point where the meanings of the terms will be so crystal clear that there will be one single interpretation, as it is the case with $2 + 2 = 4$, pupil Kappa shows that it is possible to stretch also this proposition by stretching the meaning of "addition". To this aim it is envisaged a generalized notion of addition which could be called *addition as package*. The usual addition is recovered from that as the very special case of packing where the weight of the covering material is zero.
- Of course, it is also in this case compelling the closeness with themes from Kripke's *Wittgenstein on Rules and Private Language* of 1982, especially with the mathematical example Kripke gives to support his interpretation of the rule-following paradox.

I SHALL STRETCH "STRETCHING" III

- From our point of view, beyond looking forward, it is also interesting to look backward: it is in fact immediate to think to Popper [1946], where we encountered an analogous variation on the operation of addition, but of course absolutely lacking any skeptic trace.
- Popper's speculation on the applicability of logic and arithmetic to reality was, so to speak, the breach through which Lakatos could insert his quasi-empirical proposal skepticism colored.

INTRODUCTION THE LOGIC OF $\mathcal{P}\&\mathcal{R}$ POPPER COMES INTO PLAY REMARKS ON POPPER'S LOGICAL ENQUIRIES BA

I SHALL STRETCH "STRETCHING" III

- Arguing about the cylinder, and wondering which kind of counterexample to the "Cauchy proof" it is, pupil Gamma claims that the falsity of "there is a diagonal of the circle that does not create a new face" entails the truth of "all diagonals of the circle create a new face" (p. 44).
- Lakatos emphasizes that in this way the universal quantifier underwent a *modest stretching*, consisting in removing the existential import from its meaning, so that it no longer applies only to non-empty classes. This was an important event, since it draws attention on the possibility that also logical notions experience some shifts of meaning.

BEING CAUTIOUS

- Lakatos's insight, in this case, seems to foreshadow Etchemendy [1990], where the existential quantifier was included among the variable expressions, identifying its satisfaction domain with all subcollections of the universe.
- As a result of this one gets that the sentence $\exists x \exists y(x \neq y)$ would be logically true iff every subcollection of the universe contained at least two elements. This last statement, however, is of course false (there are the singletons). Thus, rightly, we get that $\exists x \exists y(x \neq y)$ isn't logically true.

BEING CAUTIOUS II

- Lakatos, however, does not follow that line of thought: pupil Theta says that concept-stretching has to stop at the point where it opens the way to irrationalism, and concludes:

  > *We may have to find out which are those terms whose meaning can be stretched only at the cost of destroying the basic principles of rationality (p. 103).*

- And Lakatos notes that the most interesting results in this direction were Popper's papers of the late 40s from which it follows that "one cannot give up further logical constants without giving up some basic principles of rational discussion" (p. 104).

... BUT STILL CRITICAL

- In a footnote at p. 123, Lakatos observes that an unsatisfactory trait of Popper's treatment of logical form is the unsufficient attention devoted to the important problem of translatory definitions.
- The limit of Popper's idea is to search for a definition of valid inference depending *only* on the list of formative signs.
- > [V]alidity of an intuitive inference depends also on translation of the inference from ordinary (or arithmetical, geometrical, etc.) language into the logical language: it depends on the translation we adopt.

... BUT STILL CRITICAL II

- To overcome Tarski's puzzle in the final part of his 1936 paper "On the notion of logically following", Popper tried to reverse Tarski's order of priority, taking the notion of "derivability" as primitive, and showing that those signs are *logical*, or *formative*, which can be defined with the help of that primitive concept.
- To find the correct order of prioriy is *one* problem, which Lakatos doesn't appreciate so much.
- The case of Euler's Conjecture –which was of the form "All A's [polyhedrons] are B's [Eulerian]"– is in fact evidence that assessing logical validity does not hinge *only* on the list of formative signs – in this case "all" and "implication" –.
- The example of the cylinder showed that deforming "A" entailed also a deformation of logical terms.

THANKS

FOR YOUR ATTENTION

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Popper and the Role of Inference Rules in Logic

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An early proponent of natural deduction?

Replies to my critics

For my main intention was to simplify logic by developing what has been called by others „natural deduction“. I suppose that as an effort to build up a simple system of natural deduction (a commonsense logic, as it were), my papers were just a failure.

Overview

1. Biography
   - Vienna
   - Christchurch
   - London
2. Articles
   - Boolean Algebra
   - On Derivation and Proof
   - On Systems of Rules of Inference (1946/7)
   - A Note on the Classical Conditional

Vienna years

- Studies calculus and natural numbers under Hans Hahn in 1922. The course ends with *Principia Mathematica*.
- Reads Carnap's *Logical Syntax* in August 1934.
- Met Tarski and other Polish logicians at Prague conference in 1934 and befriends Tarski.

Axiome, Definitionen und Postulate der Geometrie (1928)

It is to be hoped that in the near future a commonly accepted symbolic language will be established at least in logic. Not everyone who is not especially concerned with logic can be burdened with a new symbolic language for every new logistic work. Therefore I have obviously not used the „logical calculus", even though the subject lent itself to it.

Christchurch

- Popper arrives in March 1937 in Christchurch for a position of lecturer at Canterbury College.
- He teaches 2 courses in logic, and 1 in ethics, history of philosophy and introduction to philosophy.
- He planned to write a logic textbook $\sim$ 1937/8

John Niemeyer Findlay

J. N. Findlay (1903-1987) was a South-African philosopher who taught at the University of Otago, Dunedin, from 1931 to 1945. Teacher of A.N. Prior.

- Popper probably collaborated with John Findlay on a paper on Gödelian sentences.
- Thought about applying for Findlay's chair after Findlay left.
- Discussed the Gödel article with Bernays in 1946.

Henry George Forder

H.G. Forder (1889-1981) was a New Zealand mathematician working mainly on geometry at Auckland University College.

- Extensive correspondence from 1943 to 1946.
- They discuss logic, foundations of mathematics and quantum physics.
- Forder helps Popper with his articles on boolean algebra.

Henry George Forder

Letter to H.G. Forder, May 7th 1943

„You describe yourself as a disciple of Hilbert, emphasizing the undoubtedly profound advance made by Hilbert beyond Principia Mathematica. I agree with your judgement. I may perhaps describe myself as a disciple of my friend Tarski whose methods, I believe, carry him nearly as far beyond Hilbert (in the direction indicated by you, i.e. towards the ‚inhaltlich', i.e. ‚purporting' or ‚contentual' or ‚designational' or ‚denoting function of a formal calculus') as Hilbert has gone beyond Russell."

Rudolf Carnap

Rudolf Carnap (1891-1970) fled in 1935 and taught at the University of Chicago from 1936 to 1952.

- Carnap sends Popper his books „*Introduction to Semantics* (1942)“ and „*Formalization of Logic* (1943)“.
- Popper replies to Carnap with remarks and comments on Carnap's books.

London

Popper arrives in London in 1946 and publishes soon after the following articles:

- Logic without Assumptions (1947)
- New Foundations for Logic (1947)
- Functional Logic without Axioms or Primitive Rules of Inference (1947)
- On the Theory of Deduction 1 & 2 (1948)
- The Trivialization of Mathematical Logic (1949)

Paul Bernays

P. Bernays (1888-1977) teaches at the ETH Zurich.

- Bernays and Popper meet in Zürich in December 1946 and discuss the possibility of publishing an article on logic together.
- Popper borrows copies of articles of Church, Hertz, Gentzen, Glivenko, Tarski and Jaskowski from Bernays.
- Popper writes an article with contributions from both him and Bernays. The finished article „*On Systems of Rules of Inference*“ never gets published.

L.E.J. Brouwer

The nature of the relationship between Popper and Brouwer is discussed neither in the biography of van Dalen nor in that of Hacohen.

- Meeting and discussions in July 1947.
- Popper publishes 2 articles through Brouwer (Functional Logic without Axioms or Primitive Rules of Inference, On the theory of Deduction I& II).
- Brouwer repeatedly meets with Popper in London in the following years.

Hacohen: Popper. 1902-1945 The formative years.

He heard first in 1920 about Brouwer and intuitionist mathematics. Brouwer seemed to him to challenge mathematical rationality: „I was stunned, found it irritating and depressing, but could do nothing with it.“. On Hahn's authority he trusted the *Principia*, „and looked at Weyl and Brouwer with suspicion“.

Luitzen E. J. Brouwer

Dedication of „Int. Betrachtungen über den Formalismus“

To Karl Popper. In friendship and remembrance of meetings in Holland and England and of discussions on important subjects.

Dedication of „Consciousness, philosophy and mathematics“

To Karl Popper In friendship and high esteem. 5.II.1949

Letter from Brouwer to Popper, January 19th 1948

The more I read and think your paper over, the more I get impressed by its importance

In 1948 Brouwer writes a letter of recommendation for Popper.

Brouwer to Harold Jeffreys, May 11th 1948

(iii) Mathematical logic, where Popper plays a prominent part in the complete renewal this science is undergoing just now. In particular his papers on derivation and negation which appeared about the end of 1947, I think will be consulted and quoted during a generation.

Boolean Algebra (1943)

Boolean Algebra (∼1943)

In 1943 Popper writes a series of articles on boolean algebra:

- Extensionality in a Rudimentary Boolean Algebra
- An Elementary Problem of Boolean Algebra
- Completeness and Extensionality of a Rudimentary Boolean Algebra
- Postulates for Boolean Algebra
- Simply Independent Postulates for Boolean Algebra

We will discuss „Extensionality in a Rudimentary Boolean Algebra“.

Dating

Letter to Carl Hempel, July 5th, 1943

„I hope to have something ready for the ‚Journal‘ in about three or four months.“

Letter to H.G. Forder, July 21st 1943

I am just about to finish a paper on Boolean Algebra and extensionality. In this paper, I discuss some problems of „=“ in a purely technical manner. I should be extremely grateful if you would go through this paper of mine before I send it to the „Journal“.

Letter to H.G. Forder, August 11th 1943

[...] I want to take a look at them before finishing my paper on „Extensionality in a Rudimentary Boolean Algebra“.

Equality and Identity

Popper distinguishes two different relations of „sameness“:

- Equality (denoted $=$)
- Identity (denoted $\equiv$)

Identity is the stronger of the two relations. Both equality and identity are equivalence relations, but only for identity does the following rule of inference hold:

$$\frac{x \equiv y \qquad \phi(x)}{\phi(y)} \text{ (Subst)}$$

Admissibility of (Subst)

The question referred to in the title of the paper is:

Admissibility of (Subst)

Given a formal calculus of equality, e.g. a suitable formalization of Boolean Algebra, under what circumstances is the (Subst) rule admissible.

Popper takes the admissibility of (Subst) to be a criterion for the *extensionality* of a system. (He does not use this terminology).

Relevance for his logic

These technical investigations informed his definitions of the logical constants, i.e.:

$$a \wedge b \vdash c \leftrightarrow a, b \vdash c \tag{1}$$

or

$$d, a \wedge b \vdash c \leftrightarrow d, a, b \vdash c \tag{2}$$

They also qualify a statement in a letter to Quine.

The inception of his „metalinguistic calculus“?

Letter to H.G. Forder, May 7th 1943

Regarding your question „Is there a formal logic of deduction where instead of $p \supset q$ we have ‚from p we can prove q‘ “, my reply is „yes“. [...] we can even interpret the Boolean Algebra as such a system:Interpret the variables of the Boolean Algebra as variable *names* of sentences; interpret „$a + b$“ as the descriptive *name* of the disjunction of the sentences designated by „a“ and and by „b“;[...] then *every Boolean theorem* of the form „$\ldots \subset$ —“ can be interpreted as „from . . . follows —“.[...] This is a problem in which I have been much interested in connection with my probability theory. I call this interpretation of the Boolean algebra the „meta-propositional calculus“, or the „*calculus of propositional names*“.

Boolean Algebra (∼1943)

Letter to Rudolf Carnap, July 5th, 1943

„I received the ‚Formalization of Logic' last week." [...] „In §§26 ff. you define a calculus which is, in a certain sense, logically stronger than PC. But it is not richer in theorems. The idea that any ‚addition' to, or ‚strengthening' of the calculus must lead to additional theorems is, of course, shown by you to be an unwarranted prejudice. It is this deep-rooted prejudice which has led us all to believe that, since the PC is complete regarding theorems, it is impossible to strengthen it further." [...] „What is needed is perhaps a more detailed comparison of the various senses of ‚completeness' than the one on p. 99."

On Derivation and Proof (1947)

Articles

After writing „On Systems of Rules of Inference“ (cp. next section) Popper becomes interested in the difference between derivation and demonstration.There exist three drafts of articles he intended to write.

- Derivation and Demonstration in Propositional and Functional Logic
- The Propositional and Functional Logic of Derivation and of Demonstration
- ⟨Untitled manuscript⟩

Dating

Letter to Bernays, December 1946 – February 1947

I have continued working, mainly on „Derivation versus Demonstration (or Proof)“, and have obtained some really interesting results: — I believe that you will admit, when you hear them, that the distinction is indeed quite important. But that enquiry presupposes the one on deducibility.

The distinction between derivation and demonstration

Untitled manuscript

The relation between derivation and demonstration has been interpreted, ever since Aristotle, as one between genus and species; that is to say, demonstration has always been interpreted as a *kind of derivation*. [. . .]The present paper attempts to show that this view is misleading, and that it prevents us from a clear understanding of derivation as well as demonstration; that is to say, from an understanding of logic. According to the theory here to be developed, demonstration is a procedure of which derivation is always a part — the most conspicuous part — but never the whole; a demonstration always contains a derivation [. . .], but it never is a derivation.

Rules of Demonstration

Consider a derivation in the system of PM:

$$\vdash \phi \qquad (1)$$
$$\vdash \phi \supset \psi \qquad (2)$$
$$\vdash \psi \qquad (\text{MP } 1,2)$$

Modus ponens appears here not as a derivation rule, but as a *rule of demonstration* or *rule of proof*.
In linear notation Popper would write this rule of proof as:

$$(\vdash \phi \ \ \& \ \vdash \phi \supset \psi) \Rightarrow \vdash \psi$$

Rules of Derivation

Modus ponens as a derivation rule is symbolized by Popper variably as:

$\phi, \phi \supset \psi / \psi$ $\qquad$ $\phi, \phi \supset \psi \rightarrow \psi$ $\qquad$ $\dfrac{\phi \qquad \phi \supset \psi}{\psi}$

Only the first version appears in published articles, the other versions appear in his notes and drafts.

Rules of Demonstration and Refutation

Popper considered rules of demonstration and refutation explicitly. E.g.:

$$\frac{\dfrac{\phi}{\psi} \qquad \dfrac{\phi}{\neg\psi}}{\dashv \phi} \qquad\qquad \frac{\dfrac{\phi}{\psi} \qquad \dfrac{\neg\phi}{\psi}}{\vdash \neg\neg\psi}$$

The left rule is a rule of refutation containing the two derivations ϕ/ψ and $\phi/\neg\psi$.

Untitled manuscript

[. . .]demonstration is a procedure of which derivation is always a part — the most conspicuous part — but never the whole; a demonstration always contains a derivation [. . .], but it never is a derivation.

The Separation of Derivation and Demonstration

Derivation and Demonstration in Propositional and Functional Logic

Logic is (a) the theory of deduction and derivation, (b) the system of L-true (logically demonstrable, or „asserted“) propositions.If we look at it mainly as (b), then we obtain (a) as a by-product. For each L-true conditional yields, together with the *modus ponendo ponens*, a valid rule of derivation[. . .].But if (a) is our aim — and for the logician, rather the more important of the two aims —, then it is not clear why we should reach it by the roundabout way *via* (b). Why should we not develop the system of valid rules of derivations in a more direct fashion, that is to say, independently of the development of the L-true propositions (including those which are „primitive“)?

The Separation of Derivation and Demonstration

The Propositional and Functional Logic of Derivation and of Demonstration

We thus operate with two connected, but precisely distinct systems — the system of derivation rules, and the system of demonstrable statements. Or, in other words, we make a clear distinction between the *logic of derivation* and the *logic of demonstration*, and we develop the former independently from the latter.

Ibid.

Gentzen [. . .] does not keep the system of derivations distinct from that of demonstration; on the contrary, he operates without any clear distinction between them, with rules which, in our system, are clearly distinguished into *rules of derivation* and *rules of demonstration*.

The Separation of Derivation and Demonstration

The Propositional and Functional Logic of Derivation and of Demonstration

We shall not make use of absolute derivation rules such as those used by Carnap, which state that certain statements or formulae are derivable without premises [...]; we do not consider these as „genuine" rules of derivation, but rather as a different way of formulating axiom schemata, and it is our aim to avoid all axioms, however formulated: this is precisely why we confine ourselves to „genuine" *assuming* derivation rules [...].

This makes him more radical than Gentzen, who has no qualms introducing an axiom into NI in order to turn it into NK.

On Systems of Rules of Inference (1946/47)

On Systems of Rules of Inference

Endnote of „On Systems of Rules of Inference“

Meeting in Zürich in December 1946, the authors found that, starting from very different questions, they had constructed, independently, very similar theories which they decided to publish conjointly.

Letter to P. Bernays, December 1946 – February 1947

You probably remember our discussion, and your kind [...] suggestion, to publish a small article on „Deducibility“ together.

Letter to P. Bernays, March 3rd 1947

Here is the article.[...] I propose to call the article: „On Systems of Rules of Inference.“ The title is not very good, but so far I couldn't think of a better one.

Why was the article never published?

Letter from Bernays, May 12th 1947

Your articles were naturally of great interest to me. It is just that with regards to the general tendency of the observations, I did not find such an unanimity with my own attitudes as I had expected after our discussions which we had here.

Separation of Logical Constants

On Systems of Rules of Inference

We propose to characterize the logical properties of compound statements by the method of laying down, *for each separately*, primitive rules of inference. (Each of these rules determines the deductive power, as it were, of the compound that occurs in it.)

This requires more than merely the fact that only one logical constant appears in an inference rule. Logical constants have to be characterized by a set of rules strong enough to deduce any valid formula which contains only that logical constant. (Cp. Peirce's law).

The rules of Implication

The set of rules for implication therefore contains a version of Peirce's rule.

- $a_1, \ldots, a_n / b_1 \supset b_2 \leftrightarrow a_1, \ldots, a_n, b_1 / b_2$
- $a_1, \ldots a_n, b_1 \supset b_2 / b_1 \rightarrow a_1, \ldots a_n / b_1$

On Systems of Rules of Inference

(Each of these rules determines the deductive power, as it were, of the compound that occurs in it.)

Completeness of Rules

On Systems of Rules of Inference

Every valid rule of inference R which can be expressed in our system can be obtained in it by employing [...] merely those primitive rules which refer to the compounds occurring in R.
In order to make each of the primitive rules (or group of rules which introduce *one* of the various compounds) complete in this sense, we have to make the system redundant.

A pure system of derivation

On Systems of Rules of Inference

But a purely derivational logic can be monistic. It can ignore the logically true statements; but it allows us, after we have build up an independent pure logic of derivation, to introduce later the logic of demonstration, or of the logiclly true statements, by simply adding a definition of demonstrablity [...].

Pure Rules of Derivation

On Systems of Rules of Inference

A rule of inference is called „pure“ if and only if we can construct, in some object language L containing non-tautological statements [...] an example which satisfies the rule (in a non-vacuous way if the rule is conditional), and which, moreover, satisfies the requirement that all statements of the example, whether compounds or components, are non-logical.

This excludes $a/b \supset b$ but also $(a/b \ \& \ \neg a/b) \rightarrow c/b$.

A Note on the Classical Conditional (1952)

A Note on the Classical Conditional

Instead of justifying the inference rules of implication by means of an explanation in terms of truth, Popper tries to establish the truth table using inference rules.

A note on the Classical Conditional

There are two rules which appear to be acceptable to the most sensitive analysts of the conditional, (1) the so called *deduction-theorem* and (2) the *modus ponendo ponens*

(1) If a/b, then $\vdash a \supset b$ (2) $a, a \supset b/b$

Ibid.

It is the contention of this note that [...] these two rules suffice for the derivation of the truth-table characterization of the classical conditional from that of classical negation.

The derivation of Line II

Popper establishes the second line of the following truth table of the conditional in the following way:

Line	a	b	$a \supset b$
I	T	T	T
II	T	F	F
III	F	T	T
IV	F	F	T

Using $a, a \supset b / b$ and assuming b to be false, either a or $a \supset b$ have to be false, hence if a is true then $a \supset b$ is false.

Intuitionist Implication

A Note on the Classical Conditional

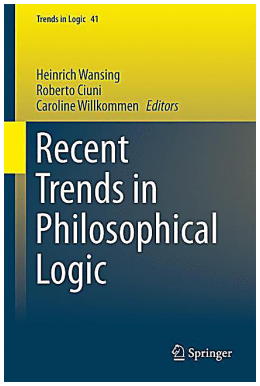
While contending that the truth table of the classical conditional does not, upon closer inspection, conflict with the usages of an „ordinary language“,[...], I am very ready to admit with the Intuitionists (Brouwer, Heyting) that „ordinary language“ usages involve us into difficulties when problems of infinity are involved, and that we may have to sacrifice classical negation, with its characteristic truth table [...].

Conclusion

Conclusion

- Popper used technical investigations in formal logic for genuine philosophical ends. E.g. the analysis of the difference between equality and identity, or between derivation and demonstration.
- He concerned himself with the important and fundamental problems in logic: logicality, weaker notions of negation and different concepts of implication: classical, intuitionistic and strict.
- He finds the formulations of logic based on axioms/primitive propositions unsatisfactory and tries to formulate logic in such a way that derivation is primary and demonstration secondary.

Popper on Logic



- Springer, Trends in Logic
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From the dynamic of informational time and space to objects and propositions : an Husserlian reading of Church and Gentzen

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Husserl's programme : to complete formal logic by a transcendental logic

- For Husserl, the objects and judgments of formal logic are only an idealization, an abstraction from the rational process (clarification, distinction, bringing signification to consciousness up to evidence) :
  - human reasoning only exists *in time*, as a process directed toward a target
  - along that process the targeted objects and judgments (*noèma*) are not real : not stable, not identical.
  - only the *acts* of consciousness composing the intentional *process* of thought itself (the *noèsis*) are real.
- To found and explain Logic, need to relate formal logic (idealized, artificial) to (real) reasoning, thus need to *complete Formal logic* by a theory of the events of the rational process.
- Such a theory aims to catch the objective part of the subjective rational process (so pretends to differ from psychology),
- How to reach this aim ? Which methodology ?

Husserl's methodology

- 1929 : Husserl, *Formal logic and Transcendental logic*
- Phenomenological method :
  - Suspension of our belief to the reality of objects and beings that appear to us (they are not independent to the only thing which is real : the rational process itself)
  - Investigate the objective part of the rational process namely the *possibility conditions* (transcendentalism) of the *transformations* of concepts/objects/propositions along the rational process
- Classical objections against husserlian phenomenology (as well as Intuitionism) : by focusing his idealism on *consciousness*, Husserl fails in his pretension to avoid psychology.
- To overcome that critic : concentrate on the minimal fact that the *noèsis* is an informational process, hence investigate the *possibility conditions* for a pure *transformation* of information to generate meaning.

Methodological alternatives toward the transcendental programme

In parallel (and almost simultaneously) to Husserl, two theoretical propositions put processuality on the center of the logical stage (without any resort to consciousness processuality) :

- A. Church : attempt to rebuild logic from pure processuality (lambda-calculus).
- G. Gentzen : focus on the process of "analytization" of proofs (normalization, cut-elimination)

- A theory of pure informational time :
- Church's Lambda-calculus (+/-1930 . . .)
  - Logical antinomies and the crisis of the set theoretical ontology
  - Refound ontology in space and time, starting from the notion of Function
  - A theory of Functional action presupposes a theory of informational processes
- "My" reading of Lambda-calculus :
  - an axiomatic for pure informational events
  - a transcendental approach of logical, mathematical ontology/semantics
  - a personal philosophical reading . . .

Criteria for the success of an ontological theory

1. Permanence
2. Separation (of individuals)
3. Variety of rôles (personalities, characters)

Classical objections against the project itself toward Ontology (F. Wolff)

- Heraclitean objection.
  - Being and becoming (Being is but a fiction).
  - In order to face this objection : becoming left open the possibility of invariants.
- Phenomenist objection.
  - We only know what appears to us (appearances cannot be overcome)
  - In order to face this objection : transcendentalism (possibility conditions of invariants appearance)
- Nominalist objection.
  - Unreasonable (arbitrary) power of langage.
  - In order to face this objection : refounding langage upon a proto-linguistical instance (actional efficiency of signs).

Toward a logico-mathematical ontology founded on time and space

1. Church approach of ontology targets the satisfaction of the three criteria (Permanence, Separation of individuals, Variety of rôles) and the overcoming the the three objections :
   - 1.1 In order to overcome the heraclitean objection keeping satisfying the three criteria :
     - 1.1.1 Start from processuality itself, trying to identifying their invariants
     - 1.1.2 Observe, beyond the identifications induced by forgetting the variance, which actions survive *separated*
     - 1.1.3 Among them, identify *rôles* (characters)
   - 1.2 Nominalist objection will be overcome :
     - 1.2.1 informational actions are proto-linguistical, before any signification (existentialism)
     - 1.2.2 but they will authorize the occasional emergence of signification
   - 1.3 Phenomenist objection will be overcome :
     - 1.3.1 trace back up to possibility conditions of signifiant informational processes
     - 1.3.2 point of departure : **elementary informational events, purely spatio-temporal**

What is Lambda calculus ?

- Explained to a mathematician ?

  An attempt to refound Logic from the notion of function (after antinomies in Cantorian/Fregean Logic)

- Explained to a computer scientist ?

  A programming language

- Explained to a philosopher ?

  A theoretical proposition to found the logical-mathematical ontology upon :
  - the idea of process
  - from a given approach of the idea of elementary informational events
  - which satisfies the three criteria (permanence, separation of individuals, variety of rôles)
  - and avoid the three objections (heraclitean, phenoménist, nominalist).

Process and Events

Church introduces espace and time, as such, in Logic :

- Informational processes (transformation of information) :
  - not through an external "treatment"
  - but as an internal transformation
- Space and Time : *events*

Events

Notion of event :

- Events and Space : events are always *situated*, they occur somewhere (*abstract* space versus *concrete* space)
- Events and Time : *relative* time versus *absolute* time

A. N. Whitehead, *An Enquiry Concerning The Principles Of Natural Knowledge*, Cambridge UP, Cambridge, 1919, p. 64.

> *« The essence of an object does not depend on its relations, which are external to its being. [. . .] its self-identity is not wholly dependent on its relations. But an event is just what it it, and is just how it is related ; and it is nothing else.*
> *Thus objects lacks the fixedness of relations which events possess, and then space and time could never be a direct expression of their essential relations [though fixedness is essential for events relations]. »*

Church's approach of the notion of event

- Questions about the notion of event and Church's answers :
  1. What will we consider as **an elementary informational event** ? Church's answer : the arrival of an information in a place (*variable*), i.e. substitution.
  2. How to describe a **complex situation** (context) where the event occurs ?
     - How to explicit that, in a given place, an information can be **involved in multiple contexts** ? Church's answer : By multiplying the occurrences of the same variable (each one being merged into a context).
     - So, a same information can be **involved with diferents rôles**.
     - Which "rôles" will we consider ? Church's answer : agent/patient = function/argument. Notation : $(t)u$ where t in *function* position and u in *argument* position.
- A lambda-calculus term (a "lambda-term") : a topologically complex representation of a given situation (context) in which some events may become (each event being then multiply involved and with differentiated rôles).
- A first example of a lambda-calculus term : $(x)(y)x$

Definition of lambda-calcul

1. Static (definition of lambda-terms)
   - variables : $x, y, z \ldots$
   - application : $(t)u$ (term t is applied to term u)
   - abstraction : $\lambda x\, t$ (the function which, to x, associates the term t)

   An example :
   $$(\lambda x \lambda y\ (y)(y)x)\lambda z\, z$$
2. Dynamic (definition of β-reduction)
   $$\ldots(\lambda x\, t)u \ldots \quad \rightsquigarrow \quad \ldots t[u/x] \ldots$$
3. The space in which events occur is an abstract space (η-equivalence) :
   - for bound variables : the name of places does'nt matter, only the difference of places does.
   - for free variables : different names cannot be identified (postulates about space : avoid ubiquity and avoid the possibility of two individuals staying simultaneously in the same place)

Recording the criteria for an ontology

1. Permanence
2. Separation of individuals
3. Variety of rôles (Personalities, Characters)

1. Permanence

- Individuals in time
- A simple example : arithmetical elementary computation
- Equivalence beyond transformations
- Confluence (Church-Rosser)
- The arithmetical example is too simple : in lambda-calculus the 'individuals" targeted are (more generally) *generic actors* (they perform generic actions : algorithms)

2. Separation of individuals

- External separability : computational consistency (identification "forced" by the dynamic is not too strong).
- External separability : computational consistency (identification "forced" by the dynamic is not too strong).
  - in particular, distinct places remain distinct : space is space
  - space is infinite : an infinite number of places
- Internal separability (Böhm's theorem)

3. Variety of rôles (Personalities, characters)

"Tell me how you act, I will tell who you are".

Various possible points of view on action : a description of any behavior may be more or less complete/partial and be made at various scales (depending on the "screen" chosen to measure the effects of the behavior) . . .

- "Character" : idea of a particular behavior. For example :
  - Identity : $\lambda x\, x$
  - Iterators : $\lambda x \lambda f\, (f)(f)(f)x$
  - Auto-applicator : $\lambda x\, (x)x$
- Effects upon the "extensional screen" (the set of normal terms = results, outputs)
- Common behavior (homogenous reaction to "tests" from a given set of tests) : notion of types (+/- formulas, propositions)

Other possible axiomatizations of time and space of processes

Today : beyond Church

1. New rôles : beyond function and argument (interaction versus action)
2. Non determinism
3. Extensions : Entanglement (Pablo Arrighi's talk), control
4. Decomposition of the "elementary" events (Linear Logic).
   A decomposition of substitution, i.e. of events happening :
   - Refined rôles (agent/patient) : multiplicative/additive
   - Aspectual distinction : perfective (accomplished) / imperfective
     - Non linear effects
     - Stratification (unvisible without the linear magnifying glasses) : a some given instant, an event happens *at a given level*.
     - Along time, informations may change of layer etc

Conclusion on objects (individuals) :

- transcendental approach
- plurality of axiomatics, plurality of ontologies

SECOND PART.
GENTZEN : FROM OBJECTS TO JUDGMENTS
(Next time ?)

DEUXIÈME PARTIE : DES OBJETS AUX JUGEMENTS

Logique transcendantale : des objets aux jugements

- Hilbert
  - Théorie de la démonstration
  - Les démonstrations/preuves comme textes
- Gentzen (+/-1930)
  - Les preuves dans le temps
  - Les différents états d'une preuve
  - L'évidence selon Gentzen : la vérité directement visible.
  - notion de preuve analytique (preuve où tout énoncé présent s'obtient comme sous-énoncé de l'énoncé prouvé),
  - en d'autres termes (moins linguistiques) :
    - une preuve où ne figure aucun concept qui ne vienne du problème résolu (où tout concept présent s'obtient en analysant le problème résolu) ;
    - une preuve sans abstraction extrinsèque : les justifications du théorème (i.e. l'argumentation) correspond directement à ce qui se *voit* dans l'énoncé prouvé.

L'évidentiatation selon Gentzen :

- analytiser une preuve = la transformer en une preuve analytique (du même théorème)
- l'évidentiation = le processus d'analytisation des preuves.

Resultat fondamental de Gentzen : toute preuve en logique du premier ordre peut être convertie en une preuve analytique (du même théorème). Gentzen :

- définit un processus de transformation des preuves,
- démontre que ce processus termine toujours,
- et qu'il se termine en produisant une preuve analytique (de même conclusion).

Dans la littérature logique, la dénomination usuelle de ce processus d'évidentiation diffère selon les systèmes de représentation de preuves considérés :

- "normalisation des preuves" en *Deduction naturelle*
- "élimination des coupures" en *Calcul des séquents*

Évidence hors de la logique du premier ordre ?

En dehors de la logique du premier ordre :

- nous perdons la "prédicativité" (Poincaré), en sorte qu'*analytique* ne peut conserver le même sens.
- l'analytisabilité des preuves n'est plus garantie.
- mais tentatives récentes pour dégager une notion plus subtile d'evidence : la généricité (cf. Paolo Pistone)

Processus d'évidentiation et processus informationnel

- 1969 : "Correspondance Preuves-Programmes" (Curry-Howard)
- L'évidentiation n'est qu'un cas particulier de processus informationnel
- Expliquer la logique à partir de l'espace-temps

Calculus of Natural Calculations

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Motivation

1. *Gentzen:* I intended first to set up a formal system which comes as close as possible to actual reasoning. The result was a '*calculus of natural deduction*'.

   (The Collected Papers of Gerhard Gentzen, p. 68)

   The technically convenient Sequent Calculus is introduced only after Natural Deduction.
2. *relevance:* The advantage of such natural calculi:
   - *safety:* The inevitable gap between informal reasoning and its formalisation is minimised. We obtain safety about the formalisation.
   - *philosophy:* Formal investigation of informal properties of proofs is philosophically justified. We may investigate natural reasoning (in contrast to technically convenient reasoning).

Motivation

1. *Natural Deduction:* Is pretty close to informal argumentations with respect to reasoning with statements.
2. *reality:* But mathematicians do not only argue with statements, they also calculate in proofs with mathematical objects.
   Standard formalisation of calculation via equality statements is not natural.
3. *Natural Calculation:* Omit the standard rules for equality statements and introduce term rules which allow to calculate with terms.

Calculations in Proofs

Informal Mathematical Reasoning with Terms

1. *calculation:* $t_1 \overset{\text{(Justification)}}{=} t_2 \overset{\text{(Justification)}}{=} t_3 \overset{\text{(Justification)}}{=} \dots t_n$
2. *evaluation:* By the transitivity of equality, we conclude $t_1 = t_n$.

Example: induction step for proving $n + 0 = 0 + n$

1. *announcement:* We have to show $S(n) + 0 = 0 + S(n)$.
2. *calculation:*

$$S(n) + 0 \overset{(x+0=x)}{=} S(n) \overset{(x+0=x)}{=} S(n+0) \overset{(IH)}{=} S(0+n) \overset{(x+S(y)=S(x+y))}{=} 0 + S(n)$$

3. *evaluation:* By transitivity of equality, $S(n) + 0 = 0 + S(n)$.

Calculations in Proofs

Informal Mathematical Reasoning with Terms

1. *calculation:* $t_1 \overset{\text{(Justification)}}{=} t_2 \overset{\text{(Justification)}}{=} t_3 \overset{\text{(Justification)}}{=} t_4$
2. *evaluation:* By the transitivity of equality, we conclude $t_1 = t_4$.

Usual Formalisation (Natural Deduction)

$$\dfrac{\dfrac{\dfrac{\text{Reason}}{t_1 = t_2} \quad \dfrac{\text{Reason}}{t_2 = t_3}}{t_1 = t_3} \quad \dfrac{\text{Reason}}{t_3 = t_4}}{t_1 = t_4}$$

Calculations in Proofs

Informal Mathematical Reasoning with Terms

1. *calculation:* $t_1 \overset{\text{(Justification)}}{=} t_2 \overset{\text{(Justification)}}{=} t_3 \overset{\text{(Justification)}}{=} t_4$
2. *evaluation:* By the transitivity of equality, we conclude $t_1 = t_4$.

Alternative Formalisation (Natural Deduction)

$$\dfrac{\dfrac{\text{Reason}}{t_1 = t_2} \quad \dfrac{\dfrac{\text{Reason}}{t_2 = t_3} \quad \dfrac{\text{Reason}}{t_3 = t_4}}{t_2 = t_4}}{t_1 = t_4}$$

Calculations in Proofs

Informal Mathematical Reasoning with Terms

1. *calculation:* $t_1 \overset{\text{(Justification)}}{=} t_2 \overset{\text{(Justification)}}{=} t_3 \overset{\text{(Justification)}}{=} t_4$
2. *evaluation:* By the transitivity of equality, we conclude $t_1 = t_4$.

Intended Formalisation - Calculation

$$(=)\ \frac{t_1 \qquad \text{Justification}}{(=)\ \dfrac{t_2 \qquad \text{Justification}}{(=)\ \dfrac{t_3 \qquad \text{Justification}}{t_4}}}$$

Calculations in Proofs

Informal Mathematical Reasoning with Terms

1. *calculation:* $t_1 \overset{\text{(Justification)}}{=} t_2 \overset{\text{(Justification)}}{=} t_3 \overset{\text{(Justification)}}{=} t_4$
2. *evaluation:* By the transitivity of equality, we conclude $t_1 = t_4$.

Intended Formalisation - Evaluation

$$(=)\ \frac{[t_1]^1 \qquad \text{Justification}}{(=)\ \dfrac{t_2 \qquad \text{Justification}}{(=)\ \dfrac{t_3 \qquad \text{Justification}}{\dfrac{t_4}{t_1 = t_4}\,(1)}}}$$

Analysis: Natural Aspects

1. *manipulation of terms:* Calculations with terms and not only argumentations with formulae.
2. *no redundant statements:* No explicit formulation of redundant statements as $t_1 = t_3$ or $t_2 = t_4$.
3. *linearity & extendability:* Preservation of the linear character of calculations and of the possibility to extend calculations.
4. *evaluation of calculations:* The argumentation step of evaluating the calculation (concluding that $t_1 = t_4$) is explicitly formalised.

Formal Introduction of Natural Calculations

The calculus of Natural Calculations is an extension of Natural Deduction:

- with the usual rules for connectives and quantifiers
  (for classical as well as for intuitionistic logic)
- without the standard rules for identity
- new rules for the treatment of terms

New Rules - Atomic Derivation, Equality Statements

1. *new atomic derivation:* Every term t is a derivation.

2. *justified calculation (positive ; negative):* (Elimination of $=$)

$$(E=)\ \frac{r(t) \qquad t = s}{r(s)} \quad ; \quad (E=)\ \frac{s = t \qquad r(t)}{r(s)}$$

   Substitution of some occurrences of t in r.

   The equations are called direct justification of the inference step, the subtrees above justification.

3. *evaluation:* (Introduction of $=$)

$$(I=)\ \frac{\begin{array}{c}[t]\\ \vdots\\ s\end{array}}{t = s}$$

   The discharge of t is mandatory.

Remarks

1. *discharging terms:* We discharge the upmost term in a calculation to indicate that the calculation is finished and the result is used.
2. *negative rules:* We formulated negative rules to reflect the symmetry of equality.
3. *substructural rules:* We allow the substructural manipulation of terms.
4. *left side labelling:* We label the calculation steps on the left side. This allows to distinguish them easily from traditional inference steps.

New Rules - Auxiliary Calculations

Some more rules needed:

4. *auxiliary calculation - term (positive ; negative):*

$$(\mathsf{A}_{\mathsf{trm}})\ \frac{r(t) \qquad \begin{matrix}[t]\\ \hline\hline s\end{matrix}}{r(s)} \quad ; \quad (\mathsf{A}_{\mathsf{trm}})\ \frac{\begin{matrix}[s]\\ \hline\hline t\end{matrix} \qquad r(t)}{r(s)}$$

   Substitution of some occurrences of t in r.

5. *auxiliary calculation - formula (positive ; negative):*

$$(\mathsf{A}_{\mathsf{fml}})\ \frac{\phi(t) \qquad \begin{matrix}[t]\\ \hline\hline s\end{matrix}}{\phi(s)} \quad ; \quad (\mathsf{A}_{\mathsf{fml}})\ \frac{\begin{matrix}[s]\\ \hline\hline t\end{matrix} \qquad \phi(t)}{\phi(s)}$$

   Substitution of some occurrences of t in r.

   The side calculations are called justification of the inference step.

An Example

We illustrate the Calculus of Natural Calculations by an example:

- Addition with 0 is commutative in Peano Arithmetics (PA).

  Formally: $PA \vdash \forall x.x + 0 = 0 + x$

We focus on the formal calculations; the partial derivations are simplified.

Example: Commutativity of Addition with 0 in PA

Statement: $A_1, A_2, IS \vdash \forall x : x + 0 = 0 + x$

- Proof by induction on x.
- Infer $\phi(0) \doteq 0 + 0 = 0 + 0$.

$$(I=)\ \frac{[0+0]}{0+0=0+0}$$

- Assume IH $\doteq \phi(x) \doteq x + 0 = 0 + x$; infer $\phi(S(x)) \doteq S(x) + 0 = 0 + S(x)$:

$$(I=)\ \frac{(E=)\ \dfrac{(E=)\ \dfrac{A_2}{S(0+x)=0+S(x)}\quad \text{IH}}{S(x+0)=0+S(x)}}{S(x)+0=0+S(x)}\qquad (E=)\ \frac{(E=)\ \dfrac{[S(x)+0]}{S(x)}\quad A_1}{A_1}$$

$A_1 : \forall x.\, x + 0 = x \quad ; \quad A_2 : \forall xy.\, x + S(y) = S(x + y)$

IS = induction schema

Example: Commutativity of Addition with 0 in PA

Statement: $A_1, A_2, IS \vdash \forall x : x + 0 = 0 + x$

We have:

- $A_1, A_2 \vdash \phi(0)$
- $A_1, A_2, \phi(x) \vdash \phi(S(x))$

By an application of induction schema, we easily obtain:

- $PA \vdash \forall x.x + 0 = 0 + x$

□

$A_1 : \forall x.\, x + 0 = x \quad ; \quad A_2 : \forall xy.\, x + S(y) = S(x + y)$

IS = induction schema

Improved Substitution

Before discussing the properties of Natural Calculations, we investigate the notation of substitution:

1. The disadvantage of the traditional notation.
2. Introduction of nominal forms and of a general substitution function.
3. Improved notation of substitution via elimination forms.

Traditional Substitution

1. *traditional notation:* We used the traditional notation $r(t) / r(s)$ and $\phi(t) / \phi(s)$ for substitution. (marking t / replacing t by s)
2. *problems:* There are some drawbacks as there is no syntactic entity reflecting the position of the involved terms:
   - Dependence on the order of reading.
   - If we replace different occurrences of t twice by s in r, we obtain $r(s) \neq r(s)$.
   - We have to understand informal restrictions on the occurrences (as some occurrences of t, all (free) occurrences of t, exactly one occurrence of t, etc.)
3. *improvement:* We improve the notation of substitution with the help of nominal forms. (Idea due to Schütte.)

Introduction of Nominal Forms

Nominal forms are a generalisation of terms and formulae:

1. *nominal symbols:* The alphabet is extended by countable many nominal symbols $*_k$ ($k \in \omega$). $*$ abbreviates $*_0$.
2. *nominal terms:* Term generation is extended by a new rule:
   - $*_k$ is atomic nominal term.

   Metavariables: $\mathtt{t}$, $\mathtt{s}$ etc.

   Example: $*_5 + (*_7 + 6)$, $S(*)$, $5 + x$
3. *nominal formulae:* Generated according to the usual rules, but with respect to nominal terms.

   Metavariables: $\mathtt{A}, \mathtt{B}$ etc.

   Caveat: $\forall * .\mathtt{A}$ is not a nominal formula. (As $*$ represents here variables and not arbitrary terms.)

General Substitution Function

We define a general substitution function with respect to nominal terms and nominal formulae:

1. *nominal terms:* $\mathtt{t}[\mathtt{t}_0, \ldots \mathtt{t}_n]$ is the result of the simultaneous replacement of all $*_k$ by $\mathtt{t}_k$ in $\mathtt{t}$. (recursive definition)

   Example: $*_1 + (*_1 + *_2)[5, *] \eqsim * + (* + *_2)$, $S(*)[5 + *] \eqsim S(5 + *)$, $5 + x[*_1 + *_2] \eqsim 5 + x$
2. *nominal formulae:* $\mathtt{A}[\mathtt{t}_0, \ldots \mathtt{t}_n]$ is the result of the simultaneous replacement of all $*_k$ by $\mathtt{t}_k$ in $\mathtt{A}$. (recursive definition)

   In the second argument, we still have a sequence of nominal terms!

Elimination Form

An elimination form is the result of eliminating some occurrences of a term in a term or formula and replacing them by suitable nominal symbols.

1. *standard terms:* A nominal term $\mathtt{t}$ is an elimination form of a term t, if there is a sequence $t_0, \dots t_n$ such that $\mathtt{t}[t_0, \dots t_n] \mathrel{\widehat{=}} t$.
   (We assume that $*_0, \dots *_n$ all occur in $\mathtt{t}$.)
   We say that the terms $t_0, \dots t_n$ are eliminated in $\mathtt{t}$.
2. *example:* $* + *$ is an elimination form of $0 + 0$, but not of $0 + 1$; the term 0 is eliminated.
   $*_0 + *_1$ is an elimination form of $0 + 0$ in which 0 and 0 are eliminated; but also of $0 + 1$ with 0 and 1 eliminated.
3. *standard formulae:* analogously

Remarks (Elimination Form)

1. *purpose:* The purpose of elimination forms is to represent the position of occurrences of terms in terms or formulae.
2. *advantage:* There is a syntactic entity corresponding with the intended substitution. Different occurrences are represented by different elimination forms.
3. *restrictions:* In particular, we can explicitly formulate the restrictions on a substitution.
   - *some occurrences of t:* arbitrary elimination form, in which t is eliminated (the only nominal symbol is $*$, occurs at least once).
   - *all (free) occurrences of t:* as before, but no (free) occurrences of t in the elimination form.
   - *exactly one occurrence of t:* as before, but there occurs only one nominal symbol in the elimination form.

Improved Rules

We illustrate the formulation of the rules with the help of nominal forms.

1. *formal notation:* justified calculation (positive ; negative)

$$(E{=})\ \frac{\mathtt{r}[t] \qquad t = s}{\mathtt{r}[s]} \quad ; \quad (E{=})\ \frac{s = t \qquad \mathtt{r}[t]}{\mathtt{r}[s]}$$

   $\mathtt{r}$ is a proper, unary nominal term.

2. *characterisation:* We may calculate from a term r to a term r', if there is a proper nominal term $\mathtt{r}$ such that $\mathtt{r}[t] \mathrel{\hat{=}} r$ and $\mathtt{r}[s] \mathrel{\hat{=}} r'$.

Structural Results

Some proof theoretical results about Natural Calculations

- Introduction of calculations and dual calculations.
- Redundancy of negative auxiliary calculations.
- Variants of calculations and the linearisation of calculations.

Useful Terminology

- A calculation **C** is a derivation starting with the introduction of a term and consisting of some calculation steps; together with their direct justifications (equality statements or auxiliary calculations); ends with a term.

⇝ Useful to investigate the new possibilities of the calculus without considering the proof trees (of formulae) allowing to infer the direct justifications.

- *result:* The result **Res**(**C**) of a calculation **C** is the equality statement $t = s$ such that t is the primary term and s is the final term in the calculation.

⇝ In an evaluation step, we discharge the primary term t and introduce the result of the calculation.

Calculations in Derivations

- *calculation in a derivation:* We can identify a calculation inside an arbitrary derivation.

$$\begin{array}{c} \mathbf{D}_0 \dots \mathbf{D}_{n-1} \\ \mathbf{C} \\ \mathbf{F} \end{array}$$

- *remaining derivation:* Due to the side conditions of the introduction of the universal quantifier, **F** alone is, in general, not a derivation. (Dependence on the subderivations!)
- *independence of calculations:* **C** is independent of the subderivations.
- *same justification:* If $\mathbf{C}'$ has the same direct justifications (or less) and the same result as **C**, then we may exchange them without loss.

Dual Calculations

- *dual calculation:* A dual calculation $\mathbf{C}^d$ of a calculation $\mathbf{C}$ begins with the final term of $\mathbf{C}$, calculates in the inverse direction, and uses the same direct justifications.
- *proposition:* Every calculation has a dual calculation; the result of the calculation is inverted.
- *proof:* Calculate bottom up and switch the positions of the justifications. For example:

$$\frac{\mathtt{r}[t] \qquad t = s}{\mathtt{r}[s]} \quad \rightsquigarrow \quad \frac{t = s \qquad \mathtt{r}[s]}{\mathtt{r}[t]}$$

- *observation:* We use the same nominal term $\mathtt{r}$ for characterising the calculation steps (independent of the order of reading).

Consequence (Dual Calculations)

- *consequence:* Negative auxiliary calculations (term and formulae) are redundant.
- *proof:* Exchange negative auxiliary calculation steps with positive calculation steps using the dual side calculation. For example:

$$\begin{array}{c} [s] \\ \mathbf{C} \\ t \qquad \mathtt{r}[t] \end{array}_{(\mathsf{A}_{\mathsf{trm}})} \quad \rightsquigarrow \quad \begin{array}{c} [t] \\ \mathbf{C}^d \\ \mathtt{r}[t] \qquad s \end{array}_{(\mathsf{A}_{\mathsf{trm}})}$$

$$\mathtt{r}[s] \qquad\qquad\qquad\qquad \mathtt{r}[s]$$

- *termination:* While transforming $\mathbf{C}$ into $\mathbf{C}^d$, the positions of justifications switch, but nevertheless strong normalisation and confluence.
- *attention:* we still need negative justified calculations and positive auxiliary calculations!

Variants of Calculations

- $\mathtt{t}$-*variant:* If $\mathtt{t}$ is a proper unary nominal term, then $\mathtt{t}(\mathbf{C})$ is a calculation using the same side calculations and direct justifications, but in which all terms t in the main path are replaced by $\mathtt{t}[t]$.
- *proposition:* Every $\mathtt{t}$-variant of a calculation $\mathbf{C}$ is a calculation; if $t = s$ is the result of $\mathbf{C}$, then $\mathtt{t}[t] = \mathtt{t}[s]$ is the result of $\mathtt{t}(\mathbf{C})$.
- *proof:* We have to check each calculation step. For example:

$$(E{=})\ \frac{\mathtt{r}[t] \qquad t = s}{\mathtt{r}[s]} \quad \rightsquigarrow \quad (E{=})\ \frac{\mathtt{t}[\mathtt{r}[t]] \qquad t = s}{\mathtt{t}[\mathtt{r}[s]]} \quad \rightsquigarrow \quad (E{=})\ \frac{\mathtt{t}[\mathtt{r}][t] \qquad t = s}{\mathtt{t}[\mathtt{r}][s]}$$

Linear Calculations

- *consequence:* We may transform every calculation into a linear calculation (without auxiliary calculations with respect to terms).
- *proof:* It is sufficient to investigate positive auxiliary steps (term), as negative auxiliary steps are already eliminated.

$$(\mathsf{A}_{\mathsf{trm}})\ \frac{\mathtt{r}[t] \qquad \begin{matrix}[t]\\ \mathbf{C}\\ s\end{matrix}}{\mathtt{r}[s]} \quad \rightsquigarrow \quad \begin{matrix}\mathtt{r}[t]\\ \mathtt{r}(\mathbf{C})\\ \mathtt{r}[s]\end{matrix}$$

- *attention:* we still need positive auxiliary calculations for formulae, but again these may be linear.

Proof Theoretical Results

Presupposing that every calculation is linear, we easily prove some standard proof theoretical results:

- *Completeness:* The calculus of Natural Calculations is sound and complete with respect to classical and intuitionistic logic.
- *Normalisation:* We may extend the normalisation results provided by Pravitz.

Soundness

- *soundness of calculations:* The result of every linear calculation is provable with the help of standard identity rules under the assumption of the direct justifications.
- *proof:* By induction over the length of the calculation.
- $n = 0$*:*

$$t_0 \quad \rightsquigarrow \quad \frac{}{t_0 = t_0}$$

- $n + 1$*:*

$$\frac{\begin{array}{c} t_0 \\ \hline\hline \mathtt{r}[s_0] \end{array} \qquad s_0 = s_1}{\mathtt{r}[s_1]} \quad \rightsquigarrow \quad \frac{\begin{array}{c} (\mathrm{IH}) \\ \hline\hline t_0 = \mathtt{r}[s_0] \end{array} \qquad s_0 = s_1}{t_0 = \mathtt{r}[s_1]}$$

With $t_n \mathrel{\widehat{=}} \mathtt{r}[s_0]$, and $t_{n+1} \mathrel{\widehat{=}} \mathtt{r}[s_1]$.

Soundness

- *soundness:* The calculus of Natural Calculations is sound with respect to classical and intuitionistic logic.
- *proof:* Replace every linear calculation by a standard derivation; in the case of auxiliary calculations for formulae, we obtain this way the standard rule of subsitutivity (with respect to formulae).
- *remark:* The proof does not depend on the rules for formulae; therefore, with respect to classical and intuitionistic logic.

Completeness

- *completeness:* The calculus of Natural Calculations is complete with respect to classical and intuitinistic logic.
- *proof:* Every usual identity rule of Natural Deduction may be transformed into a proof schema using the rules of Natural Calculations.
  For example:

$$\frac{t = s}{s = t} \quad \rightsquigarrow \quad \frac{\dfrac{t = s \qquad [s]}{t}}{s = t}$$

Maximal Formulae

- *maximal formulae:* An occurrence of an equality statement, which is introduced in the last inference step and eliminated in the next inference step, is a new kind of maximal formula. They may be eliminated trivially.
- *schematically:* The positive case:

$$\dfrac{\mathfrak{r}[t] \qquad \dfrac{\overset{[t]}{\overline{\overline{s}}}}{t = s}\,(I)}{\mathfrak{r}[s]}\,(E) \quad \rightsquigarrow \quad \dfrac{\mathfrak{r}[t] \qquad \overset{[t]}{\overline{\overline{s}}}}{\mathfrak{r}[s]}$$

- *linearisation:* After eliminating the maximal formula, we have to linearise the calculation.

Auxiliary Calculation - Formulae

We investigate auxiliary calculations for formulae.

- *hidden cuts:* An auxiliary calculation may hide a standard cut: introduction rule $\rightsquigarrow$ Auxiliary calculation $\rightsquigarrow$ elimination rule.
- *movability:* The auxiliary calculation for formulae may be moved in the derivation almost freely. Thus, we may eliminate the hidden cuts.
- *standard normalisation:* The standard normalisation procedures may be applied to the subtrees consisting only of formulae.

Auxiliary Calculation - Formulae

Towards normalisation:

- *intended position:* We can move auxiliary calculations between elimination and introduction parts.
- *atomic application:* We even may restrict auxiliary calculations to atomic formulae and avoid equations.
- *not substructural:* As we may use t-variants of the side calculations, we may restrict auxiliary calculations to affect only whole terms.
- *subsequent applications:* We may merge subsequent side calculations with respect to the same term into one side calculation.
- *independent calculations:* If P is n-ary predicate symbol, then n side calculations (at most) are sufficient, affecting the n terms of $P(t_0, \ldots t_{n-1})$.

Normalisation

- *calculations:*
  - The direct justifications are atomic conclusion of an elimination part of a derivation.
  - The result is the upmost atomic formula of an introduction part of a derivation.
  - Alternatively, the calculation is integrated in the upmost formula of an introduction part.
- *schematically:*

$$(t)\ \frac{\overset{E_0}{\vee}\ \cdots\ \overset{E_{n-1}}{\vee}}{t = s}\ (s) \qquad ; \qquad \frac{\overset{E}{\vee}\ P(t) \qquad (t)\ \frac{\overset{E_0}{\vee}\ \cdots\ \overset{E_{n-1}}{\vee}}{\vee}\ (s)}{P(s)}$$

Subatomic Normalisation

- *inside calculations:* The inner structure of calculations needs an own investigation with respect to normalisation.
- *linearity:* We already have proved that we can linearise any calculation.
- *other properties:* Are there other nice structural results to be achieved?

Subatomic Normalisation

- *problem:* There is no previously given order of the calculation steps in a calculation, as the order depends on what we want to calculate and which justifications we have.
- *example:*
  - If we have $a = f(c)$ and $f(c) = b$, we may calculate that $a = b$ and the complex term is in the middle of calculations.
  - If we have that $a = f(b)$ and $a = f(c)$ we may calculate that $f(b) = f(c)$ and the simple term is in the middle of calculation.
  - If we want to calculate $b = a$ and $f(c) = f(b)$, respectively, we have to use the dual calculations in which the order of the terms is inverted.

Subatomic Normalisation

- *problem:* The order of independent calculations may be exchanged.
- *example:*

$$\frac{(1+2)+(3+4)}{\dfrac{3+(3+4)}{3+7}} \quad \Longleftrightarrow \quad \frac{(1+2)+(3+4)}{\dfrac{(1+2)+7}{3+7}}$$

Subatomic Normalisation

- *result:* Circular calculations may be eliminated. Thus, if a term t occurs twice in a calculation, we may eliminate the calculation steps between them.
- *loss of confluence:* If we calculate from a to b, back to a and then again to b, we may chose which steps to eliminate. But the two calculations from a to b may be different.

More Calculations

- shed light on further natural extensions of the calculus

Excursus: Reflection of Properties

The properties of identity are reflected by our term rules:

- *reflexive:* Immediate introduction of the equality.
- *symmetry:* Positive and negative calculation steps.
- *transitivity:* Extendability of calculations.
- *congruence (compatibility with function symbols):* Arbitrary substructural manipulation of terms in calculation steps.

Can we formalise other properties and thereby introduce rules for other relation symbols?

Smaller-Than Relation

- *smaller than:* Assuming arithmetic with the relation symbols $\leq$ and $<$ we may allow positive, substructural "smaller-than"-calculations.
- *example:*

$$(\leq)\ \frac{(=)\ \dfrac{[x+x] \qquad x \leq x+x}{x+(x+x)} \qquad x+x=2x}{(<)\ \dfrac{x+2x \qquad\qquad 2x<2x+1}{\dfrac{x+(2x+1)}{x+x\leq 2x+1}\,(I\leq)}}$$

Technical Comments

- *elimination rules:* We extend the calculus with new elimination rules with respect to new binary relation symbols.
- *introduction rules:* Different introduction rules for all relevant relation symbols.
- *example:* In a calculus with $=$, $<$ and $\leq$:
  - Introduce $=$, if all calculation steps are $=$-calculations.
  - Introduce $<$, if at least one $<$-calculation step was done.
  - Introduce $\leq$ after arbitrary calculations.
- *truly global rules:* Introduction rules depend on all calculation steps, not only on the upmost premise and the conclusion!

More Complications

- *calculating with integers:* If we deal with integers (in contrast to natural numbers), then we have to introduce negative and positive positions in a term (via nominal forms) and adapt the calculation rules:
  1. Positive calculation steps only with respect to positive positions in a term.
  2. Negative calculation steps with respect to negative positions in a term.
- *example:*

$$\dfrac{\dfrac{2}{5-3} \quad 5<6}{\dfrac{6-3}{3}} \quad ; \quad \dfrac{2<3 \quad \dfrac{2}{5-3}}{\dfrac{5-2}{3}}$$

More Complications

- *inequality:* If we allow $\neq$-calculation steps, then the introduction rule for $\neq$ demands that only one such calculation step is done (arbitrary $=$-calculations).
- *non-sense calculations:* Non-sense calculations are possible (more than one $\neq$-step), but no suitable introduction rules for them.

Non-Standard Calculations

- *side conditions:* calculations presupposing $\in$-transitivity:

$$\dfrac{\dfrac{\dfrac{[t] \qquad t \in s}{s} (\in) \qquad s \in r}{r} (\in) \qquad \in -\mathrm{trans}(r)}{t \in r}$$

- *circular calculations:* circular calculation schema:

$$\dfrac{\dfrac{\dfrac{\dfrac{[a] \qquad a \subseteq b}{b} \qquad b \subseteq c}{c} \qquad c \subseteq a}{a}}{a = c}$$

- *philosophical example:* calculations presupposing side conditions:

$$\dfrac{\dfrac{\dfrac{[\text{Adam}] \qquad \text{B(Adam, Bertram)}}{\text{Bertram}} \qquad \text{B(Bertram, Caesar)}}{\text{Caesar}} \qquad \text{Adam} \neq \text{Caesar}}{\text{B(Adam, Caesar)}}$$

Integration of Meta Argumentations

- *bridge rules:* integration of meta argumentations via the following brigde rules:

$$\dfrac{\dfrac{[\phi_0], \ldots [\phi_{n-1}]}{\psi}}{\phi_0, \ldots \phi_{n-1} \Rightarrow \psi} \quad ; \quad \dfrac{\phi_0 \quad \ldots \quad \phi_{n-1} \quad \vec{\phi} \Rightarrow \psi}{\psi}$$

- *sequent calculus:* Further extension of Natural Deduction by the sequent calculus rules appears as a natural extension for dealing with meta statements.
- *more rules:* More meta logical rules possible as in case of calculations.

Conclusion

1. *Natural Calculation:* We have seen an extension of Natural Deduction allowing to calculate with terms and providing a natural formalisation of argumentations dealing with equalities.
2. *structural results:* We have seen some proof theoretical results: completeness with respect to classical and intuitionistic logic, normalisation on the level of formulae. Additionally: problems of normalisation on the subatomic level.
3. *more calculations:* We have seen that the presented framework is able to formalise even more kinds of calculations and argumentation schemata found in every day mathematics.

Last Slide

Thank You!

Decidable logics, the axiom rule, and (non) cut-elimination

Gilles Dowek
(joint work with Ying Jiang)

Two kinds of logics

Undecidable and decidable ones

The axiom rule (hypothetical deduction) makes the difference

But introduction rule, automaton, Curry-(de Bruijn)-Howard correspondence, cut, cut elimination common concepts
If apropriately generalized

I. Introduction rules and automata, in general

Introduction rule

Given a well-founded order $\prec$

$$\frac{s_1 \ \ldots \ s_n}{s'}$$

an introduction rule if $s_1 \prec s'$, ..., $s_n \prec s'$

Examples

$$\frac{even(x)}{odd(a(x))}$$

$$\frac{A \quad B}{A \wedge B}\ \wedge\text{-intro}$$

$$\frac{A \wedge B}{A}\ \wedge\text{-elim}$$

Automaton

A (finite in conclusions) inference system containing introduction rules only

Provability decidable: finite search space

An example

$$\frac{even(x)}{odd(a(x))} \qquad \frac{odd(x)}{even(a(x))} \qquad \frac{}{even(\varepsilon)}$$

$odd(a(a(a(\varepsilon))))$ provable
$even(a(a(a(\varepsilon))))$ not provable in *even*

An example

$$\frac{even(x)}{odd(a(x))} \qquad \frac{odd(x)}{even(a(x))} \qquad \frac{}{even(\varepsilon)}$$

$odd \xrightarrow{a} even$ $\qquad$ $even \xrightarrow{a} odd$ $\qquad$ *even* final

$odd(a(a(a(\varepsilon))))$ provable: *aaa* recognized in *odd*
$even(a(a(a(\varepsilon))))$ not provable: *aaa* not recognized in *even*

II. The Curry-(de Bruijn)-Howard correspondence, in general

How to represent a finite state automaton?

$odd \xrightarrow{a} even$	$even \xrightarrow{a} odd$	$even$ final
$\dfrac{even(x)}{odd(a(x))}$	$\dfrac{odd(x)}{even(a(x))}$	$\dfrac{}{even(\varepsilon)}$
$\dfrac{even}{odd}\, a$	$\dfrac{odd}{even}\, a$	$\dfrac{}{even}\, \varepsilon$

Proofs labeled with propositions, rules names, and both

$$\dfrac{\dfrac{\dfrac{\dfrac{}{even}}{odd}}{even}}{odd} \qquad \begin{array}{l} \varepsilon \\ a \\ a \\ a \end{array} \qquad \dfrac{\dfrac{\dfrac{\dfrac{}{even}\,\varepsilon}{odd}\,a}{even}\,a}{odd}\,a$$

proof-checking still decidable

conclusion still computable (*)

(*) determinism: otherwise being a conclusion decidable

A linear notation for proofs labeled with propositions

$$\begin{array}{l} \varepsilon \\ a \\ a \\ a \end{array}$$

$a(a(a(\varepsilon)))$
Proof-term
Being a conclusion decidable: $a(a(a(\varepsilon))) : odd$ decidable

The set of pairs $\pi : A$ such that π ~~has type A~~ is a linear representation of a proof of A decidable

Why?

Transform each rule

$$\frac{s_1 \ \ldots \ s_n}{s'} f$$

into

$$\frac{\pi_1 : s_1 \ \ldots \ \pi_n : s_n}{f(\pi_1, \ldots, \pi_n) : s'}$$

A automaton that proves exactly the pairs $\pi : A$ such that π is a linear representation of a proof of A

This automaton: type-checking algorithm of the inference system

Inductively defined set: projection of a decidable set

An example

$$\frac{even}{odd}\, a \qquad \frac{odd}{even}\, a \qquad \frac{}{even}\, \varepsilon$$

$$\frac{x : even}{a(x) : odd} \qquad \frac{x : odd}{a(x) : even} \qquad \frac{}{\varepsilon : even}$$

$$\frac{even(x)}{odd(a(x))} \qquad \frac{odd(x)}{even(a(x))} \qquad \frac{}{even(\varepsilon)}$$

Another example

$$\frac{A \wedge B}{A}\ fst \qquad \frac{A \wedge B}{B}\ snd \qquad \frac{A \Rightarrow B \quad A}{B}\ app \qquad \frac{}{P \wedge (P \Rightarrow Q)}\ c$$

$$\frac{\pi : A \wedge B}{fst(\pi) : A} \qquad \frac{\pi : A \wedge B}{snd(\pi) : B} \qquad \frac{\pi_1 : A \Rightarrow B \quad \pi_2 : A}{app(\pi_1, \pi_2) : B} \qquad \frac{}{c : P \wedge (P \Rightarrow Q)}$$

$$\dfrac{\dfrac{\dfrac{}{P \wedge (P \Rightarrow Q)}\ c}{P \Rightarrow Q}\ snd \qquad \dfrac{\dfrac{}{P \wedge (P \Rightarrow Q)}\ c}{P}\ fst}{Q}\ app$$

$$app(snd(c), fst(c)) : Q$$

Curry-(de Bruijn)-Howard correspondence

(minus trivial details about bound variables specific to Natural deduction)

III. Cuts, in general

(General) cuts

$$\dfrac{\dfrac{\cdots}{s_1}\ R_1\ \text{(intro)} \quad \ldots \quad \dfrac{\cdots}{s_n}\ R_n\ \text{(intro)}}{s'}\ R'\ \text{(non-intro)}$$

π cut-free: no cuts in π

An inference system has the cut-elimination property if every proof can be transformed into a cut-free proof

A theorem

A cut-free proof contains introduction rules only

Induction over proof structure

$$\dfrac{\dfrac{\pi_1}{s_1} \quad \ldots \quad \dfrac{\pi_n}{s_n}}{s'}\ R'$$

π_1, ..., π_n: introduction rules only
π_1, ..., π_n end with introduction rules
R': introduction rule

A corollary

In an inference system that has the cut-elimination property provability is decidable

Drop the non-introduction rules, preserving provability

An automaton

IV. Finite domain logic

Finite domain logic

Natural deduction tailored to prove the propositions valid in a given finite model $\mathcal{M}$

- a constant for each element in the model (and no other function symbols)
- $A \Rightarrow B$ abbreviation for $\neg A \vee B$
- negation pushed to atomic propositions (using de Morgan's laws)

- $\forall$-intro and $\exists$-elim rules replaced by enumeration rules (ω-rules?)

$$\frac{\Gamma \vdash (c_1/x)A \quad \ldots \quad \Gamma \vdash (c_n/x)A}{\Gamma \vdash \forall x\ A}\ \forall\text{-intro}$$

$$\frac{\Gamma \vdash \exists x\ A \quad \Gamma, (c_1/x)A \vdash C \quad \ldots \quad \Gamma, (c_n/x)A \vdash C}{\Gamma \vdash C}\ \exists\text{-elim}$$

- atom rule

$$\frac{}{\Gamma \vdash L}\ \text{atom if } L \in \mathcal{P}$$

  $\mathcal{P}$ finite set containing P or $\neg P$, for each closed atomic P
- no rules for implication and negation

$$\frac{}{\Gamma, A \vdash A}\ \text{axiom}$$

$$\frac{}{\Gamma \vdash L}\ \text{atom if } L \in \mathcal{P}$$

$$\frac{}{\Gamma \vdash \top}\ \top\text{-intro}$$

$$\frac{\Gamma \vdash \bot}{\Gamma \vdash A}\ \bot\text{-elim}$$

$$\frac{\Gamma \vdash A \quad \Gamma \vdash B}{\Gamma \vdash A \wedge B}\ \wedge\text{-intro}$$

$$\frac{\Gamma \vdash A \wedge B}{\Gamma \vdash A}\ \wedge\text{-elim}$$

$$\frac{\Gamma \vdash A \wedge B}{\Gamma \vdash B}\ \wedge\text{-elim}$$

$$\frac{\Gamma \vdash A}{\Gamma \vdash A \vee B}\ \vee\text{-intro}$$

$$\frac{\Gamma \vdash B}{\Gamma \vdash A \vee B}\ \vee\text{-intro}$$

$$\frac{\Gamma \vdash A \vee B \quad \Gamma, A \vdash C \quad \Gamma, B \vdash C}{\Gamma \vdash C}\ \vee\text{-elim}$$

$$\frac{\Gamma \vdash (c_1/x)A \quad \ldots \quad \Gamma \vdash (c_n/x)A}{\Gamma \vdash \forall x\ A}\ \forall\text{-intro}$$

$$\frac{\Gamma \vdash \forall x\ A}{\Gamma \vdash (c_i/x)A}\ \forall\text{-elim}$$

$$\frac{\Gamma \vdash (c_i/x)A}{\Gamma \vdash \exists x\ A}\ \exists\text{-intro}$$

$$\frac{\Gamma \vdash \exists x\ A \quad \Gamma, (c_1/x)A \vdash C \quad \ldots \quad \Gamma, (c_n/x)A \vdash C}{\Gamma \vdash C}\ \exists\text{-elim}$$

Cut elimination

Business as usual
Drop all the elimination rules
An automaton

Drop the axiom rule (context always empty)

An automaton

$$\frac{}{\vdash L}\ \text{atom if } L \in \mathcal{P}$$

$$\frac{}{\vdash \top}\ \top\text{-intro}$$

$$\frac{\vdash A \quad \vdash B}{\vdash A \wedge B}\ \wedge\text{-intro}$$

$$\frac{\vdash A}{\vdash A \vee B}\ \vee\text{-intro}$$

$$\frac{\vdash B}{\vdash A \vee B}\ \vee\text{-intro}$$

$$\frac{\vdash (c_1/x)A \quad \ldots \quad \vdash (c_n/x)A}{\vdash \forall x\ A}\ \forall\text{-intro}$$

$$\frac{\vdash (c_i/x)A}{\vdash \exists x\ A}\ \exists\text{-intro}$$

Proof search or model checking?

Proving implication

Additive rules

$$\frac{\vdash B}{\vdash A \Rightarrow B}$$

$$\frac{A \vdash}{\vdash A \Rightarrow B}$$

instead of the multiplicative

$$\frac{A \vdash B}{\vdash A \Rightarrow B}$$

How do you know that "if the Sun shines, then the Sun shines"?
(a different proof every talk)

V. Natural deduction

Natural deduction (for Predicate logic)

Undecidable

Can it have the cut elimination property?

No

Natural deduction (for Predicate logic)

Introduction rules: $\Rightarrow$-intro, $\forall$-intro, $\wedge$-intro... and axiom

$$\frac{}{\Gamma, A \vdash A}\ \text{axiom}$$

Non-introduction rules: $\Rightarrow$-elim, $\forall$-elim, $\wedge$-elim...

$$\frac{\dfrac{}{P \wedge Q \vdash P \wedge Q}\ \text{axiom}}{P \wedge Q \vdash P}\ \wedge\text{-elim}$$

is a (general) cut but not a (specific) cut

VI. Saturating inference systems (and attempting to do so)

When an inference system does not have the cut-elimination property

$$\frac{}{R(a(x))} \qquad \frac{R(x)}{Q(a(b(x)))} \qquad \frac{Q(a(x))}{P(x)}$$

$$\frac{\dfrac{\dfrac{}{R(a(b(\varepsilon)))}}{Q(a(b(a(b(\varepsilon)))))}}{P(b(a(b(\varepsilon))))}$$

Transform it in such a way it does

Saturation
Add the derived rule

$$\frac{R(x)}{P(b(x))}$$

The proof

$$\frac{\dfrac{\dfrac{}{R(a(b(\varepsilon)))}}{Q(a(b(a(b(\varepsilon)))))}}{P(b(a(b(\varepsilon))))}$$

then reduces to

$$\frac{\dfrac{}{R(a(b(\varepsilon)))}}{P(b(a(b(\varepsilon))))}$$

Cut elimination $\longrightarrow$ drop the non introduction rules $\longrightarrow$ automaton $\longrightarrow$ decidability

Attempting to saturate Natural deduction

$$\dfrac{\dfrac{}{P \wedge Q \vdash P \wedge Q}\ \text{axiom}}{P \wedge Q \vdash P}\ \wedge\text{-elim}$$

Add the derived rule

$$\dfrac{}{\Gamma, A \wedge B \vdash A}$$

would generate an infinite number of rules
Instead

$$\dfrac{\Gamma, A \vdash D}{\Gamma, A \wedge B \vdash D}$$

(Gentzen style) sequent calculus

Attempting to saturate Natural deduction

Sequent calculus still has an non-introduction rule: contraction
Saturation: Kleene style sequent calculus

Still non-introduction rules
Saturation: Vorob'ev-Hudelmaier-Dyckhoff-Negri style sequent calculus

Still non-introduction rules...

First steps of a hierarchy of sequent calculi
Each proves the decidability of a larger fragment of Predicate logic

The axiom rule

Not needed in finite domain logic

Comes from the will to reason generically on elements of an infinite domain

$$\forall x\ (P(x) \Rightarrow P(x))$$

Introduces non-eliminable cuts (elimination rules, etc.) and undecidability

Attempting to eliminate these cuts yields a hierarchy of sequent calculi

Decidability: a weakness or a strength?

Introduction rule, automaton, Curry-(de Bruijn)-Howard correspondence, cuts, cut elimination (appropriately generalized): broad notions that apply both to undecidable and decidable logics

The problem of semantic completeness in proof-theoretic semantics

(joint work with Thomas Piecha)

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Proof-theoretic semantics

- Semantics of proofs
- Semantics in terms of proofs

There must be elementary proofs in terms of which proofs in general are interpreted.

These are normally proof steps, which generate canonical proofs.

Two conceptions of proof-theoretic semantics

- The standard unidirectional approach
  - Model natural deduction
  - Introduction rules primary
  - Implication transformational: transmission view
    Thus not elementary, but reductive
  - Labelled as $x : A \vdash t : B$
- The bidirectional approach
  - Model sequent calculus
  - Both right and left introduction rules primary
    (actually: either right or left)
  - Implication basic
    Thus elementary, and thus fundamental
  - Labelled as $f : (A \vdash B)$

Standard proof-theoretic semantics

Here I deal with the first conception, following Prawitz who is strongly in favour of it. This corresponds to almost all of the intuitionistic tradition.

Idea: Definition of validity, but not of formulas, but of proofs

Semantic question: Do the valid proofs generate all and perhaps exactly the valid formulas of intuitionistic first-order logic?

This is a sort of Tarski-style semantical approach.

Instead of structures and models: Atomic systems S, which are sets of atomic rules of the form $\dfrac{A_1, \ldots, A_n}{B}$.

Definition: S-Validity of proofs

(0) *Every closed proof in S is S-valid.*

(I) *A closed canonical proof is S-valid, if its immediate subproof is S-valid.*

(II) *A closed noncanonical proof is S-valid, if it reduces to a S-valid canonical proof.*

(III) *An open proof* $\begin{array}{c} A_1 \ldots A_n \\ \mathcal{D} \\ B \end{array}$ *is S-valid, if for every $S' \geq S$*

and for every list of closed S'-valid proofs $\begin{array}{c} \mathcal{D}_i \\ A_i \end{array}$

$(1 \leq i \leq n)$, the proof $\begin{array}{c} \mathcal{D}_1 \quad \mathcal{D}_n \\ A_1 \ldots A_n \\ \mathcal{D} \\ B \end{array}$ *is S'-valid.*

Prawitz's completeness conjecture of 1971

$A_1, \ldots, A_n \vdash_I B$ iff for every S there is an S-valid proof of B from $A_1, \ldots, A_n$.

According to Prawitz, there is a strong intuition behind this idea.

This intuition is based on the fact that one cannot imagine stronger elimination rules than the standard ones, which are valid.

Claim: This conjecture is false.

What makes one skeptical?

Validity of consequences and thus of rules sounds like admissibility.

A rule $\frac{A}{B}$ is admissible if every closed proof of A can be transformed into a closed proof of B.

Now we know that in intuitionistic logic, admissibility does not coincide with derivability.

Admissibility vs. derivability in intuitionistic logic

In the purely implicational fragment of intuitionistic logic, admissibility coincides with validity (Mints).

However, if negation and disjunction are present, then there are admissible, but non-derivable rules. The most prominent (and simplest) example is Harrop's rule:

$$\frac{\neg A \to (B \vee C)}{(\neg A \to B) \vee (\neg A \to C)}$$

A related example is Mints's rule

$$\frac{(A \to B) \to (A \vee C)}{((A \to B) \to A) \vee ((A \to B) \to C)}$$

Crucial: Mixture of implication and disjunction in one rule.

Claim: Harrop's rule can be validated

under very weak and plausible conditions.

$$\frac{\neg A \rightarrow (B \vee C)}{(\neg A \rightarrow B) \vee (\neg A \rightarrow C)}$$

Basic principles of validity semantics

Rather than dealing with full validity semantics, we deal with certain conditions that hold for validity semantics.

Our result therefore extends to all semantics which satisfy these conditions.

($\wedge$) $\models_S A \wedge B$ iff $\models_S A$ and $\models_S B$

($\vee$) $\models_S A \vee B$ iff $\models_S A$ or $\models_S B$

($\rightarrow$) $\models_S A \rightarrow B$ iff $A \models_S B$

($\models$) $\Gamma \models A$ iff for all S: if $\models_S \Gamma$ then $\models_S A$

($\models_S$) If $\Gamma \models A$ and $\Gamma, A \models_S B$ then $\Gamma \models_S B$

The generalized disjunction property

GDP($\Vdash$): If $\Gamma \Vdash A \vee B$, where $\vee$ does not occur positively in Γ, then either $\Gamma \Vdash A$ or $\Gamma \Vdash B$.

Lemma. If *GDP*($\models_S$) for every S, then Harrop's rule is valid under substitution.

Therefore, if we can prove the general disjunction property for $\models_S$, we have refuted Prawitz's completeness conjecture.

Eliminating disjunctions from negated formulas

- $\neg(A \vee B) \dashv\vdash \neg A \wedge \neg B$
- $\neg(A \wedge B) \dashv\vdash \neg(\neg\neg A \wedge \neg\neg B)$
- $\neg(A \rightarrow B) \dashv\vdash \neg\neg A \wedge \neg B$

This gives us the following:

$\neg A \models_S B \vee C$
$\Longrightarrow A' \models_S B \vee C$ for disjunction-free A'
$\Longrightarrow A' \models_S B$ or $A' \models_S C$ by *GDP*($\models_S$)
$\Longrightarrow \neg A \models_S B$ or $\neg A \models_S C$

Proving the generalized disjunction property

(Conditionalisation) For every S there is a set of disjunction-free formulas S^* such that for all Γ and A: $\Gamma \models_S A$ iff $\Gamma, S^* \models A$.

Conditionalisation is equivalent to the monotonicity of $\models_S$ with respect to S.

Lemma. Suppose Prawitz's completeness conjecture is true. Then Conditionalisation implies $GDP(\models_S)$ for every S.

Final result: Suppose Prawitz's completeness conjecture is true. Then $GDP(\models_S)$ for every S.

Therefore we have refuted Prawitz's completeness conjecture under the supposition that Prawitz's completeness conjecture is true, which means that we have refuted it outright.

Remarks

- One might challenge the assumption of monotonicity of $\models_S$ with respect to S.
- However, there is no convincing alternative semantics.
- Moreover, this means challenging the generalized disjunction property which one probably would like to have in any case.
- We do have completeness for the purely implicational fragment. We even have completeness for formulas, in which disjunction occurs only positively.
- In any case the mixing of implication and disjunction in one and the same formula is essential.

The intuition about elimination rules

Intuition: The standard elimination rules are the strongest elimination rules one can think of, given the standard introduction rules.

This is no objection to the result obtained:

- The counterexample (Harrop's rule) is not an elimination rule. It rather contains two constants (implication and disjunction) within a single rule.
- It is the interaction of logical constants which produces the negative result.

The standard elimination rules are the strongest valid elimination rules

Result: Every elimination rule, which is valid, is derivable using the standard elimination rules.

General form of introduction and elimination rules

General form of introduction rules:

$$\frac{\Sigma_{i1} \quad \ldots \quad \Sigma_{i\ell}}{A_{i1} \quad \ldots \quad A_{i\ell}} \; c$$

also written as $\dfrac{\Gamma_i}{c}$ $(0 \leq i \leq n)$

General form of elimination rules:

$$\frac{\Pi_{ji} \quad \Pi_{jk}}{c \quad B_{j1} \quad \ldots \quad B_{jk}} \; B_j$$

also written as $\dfrac{c}{\Delta_j}$ $(0 \leq j \leq m)$

The *canonical* elimination rule for c is

$$\frac{\Gamma_1 \quad \Gamma_k}{c \quad C \quad \ldots \quad C} \; C$$

General form of introduction and elimination rules

General form of introduction rules:

$$\frac{\Sigma_{i1} \quad \ldots \quad \Sigma_{i\ell}}{A_{i1} \quad \ldots \quad A_{i\ell}} \; c$$

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General form of elimination rules:

$$\frac{\Pi_{ji} \quad \Pi_{jk}}{c \quad B_{j1} \quad \ldots \quad B_{jk}} \; B_j$$

also written as $\dfrac{c}{\Delta_j}$ $(0 \leq j \leq m)$

That $\dfrac{c}{\Delta_j}$ satisfies the reduction criterion means $\Gamma_i \vdash \Delta_j$ for every i. Thus, by using the canonical elimination rule for c, we can derive Δ_j from c.

The maximality result

Theorem. Any valid rule, which has the form of an elimination rule, namely the form $\frac{c}{\Delta}$, is derivable using the canonical elimination rule.

The proof uses the following *structural completeness* result:

Every structurally valid rule is structurally derivable, (i.e., derivable without using rules for logical constants).

(This is essentially a variant of the completeness for the implicational fragment.)

Proof of the maximality result

Proof. Suppose $\frac{c}{\Delta}$ is valid. As the introduction rules $\frac{\Gamma_i}{c}$ for c are trivially valid, we obtain that $\frac{\Gamma_i}{\Delta}$ is valid for every i, since the composition of valid rules is valid. Then by structural completeness, $\frac{\Gamma_i}{\Delta}$ is derivable for every i, which means that the reduction criterion is met for the elimination rule $\frac{c}{\Delta}$. This means that $\frac{c}{\Delta}$ is derivable (by structural means) from the canonical elimination rule for c.

Summary of *restricted completeness*

- Let c be a connective, for which introduction rules as well as the canonical elimination rule are given. Then any valid rule of the form $\dfrac{c \quad \vdots \cdots \vdots}{\cdots}$ is derivable.
- Suppose $c, \Gamma \vdash q$ is a valid consequence, where Γ does not contain any logical constant. Then this consequence is derivable (since it represents an elimination inference).
- In other words: Every structural statement that we can infer validly from c, can be derived from c.
- This can be viewed as a *restricted form of completeness*: We have completeness for the valid structural consequences of c (but not for every statement in which c might be involved).

Global summary

- Prawitz completeness conjecture is false.
- However, there is some restricted form of completeness.

What Is To be Done?

- Is there any weaker validity concept that would render intuitionistic logic complete?
- What is the logic characterized by the given concept of validity?
- Should one give up and leave standard uni-directional semantics in favour of a bi-directional one?

On propositional variables:

the atomic and the parametric view

Paolo Pistone

Beyond Logic 2017

Cerisy, May 22-27, 2017

Outline

1 Introduction

2 The atomic view

3 The parametric view

What is a proof of $p \to p$?

Constructive proof-theory: intuitionistic logic, *BHK*, realizability, ...
Two active traditions in proof theory:

Atomist semantics

- Kreisel 1961, Gabbay 1976
- Prawitz's proof-theoretic validity (1971)
- Dummett's "justification of logical laws" (1991)
- Proof-theoretic semantics (Schroeder-Heister, Piecha, Sandqvist, ...)

Parametric semantics

- Girard's reducibility candidates (1971)
- Parametric polymorphism (Reynolds, Strachey, Plotkin,...early 80's)
- Functorial semantics (Bainbridge, Scott, Scedrov, ... early 90's)
- (Bi)fibrational semantics (Hermida, Ghani, Reddy, ... last 10 years)

Outline

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Motivations

Compositionality + non circularity ⇒ Molecularity

If the intuitionistic explanations of the logical constants and, more generally, of the meanings of mathematical statements are to be considered as constituting a coherent theory of meaning for the language of mathematics, then the notion of proof which is appealed to must be such that we can fully grasp the concept of a proof of any constituent of a given sentence in advance of grasping that of a proof of that sentence. (Dummett 1977)

Compositionality demands that the relation of dependence imposes upon the sentences of the language a hierarchical structure deviating only slightly from being a partial order. (Dummett 1991)

Proof-theoretic semantics

Sequent-based semantics

Typically, **Kripke models**:

- $w \Vdash A \wedge B$ *if* $w \Vdash A \wedge w \Vdash B$
- $w \Vdash A \vee B$ *if* $w \Vdash A \vee w \Vdash B$
- $w \Vdash A \to B$ *if* $\forall w' \geq w$ $(w' \Vdash A \Rightarrow w' \Vdash B)$

Soundness: $\vdash_{LJ} A \Rightarrow \Vdash A$

Completeness $\Vdash A \Rightarrow \vdash_{LJ} A$

Kripke structures $\simeq$ Heyting Algebras

Proof-based semantics

Typically, **BHK/Realizability**:

- $u \Vdash A \wedge B$ *if* $u = (u_1, u_2)$ with $u_1 \Vdash A \wedge u_2 \Vdash B$
- $u \Vdash A_1 \vee A_2$ *if* $u = (i, u_i)$ with $u_i \Vdash A_i, i = 1, 2$
- $u \Vdash A \to B$ *if* $\forall v$ $(v \Vdash A \Rightarrow uv \Vdash B)$

Soundness: $\begin{matrix}\mathscr{D}\\ A\end{matrix} \Rightarrow \mathscr{D}^- \Vdash A$

Completeness $u \Vdash A \Rightarrow u = \mathscr{D}^-$ for some $\begin{matrix}\mathscr{D}\\ A\end{matrix}$

BHK/Realizability $\simeq$ *CCC* categories

Atomist semantics: first attempt

Atomic base: $S = \{\ldots,\ p_1, \ldots, p_n \vdash p\ ,\ \ldots\} \quad \Rightarrow \quad (\mathcal{S}, \subseteq)$ poset

Every $S \in \mathcal{S}$ defines a **Kripke structure** $(\mathcal{M}_S, \subseteq)$:

- $S \Vdash p$ *if* $\vdash_S p$
- $S \Vdash A \wedge B$ *if* $S \Vdash A \wedge S \Vdash B$
- $S \Vdash A \vee B$ *if* $S \Vdash A \vee S \Vdash B$
- $S \Vdash A \to B$ *if* $\forall S' \supseteq S$ $(S' \Vdash A \Rightarrow S' \Vdash B)$

A is **valid** if $\emptyset \Vdash A$ (i.e. if all $S \Vdash A$ by monotonicity)

Actually a unique structure $(\mathcal{S}, \subseteq)$.

Atomist semantics: first attempt

Atomic base: $S = \{\ldots,\ \dfrac{p_1, \ldots, p_n}{p},\ \ldots\} \quad \Rightarrow \quad S(p) = Sat\{\ \dfrac{\mathscr{D}}{p}\ S\text{-der.}\}$

Every $S \in \mathcal{S}$ defines a **Kripke structure** $(\mathcal{M}_S, \subseteq)$:

- $S \Vdash p$ *if* $\vdash_S p$
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A is **valid** if $\emptyset \Vdash A$ (i.e. if all $S \Vdash A$ by monotonicity)

Actually a unique structure $(\mathcal{S}, \subseteq)$.

Every upwards closed $s \subseteq \wp(\mathcal{S})$ defines a **realizability** interpretation

- $a \Vdash_s p$ *if* $a \in S(p)$
- $u \Vdash_s A \wedge B$ *if* $u \rightsquigarrow (u_1, u_2)$, with $u_1 \Vdash_s A \ \wedge\ u_2 \Vdash_s B$
- $u \Vdash_s A_1 \vee A_2$ *if* $u \rightsquigarrow (i, u_i)$ with $u_i \Vdash_s A_i, i = 1, 2$
- $u \Vdash_s A \to B$ *if* $\forall v$ $(v \Vdash_{s'} A \Rightarrow u(v) \Vdash_{s'} B)$

$\dfrac{\mathscr{D}}{A}$ is **valid** if $\mathscr{D}^- \Vdash_\emptyset A$ (i.e. if all $S \Vdash A$ by monotonicity)

Actually a unique model $(\mathcal{S}, \subseteq)$.

A dualist semantics

Atoms p, q, r correspond to sets of indistinguishable tokens.

Remark: if $f \Vdash_S p \to A$ and $a, b \Vdash_S p$, then $f(a)$ and $f(b)$ are the same up to permutation.

Non-atomic formulas correspond to sets of possibly distinct proofs.

Example: $\mathtt{F}_1, \mathtt{F}_2 \Vdash (A \vee A) \to (A \vee A)$, where

$$\mathtt{F}_1 = \lambda u.\mathtt{case}u(\lambda x_1.\lambda x_2.\mathtt{in}_1 x_1)(\lambda x_1.\lambda x_2.\mathtt{in}_2 x_2)$$
$$\mathtt{F}_2 = \lambda u.\mathtt{case}u(\lambda x_1.\lambda x_2.\mathtt{in}_2 x_2)(\lambda x_1.\lambda x_2.\mathtt{in}_1 x_1)$$

Atomic test?

$$t \Vdash A \to C \quad \Rightarrow \quad t[B/p] \Vdash A[B/p] \to C \quad \text{for any } B$$

A dualist semantics

Atoms p, q, r correspond to sets of indistinguishable tokens.

Remark: if $f \Vdash_S p \to A$ and $a, b \Vdash_S p$, then $f(a)$ and $f(b)$ are the same up to permutation.

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Atomic test?

$$\mathtt{AT}[X](f_1^{A\to C}, f_2^{A\to C}, x^{A[X/p]}) = \begin{cases} f_1(x) & \text{if } X \text{ is atomic} \\ f_2(x) & \text{otherwise} \end{cases}$$

For any B, $\mathtt{AT}[B] \Vdash A[B/X] \to C$.

In the language of **Fat**, does $\mathtt{AT} \Vdash \forall X(A \to C)$?

Atomic test?

$$\dfrac{\begin{array}{cc}\overset{n}{[A]} & \overset{n}{[A]} \\ \mathscr{D}_1 & \mathscr{D}_2 \\ C & C\end{array}}{A \to C}\ \texttt{Atest}, n$$

Atomic test?

$$\dfrac{\dfrac{\begin{array}{cc}\overset{n}{[A[B/p]]} & \overset{n}{[A[B/p]]} \\ \mathscr{D}_1 & \mathscr{D}_2 \\ C & C\end{array}}{A[B/p] \to C[B/p]}\ \texttt{Atest}, n \qquad \begin{array}{c}\mathscr{D} \\ A[B/p]\end{array}}{C[B/p]}$$

Atomic test?

$$\dfrac{\dfrac{[A[\overset{n}{B}/p]]}{\begin{array}{c}\mathscr{D}_1\\ C\end{array}} \quad \dfrac{[A[\overset{n}{B}/p]]}{\begin{array}{c}\mathscr{D}_2\\ C\end{array}}}{A[B/p] \to C[B/p]}\,\texttt{Atest}, n \qquad \begin{array}{c}\mathscr{D}\\ A[B/p]\end{array}}{C[B/p]}$$

$$\begin{array}{c}\mathscr{D}\\ A[q/p]\\ \mathscr{D}_1\\ C[q/p]\end{array}$$

if $B = q$ is atomic

Atomic test?

$$\dfrac{\dfrac{[A[\overset{n}{B}/p]]}{\begin{array}{c}\mathscr{D}_1\\ C\end{array}} \quad \dfrac{[A[\overset{n}{B}/p]]}{\begin{array}{c}\mathscr{D}_2\\ C\end{array}}}{A[B/p] \to C[B/p]}\,\texttt{Atest}, n \qquad \begin{array}{c}\mathscr{D}\\ A[B/p]\end{array}}{C[B/p]}$$

$$\begin{array}{c}\mathscr{D}\\ A[q/p]\\ \mathscr{D}_1\\ C[q/p]\end{array}$$

if $B = q$ is atomic

$$\begin{array}{c}\mathscr{D}\\ A[B/p]\\ \mathscr{D}_2\\ C[B/p]\end{array}$$

if B is not atomic

Incompleteness

Theorem[Gabbay 1976] Validity is not complete

$$\Vdash (p \to A \vee B) \to (p \to A) \vee (p \to B)$$

Idea of the proof. There exists a **minimum extension** satisfying p. If $S \vdash p \to A \vee B$, take $S \cup \{\vdash p\} \supseteq S$. It satisfies either A or B and is contained in any $S' \supseteq S$ satisfying p.

Corollary Validity is not substitution-closed

Proof. $\nVdash (C \vee D \to A \vee B) \to (C \vee D \to A) \vee (C \vee D \to B)$

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?

10 / 20

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Corollary Validity is not substitution-closed

Proof. $\nVdash (C \vee D \to A \vee B) \to (C \vee D \to A) \vee (C \vee D \to B)$

Theorem Validity is not complete.

$$\mathtt{GAB} \Vdash (p \to A \vee B) \to (p \to A) \vee (p \to B)$$

$$\mathtt{GAB}(u, f, g) = \begin{cases} \lambda x.\mathsf{case}(u(x), f, g) & \text{if } u(a) \rightsquigarrow (1, u_1(a)) \\ \lambda x.\mathsf{case}(u(x), f, g) & \text{if } u(a) \rightsquigarrow (2, u_2(a)) \end{cases}$$

If $u \Vdash_S p \to A \vee B$, then u is "constant" on p: if $u(a) \rightsquigarrow (i, u'(a))$, then $u(x) \rightsquigarrow (i, u'(x))$.

Corollary Validity is not substitution-closed.

Proof. Take $D = (A \vee A) \to (A \vee A)$; then $\mathtt{GAB} \nVdash (D \to A \vee B) \to (D \to A) \vee (D \to B)$.

Schematic validity

$\vDash A$ if for any formulas $\vec{B}$, $\Vdash A[\vec{B}/\vec{p}]$.

Compatible with molecularism? (Goldfarb 2006)

Theorem [Goldfarb 2006] Schematic validity is complete.

Idea of the proof. If A is not derivable, define an instance and a countermodel starting from a Kripke countermodel.

Schematic validity

$u \vDash A$ if for all formulas $\vec{B}$, $u \Vdash A[\vec{B}/\vec{p}]$.

Compatible with molecularism? (Goldfarb 2006)

Theorem [Goldfarb 2006] Schematic validity is complete.

Idea of the proof. If A is not derivable, define an instance and a countermodel starting from a Kripke countermodel.

Might be incomplete?

Idea: construct an incorrect realizer of the (valid) formula

$(p \to A \vee B) \to ((p \to A) \vee (p \to B)) \vee (p \to A \vee B)$

$\mathtt{P} := \lambda u.\mathtt{AT}(\mathtt{in}_1(\mathtt{GAB}(u)), \mathtt{in}_2(u))$

Outline

1 Introduction

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Motivations I

If we reject the belief that it is necessary to run through individual cases and rather make it clear to ourselves that the complete verification of a statement means nothing more than its logical validity for an arbitrary property, we will come to the conclusion that impredicative definitions are logically admissible. (Carnap 1983)

From proof theory

Extend **BHK** to second order

[...] *everything works as if the rule of universal abstraction (which forms functions defined for every type) were so uniform that it operates without any information at all about its arguments.* (GLT 1989)

From computer science

Describe **polymorphic** programs

$\texttt{Map}(f, L)$ = "list of all $f(a)$, for a in L", where
$\texttt{f} : \alpha \to \beta$, $\texttt{L} : \texttt{List}[\alpha]$

$\texttt{Map} : \texttt{List}[\alpha] \to \texttt{List}[\beta]$, for all α, β (Strachey 1967)

Intuitively, a parametric polymorphic function is one that behaves the same way for all types [...] (Reynolds 1983)

13 / 20

Motivations II

$t \Vdash \forall XA$ iff for any B, $tB \Vdash A[B/X]$

14 / 20

Motivations II

$t \Vdash \forall XA$ iff for any B, $tB \Vdash A[B/X]$

Parameterization 1: schematic validity is circular!

Motivations II

$t \Vdash \forall XA$ iff for any B, $tB \Vdash A[B/X]$

Parameterization 1: schematic validity is circular!

Model theory:

"A is true under interpretation η" (Tarski) (assignment of sets to variables)

14 / 20

Motivations II

$t \Vdash \forall X A$ iff for any $B, \mathcal{C}_B$, $tB \Vdash A[\mathcal{C}/X]$

Parameterization 1: schematic validity is circular!

Proof theory:

> "$u \Vdash A$ under interpretation η" (**Girard/Krivine**) (assignment of sets of proofs to variables)

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Focus on **formulas with parameters**

14 / 20

Motivations II

a proof of $\forall XA$ is a function from $\mathcal{C}s$ to proofs of $A[\mathcal{C}/X]$

Parameterization 1: schematic validity is circular!

Proof theory:

"$u \Vdash A$ under interpretation η" (**Girard/Krivine**) (assignment of sets of proofs to variables)

Parameterization 2:

Focus on **formulas with parameters**

Motivations II

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Proof theory:

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Parameterization 2:

- (Girard 1971, Harper&Mitchell 1999) Non-uniform programs are paradoxical!

$$\mathrm{J}[X] = \begin{cases} u & \text{if } X = \sigma \\ v & \text{otherwise} \end{cases}$$

Focus on **formulas with parameters**

Motivations II

a proof of $\forall XA$ is a function from $\mathcal{C}s$ to proofs of $A[\mathcal{C}/X]$

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Focus on **formulas with parameters**

Parameterization 2:

- (Girard 1971, Harper&Mitchell 1999) Non-uniform programs are paradoxical!

$$\mathtt{J}[X] = \begin{cases} u & \text{if } X = \sigma \\ v & \text{otherwise} \end{cases}$$

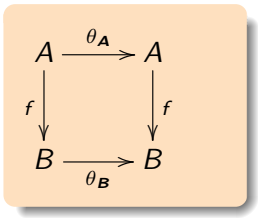
- similarly, System F + `AT` will produce paradoxes (while **Fat** + `AT` doesn't?)

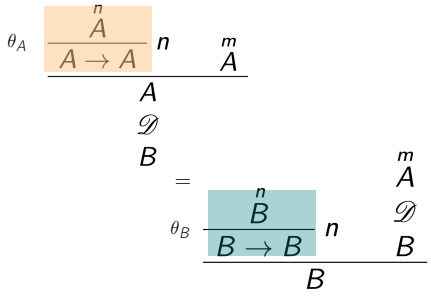
Intuition 1: natural transformations

$$\theta_X : X \to X$$

Intuition 1: natural transformations

$$\theta_X : X \to X$$

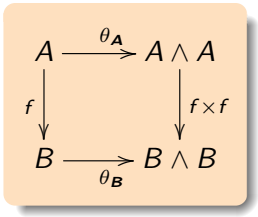


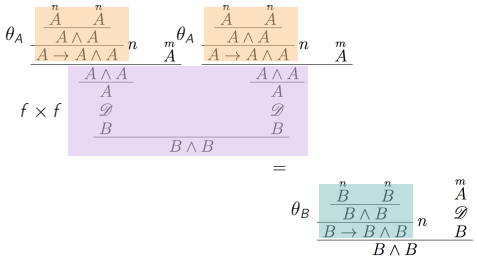


$$b = f(a) \;\Rightarrow\; \theta_B(b) = f(\theta_A(a))$$

Intuition 1: natural transformations

$$\theta_X : X \to X \wedge X$$

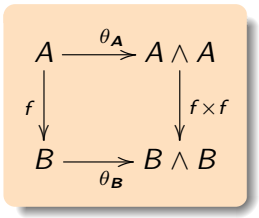




$$b = f(a) \;\Rightarrow\; \theta_B(b) = (f \times f)(\theta_A(a))$$

Intuition 1: natural transformations

$$\theta_X : X \to X \wedge X$$



$$b = f(a) \;\Rightarrow\; \theta_B(b) = (f \times f)(\theta_A(a))$$

Intuitively, a **functor** $F[\vec{X}]$ is a way to denote a formula $F[\vec{A}]$ "uniformly" for any choice of $\vec{A}$.

Intuitively, a **natural transformation** $F[\vec{X}] \overset{\theta_{\vec{X}}}{\to} G[\vec{X}]$ is a way to transform a proof of $F[\vec{A}]$ into a proof of $G[\vec{A}]$ "uniformly" for any choice of $\vec{A}$.

Intuition 2: binary relations

θ_A and θ_B should be indistinguishable: no binary relation $R \subseteq A \times B$ and aRb such that $\theta_A(a) R \theta_B(b)$ fails.

Base level: $[\![A]\!]_0^{\eta}$ is an object (under interpretation η):

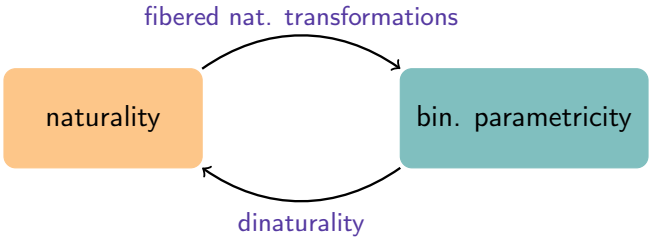
- $u \in [\![A \wedge B]\!]_0$ if $u = (u_1, u_2)$ with $u_1 \in [\![A]\!]_0, u_2 \in [\![B]\!]_0$
- $u \in [\![A_1 \vee A_2]\!]_0$ if $u = (i, u_i)$ with $u_i \in [\![A_i]\!]_0$
- $u \in [\![A \to B]\!]_0$ if $\forall v (v \in [\![A]\!]_0 \Rightarrow u(v) \in [\![B]\!]_0$

Relational level: $[\![A]\!]_r$ is a relation over $[\![A]\!]_0^{\eta_1}$ and $[\![A]\!]_0^{\eta_2}$:

- $(u, v) \in [\![A \wedge B]\!]_r$ if $u = (u_1, u_2), v = (v_1, v_2)$ with $(u_1, v_1) \in [\![A]\!]_r, (u_2, v_2) \in [\![B]\!]_r$
- $(u, v) \in [\![A_1 \vee A_2]\!]_r$ if $u = (i, u_i)$, $v = (i, v_i)$ with $(u_i, v_i) \in [\![A_i]\!]_r$
- $(u, v) \in [\![A \to B]\!]_r$ if $\forall w, z ((w, z) \in [\![A]\!]_r \Rightarrow (u(w), v(z)) \in [\![B]\!]_r$

Parametric semantics: $u \in [\![A]\!]_0$ is **parametric** if $(u, u) \in [\![A]\!]_r$

Functorial proof-theoretic semantics



1. a formula $A[\vec{X}]$ is a way to denote an object $[\![A]\!]_0$ "uniformly" in $\vec{X}$
2. a proof of $A[\vec{X}] \to B[\vec{X}]$ is a way to transform a proof of $A[\vec{X}]$ into a proof of $B[\vec{X}]$ "uniformly" in $\vec{X}$.

1. a formula $A[\vec{X}]$ is a way to denote a relation $[\![A]\!]_r$ "uniformly" in $\vec{X}$
2. a proof of $A[\vec{X}] \to B[\vec{X}]$ is a way to transform related proofs of $A[\vec{X}]$ (i.e. elements of $[\![A]\!]_r$) into related proofs $B[\vec{X}]$ (i.e. elements of $[\![B]\!]_r$)"uniformly" in $\vec{X}$.

The abstraction theorem and its inverse

Theorem[Reynolds 1984] Every derivation in LJ^2 is parametric.

Remark: no parametric family $(A+B)^X \xrightarrow{\theta_X} A^X + B^X$

Theorem[Π^1-completeness] For A a Π^1 formula, if u is a parametric realizer of A, then $NF(u)$ exists and is the translation of an intuitionistic derivation of A.

Incompatible foundations?

What is a proof of $p \to p$?

A simple proof **because p must be simple**

- validity for closed formulas
- inductive structure of formulas
- schematicity makes sense (required?)
- non-parametricity makes sense

A simple proof **because p can be complex**

- validity for formulas with parameters
- inductive structure of functors (formulas with parameters)
- ignorance about p
- parametricity

Thank you

Methodological remarks on completeness and incompleteness theorems

V. Michele Abrusci
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Workshop *Beyond Logic*, Cerisy-La-Salle, May 25th, 2017

Introduction, 1

The first methodological remark is the following one: completeness and incompleteness theorems - stated in mathematical logic during last century - may be considered as answers given to a general philosophical question when the concepts occurring in the question receive a particular format i.e. a rigorous definition satisfying some natural conditions.

My talk contains

- the presentation of a general philosophical question,
- the presentation of completeness and incompleteness as *answers* to this question (under the use of particular formats for the involved concepts).

Introduction, 2

During my talk I will expose other methodological remarks which I think are relevant from a philosophical point of view. Some of these methodological remarks have been inspired by

- the developments of proof theory and linear logic
- the discussions during the meetings of LIGC (including the Cerisy meeting in 2006)
- the recent stimulating Girard's books on logic.

The aim of this talk is to contribute to the understanding of main theorems of mathematical logic.

3

Section 1. A general philosophical question

4

A general philosophical question, 1: presuppositions

Presuppositions of the question (these presuppositions are also at the basis of logical investigations):

- a (naive) notion of *proposition*, a sentence which may be accepted or rejected,
- a (naive) notion of *proof* of a proposition A, something that allows to accept the proposition A, and so the (naive) notion of *provablity* of a proposition A i.e. *there is a proof of* A,
- a (naive) notion of *refutation* of a proposition A, somethin that allows to reject the propostion A, and so the (naive) notion of *refutability* of a proposition A i.e. *there is a refutation of* A.

5

A general philosophical question, 2: formulation

The general philosophical question is about *provability* and *refutability* of propositions:

Question: Is it true that "*For every proposition* A*, either* A *is provable or* A *is refutable*"?

i.e. is by using a modern terminology : it is true that there a *duality* between *proability* and *refutability* of propositions?

The question may be reformulated in several equivalent ways.

6

A general philosophical question, 3: other formulations

Some examples of equivalent reformulations of the question:

- is it true that "*For every proposition A, if A is unprovable then A is refutable*"? i.e is it true that, when there is no proof of a proposition A, there is a refutation of A? i.e. is it true that there is a refutation of A when there is a lack of proofs of A?

- is it true that "*For every proposition A, if A is irrefutable, then A is provable*"? i.e. is it true that, when there is no refutation of a proposition A, there is a proof of A? i.e. is it true that there is a proof of A when there is a lack of refutations of A?

7

A general philosophical question, 4: format of concepts

In order to solve this general philosophical question, we need to define - in a rigorous way - the concepts involved in the question, i.e. we need to specify a *format* F (i.e. a rigorous definition) of the following concepts:

- the concept of proposition, i.e. a format F of propositions,

- the concept of proof, i.e. a format F of proofs (of the propositions in the format F),

- the concept of refutation, i.e. a format F of refutations (of the propositions in the format F).

8

A general philosophical question, 5: application to a format

A format of propositions, proofs and refutations does not need to be the format of *all the possible* propositions, proofs and refutations; a format may concern only a class of propositions, proofs, refutations.

When we give a format F of propositions, proofs and refutations, the general philosophical question may be applied to the format F, as follows:

It is true that, for every proposition A in the format F, either A is provable by means of proofs in the format F or A is refutable by means of refutations in the format F?

Every format F must satisfy a general condition in order to be accepted.

9

A general philosophical question, 6: condition to accept a format

The condition to be satisfied in order to accept a format F is the following:

no proposition in the format F may be both provable by means of proofs in the format F and refutable by means of refutations in the format F,

i.e. *no proposition in the format F has both a proof in the format F and a refutation in the format F*.

This condition express the non-contradiction principle, and means that we are considering propositions, proofs and refutations *independently from the time*, i.e. in a single instant or (if you prefer) constant along all the development of the time, i.e. we are accepting one of the main features of classical logic.

10

A general philosophical question, 7: other formulations of the condition on the formats

The condition to be satisfied in order to accept a format F may be formulated as follows: *for every proposition A in the format F, either A is unprovable in the format F or A is irrefutable in the format F* (whereas the general philosophical question applied to the format F is: *It is true that, for every proposition A in the format F either A is provable in the format F or A is refutable in the format F*?) i.e.

- *for every proposition A in the format F, if A is provable in the format F then A is irrefutable in the format F*,

- *for every proposition A in the format F, if A is refutable in the format F then A is unprovable in the format F*.

11

A general philosophical question, 8: positive and negative statements

First remark:

- *provability* in a format F, and *refutability* in a format F, are *existential statements*, i.e. (as well explained by investigations in proof-theory and in particular in linear logic) are *positive statements*, i.e. these statements are proved in a non-reversible way;

- *unprovability* in a format F, and *irrefutability* in a format F, are *universal statements*, i.e. (as well explained by investigations in proof theory and in particular in linear logic) are *negative statements*, i.e. these statements are proved in a reversible way.

12

A general philosophical question, 9: truth and falsehood

Another remark:

- the *provability* of a proposition A (in a format F) and the *irrefutability* of A (in the format F) are two ways to state that A *is true*, the first one is a *positive* way, the other one is a *negative* way (A is true since A wins against every refutation, i.e. A is true since A cannot be refuted);

- the *refutability* of a proposition A (in a format F) and the *unprovability* of A (in the format F) are two ways to state that A *is false*, the first one is a *positive* way, the other one is a *negative* way (A is false since A cannot be proved).

13

A general philosophical question, 10: positive and negative

So, given a format F of propositions, proofs and refutations, we get two ways to define *truth* (*provability* and *irrefutability* in the format F) and two ways to define *falsehood* (*refutability* and *unprovability* in the format F).

Of course, we know that - when a concept may be defined both in a positive way (existential statement) and in a negative way (universal statement):

- it is trivial that the positive definition implies the negative one,

- the inverse implication it is not trivial - and is in some cases false.

Example: negative and positive definition of infinite set (the fact that positive definition implies the negative definition is trivial, whereas we need Choice Axiom to state that the negative definition implies the positive one).

14

A general philosophical question, 11: completeness theorems

The *positive answer* to the general philosophical question, applied to a format F of propositions, proofs and refutations:

- is the proof that in the format F the *negative definition of truth* (irrefutability in the format F) *implies the positive definition of truth* (provability in the format F) i.e that in the format F the *negative definition of falsehood implies the positive definition of falsehood*;

- establishes a theorem stating the equivalence between provability and irrefutability in the format F - i.e. between a positive definition of truth and a negative one: this kind of theorem is called *completeness theorem* for the format F.

15

A general philosophical question, 12: incompleteness theorems

The *negative answer* to the general philosophical question, applied to a format F of propositions, proofs and refutations:

- is the proof that in the format F there is a proposition A s.t. A is both irrefutable and unprovable, i.e. the negative definition of truth (irrefutablity) does not implies positive definition of truth (provability) and the negative definition of falsehood does not imply the positive definition of falsehood;

- establishes a theorem stating the failure of the equivalence between provability and irrefutability in the format F: this kind of theorem is called *incompleteness theorem* for the format F.

16

A general philosophical question, 13: negation

When a format F of propositions, proofs and refutations is closed under negation , i.e. when for every proposition A in the format F also the proposition $\neg A$ is in the format F and every refutation of A is a proof of $\neg A$, then refutability of a proposition A is equivalent to the provability of $\neg A$, so that:

- a completeness theorem states that *for every proposition A in the format F , either there is a proof of A in the fromat F or there is a proof of $\neg A$ in the format F*,

- an incompleteness theorem states that *there is a proposition A in the the format F s.t. both A and $\neg A$ are unprovable in the format F*.

17

Section 2. Some formats for the general philosophical question

18

Some formats for the general philosophical question

I will present well-known completeness and incompleteness theorems:

- completeness theorem for the first-order logic, and Π^1-completeness theorem,
- incompleteness theorem for (extensions) of first-order Peano Arithmetic,
- Σ^1-incompleteness theorem and incompleteness theorem for second-order logic,

as *answers* to the general philosophical question applied to some *formats*.

19

Completeness theorem for first-order logic, 1: preliminaries - formulas and propositions

Usually, in first-order logic one deals with first-order *formulas*; but first-order formulas - even if they are closed formulas - are not *propositions* since the value of a (closed) first-order formula depends on the value of propositional letters, individual symbols, predicate symbols, function symbols ... and depends firstly on the choice of a non-empty set X where we find these values. So, given a (closed) first-order formula B, it is better to consider a variable for sets X and all the propositional letters, individual symbols, predicate symbols and function symbols occurring in B as *variables* occurring in B.

A (closed) first-order formula B becomes a proposition when we put before B one quantifier (universal or existential) for each variable occurring in B: this proposition is a second-order proposition and a logical proposition (since it contains logical concepts only).

20

Completeness theorem for first-order logic, 2: preliminaries - Π^1 and Σ^1 propositions

Let B be a closed first order formula:

- by $\underline{\forall}(B)$ we denote the *universal closure* of B (i.e. before B there is an universal quantifier froreach variable occurring in B): intuitively, the logical proposition $\underline{\forall}(B)$ says that B is true for every value of its variables, i.e. that B is a first-order *tautology*;

- by $\underline{\exists}(B)$ we denote the *existential closure* of B (i.e. before B there is an existential quantifier for each variable occurring in B): intuitively, the logical proposition $\underline{\exists}(B)$ says that B is true for some value of its variables, i.e. that B is a first-order *satisfiable* formula.

21

Completeness theorem for first-order logic, 3: the format (propositions and proofs)

The format of first-order logic is the the following format for the concepts involved in the general philosophical problem.

- *Propositions*: the logical propositions belonging to Π^1, i.e. the propositions of the form $\underline{\forall}(B)$ where B is a closed first-order formula (inside a well-defined first-order language).

- *Proof* of a proposition $\underline{\forall}(B)$: a logical derivation of B inside a well-defined calculus for first-order classical logic (Hilbert's system, or Sequent Calculus, or Natural Deduction calculus, ...). Remark that (since $\forall$ is a negative operator, i.e. with reversible rules) a proof of a closed first-order formula B gives the proof of $\underline{\forall}(B)$ and viceversa; so B is provable in first-order logic iff $\underline{\forall}(B)$ is provable in second order logic.

22

Completeness theorem for first-order logic, 4: the format (refutations)

- *Refutation* of a proposition $\underline{\forall}(B)$: values for the variables occurring in B s.t. B becomes false, i.e. countermodels of B, i.e. models of $\neg B$, inside a well defined semantics for first-order classical logic (Tarski semantics). Remark that, when there is a countermodel of a closed first-order formula B, then we discovery that $\underline{\exists}(\neg B)$ is true i.e. that $\underline{\forall}(B)$ is false.

Remark that this format is not closed under negation (the negation of a proposition $\underline{\forall}(B)$ belonging to Π^1 is the proposition $\underline{\exists}(\neg B)$ not belonging to Π^1), and that in this format proofs are defined in a syntactical way whereas refutations are defined in a semantical way.

23

Completeness theorem for first-order logic, 5: the format satisfies the condition

Soundness theorem of first-order logic says that (w.r. to a given well defined calculus and a given well-defined semantics): *for every closed first-order formula B, either B is logically underivable or B has no countermodel, i.e. if B is logically derivable then B is a logical tautology*

This statement is equivalent to the following one: *for every proposition $\underline{\forall}(B)$ belonging to Π^1, either B is logically underivable or B has no countermodel* i.e.

for every proposition $\underline{\forall}(B)$ belonging to Π^1, either $\underline{\forall}(B)$ is unprovable in the format for first-order logic or $\underline{\forall}(B)$ is irrefutable in the format for first-order logic

i.e. it says that the format for first-order logic satisfies the condition.

24

Completeness theorem for first-order logic, 6: positive answer to the general philosophical problem applied to the format

As we know, completeness theorem for first-order logic says that (w.r. to a given well-defined calculus and a given well-defined semantics): *For every first-order formula B, either B is logically derivable or B has a counter-model, i.e. if B is a tautology then B is logically derivable* . This statement is equivalent to the following one: *for every proposition $\underline{\forall}(B)$ belonging to Π^1, either B is logically derivable or B has a countermodel* i.e.

for every proposition $\underline{\forall}(B)$ belonging to Π^1, either $\underline{\forall}(B)$ is provable in the format for first-order logic or $\underline{\forall}(B)$ is refutable in the format for first-order logic

i.e. it gives the positive answer to the general philosophical question, applied to the format for first-order logic.

25

Completeness theorem for first-order logic, 7: Π^1-completeness

Completeness theorem for first-order logic may be reformulated as Π^1-completeness theorem:

for every proposition $\underline{\forall}(B)$ belonging to Π^1, if $\underline{\forall}(B)$ is true (i.e. if B has no countermodel) then $\underline{\forall}(B)$ is provable in the format for first-order logic

and therefore we get (by soundness theorem) that

for every proposition $\underline{\forall}(B)$ belonging to Π^1, $\underline{\forall}(B)$ is true iff $\underline{\forall}(B)$ is provable in the format for first-order logic .

26

Completeness theorem for first-order logic, 8: refinements

As we remarked above, the format for first-order logic deals with a *syntactical* notion of proof and a *sematical* notion of refutation; and this in agreement with the paradigm traditionally adopted in logic and in philosophy of logic (during last century): proofs are always syntactical objects, refutations belong always to semantics. But, we know that a very important improvement of completeness theorem for first-order logic (due to Schutte, and based on sequent calculus) leads to revise this paradigm:

For every first-order formula B, there is a syntactical object s.t. either this object is a logical derivation of B or this object gives at least a counter-model of B.

We will show below other formats where refutations are of syntactical nature.

27

Incompleteness theorem for first-order Peano Arithmetic, 1: the format

Let us consider the following format of propositions, proofs and refutations:

- *Propositions*: a proposition is a closed formula of the language of first-order Peano Arithmetic,
- *Proofs*: a proof of a closed formula A of the language of first order Peano Arithmetic is a derivation of A from Peano Axioms
- *Refutations*: a refutation of a closed formula A of the language of first-order Peano Aritmetic is the fact that A is false in the standard model for Peano Axioms.

28

Incompleteness theorem for first-order Peano Arithmetic, 2: negative answer to the general philosophical question applied to the format

Of course, this format satisfies the condition on the formats, under the hypothesis that Peano Axioms are consistent (so that there are models of Peano Axiom): indeed, every closed formula derivable from Peano Axioms is true in the standard model for Peano Axioms.

(First) incompleteness theorem for first-order Peano Arithmetic states (under the hypothesis that Peano Axions are consistent): *There are closed formulas A in the language of first-order Peano Arithmetic, such that A is not derivable from Peano Axioms and A is true in the standard model for Peano Axioms*

This statement is clearly the negative answer to the general philosophical question applied to the format above considered.

29

Incompleteness theorem for first-order Peano Arithmetic, 3: lack of proofs, lack of refutations

As we remarked in the first part of the talk, an incompleteness theorem may be considered as a theorem saying that in the format there is a lack of proofs (for some trrefutable propositions), or as a theorem saying that in the format there is a lack of refutations (for unprovable propositions). This holds also for first.order Peano Arithmetic.

The reaction to this lack of proofs has been to get a completeness theorem by *a considerable extension* of the format of proofs, i.e. by taking as proofs of a closed formula of first-order Peano Aritmetic a derivation of the formula inside ω-logic (with a constructive ω-rule).

The reaction may be also to get a completeness theorem by *a considerable extension* of the format of refutations ... (not simply by using non-standard models for Peano Aritmetic).

30

Incompleteness theorem for first-order Peano Arithmetic, 4: another format

Let us consider the following format where both proofs and refutations are syntactical objects (indeed, the format used implicitly by Gödel in its proof):

- *Propositions*: closed formulas of the language of first-order Peano Arithmetic,
- *Proofs*: proofs of a closed formula A of the language of first order Peano Arithmetic are the derivations of A from Peano Axioms,
- *Refutations*: refutations of a closed formula A of the language of first-order Peano Aritmetic are the derivations of $\neg A$ from Peano Axioms.

31

Incompleteness theorem for first-order Peano Arithmetic, 5: another formulation

This format satisfies the condition on the formats, from the hypothesis that Peano Axioms are consistent (i.e. underv the hypothesis that it is impossible to prove both A and $\neg A$ from Peano Axioms).

We may formulate (first) incompleteness theorem for first-order Peano Arithmetic (under the hypothesis that Peano Axioms are consistent) as a negative answer to the general philosophical question applied to this format:

There are closed formulas A in the language of first-order Peano Arithmetic, such that A is not derivable from Peano Axioms and $\neg A$ is not derivable from Peano Axioms.

in this sense, we may say that incompleteness theorem for first-order Peano Aritmetics is independent from semantical considerations.

32

Σ^1-incompleteness theorem and incompleteness theorem of second-order logic, 1: the format

Let us consider an arbitrary consistent system T for second-order logic. With respect to T, we may define the following T-*format* of propositions, proofs and refutations (where propositions are logical propositions, and both proofs and refutations are syntactical objects):

- *Propositions*: the logical propositions belonging to Π^1, i.e. of the form $\underline{\forall}(B)$ where B is a closed first-order formula.
- *Proof* of a proposition $\underline{\forall}(B)$: a derivation of B inside T
- *Refutation* of a proposition $\underline{\forall}(B)$: a derivation of $\underline{\exists}(\neg B)$ inside T.

33

Σ^1-incompleteness theorem and incompleteness theorem of second-order logic, 2: remarks

Every T-format satisfies the condition on the formats, under the hypothesis that T is consistent: indeed, if there is a proof of $\underline{\forall}(B)$ and a refutation of $\underline{\forall}(B)$, then there is a derivation in T of B (so that also a derivation in T of $\underline{\forall}(B)$) and a derivation in T of $\underline{\exists}(\neg B)$, i.e. a contradiction.

Remark that - in a T-format - refutations are derivations in T of propositions belonging to Σ^1.

As a consequence of Completeness Theorem for first-order logic, we get (as we explained above) the Π^1-completeness theorem; as a consequence of Incompleteness Theorem for Peano Arithmetic, one obtains the Σ^1-incompleteness theorem stating: *There is no consistent system T s.t. every true proposition belonging to Σ^1 is derivable from T*.

34

Σ^1-incompleteness theorem and incompleteness theorem of second-order logic, 3: remarks

As a consequence of Σ^1-incompleteness theorem, we get immediately an Incompleteness Theorem for second-order logic: *There is no consistent system T s.t. every true logical proposition is derivable from T*.

Σ^1-incompleteness theorem may be formulated as a negative answer to the general philosophical question applied to the T-format, for *any* consistent system T of second-order logic.

Indeed, Σ^1-incompleteness theorem says that, if T is a consistent system for second-order logic, then there are first-order formulas $\neg B$ such that $\underline{\exists}(\neg B)$ is true but $\underline{\exists}(\neg B)$ is not derivable in T.

35

Σ^1-incompleteness theorem and incompleteness theorem of second-order logic, 4: a negative answer to the general philosophical question, applied to the format

Now, the statement $\underline{\exists}(\neg B)$ *is true* is equivalent to the statement B *ha a countermodel* and so (by Completeness Theorem for first-order logic) to the statement B *is not derivable in* T.

Therefore, we may reformulate Σ^1-incompleteness theorem as follows: *if T is a consistent system for second-order logic, then there are first-order formulas B such that B is not derivable in T and $\underline{\exists}(\neg B)$ is not derivable in T* , i.e.:

if T is a consistent system for second-order logic, then in the T-format there are propositions $\underline{\forall}(B)$ s.t. $\underline{\forall}(B)$ is unprovable and $\underline{\forall}(B)$ is unprovable.

36

Σ^1-incompleteness theorem and incompleteness theorem of second-order logic, 5: lack of proofs, lack of refutations

Remark that Σ^1-incompleteness theorem says that, for every consistent system T of second-order logic, we have a lack of proofs and a lack of refutations in the T-format.

In particular, in every consistent system of second-order logic:

- when $\underline{\exists}(\neg B)$ is not derivable, one cannot say that there is a derivation of $\underline{\forall}(B)$ i.e. a derivation of the first-order formula B;
- when $\underline{\forall}(B)$ is not derivable (i.e. when B is not derivable), one cannot say that there is a derivation of $\underline{\exists}(\neg B)$.

37

Σ^1-incompleteness theorem and incompleteness theorem of second-order logic, 6: a consequence for first-order logic

Question: is it possible to have a completeness theorem for first-order logic, with respect to the following format where both proofs and refutations are syntactical objects?

The format:

- *Propositions*: the logical propositions belonging to Π^1, i.e. of the form $\underline{\forall}(B)$ where B is a closed first-order formula (inside a well-defined first-order language).
- *Proof* of a proposition $\underline{\forall}(B)$: a logical derivation of B inside a well-defined calculus for first-order classical logic (Hilbert's system, or Sequent Calculus, or Natural Deduction calculus, ...).

38

Σ^1-incompleteness theorem and incompleteness theorem of second-order logic, 7: a consequence for first-order logic

- *Refutation* of a proposition $\underline{\forall}(B)$: a logical derivation of FALSE from instances of B inside the same calculus.

The answer is NOT, as a consequence of Σ^1-incompleteness theorem.

Indeed, in this format a proposition $\underline{\forall}(B)$ is refutable iff $\underline{\exists}(\neg B)$ is derivable ... (Details are omitted...).

39

Thank you for your attention!

40

A hint to answer the question "What is logic?"

—

"Proof-formation precedes proposition-rule formation"

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Preliminary version, from a talk at the Cerisy Meeting May 2017

Abstract

We critically discuss the contemporary notions of "formal logical language" and of "formal proof", and we remind ourselves that "formal proof" misses (or veils) some important points regarding what is logic. Instead of the traditional notions, formal language and formal proofs, as the central, we emphasize importance of the notion of "proof-formation", which determines (decides or commits to) rules and propositions. This would provide us with a different view of logic that "proof-formation precedes proposition-rule formation".

> Russellian signs veil the important forms of proof as it were to the point of unrecognizability, as when a human form is wrapped up in a lot of cloth.
>
> —Wittgenstein, *Remarks on the Foundations of Mathematics*

Introduction

What is logic? This is, in our opinion, one of the oldest and of the newest important questions in logic. We would like to point out that it is not so easy to answer this question. To answer the question "what is logic?" would involve various different issues. For example: Is logic related to the way of our thinking or independent of it? Does it give certification of our argumentation? Is logic classified by deduction, induction and abduction as Peirce did? If so, why deduction, or the logical consequence relation, has been considered important among the three by the "logicians"? How can we understand cultural differences of logic, e.g. in the historical contexts of Aristotle, logics in the European and Islamic worlds and of Indian-East Asian logics? How about differences of logic studies among different academic communities, such as philosophy, mathematics, computer science, AI, psychology, neuro-science, and others?

Although there are many issues to be discussed to answer the question, we here take only one topic: "proofs precede propositions". We would like to sketch our view of "proof formation", in contrast to the 20th century view of "formal proofs". In our opinion, the discussion on this topic could provide us with a hint to answer the question "what is logic?", from a standpoint of philosophy of proofs. We would like to remark that the 20th century notions of "formal logical language" and "formal proof" may miss (or veil) some important points regarding what is logic. In particular, the formal proofs are defined by means of the formal proposition and rules which are pre-defined, (and semantics of proofs depends on semantics of propositions or on semantics of rules). We challenge to emphasize different views, especially, the view that "proof-formation precedes proposition-rule formation". This view would help, in our opinion, remove the veil of fixed formal language and formal proofs from logic, and help us reveal what is logic.

1 "From propositions to proofs" or "from proofs to propositions"? Formal-proofs vs proof-forms

We introduce our view that the "proof formation" comes first and the proof formation activity determines which rules of logic we follow. For simplicity of the argument we mainly consider only limited alternatives of logical grammar rules. For simplicity we consider classical, intuitionistic and sub-structural (linear) logical rules as possible alternatives of the rules below.

The notion of modern logic depends on the notions of formal logical language and of formal proof. The typical formality here consists of the inductive rules-based generative rules, in fact "formal logical propositions" are inductively generated by inductive rules (which may be called grammar in the generative linguistic sense). Then the closed set of formal-proofs is given by a fixed set of generative rules, which are called logical inferences and axioms (with or without another generative set of extra-logical, e.g., mathematical, axioms). From this point of view the basic logical expressions are inductive structures, or more precisely those of inductive data-types. This modern notion of formal proofs presumes that "propositions precedes proofs". (The semantic characterization of logical consequence relation has more or less the same steps, from interpretations of propositions to the soundness-completeness properties of inductively defined "provability".)

We would like to express limitations of the notion of formal proofs based on formal universal language and pre-fixed rules and formal propositions.

Brouwer and his intuitionist school claim that the proof constructions are not closed by any fixed language nor fixed rules. In this sense, we share some claims with them. However, we do not go into the issues of mental constructions of proof. We rather keep ourselves our language-game based situational externalist position. The formal proofs presume fixed rules for proving, but the proof-formations could proceed the decisions as to which rules, for example, either classical or intuitionist or other rules at each step of proof-formation, in our view. Rules and norms could be

revised in the next step of proof-formation. The revisability/modifiability of rules is, in our opinion, an important nature of proof-formation in general. (For example, a commitment to a new proof formation of $30 \times 30 = 900$ may be a commitment to a new rule and new concept, e.g., use of exponentiation in our ordinary life, rather than multiplication. This is not canceling the multiplication rules, but connecting new exponentiation rules to multiplication rules and to counting: in this way the network of norms is revised. We do not enter this topic in detail though.) A new proof formation is to commit to a new rule; how to calculate is changed (e.g., from "how one should do with multiplication" to "how one should do with exponentiation"). A new calculation rule is taken even for the name result. A new commitment to a rule may be forming a concept of exponentiation. Possibility of revisions of rules would be a requirement for proof formation.

The following is a sort of simplest example from dialogical/game-form proof formation. Our emphasis here is that proof-game-formation does not presume logical rules/grammar fixed. This standpoint is shared with Girard's Ludics. Ludics does not presume pre-fixed formation of proposition nor rules for proof-formations. This is also shared with our standpoint that "proof-formation precedes proposition-formation" which is contrasted to the modern notion of "formal proof" which presupposes "proposition-formations precedes formal proof-formation".

I would say that there are cases in which the logical rules are not pre-fixed before the proof-formation activities. Here, a proof formation has the aim to not to loose the debate against the opponent.

Let's consider "A or not A", or say "God exists or God does not exists". A dialogue of the debate between the opponent and proponent could be seen as a game semantics way, but now the logical rules, or the norms to follow, are determined only with the game-playing, which is different from the fixed rule game.

- The opponent asks me: would you defend "God exists" or "God does not exist"?
- I respond: I defend "God exists". And I ask the opponent which she/he defends.
- Opponent responds: I defend "God does not exists".
- I say: I change my mind and I defend "God does not exist".

When this changing my mind is allowed, at this moment it is a commitment to take the classical logic rule, and if the opponent also agrees to this commitment, the two claim the same, and I do not loose the debate (usually). (If the opponent forms the third proof of Thomas Aquinas, I can do the same.) When changing my mind is not allowed, one needs to keep the intuitionistic logical rules at the moment.

> Why should I not say: in the proof I have won through to a *decision*?
>
> The proof places this decision in a system of decisions.
>
> (Wittgenstein, RFM III-28)

Je simule les questions et réponses de l'Opposant

$A \vdash A$

Je réponds à l'Opposant avec l'autre choix "A"

$A \vdash$

L'Opposant contre-attaque

$\vdash$ non-A

Je réponds

Lequel défendez-vous, A ou non-A ?

L'Opposant pose la question

Dieu existe ou il n'existe pas

$\vdash$ A ou non-A

Je défendrai

In fact, starting assumptions and rules are revised depending on the situation in the dispute, or on the situation in rational reasoning, but still we form a proof: This formation activity is not a "formal proof" of a fixed logical grammar/rules in the traditional sense, which is not revisable. The issue would be the choice of logical grammar, in the similar way to the grammar choice in the case of realist/anti-realist issues of the case of Dummett, but revisability, at any time during a proof formation, of the underlying grammar rule is important for us (which is different from the case of Dummett). (Revision of the defense statement from "God exists" to "God does not exist" has changed the meaning of the statement that was the starting statement of this dispute, namely the meaning of "God exists or God does not exist". The statement of the meaning is keeping revised as the meaning of the statement is in fact the proof being revised and formed. Here, we have used the word "meaning" naively; more strictly speaking, this is the change of concept-formation, if we stick to Wittgenstein's wordings.)

Although we do not show enough textual evidence in this (version of the) paper, our view is closely related to Wittgenstein's view of mathematical proofs in his RFM. Wittgenstein often emphasizes that proving is a decision of rule and a commitment to a norm, and proving is a concept formation and networking of norms. (The view that proving is a networking to form a new concept will be also mentioned when we discuss Husserl later.) Here he restricts his attention to the "mathematical" propositions. He considers a proof of, say, $5 + 3 = 8$, and tells us how to calculate this is a "proof".[1] and calculation in different ways are different proofs. There is no mathematical proposition without proof. There are

[1] Would one say that someone understood the proposition "$563 + 437 = 1000$" if he did not know how it can be proved? Can one deny that it is a sign of understanding a proposition if a man knows how it could be proved?

Wittgenstein: Each proof proves not merely the truth of the proposition proved, but also that it can be proved "in this way".

The interrogator: But, this latter can also be proved in another way, and in doing so.

no two proofs for one single mathematical proposition because each proof is a mathematical proposition. The ending proposition (theorem) can be deleted from a proof unless the proof still "shows" how to calculate and how to apply. This is exactly the case when the theorem (ending proposition) is algebraic such as the distributive law or the commutative law of arithmetic. A proof makes formation which is referred to paradigm, model and patterns. Wittgenstein keeps a strikingly coherent view of arithmetical proposition-calculation-proof from *Tractatus* through the mid period to the late period (although the mathematical propositions is called the pseudo-propositions in *Tractatus*).[2]

To make sense of these "proof-formation" games, there is a "frame" for all possible moves, which makes sense for the decision of the next move. (Spending time on a boat on the sea makes no sense for a mountain-climbing activity game.) A natural candidate of this basic frame to make sense of proof-formation games would be some framework of restricted use of natural deduction or syllogism or quantificational logic, which we discuss in the next Section, partly.

Proof-formation sets/revises a network of norms in various ways in our opinion. (In contrast to this, the traditional "formal-proofs" have no relations to norms and norm-revisions, although they might have some connection to formal truth and formal satisfaction.)

Although I do not enter this in detail in this paper and leave this in the subsequent papers, non-prefixed logical grammars are chosen, say at every step, during proof formations and we could keep pure proof formation for practical reasoning, too. Many philosophers think practical reasoning is not directly related to the notions of formal logical consequence, hence formal proof. Something wrong here is the traditional notion of "formal-proof". Then, we could capture rational and practical reasoning by the proof-formation activities, without considering a domain of practical logical reasoning independent of the basic logical proof formation activities.

Wittgenstein: Yes, but the proof proves this in a particular way and in doing so proves that it can be demonstrated in this way. (RFM III 59)

A proof is a mathematical entity which cannot be replaced by any other.

The new proof shows (or makes) a new connection. (But in that case is there not a mathematical proposition saying that this connection exists?) (RFM III 60)

The mathematical Must is only another expression of the fact that mathematics forms concepts.

Mathematics forms a network of norms. (RFM VII 67)

[2]Wittgenstein's view on proofs is strikingly coherent from the early work, *Tractatus*, to the later work. Although the connection of the argument here to that of *The Big Typescript* [Marion-Okada 2017] is both explained here, it will be provided by the author in a subsequent paper.

Here, we only point out the following. Committing a proof (namely how to calculate) is the commitment to norm.

I go through the proof and say "Yes, this is how it *has* to be; I must fix the use of my language in *this* way." (RFM III 30)

A mathematical proof moulds our language. (RFM III 71)

The following is usually considered a "practical" reasoning, or practical Modus Ponens in practical reasoning (which has been often claimed "not captured by formal logical inferences") as to how one *should* act.

(1) I desire that my tooth decay is cured.

(2) If I desire that my tooth decay is cured, it is necessary that I have a dentist appointment.

(3) I should have a dentist appointment.

I would say that this is just another proof-forming of proof-based logic with the usual Modus Ponens formation. We do not need propositional attitude forms or deontic modality to form proof.

(1) I have a dentist appointment.

(2) If I have a dentist appointment, then my tooth decay is cured.

(3) My tooth decay is cured.

My forming this proof *is* my decision and commitment to curing my tooth decay. There may be a different proof formation, with tooth-extraction or medication. Commitment to a different proof would show how to calculate it, namely, how to proceed it, i.e., how to do with the tooth. Commitment to this proof sets a norm which I should follow. I should follow this procedure which I have accepted as a proof formation. By accepting the proof, I form a proof networking with practical life, and you commit yourself with (1) to determining this is the proof. Now, I know what means "I cure my tooth decay". This means this proof. I might have had a desire or expectation that my tooth is cured. But, the exact desire and expectation is clearer only after a proof is formed where the norm is set and connected to our practical life as to what I should do. The (my) proof determines the norm and I commit myself to go to a dentist. This norm is the norm of what to do, but also the norm of what "to go to a dentist" means, etc.

This is forming the "inferential networking" of "my curing a tooth decay" to others. When a proof-network is forming as above, one has the commitment as to what one proves and the proof is very often the matter of her/his matter of decision. And it set the norm regarding going to dentist and makes her/him know/believe what means going to a dentist at the same time. When she/he commits to the proof, she/he finds another

Here, one could say that the classical logical implication does not fit for (2) above. It is not the classical logical implication. Intuitionistic implication would serve better? No, neither fits (note that there is no difference as the grammar between the intuitionistic Modus Ponens and the classical.) In fact, it is linear logical implication involved which expresses a state-change. The state of having a dentist appointment changes to the state of the tooth decay being cured. This is also understood as the resource-sensitive implication. The state of having an appointment is consumed

to generate the tooth's being cured. We do not discuss these implicit linear logical proof formations in it further. (we shall discuss this elsewhere), but what we would like to say here is that the so called practical reasoning or practical rationality about what one *should* act, can be considered based of non-practical *proof-formation* with revising rules/grammars from classical to intuitionistic to linear and others. Normativity comes with one's commitments with proof formation (of normal proof in the sense of logic).

Remark on the flexible choices between classical-intuitionistic-l logical rules implicit in the controversy in cognitive psychology of logical inferences, and the relation to our view. Although the main controversy has been about mental process models of logical proving, there is an important reading of this psychological modeling issues from the point of view of the logical grammar choices in our context.

There has been the central controversy of cognitive psychology of logical inferences since 1970s, between the Mental Model and Mental Logic. This controversy was the cognitive psychological version of the foundational Semantics-vs-Syntax debate as well as the Classical Logic vs Intuitionistic Logic debate several decades before in logic, as pointed out in [Grialou-Okada 2003]. The Mental Logic process model takes an intuitionistic natural deduction proof-formation process model as the basic process model. Braine claims that the natural deduction inference process is "natural" as Gentzen claims. (It was claimed in both introductions of their papers!). Gentzen's notions of normal proof and of normalizations work properly (with the subformula property) not for classical proofs but intuitionistic proofs. So, Gentzen rather suddenly moves from the topic of natural deduction proof formations to sequent calculus in order to introduce the notion of normal proof formation of classical logic. But, it is important to see the implicit difference between the natural deduction "proof-formation" which is essentially claimed to be the rules of our logical thinking and the sequent calculus which is (rather artificial as thinking rules, or just formation for counter-examples) formalism of "formal-proof" system. Although Gentzen does not explain explicitly the notions of normal proof and of normalization with "natural" deduction because he is in a hurry to move to sequent calculus formal proofs in the paper, it is much more clear that Gentzen knows the normal form theorem and normalization theorem of intuitionistic proofs perfectly. (It is given explicitly by Prawitz much later by reading Gentzen.) Gentzen's normal proof formation of natural deduction or natural rules of thinking is a striking claim and example of non-formal but proof-formation properly with modern logic. This intuitionistic proof-formation of natural deduction could be considered a basis of normal and normative proof formation (especially if we take the standpoint that deductive proof-formations are related to practical-normative reasoning, as briefly discussed above). This could be a basis but, as I claimed above, the proof formation is not determined by fixed universal rules, neither intuitionistic nor classical, for example. Depending on the situation, refutational-proof formation is introduced, or double-negation

is eliminated, while in other situations it is forbidden, not allowed. Some other restrictions could be set, e.g. some substructural rule restrictions (in the case of state-transition style logical implications, for example).

The Mental Logic School of cognitive psychology of logical inference and proof took that natural deduction proof processes are the basis of our deductive mental processes. It is also in harmony with the above setting of non-deterministic formal language. First of all, this is a mental process model of logical thinking, and not fixed formal language and formal proofs. Second, no determination of choice of logical rules because although intuitionistic natural deduction rules are the basic frame, de Morgan equivalent rules are allowed to be taken as rules when needed. Third, proof-formations are within a small range of logical complexity because the process model is concerned with our real-feasible mental processes.

2 Proof-formations as graphic network formations

To see how the proof formation is different from "formal-proofs" of formal logical-linguistic language, one direction of our examinations could be that of diagrammatic-graphic proof formations or proof-network formation.

Peirce, independently of Frege, formulated first order (quantified) predicate logic. The propositions and proofs are represented by his diagrammatic-graphic system, called Existential graphs (E-graphs). One can see the representation system of E-graphs for representing quantified first order logical formulas are defined "inductively" too, as usual inductively defined linguistic formulas. But, the nature of "inductivity" is different from the purely formal-linguistic ones. The formation of graphs for simple formulas indeed have visual assistance of the logical relations of the formulas. In this sense, this line of diagrammatic representation system of first order is the same line as those of his previous work on diagrammatic representation system of syllogisms, such as Euler diagrammatic systems. The underlying idea is that logical representations and deductive proof operations of ordinary life are restricted, and the diagrammatic versions of logic are of effective use. On the other hand, Existential-graph representations of quantified formulas and E-graph-based deductions seems not realistic when the formulas and deductions are complicated; for example, too many edges in a complicated way to visually understand them. For (an extreme) example, consider when one would like to do some metamathematics such as consistency proof of fist order Peano Arithmetic or model theoretic completeness proof or even soundness proof of first order logic, choosing Peirce's E.

It does not mean the demerit nor defect of Existential-graphs or graphic representations and operations of proof formations in first order logic. On the contrary, we would count this difficulty as a big merit. The graphic representations unveil the formal idealism of formal-language and formal proofs; they show, more closely than symbolic-linguistic representations, what are logical representations (e.g. infinite applicability of formal rules). The graphic-diagrammatic proofs would

reveal what is logic in our real situations of representation of propositions and operational procedures of inferring with them.

Independently of Existential-graph construction-manipulation processes, Peirce also strongly emphasizes the necessary or indispensable role of diagrams in proofs. For example, diagrammatic proofs with auxiliary lines in Euclidian geometry. As Tarski later formulated Euclidian geometry in the first order predicate logic, one can develop Euclidean proofs as "formal-proofs". But, based on Peirce's view this line of proof-developments inside the formal language would not capture geometrical proof formations (in his notation, "theorematic" deductive proof procedure).

The syllogistic reasoning and its slight modification is known to cover a quite large part of our logical inference with ordinary language. This suggests that the ideal generalization to the first order predicate logic language is too general for our own logical reasoning. Here, for the use of the syllogistic part of logic, it is important to see how diagrammatic inference is still important in comparison with ideal linguistic syllogism. A recent examination of ours shows that it is easier for the subjects to infer with Euler diagrammatic and Venn diagrammatic than with purely linguistic tasks, and that among the two major diagrammatic logics, it is easier with Euler than with Venn (see [Mineshima-Sato-Takemura-Okada 2014]; Euler diagram has more inferential features while Venn is model-checking or classical semantics. We shall discuss this elsewhere further). The essence of syllogistic proof formation is not dealing with circles, but edge-manipulation of the underlying (forest) graphs (see [Mineshima-Okada-Takemura, 2012]). In this sense, proof-formation in our ordinary life is a manifold formation, which we introduce in this Section.

We cannot presume easily that professional mathematical working proof formation activities can be captured by "formal language" as a making "formal proofs", either. The usual mathematical activities have rather shallow depths of logical complexity, namely shallow nested depth of, e.g. quantifier-alternatives or nested depth of logical implications, even though they have rich depths of mathematical complexity. It is simply because one cannot use formally-logical complex expressions in ordinary life and in mathematical communications. When some complex logical expressions are required, a new tool such as diagrammatic proof-formation grammar rules are formed with concept formations. Diagrammatic representations and proof formations are essential as the grammar of category theory. We shall discuss this elsewhere.

There are radical, or I would say revolutionary movements of reconsideration of proof formation of logic. One of the representatives of such is linear logic and its various versions, all initiated by Jean-Yves Girard. In our context of this Section, I would remind the readers that one of the main underlying ideas of linear logic, when Girard published his first paper on linear logic, was exactly the anti-inductive notion of logic that is the notion of proofnet, a graphic characterization of proof-formation of logic.

Even the linguistic proof can be seen as a picture or proof-picture. Wittgenstein, in the writing already discussed in the previous Section, referred to proof-pictures. For example, he expressed it as follows.

> The proof must be our model, our picture, of how these [arithmetical] operations have a result.
>
> The 'proved proposition' expresses what is to be read off from the proof-picture. (RFM III 25)

We would propose to read Husserl's notion of manifold in his universal arithmetic as a relational graphic proof network formation where each edge is a proof step.

Characterizing the real number theory, Hilbert applied his axiomatic method to arithmetic in 1900, where Hilbert listed axioms including the continuity axioms, while Husserl (in 1901) deleted the continuity axioms and changed the way of algebraic equational rules of Hilbert to term-rewrite rules. (See [Okada 2013], [Hartimo-Okada 2016].)

While Hilbert's continuity axiom (completeness axiom) requires the maximum model of the axioms, The definiteness criterion which Husserl imposes requires that the rewrite rules are convergent (confluent and terminating, so that any term is reduced to a unique irreducible form). Hence, the underlying idea is the same as (Knuth–Bendix) completion. Hilbert's completeness axiom requires that the model is the maximum model, while Husserl's definiteness criterion implicitly produces the minimum model (although the notions of model or semantics are not in Husserl). For example, in his 1901 manuscripts in which he discussed his universal arithmetic most intensively, Husserl's manifold is concept-formation in the sense that the relational networking of proofs makes the "concepts" (or significative intentions) of it, and is clearer (fulfilled) in the definite manifold. (see [Okada 2013]). Both Hilbertian completion to the maximum (from the view outside the proving grammar), and Husserlian completion to the minimum within the proving grammar are very much graphic or spatial, in our opinion, although this issue should be discussed further. Husserl's definiteness requirement of the possible proof-path graph (manifold) is purely graph-theoretic.

He considers the whole possible equational inference move-relations as a manifold, by a relational graphic view. No terms or equational propositions are pre-assumed as terms or propositions, but those are only positions/locations or relational edges of this possible proof-formations (open) graph, called manifold (in Husserl's sense). This may be understood as one example of "proof-formation precedes propositions".

> Each object, formally considered, is the mere locus/position [Stelle] in the network of relations [Relationsgewebe], i.e., in the relational form, where objects can be situated; and the relational form must be so rigid – it must be formally differentiated to the last degree – that it unambiguously [eindeutig] determines each locus/position in

> relation to the other loci/positions. If indeterminacies still remain here, then there would also be the possibility of formally characterizing the network of relations further. (1901, Hu XII 475, transl. by D. Willard)

His manifold is not just a set-theoretic model structure (which can denote extensional relations of each operator term); a possible proof is a process networking terms and connecting to other proofs. A concept expressed by a term is more clarified with the networking. When a new term is entered in the proof formations, the rules are extended and connected to the already networked manifold with the requirement of definiteness. He keeps this basic standpoint of manifold until FTL (1929)[3].

We would propose a dynamic reading of Husserl's notion of definite proof (of his 1901 manuscripts). When the constructor terms are pre-given before any proof-formation, he calls such manifold "constructible manifold" or "mathematical manifold". But for our discussion, a general (not constructible) manifold would be more interesting than constructor-based manifold, where the proof-formation steps re-set rules which should follow, such as Knuth–Bendix completion, to get normal terms, which serve as a standard/norm. It is also the case that Husserl converts Hilbert's algebraic equational rules to his calculable rewrite rules for his definite proof formations, namely convergent rewrite proofs. He seems to assume this type of completion partly. He expresses the normal (in his terminology, irreducible) terms as the standard (Etalons); in the case of constructor-based manifold, the standards are already pre-fixed (as the number series). In the case of non-constructible manifold, proof formations and standards are formed together, which would fit our proof formation discussed in this paper. It is also noted that both Husserl in 1901 and Knuth–Bendix in 1969 emphasize the static pre-procedures of completion. However, Husserl works on non-prefixed formal languages, and the main aim of the definite manifold theory of Husserl is to set universal arithmetic, which is linguistically open and extendable. New terms of new concepts are to be networked my means of proof-network formation, and the new concepts located by the new network become clear when the graphic structure of the network is transformed by the choice of rules (completion).

The irreducible terms are normal and standard terms and serve as the value field; they are the field of the standard model for the extensional interpretation of (universal) arithmetic. We would like to emphasize that the Husserlian proof-network and rewrite proof-network are the network of proof-formations and cannot be captured by model theory or value/denotation-based semantics. Note that the reverse direction is obvious; from the proof-formation network, one could have a standard semantic interpretation by "projection to the normal terms".[4]

[3] E. Husserl: *Formale und transzendentale Logik. Versuch einer Kritik der logischen Vernunft*, Niemeyer, Halle 1929.

[4] In this version, we had no space to present another aspect of our semantics implicit in the proofs. We can, for example, set up sound-complete semantics to the formal proofs (e.g. from [Okada 1997] to [Okada-Takemura 2004, 2007]) and can induce the usual model-theoretic soundness and completeness

3 Concluding remark

We considered some aspects of "proof-formation", in contrast to formal proofs with fixed formal language and fixed formal logical rules. We emphasized that proof formation involved norm formation and rule-decisions. Viewing proof formation as graphic-diagrammatic network formation would be helpful in considering a non-formal notion of proofs. We pointed out that our view of our proof-formation had some common basis with some writings of Wittgenstein, and also would be related to some of Husserl and Peirce. We stated our view of proof-formation as "proof-formation precedes proposition-rule formation". We believe that this view would help remove the veil of pre-fixed formal language and formal proofs from logic and help us reveal what is logic.

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from it by the projection to the provability-value field. This is the matter as to how the order of definitions and lemmas the computability-reducibility argument of normalization are arranged so that the argument can be separated to soundness and completeness of proofs semantics. This tells us which of the dual rules need to be chosen as the definition of the computability predicate. This can be done for the various semantics to proof-normalizations directly, including higher order cases, without needs of trick up to weak-normalization. Then, by projecting the whole argument to the provability-dimension (one-dimensional) from the proofs-dimension (two-dimensional), one gets the usual semantic completeness and normal-form theorem for the provability level. This is hence another example of how the projection of the "proof-normalization complex" (or manifold) to the value-fields automatically provides the semantics (and normal-form theorem) for provability. However, implicit typing is already given in this semantics for proofs, as in the case of "constructible manifold" of Husserl, mentioned above. We discuss this elsewhere.

ed., as appendix VI. (See also Hilbert & Ackermann, *Principles of Theoretical Logic*, 1928; Lecture note by Hilbert 1917.)

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Why complexity theorists should care about philosophy

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Computational Complexity

Once upon a time, people asked (and answered) the following question:

- What is a computable function?

That's all good in theory, but once first computers were built and in use, people realised there was another important question, namely:

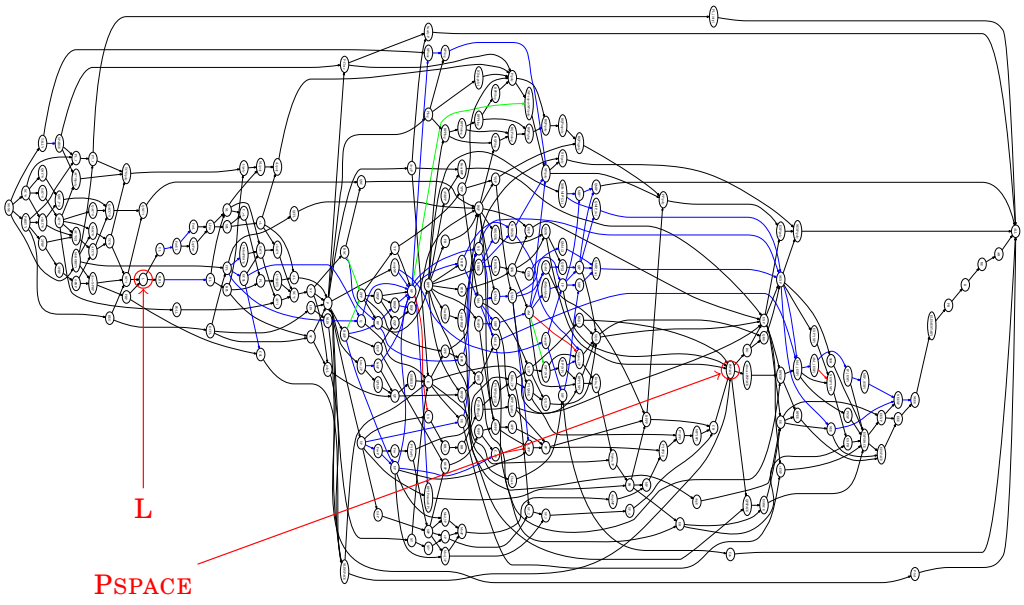
- What is an *efficiently* computable function?

I.e. what if we wanted the answer to be produced within our lifetimes (well, quicker than that really if the result is to be used somehow).

- This somehow marked the birth of *computational complexity*: three papers addressed this question within a year.
  (Cobham 1965; Hartmanis and Stearns 1965; Edmonds 1965)

And now let's fastforward.

Complexity Theory, Today (well, in 2006)



Source: Complexity Zoo; https://complexityzoo.uwaterloo.ca/Complexity_Zoo

Complexity Theory, Today

- A number of separation results were obtained, most of them in the 70s. But a lot of questions remain open. For instance: we know $\mathrm{L} \subsetneq \mathrm{PSPACE}$, but we don't know which of these inclusions are strict: $\mathrm{L} \subset \mathrm{P} \subset \mathrm{NP} \subset \mathrm{PSPACE}$.
- In fact, the three more important results are *negative results* (called *barriers*) showing that known proof methods for separation of complexity classes are inefficient w.r.t. currently open problems. They are: relativisation (1975), natural proofs (1995), and algebrization (2008).
- **Thus: no proof methods for (new) separation results exist today.**
- (Proviso) One research program (but one only) is considered as viable for obtaining new results: Mulmuley's *Geometric Complexity Theory* (GCT). However, according to Mulmuley, **if** GCT produces results, it will not be during our lifetimes (and maybe not our childrens' lifetime either*), since it requires the development of much involved new techniques in algebraic geometry.

Barriers in Computational Complexity.

Morally, there are two barriers (here for P vs. NP):

- Relativization/Algebrization: Proof methods that are oblivious to the use/disuse of oracles are ineffective.
  - There exists oracles $\mathscr{A}, \mathscr{B}$ such that:

$$\text{PTIME}^{\mathscr{A}^{\sim}} \neq \text{NPTIME}^{\mathscr{A}}$$

$$\text{PTIME}^{\mathscr{B}} = \text{NPTIME}^{\mathscr{B}^{\sim}}$$

- Natural Proofs: Proof methods expressible as (**Constructible**, Large) predicates on boolean functions are ineffective.
  - A natural proof of PTIME $\neq$ NPTIME implies that no pseudo-random generators (in P) have exponential hardness.

Conclusion: Lack of proof methods for separation. But why?

Barriers as Guidelines

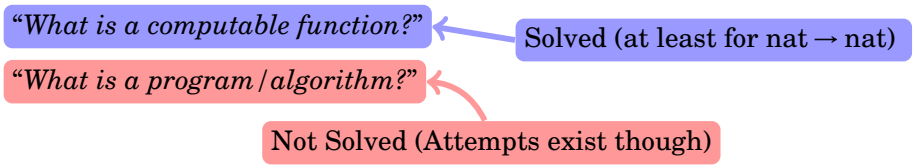
State of the Art in Complexity (Separation Problem): Barriers.

- Relativization/Algebrization: Proof methods that are oblivious to the use/disuse of oracles are ineffective.
  - Separation proof methods should depend on the computational principles allowed in the model.
- Natural Proofs: Proof methods expressible as (**Constructible**) predicates on boolean functions are ineffective.
  - Separation proof methods should not "quotient" the set of programs too much. (by definition, complexity classes are predicates on boolean functions)

Conclusion: Lack of proof methods for separation.

Thus arguably due to the following:

(Note Moschovakis already argues along these lines, but does not discuss barriers)



Digression: How to overcome barriers

- Mulmuley's program breaks *Largness*, i.e. it aims at developing techniques to prove lower bounds for a specific problem, i.e. the techniques are problem-dependent. (Mainly, GCT works with the determinant vs permanent problem in arbitrary characteristic and advocates for the use of techniques from algebraic geometry.)
- I want to break *constructivity*. (keeping the possibility of breaking largeness as well).

A more general issue

P.J. Denning, in *The Field of Programmers Myth*, Comm. ACM 47 (7), 2004:

> *We are captured by a historic tradition that sees programs as mathematical functions [...].*

- The notion of computable functions is a very bad measure of the expressivity of a model of computation. E.g. Neil Jones' *Life without cons*.

Program class	Data order 0	Order 1	Order 2	Order 3	Limit
RW, unrestricted	REC.ENUM	REC.ENUM	REC.ENUM	REC.ENUM	REC.ENUM
RWPR, fold only	PRIM.REC	PRIM^1REC	PRIM^2REC	PRIM^3REC	$\text{PRIM}^\omega\text{REC}$
RO, unrestricted	PTIME	EXPTIME	EXP^2TIME	EXP^3TIME	ELEMENTARY
ROTR tail recursive	LOGSPACE	PSPACE	EXPSPACE	EXP^2SPACE	ELEMENTARY
ROPR, fold only	LOGSPACE	PTIME	PSPACE	EXPTIME	ELEMENTARY

- More generally, complexity is a bad measure of the expressivity. Somehow, it is erroneous to think that characterising a specific class of functions, e.g. Ptime, means we understood something about complexity.
- This functional point of view can explain why we are not able to generalise the notion of complexity to higher-order functions / concurrent computation.

Better foundations

- Hypothesis: Current lack of proof methods for separation is due to a lack of adequate **mathematical** foundations.
- Suppose there exists adequate mathematical foundations.
  I.e. (this is objectively very fuzzy) for every computation $\mathscr{C}$ there exists a mathematical object $\|\mathscr{C}\|$ with $\|\cdot\|$ injective.

Claim

There are no barriers for the set of proof techniques based on such foundations.

- The argument is simple. Injectivity implies that if all such proof methods can be shown ineffective, it amounts to prove that separation is undecidable.

Better foundations

- Hypothesis: Current lack of proof methods for separation is due to a lack of adequate **mathematical** foundations.
- Suppose there exists adequate mathematical foundations.
  I.e. (this is objectively very fuzzy) for every computation $\mathscr{C}$ there exists a mathematical object $\|\mathscr{C}\|$ with $\|\cdot\|$ injective, up to some trivial equivalences (e.g. renaming of control states).

Claim

There are no barriers for the set of proof techniques based on such foundations.

- The argument is simple. (Quasi-)injectivity implies that if all such proof methods can be shown ineffective, it amounts to prove that separation is undecidable.

What is a computation/algorithm?

Several proposals.

- Turing
- Kolmogorov
- Gandy
- Moschovakis
- Gurevich.

  An ASM is a sequence of "updates" to be applied on a model of first-order logic over a fixed signature. An update is defined as either (1) a generalised assignment $f(s_1,\dots,s_n) := t$, where f is any function symbol and the s_i and t are arbitrary terms, or (2) a conditional **if** C **then** P or **if** C **then** P **else** Q, where C is a propositional combination of equalities between terms and P,Q are sequences of updates, or (3) a parallel composition of sequences of update.

Critic of Gurevich approach

- Gurevich.

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- From a point of view of capturing the notion of computation: arguably satisfying for sequential, probabilistic computation, but it is unclear if it generalises well to, e.g., cellular automata, continuous time.
- From the point of view of our project: ad-hoc objects, not based on well-founded mathematical theory. In fact, ASM may be described as generalised pseudo-code.

What is a computation/program/algorithm?

From a philosophical point of view, very few work tackle this question (at least, I could not find many). It is even more actual, with the development of new models of computation (e.g. quantum, biological).
As a starting point for the reflexion, let us consider the following questions:

- Is the universe just a big computation?
- If I let a rock fall from the top of a tower, is this a computation? If not, why?
- What about if I let a rock fall from the same tower, but depending on the initial height it activates a number n of mechanical apparatus that release a number m of balls? (e.g. the rock activates levers every meter, with the lever at height k releasing 2k+1 balls)
- What about a similar experiment where flowing water activates some mill equipped with a similar apparatus? (Is this a computation on streams?)
- Fix a mass to a spring, let go, and write down the oscillations. Is this a computation?

Where I stand

It seems important to distinguish between several different notions.

- Distinguish between experiments and a computation.
  - Differ in their intention: test the theory vs. use the theory to (locally) predict the outcome.
- Distinguish between a computation and a program.
  - Differ in their abstraction: mechanical processes / Electric signals vs. some flow of information.
  - A program is somehow distinguished from its physical realisation – the computation. I.e. one can *run* a program several times, producing several computations. However, it is bound to a model of computation (i.e. turing machines, automata, etc.).
- Distinguish between a program and an algorithm.
  - An algorithm is an abstraction of programs, free of models of computation. E.g. Sieve of Eratosthenes.
  - Very difficult task, which we will not consider here.
    cf. Blass, Derschowitz, Gurevich *When are two algorithms the same?*.

What is a program?

Informal Definition

A program is a dynamical process possibly involving exchange/duplication/erasure/modification of information.

This principle underlies a number of work.

[Complexity] Implicit Computational Complexity.
Size-change termination (Lee, Jones, Ben-Amram), mwp-polynomials (Jones, Kristiansen), Loop peeling (Moyen, Rubiano, Seiller).

[Semantics] Dynamic Semantics
Geometry of Interaction (Girard), Game Semantics (Abramsky/Jagadeesan/Malacaria, Hyland/Ong), Interaction Graphs (Seiller).

[Compilation] Compilation techniques.
Work by U. Schöpp (cf. Habilitation thesis), Loop peeling (Moyen, Rubiano, Seiller)

[VLSI design] Synthesis methods for VLSI design.
Geometry of Synthesis programme (Ghica).

What is a program?

Informal Definition

A program is a dynamical process possibly involving exchange/duplication/erasure/modification of information.

In fact a formalisation of this idea, Girard's Geometry of interaction, was intended as a proposal for mathematical foundations.

> *This paper is the main piece in a general program of mathematisation of algorithmics, called geometry of interaction. We would like to define independently of any concrete machine, any extant language, the mathematical notion of an algorithm (maybe with some proviso, e.g. deterministic algorithms), so that it would be possible to establish general results which hold once for all.*
>
> *Girard, Geometry of Interaction II (1988)*

What is a program?

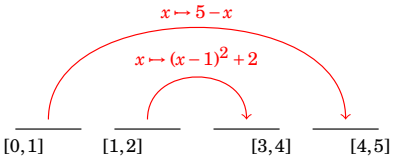
Informal Definition

A program is a dynamical process possibly involving exchange/duplication/erasure/modification of information.

- At first (technically) limited to sequential, deterministic, computation (may explain why it was somehow forgotten/discarded);
- New approach – Interaction Graphs – bypasses these limitations and allows for modelling many aspects of computation. Technically, we replace operators (i.e. bounded linear operators acting on Hilbert spaces) by *graphings*, obtaining a model which is both more general and more tractable.

What's a graphing?

- Pick a directed graph.
- Replace vertices by measurable sets, e.g. intervals on the real line.
- Decide *how* (i.e. which element of $\mathfrak{m}$) the edges map sources to targets.



The parameters of the construction:

- A measure space $(X, \mathscr{B}, \mu)$;
- A monoid $\mathfrak{m}$ of measurable maps $X \to X$ – called a **microcosm**;
- A monoid Ω;
- A type of graphing (e.g. deterministic, probabilistic);
- A measurable map $g : \Omega \to \mathbf{R}_{\geqslant 0} \cup \{\infty\}$.

Models of computation and logic

Logic	Lambda-calculus	Interaction Graphs

Proofs "Pararoofs"	Terms "Paraterms"	Winning Graphings Graphings
Cut rule	Application	Feedback
Normalisation cut-elimination	Reduction β-rule	Execution Compute paths

"Proofness" Correctness criterion	Orthogonality $t \perp E(\cdot)$ iff $E(t)$ SN	Orthogonality Complicated measurement
Formulas Proofs($A^{\perp}$) = Tests(A)	Types Realisability constr.	"Conducts" $C = T^{\perp}$, (iff $C = C^{\perp\perp}$)

Hierarchies of models

Complexity Constraints

Quantitative Aspects
(e.g. probabilities, effects)

Theorem (Seiller, APAL 2017)

For every monoid of measurable maps $\mathfrak{m}$ *(and every monoid* Ω*, and every measurable map* $g : \Omega \to \mathbf{R}_{\geqslant 0} \cup \{\infty\}$*), the set of* $\mathfrak{m}$*-graphings defines a non-degenerate model of Multiplicative-Additive Linear Logic .*

Geometric Measurement
(Ihara/Ruelle Zeta Functions)

AT LEAST!

Constraints on Graphings
(e.g. deterministic: (partial) measured dynamical systems,
probabilistic: (discrete time) Markov processes)

Hierarchies of models

Complexity Constraints

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All Geometry of Interaction constructions are recovered as specific cases

Operators in C* / von Neumann algebras (1989,1990,2011)

Unification/Resolution clauses / Prefix Rewriting (1995,2016)

Microcosms: Geometric Aspect of Complexity

We can define microcosms

$$\mathfrak{m}_1 \subset \mathfrak{m}_2 \subset \cdots \subset \mathfrak{m}_\infty \subset \mathfrak{n} \subset \mathfrak{p}$$

in order to obtain the following characterisations (as the type `nat` $\to$ `nbool`).

Microcosm	$\mathbb{M}_m^{\text{det}}$	$\mathbb{M}_m^{\text{ndet}}$		$\mathbb{M}_m^{\text{prob}}$	Logic	Machines
$\mathfrak{m}_1$	REG	REG	REG	STOC	MALL	2-way Automata (2FA)
⋮	⋮	⋮	⋮	⋮	⋮	⋮
$\mathfrak{m}_k$	D_k	N_k	coN_k	P_k	(…)	k-heads 2FA
⋮	⋮	⋮	⋮	⋮	⋮	⋮
$\mathfrak{m}_\infty$	L	NL	coNL	PL	(…)	multihead-head 2FA (2MHFA)
$\mathfrak{n}$	P	P	P	PP	(…)	2MHFA + Pushdown Stack

Refines and generalises both:

- a series of characterisations of complexity classes (e.g. L, P) with operators (with Aubert) and logic programs (with Aubert, Bagnol and Pistone);
- an independent result where I relate the expressivity of GoI models with a classification of *inclusions of maximal abelian sub-algebras*:

$$\ell^\infty(\mathbf{X}) \subseteq \ell^\infty(\mathbf{X}) \rtimes \mathfrak{m} \quad \left(\subseteq \mathscr{B}(\ell^2(\mathbf{X}))\right) \quad \text{[Feldman-Moore 1977]}$$

Microcosms: Geometric Aspect of Complexity

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$\mathfrak{n}$	P	P	P	PP	(…)	2MHFA + Pushdown Stack

- Only known correspondence between infinite hierarchies of mathematical objects and complexity classes.
- Indicates a strong connection between *geometry* and complexity: cf. microcosms generalise *group actions*, use of (generalised) Zeta functions, (homotopy) equivalence between microcosms implies equality of the classes.

A Geometric Theory of Complexity

Microcosm	$\mathbb{M}_m^{\text{det}}$	$\mathbb{M}_m^{\text{ndet}}$		$\mathbb{M}_m^{\text{prob}}$	Logic	Machines
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$\mathfrak{m}_\infty$	L	NL	coNL	PL	(…)	multihead-head 2FA (2MHFA)
$\mathfrak{n}$	P	P	P	PP	(…)	2MHFA + Pushdown Stack

Conjecture

(Equivalence classes of) microcosms correspond to complexity constraints.

Conjecture

$$\mathfrak{m} \equiv \mathfrak{n} \Leftrightarrow \mathtt{Pred}(\mathfrak{m}) = \mathtt{Pred}(\mathfrak{n})$$

A Geometric Theory of Complexity

Microcosm	$\mathbb{M}_m^{\text{det}}$	$\mathbb{M}_m^{\text{ndet}}$		$\mathbb{M}_m^{\text{prob}}$	Logic	Machines
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Conjecture

$$\mathfrak{m} \equiv \mathfrak{n} \Leftrightarrow \texttt{Pred}(\mathfrak{m}) = \texttt{Pred}(\mathfrak{n})$$

Enable (co)homological invariants to prove separation , e.g. $\ell^{(2)}$-Betti numbers:

$$\texttt{Pred}(\mathfrak{m}) = \texttt{Pred}(\mathfrak{n}) \Rightarrow \mathfrak{m} \equiv \mathfrak{n} \Rightarrow \mathscr{P}(\mathfrak{m}) \simeq \mathscr{P}(\mathfrak{n}) \overset{!}{\Rightarrow} \ell^{(2)}(\mathscr{P}(\mathfrak{m})) = \ell^{(2)}(\mathscr{P}(\mathfrak{n}))$$

($\mathscr{P}(\mathfrak{m}) = \{(x,y) \mid \exists h \in \mathfrak{m}, h(x) = y\}$ is a *measurable preorder*)

Summary

- Understand the first part as a *manifesto* to start a collaborative reflexion on the question: "What is a program", in the same way researchers once tackled the question "What is a computable function?".
- The second part is my own proposition for an answer. I believe it is a (first step towards a) satisfying solution, but I expect you to challenge it.
- The last part shows (well, very quickly mentions) how this proposition reveals some geometric nature of computation/complexity which could be exploited for developing separation methods.
- In particular, the approach defines the complexity of a *program* intrinsically (i.e. as an equivalence class of group/monoid actions/acts), i.e. a definition which is not based on an arbitrary input/output behaviour.
- While I insisted on complexity issues, the whole framework comes from logic, and raises numerous questions as to which logical systems arise from these abstract models of computation.
- Although very abstract, this is related to an automatic optimisation tool (prototype) in the LLVM compiler [Moyen, Rubiano and Seiller 2017].

Representing inferences and proofs: the case of harmony and conservativity

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Beyond Logic Conference
Centre Culturel International de Cerisy-la-Salle

Inferentialist semantics

- Inferentialist semantics are inspired by the 'meaning-as-use' paradigm: the meaning of a linguistic expression is fixed by its use.
- Two crucial aspects of the use of a linguistic expression can be singled out (Dummett 1991, p. 103):
  i) *introductory* use, answering to the question 'When should I use it?';
  i) *eliminative* use, answering to the question 'What can I do with it?'.
- When priority is given to the first aspect, a *verificationist* theory of meaning is obtained. When priority is given to the second aspect, a *pragmatist* theory of meaning is obtained.
- We will focus here on the verificationist account.

Harmony

- According to the verificationist approach, in order to be meaningful, an expression $*$ has to satisfy a condition of *harmony* between its introduction and elimination rules, such that

  > *Whatever can be drawn from the conclusion of the elimination rules for $*$ should not exceed what can already be drawn from the premisses of the introduction rules for $*$.*

- When introduction and elimination rules are expressed in terms of natural deduction rules, the harmony condition corresponds to Prawitz's *inversion principle*, consisting in the possibility eliminating *local peaks* of the form $*_I/*_E$, for a given expression $*$ (Dummett 1991, p. 248).

- The harmony condition fits particularly well with expressions corresponding to propositional connectives.

Harmony: Example (conjunction)

- Consider the (usual) rules for conjunction ($\wedge$):

$$\frac{A_1 \qquad A_2}{A_1 \wedge A_2}\,\wedge_I \qquad \frac{A_1 \wedge A_2}{A_i}\,\wedge_{E_i}\ (i \in \{1,2\})$$

- They are harmonious since local peaks of the form $\wedge_I/\wedge_{E_i}$ can be eliminated:

$$\dfrac{\dfrac{\begin{array}{cc}\mathcal{D}_1 & \mathcal{D}_2 \\ A_1 & A_2\end{array}}{A_1 \wedge A_2}\,\wedge_I}{A_i}\,\wedge_{E_i} \quad \rightsquigarrow \quad \begin{array}{c}\mathcal{D}_i \\ A_i\end{array}$$

- The same holds for standard intuitionistic connectives.

Harmony: Counterexample (tonk)

- Consider the rules for the `tonk` operator (Prior 1960):

$$\frac{A_1}{A_1 \,\texttt{tonk}\, A_2}\ \texttt{tonk}_I \qquad \frac{A_1 \,\texttt{tonk}\, A_2}{A_2}\ \texttt{tonk}_E$$

- They are not harmonious since local peaks of the form $\texttt{tonk}_I/\texttt{tonk}_E$ cannot be eliminated:

$$\dfrac{\dfrac{\begin{array}{c}\mathcal{D}\\ A_1\end{array}}{A_1 \,\texttt{tonk}\, A_2}\ \texttt{tonk}_I}{A_2}\ \texttt{tonk}_E \qquad \rightsquigarrow \qquad ?$$

Harmony: Counterexample (classical negation)

- Consider the rule for classical negation (Prawitz 1965, p. 21):

$$\dfrac{\begin{array}{c}[A]^{n.}\\ \vdots\\ \bot\end{array}}{\neg A}\ \neg_I\ (n.) \qquad \frac{\neg A \qquad A}{\bot}\ \neg_E \qquad \frac{\neg\neg A}{A}\ \neg\neg_E$$

- They are not harmonious since local peaks of the form $\neg_I/\neg\neg_E$ cannot be eliminated:

$$\dfrac{\dfrac{\dfrac{\begin{array}{c}[\neg A]^{(n.)}\\ \mathcal{D}\\ \bot\end{array}}{\neg\neg A}\ \neg_I}{A}\ \neg\neg_E\ (n.) \qquad \rightsquigarrow \qquad ?$$

- From the verificationist point of view, classical logic has to be rejected (*logical revisionism*).

Conservativity

- Another way of rejecting the `tonk` operator consists in adopting a conservativity argument, (somehow) inspired by Belnap (1962): adding `tonk` to the set of standard logical connectives leads to a non-conservative extension of the initial system.
- The idea is that an expression is not meaningful – in the sense that its use leads to linguistic misunderstandings – when the rules that govern it affect the use of other expressions.
- Consider the fragment-language $\mathcal{L} = \{\rightarrow\}$ of (positive) minimal logic $\mathbf{M}$. By adding `tonk` to it, we obtain a new system $\mathbf{M}^t$ such that

  $$\vdash_{\mathbf{M}^t} P, \text{ while } \nvdash_{\mathbf{M}} P$$

  for any atom P.
- The same argument could be applied for rejecting classical logic: classical logic is not a conservative extension of minimal logic $\mathbf{M}$ (resp. intuitionistic logic).

Conservativity

- Differently from harmony, conservativity is not a *local* condition (applying to a single expression at a time), but a *global* one (applying to a set of expressions at a time).
- Moreover, conservativity is not an absolute notion, but a *relative* one, depending on the choice of the base theory.
- Consider the language $\mathcal{L} = \{\texttt{tonk}\}$ and the theory $\mathbf{T}$ defined by the rules for `tonk`. By adding $\rightarrow$ to $\mathcal{L}$ we obtain a theory $\mathbf{T}^{\rightarrow}$ which is not a conservative extension of $\mathbf{T}$:

  $$\vdash_{\mathbf{T}^{\rightarrow}} P, \text{ while } \nvdash_{\mathbf{T}} P$$

  for any atom P.
  Should then we reject $\rightarrow$ and keep `tonk`?
- If no conditions are imposed on the choice of the base theory, then conservativity condition could be used in an arbitrary way.
- Belnap (1962, p.132) fixes the base theory by choosing a system exclusively composed by structural rules (identity, weakening, contraction, permutation, transitivity).

Does harmony imply conservativity?

- Another possibility could be to choose the base theory as composed by a set of (locally) harmonious rules.

- Under this assumption, Dummett (1973a, p. 397) advances the claim that harmony implies conservativity:

  > *[...] it is plain that the requirement of consonance [i.e. harmony] may be expressed as the demand that the addition of [a] given expression to the language yields a conservative extension of it.*
  > (cf. also Dummett 1973b, p. 221)

- More precisely, Dummett's claim can be rephrased as follows:

  (1) Let T be a theory formed by a set of harmonious rules for a language $\mathcal{L}$.
  (2) Suppose to add to $\mathcal{L}$ a new expression $*$ governed by harmonious rules.
  (3) The theory T^* obtained by adding to T the rules for $*$ is a conservative extension of T.

- Is Dummett's claim correct?

A negative answer

- Prawitz (1994, p. 374) gives a negative answer:

  > *[Dummett's claim] can hardly be correct [...] because from Gödel's incompleteness theorem we know that the addition to arithmetic of higher order concepts may lead to an enriched system that is not a conservative extension of the original one in spite of the fact that some of these concepts are governed by rules that must be said to satisfy the requirement of harmony.*

Reconstruction of Prawitz's argument (1)

- A natural interpretation of Prawitz's statement is to consider (first-order) **HA** as the base theory and extending it with rules for second-order quantifiers.

The hypothesis of Dummett's claim are satisfied:

(1) As a system of rules, **HA** is harmonious:

(i) Either the rules do not create new local peaks, e.g.

$$\frac{s(x) = s(y)}{x = y}\ P_4$$

(ii) or they create local peaks that can be eliminated, e.g.

$$\dfrac{\begin{array}{cc}\Gamma & \Delta, [A(x)]^{(n.)} \\ \vdots & \vdots \\ A(0) & A(s(x))\end{array}}{A(t)}\ Ind\ (n.)$$

where t is a numeral, and $x \notin FV(\Delta)$.

Reconstruction of Prawitz's argument (1)

- A natural interpretation of Prawitz's statement is to consider (first-order) **HA** as the base theory and extending it with rules for second-order quantifiers.

The hypothesis of Dummett's claim are satisfied:

(1) As a system of rules, **HA** is harmonious:

(i) Either the rules do not create new local peaks, e.g.

$$\frac{s(x) = s(y)}{x = y}\ P_4$$

(ii) or they create local peaks that can be eliminated, e.g.

$$\dfrac{\begin{array}{cc}\Gamma & \Delta, [A(x)]^{(n.)} \\ \mathcal{D}_1 & \mathcal{D}_2 \\ A(0) & A(s(x))\end{array}}{A(0)}\ Ind\ (n.) \quad \rightsquigarrow \quad \begin{array}{c}\Gamma \\ \mathcal{D}_1 \\ A(0)\end{array}$$

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Reconstruction of Prawitz's argument (2)

(2) The rules for second-order quantifiers are (locally) harmonious. Consider the rules for the second-order universal quantification:

$$\dfrac{\begin{array}{c}\Gamma\\ \vdots\\ A[Y^n/X^n]\end{array}}{\forall X^n A}\ \forall^2_I\ (Y^n \notin FV(\Gamma)) \qquad \dfrac{\forall X^n A}{A[T^n/X^n]}\ \forall^2_E$$

where $T^n \equiv \hat{x}_1 \dots \hat{x}_n.C(x_1, \dots, x_n)$, for a certain formula C. These rules satisfy the inversion principle:

$$\dfrac{\dfrac{\begin{array}{c}\Gamma\\ \mathcal{D}\\ A[Y^n/X^n]\end{array}}{\forall X^n A}\ \forall^2_I}{A[T^n/X^n]}\ \forall^2_E \quad \rightsquigarrow \quad \begin{array}{c}\Gamma[T^n/X^n]\\ \mathcal{D}[T^n/X^n]\\ A[T^n/X^n]\end{array}$$

- A predicative version is obtained by imposing a condition on C, that is, to be a first-order formula.

Reconstruction of Prawitz's argument (3)

The conclusion of Dummett's claim is not satisfied:

(3) Adding second-order quantifiers to **HA** leads to a system $\mathbf{HA}^*$, such that $\vdash_{\mathbf{HA}^*} Cons(\mathbf{HA}) \equiv \neg Prov_{\mathbf{HA}}(\ulcorner 0 = 1 \urcorner)$, while $\nvdash_{\mathbf{HA}} Cons(\mathbf{HA})$.

- In fact, in order to obtain such a result, it is not necessary to make appeal to full second-order logic: a predicative version of the comprehension principle is already sufficient.

- This allows one to meet the constructivistic requirements of verificationism, i.e. the possibility of establishing an *effective* procedure for constructing canonical proofs of second-order quantified propositions.

Reconstruction of Prawitz's argument (3)

- Given (i) a comprehension principle restricted to arithmetical formulas, and (ii) restricting the application of *Ind* to Π^1_1 formulas, it is possible to define a truth predicate $\mathtt{T}$, such that

  $(\mathcal{T})$ $\mathtt{T}(\ulcorner A \urcorner) \leftrightarrow A$, for all $A \in \mathcal{L}(\mathbf{HA})$;

  $(\mathcal{S})$ $st(n) \wedge Prov_{\mathbf{HA}}(n) \rightarrow \mathtt{T}(n)$.

- With $\bot \equiv (0 = 1)$, we obtain

$$\dfrac{\dfrac{\dfrac{\overset{\checkmark}{st(\ulcorner \bot \urcorner)} \quad [Prov_{\mathbf{HA}}(\ulcorner \bot \urcorner)]^{(n.)}}{st(\ulcorner \bot \urcorner) \wedge Prov_{\mathbf{HA}}(\ulcorner \bot \urcorner)} \wedge_I \quad (\mathcal{T}) \quad st(\ulcorner \bot \urcorner) \wedge Prov_{\mathbf{HA}}(\ulcorner \bot \urcorner) \rightarrow \mathtt{T}(\ulcorner \bot \urcorner)}{\mathtt{T}(\ulcorner \bot \urcorner)} \rightarrow_E \quad \dfrac{(\mathcal{S}) \quad \mathtt{T}(\ulcorner \bot \urcorner) \leftrightarrow \bot}{\mathtt{T}(\ulcorner \bot \urcorner) \rightarrow \bot} \wedge_E}{\dfrac{\bot}{\neg Prov_{\mathbf{HA}}(\ulcorner \bot \urcorner)} (n.)} \rightarrow_E$$

(see Takeuti 1987, ch. 3, § 18)

Possible objections?

- Is Prawitz's argument a definitive one?
- The argument concerns a mathematical theory. By restricting the analysis exclusively to logical theories and logical connectives, Dummett's claim could still be valid (cf. Dummett 1991, p. 215 ff.).
- However, according to Sundholm (1998), there is another way to face Prawitz's argument.
- The interest of this proposal is that it involves an analysis of inferences and proofs which does not reduce to the notion of formal derivation.

A change of perspective

- Conservativity deals with theorems, not propositions (formulas).
- A theorem corresponds to a (correct) *judgment* of the form

$$\underbrace{\overbrace{A}^{Proposition} \text{ is true}}_{Judgment} \tag{1}$$

  and the inferential steps applied in order to *demonstrate* a theorem act on judgments rather propositions.
- From the point of view of a verificationist theory, what guarantees the truth of a proposition is a proof of it.
- In this way, (1) becomes

$$\text{Proof}(A) \text{ exists} \tag{2}$$

- Yet, this judgement corresponds to an *incomplete communication* (or, following Weyl's terminology, to a *partial judgment*), waiting for the exhibition of a witness.

16 / 23

The Curry-Howard correspondence

- By adopting the point of view of the *Curry-Howard correspondence*, we have that:
  (i) propositions correspond to types (or set) of objects, so that (2) means that the type A is inhabited;
  (ii) proofs correspond to constructions (or programs) expressed by λ-terms.
- Hence, an explicit reading of (2) can be given:

$$\underbrace{\overbrace{t}^{Proof\text{-}object} : \overbrace{A}^{Type}}_{Judgment} \tag{3}$$

- λ-terms of type A are objects which keep track of the rules used for proving A.
- The Curry-Howard correspondence can be seen as a way for giving a precise formulation of the BHK interpretation, establishing then a *verificationist semantics*.

17 / 23

The Curry-Howard correspondence

- Inference rules give *construction conditions* for proof-objects; in particular, introduction rules fix the conditions for obtaining *canonical* proof-objects of given propositions. E.g.

- All steps of a proof of A can be *explicitly* built-in a λ-term: in particular, the type of discharged assumptions is indicated, and the Λ-abstraction can bind it (*Church-style presentation*).
- Thus, given a judgment

$$t : A$$

by an inspection of the way in which t has been constructed, it could be verified whether the judgment is correct or not (*type checking*).

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$$\dfrac{\begin{array}{c}\Gamma, [x : A]^{(n.)} \\ \vdots \\ t : B\end{array}}{(\lambda x : A.t) : A \to B} \to_I (n.) \qquad \dfrac{t : A \to B \qquad u : A}{(t\ u) : B} \to_E$$

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Sundholm's argument

- Consider a system $\mathbf{HA}_P$ for (first-order) Heyting arithmetic, where rules are decorated with proof-terms.
- The language of $\mathbf{HA}_P$ is composed by two parts:
  1) the language of propositions: first-order (arithmetical) formulas;*
  2) the language of proof-objects: λ-terms built with operations corresponding to first-order (arithmetical) rules.
- Extending $\mathbf{HA}_P$ with rules for second-order quantifiers (as previously done), we obtain a system $\mathbf{HA}^*_P$, such that

$$\vdash_{\mathbf{HA}^*_P} t : Cons(\mathbf{HA}) \tag{4}$$

  while $\nvdash_{\mathbf{HA}_P} t : Cons(\mathbf{HA})$.
- Since the term t keeps track of the use of second-order quantifiers for proving $Cons(\mathbf{HA}_P)$, it will contain operators like Λ and $\{\ \}$. Hence, the expression $t : Cons(\mathbf{HA})$ belongs to the language of $\mathbf{HA}^*_P$, and not to the language of $\mathbf{HA}_P$.
- $\mathbf{HA}^*_P$ is a conservative extension of $\mathbf{HA}_P$.

*This corresponds to the language of **HA**.

Non-conservativity again?

- The way in which we presented typed λ-terms considers their type as already (partially) built-in. This is the way in which we usually conceive functions, that is, with the types of their inputs (and outputs) explicitly declared.
- However, there is another way of presenting typed λ-terms, i.e. by taking them pure and assigning them a type afterwards (*Curry-style presentation*).
- Instead of $(\lambda x : A.t)$, we have $(\lambda x.t)$. By suppressing the type information the action of the Λ operator (in the proof-term) becomes harmless: it can thus be dropped together with the $\{\ \}$ operator.
- The rules for second-order quantifiers become

$$\frac{\begin{array}{c}\Gamma\\ \vdots\\ t : A[Y^n/X^n]\end{array}}{t : \forall X^n A}\ \forall^2_I\ (X^n \notin FV(\Gamma)) \qquad \frac{t : \forall X^n A}{t : A[T^n/X^n]}\ \forall^2_E$$

- Extending $\mathbf{HA}_P$ with these rules, we obtain a system which demonstrates $t : Cons(\mathbf{HA})$, but where both t and $Cons(\mathbf{HA})$ belong to the language of $\mathbf{HA}_P$. The non-conservativity phenonemon is thus recreated!

Meaningful *vs* formal rules

- Can we consider the previous argument as a genuine objection to Sundholm's position?
- In fact, no. The Curry-style rules for second order quantifiers do not allow one to express the meaning conditions for these quantifiers:

  (i) they do not allow one to express what is a canonical proof-object for a proposition $\forall X^n A$;

  (ii) they do not allow one to define an evaluation step (at the level of proof-objects) corresponding to the elimination of the local peak $\forall^2_I/\forall^2_E$.
- The Curry-style rules for second order quantifiers are simply *formal* rules, acting on purely *syntactical expressions devoided of any meaning* (in other words, λ-terms *à la* Curry are just a formal way of representing computational functions, but cannot be considered as genuine proof-objects).

Conclusions

- This leads us to the following conclusions:

  • Sundholm's argument is not a vindication of Dummett's claim obtained by trivializing it, i.e. by showing that every extension of a given theory T can be rendered conservative by adding proof-objects to the language and by decorating derivations with these proof-objects.

  • On the contrary, his position consists in claiming that the notion of conservative extension makes sense only in the case of formal, non interpreted, deduction systems, where rules act only on partial (non explicit) judgments.

  • Thus, when a theory of meaning like verificationism is under consideration, harmony seems to be a more suitable condition rather than conservativity.

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Hypothetical Reasoning in the setting of Sequent Calculi

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Cerisy La Salle, BeyondLogic, 26.05.2017

Introduction

Different approaches to hypothetical reasoning (Schroeder-Heister [2016]:

1. Placeholder view.
   1. elimination by discharge;
   2. elimination by substitution.
2. No-assumption view.
3. Bidirectional view.

Introduction

What matters in the framework of sequent calculi:

- Different interpretations of the notion of a sequent.
- Different kinds of deducibility relation induced by SC:
  - on formulae (Scott's consequence; metalogical reading of $\Rightarrow$);
  - on sequents (Gentzen's consequence; trivialization of $\Rightarrow$).

Introduction

Placeholder view

In SC we are forced to treat differently both variants:

1. Elimination by discharge.
2. Elimination by substitution.

Introduction

Placeholder view – elimination by discharge

Hypotheses in SC are expressed by axioms of the form $\varphi \Rightarrow \varphi$ which naturally correspond to arbitrary assumptions in ND.

Based on the interpretation of $\Rightarrow$ as $\vdash$ where endsequent $\Gamma \Rightarrow \varphi$ expresses the fact that φ is derived from open assumptions Γ.

Introduction

Placeholder view – elimination by substitution

This approach may be expressed in two different ways:

1. Hypotheses are additional topsequents $\Rightarrow \varphi$.
2. Hypotheses are added as a context to antecedents of (all or some) topsequents.

Note that the approach with $\Rightarrow \varphi$ as additional topsequent also may express the no-assumption view. Also the reading of $\Rightarrow$ as $\rightarrow$ (Gentzen 34) is of this kind.

Placeholder view

Elimination by substitution – hypotheses as additional topsequents $\Rightarrow \varphi$.

We have two proofs (one with hypothesis φ) and we compose them:

$$\dfrac{\dfrac{\ldots \quad \Rightarrow \varphi \quad \ldots}{D_1}}{\Gamma \Rightarrow \Delta} \qquad \dfrac{D_2}{\Rightarrow \varphi}$$

$$\dfrac{\dfrac{D_2}{\ldots \quad \Rightarrow \varphi \quad \ldots}}{\dfrac{D_1}{\Gamma \Rightarrow \Delta}}$$

Placeholder view

Elimination by substitution – hypotheses added as a context to antecedents of (all or some) topsequents.

we must also find proofs of hypotheses, then we can eliminate them by cut from the antecedent of endsequent.

$$(Cut)\ \dfrac{\dfrac{D_1}{\Rightarrow \varphi} \qquad \dfrac{\dfrac{\ldots \quad \varphi, \psi \Rightarrow \psi \quad \ldots}{D_2}}{\varphi, \Gamma \Rightarrow \Delta}}{\Gamma \Rightarrow \Delta}$$

Placeholder view

Some comments:

Both approaches synatactically equivalent to two ways of formalizing theories in SC.

In both variants $\Rightarrow$ still interpreted as $\vdash$. apparently the second is worse since it explicitly applies cut. But if this solution is applied to formalization of theories, as in Gentzen [36], then we save cut elimination (all assumptions = axioms are intact in the endsequent). On the other hand in the first variant it seems that cut is not necessary but it is not true. Consider two assumptions: $\Rightarrow \varphi, \Rightarrow \varphi \rightarrow \psi$ (the same as in the simple treatment of theories – Girard).

Placeholder view

Some comments:

Note that if we take $\vdash \subseteq P^{Fin}(Seq) \times Seq$, then $\varphi \Rightarrow \varphi$ cannot be interpreted as hypotheses since they express trivial validities.

Assumptions may be expressed by other topsequents (again in particular by $\Rightarrow \varphi$ but also by other ones; again we interpret $\Rightarrow$ as $\rightarrow$)

or by rules.

Sequents versus rules

But what sequents? what rules?

ad sequents: of particular interest are ground sequents (Gentzen 36) built from atoms. Generalised cut elimination holds.

ad rules. One particular nice solution is due to Negri and von Plato (cut elimination and generalised subformula property hold) but other are also possible.

How many?

Sequents versus rules

Rule-maker theorem – Indrzejczak [2013]

For any sequent $\Gamma \Rightarrow \Delta$ with $\Gamma = \{\varphi_1, ..., \varphi_k\}$ and $\Delta = \{\psi_1, ..., \psi_n\}$, $k \geq 0, n \geq 0$ there is $2^{k+n} - 1$ equivalent rules captured by the general schema:

$$\frac{\Pi_1, \Rightarrow \Sigma_1, \varphi_1, \ ..., \ \Pi_i \Rightarrow \Sigma_i, \varphi_i \qquad \psi_1, \Pi_{i+1} \Rightarrow \Sigma_{i+1}, \ ..., \ \psi_j, \Pi_{i+j} \Rightarrow \Sigma_{i+j}}{\Gamma^{-i}, \Pi_1, ..., \Pi_i, \Pi_{i+1}, ..., \Pi_{i+j} \Rightarrow \Sigma_1, ..., \Sigma_i, \Sigma_{i+1}, ..., \Sigma_{i+j}\Delta^{-j}}$$

where $\Gamma^{-i} = \Gamma - \{\varphi_1, ..., \varphi_i\}$ and $\Delta^{-j} = \Delta - \{\psi_1, ..., \psi_j\}$ for $0 \leq i \leq k, \ 0 \leq j \leq n$.

Sequents versus rules

Rule-maker theorem – special case

If $k = n = 1$ we have the following equivalents:

1. $\varphi \Rightarrow \psi$
2. $\psi, \Gamma \Rightarrow \Delta \ / \ \varphi, \Gamma \Rightarrow \Delta$
3. $\Gamma \Rightarrow \Delta, \varphi \ / \ \Gamma \Rightarrow \Delta, \psi$
4. $\Gamma \Rightarrow \Delta, \varphi;\ \psi, \Gamma' \Rightarrow \Delta' \ / \ \Gamma, \Gamma' \Rightarrow \Delta, \Delta'$

Sequents versus rules

Rule-maker theorem – special case

for $k = 2, n = 1$ we have:

1. $\varphi, \psi \Rightarrow \chi$
2. $\chi, \Gamma \Rightarrow \Delta \ / \ \varphi, \psi, \Gamma \Rightarrow \Delta$
3. $\Gamma \Rightarrow \Delta, \varphi \ / \ \psi, \Gamma \Rightarrow \Delta, \chi$
4. $\Gamma \Rightarrow \Delta, \psi \ / \ \varphi, \Gamma \Rightarrow \Delta, \chi$
5. $\Gamma \Rightarrow \Delta, \varphi;\ \Gamma' \Rightarrow \Delta', \psi \ / \ \Gamma, \Gamma' \Rightarrow \Delta, \Delta', \chi$
6. $\Gamma \Rightarrow \Delta, \varphi;\ \chi, \Gamma' \Rightarrow \Delta' \ / \ \psi, \Gamma, \Gamma' \Rightarrow \Delta, \Delta'$
7. $\Gamma \Rightarrow \Delta, \psi;\ \chi, \Gamma' \Rightarrow \Delta' \ / \ \varphi, \Gamma, \Gamma' \Rightarrow \Delta, \Delta'$
8. $\Gamma \Rightarrow \Delta, \varphi;\ \Gamma' \Rightarrow \Delta', \psi;\ \chi, \Pi \Rightarrow \Sigma \ / \ \Gamma, \Gamma', \Pi \Rightarrow \Delta, \Delta', \Sigma$

Sequents versus rules

Negri and von Plato approach.

variant 2 (for $k = n = 1$ and $k = 2, n = 1$) – antecedent active only in contraction-preserving form, i.e:

1. $\psi, \varphi, \Gamma \Rightarrow \Delta \ / \ \varphi, \Gamma \Rightarrow \Delta$
2. $\chi, \varphi, \psi, \Gamma \Rightarrow \Delta \ / \ \varphi, \psi, \Gamma \Rightarrow \Delta$

Advantages: Cut admissibility holds when added to system G3 via Dragalin-style proof.

Disadvantages: Many preliminaries required (h-p admissibility of structural rules and invertibility of rules).

Sequents versus rules

Our preferred solution.

variant 3 (for $k = n = 1$) and 5 (for $k = 2, n = 1$) – succedent active only in pure form, i.e:

1. $\Gamma \Rightarrow \Delta, \varphi \ / \ \Gamma \Rightarrow \Delta, \psi$
2. $\Gamma \Rightarrow \Delta, \varphi; \Gamma' \Rightarrow \Delta', \psi \ / \ \Gamma, \Gamma' \Rightarrow \Delta, \Delta', \chi$

Sequents versus rules

Our preferred solution.

variant 3 (for $k = n = 1$) and 5 (for $k = 2, n = 1$) – succedent active only in pure form, i.e:

1. $\Gamma \Rightarrow \Delta, \varphi \ / \ \Gamma \Rightarrow \Delta, \psi$
2. $\Gamma \Rightarrow \Delta, \varphi; \Gamma' \Rightarrow \Delta', \psi \ / \ \Gamma, \Gamma' \Rightarrow \Delta, \Delta', \chi$

Advantages: Cut admissibility holds via Ciabattoni, Metcalfe, Montagna-style proof. More natural. No specific form of SC needed. No preliminaries needed.
Disadvantages: Worse branching factor.

Sequents versus rules

Our preferred solution – cut elimination.

The proof of cut elimination is based on two lemmata:

Lemma (Right reduction)

Let $\mathcal{D}_1 \vdash \Gamma \Rightarrow \Delta, \varphi$ and $\mathcal{D}_2 \vdash \varphi^k, \Pi \Rightarrow \Sigma$ with $d\mathcal{D}_1, d\mathcal{D}_2 < d\varphi$, and φ principal in $\Gamma \Rightarrow \Delta, \varphi$, then we can construct a proof $\mathcal{D}$ such that $\mathcal{D} \vdash \Gamma^k, \Pi \Rightarrow \Delta^k, \Sigma$ and $d\mathcal{D} < d\varphi$.

Lemma (Left reduction)

Let $\mathcal{D}_1 \vdash \Gamma \Rightarrow \Delta, \varphi^k$ and $\mathcal{D}_2 \vdash \varphi, \Pi \Rightarrow \Sigma$ with $d\mathcal{D}_1, d\mathcal{D}_2 < d\varphi$, then we can construct a proof $\mathcal{D}$ such that $\mathcal{D} \vdash \Gamma, \Pi^k \Rightarrow \Delta, \Sigma^k$ and $d\mathcal{D} < d\varphi$.

Since all additional rules have active formulae on the right only then the Right reduction lemma goes without any changes.
We need only additional work for the Left reduction lemma.

Sequents versus rules

What with quantified statements?

One may easily obtain rules for the class of universal implications of the form:

$\forall x_1 \dots x_k(\varphi_1 \wedge \dots \wedge \varphi_n \to \psi_1 \vee \dots \vee \psi_m)$,

where all φ's and ψ's are atomic formulae.

To each such universal implication, the general schema of SC rule in our favourite form will be:

$$\frac{\Gamma \Rightarrow \Delta, \varphi_1 \quad \dots \quad \Gamma \Rightarrow \Delta, \varphi_n}{\Gamma \Rightarrow \Delta, \psi_1, \dots, \psi_m}$$

Sequents versus rules

What with quantified statements?

This result may be strenghtened to the class of basic geometric formulae of the form:

$$\forall x_1 \dots x_k\big(\varphi_1 \wedge \dots \wedge \varphi_n \to \exists y_1 \dots y_l(\psi_1 \vee \dots \vee \psi_m)\big),$$

where $k \geqslant 1$, $l, n, m \geqslant 0$, φ's are atomic formulae and ψ's are either atomic formulae or finite conjunctions of atoms.

Sequents versus rules

Rules for basic geometric formulae – Braüner [2009]

for every

$$\forall x_1 \dots x_k \big(\varphi_1 \wedge \dots \wedge \varphi_n \to \exists y_1 \dots y_l (\psi_1 \vee \dots \vee \psi_m)\big),$$

there corresponds a rule of the following form:

$$\frac{\Gamma \Rightarrow \Delta, \varphi_1 \ \dots \ \Gamma \Rightarrow \Delta, \varphi_n \quad \Psi_1, \Gamma \Rightarrow \Delta \ \dots \ \Psi_m, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}$$

where no variables of $y_1, \dots, y_l$ occur in Γ, Δ, $\varphi_1, \dots, \varphi_n$, and for each $i = 1, \dots m$: Ψ_i is a set of atoms that form conjunction ψ_i.

Sequents versus rules

Rules for basic geometric formulae – Negri and von Plato

for every

$$\forall x_1 \dots x_k \big(\varphi_1 \wedge \dots \wedge \varphi_n \to \exists y_1 \dots y_l (\psi_1 \vee \dots \vee \psi_m)\big),$$

there corresponds a rule of the following form:

$$\frac{\varphi_1 \ \dots \ \varphi_n, \Psi_1, \Gamma \Rightarrow \Delta \quad \dots \quad \varphi_1 \ \dots \ \varphi_n, \Psi_m, \Gamma \Rightarrow \Delta}{\varphi_1 \ \dots \ \varphi_n, \Gamma \Rightarrow \Delta}$$

where no variables of $y_1, \dots, y_l$ occur in Γ, Δ, $\varphi_1, \dots, \varphi_n$, and for each $i = 1, \dots m$: Ψ_i is a set of atoms that form conjunction ψ_i.

Sequents versus rules

Rules for basic geometric formulae

For our preferred format of rules things are harder. Of course we can use rules of the form:

$$\frac{\Gamma \Rightarrow \Delta, \varphi_1 \quad \ldots \quad \Gamma \Rightarrow \Delta, \varphi_n}{\Gamma \Rightarrow \Delta, \psi_1, \ldots, \psi_m}$$

but this time ψ-s may be not atomic and the Right reduction lemma fails in such a case. Fortunatelly we may transform every Braüner's rule into finite set of rules of the form:

$$\frac{\Gamma \Rightarrow \Delta, \varphi_1 \quad \ldots \quad \Gamma \Rightarrow \Delta, \varphi_n}{\Gamma \Rightarrow \Delta, \psi_1, \ldots, \psi_m}$$

where for each $i \leq m$, ψ_i is a selected (atomic) element of Ψ_i, for every combination of these atoms.

Sequents versus rules

Rules for basic geometric formulae

For example, a rule:

$$\frac{\Gamma \Rightarrow \Delta, \varphi \quad \psi_1, \psi_2, \Gamma \Rightarrow \Delta \quad \psi_3, \psi_4, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}$$

is equivalent to 4 rules of the form:

$$\frac{\Gamma \Rightarrow \Delta, \varphi}{\Gamma \Rightarrow \Delta, \psi_i, \psi_k}$$

where $i = 1$ or $i = 2$ and $k = 3$ or $k = 4$.

Case Study – Identity

Some solutions:

Takeuti:

$\Rightarrow x = x$ $\qquad$ $x = y, \varphi(x) \Rightarrow \varphi[x//y]$

Negri, von Plato:

$$\frac{x = x, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta} \qquad \frac{\varphi[x//y], \varphi(x), x = y, \Gamma \Rightarrow \Delta}{\varphi(x), x = y, \Gamma \Rightarrow \Delta}$$

φ is atomic.

Case Study – Identity

Our preferred solution:

$(1 \Rightarrow=)$ $\dfrac{}{\Rightarrow x = x}$

$(2 \Rightarrow=)$ $\dfrac{\Gamma \Rightarrow \Delta, \varphi(x) \quad \Gamma \Rightarrow \Delta, x = y}{\Gamma \Rightarrow \Delta, \varphi[x//y]}$

φ is atomic.

Case Study – Identity

How to get rid with $(1 \Rightarrow =)$?

A sequent:

$\forall x(x = \tau \rightarrow \varphi) \Rightarrow \varphi[x/\tau]$, where x is not free in τ

is equivalent to $\Rightarrow x = x$

(after Kalish and Montague ND rules for $=$).

Case Study – Identity

How to get rid with $(1 \Rightarrow =)$?

A sequent:

$\forall x(x = \tau \rightarrow \varphi) \Rightarrow \varphi[x/\tau]$, where x is not free in τ

is equivalent to:

1. $\varphi[x/\tau], \Gamma \Rightarrow \Delta \ / \ \forall x(x = \tau \rightarrow \varphi), \Gamma \Rightarrow \Delta$
2. $\Gamma \Rightarrow \Delta, \forall x(x = \tau \rightarrow \varphi) \ / \ \Gamma \Rightarrow \Delta, \varphi[x/\tau]$
3. $\Gamma \Rightarrow \Delta, \forall x(x = \tau \rightarrow \varphi); \varphi[x/\tau], \Gamma' \Rightarrow \Delta' \ / \ \Gamma, \Gamma' \Rightarrow \Delta, \Delta'$

Moreover, variant 2 may be improved:

$$\frac{x = \tau, \Gamma \Rightarrow \Delta, \varphi}{\Gamma \Rightarrow \Delta, \varphi[x/\tau]}$$

where x is not free in τ, Γ, Δ.

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Intuitionist Bilateralism: Negations, Implications and some Observations and Problems about Hypotheses

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1 Harmony: Negation, Implication and Hypotheses

In standard systems of natural deduction, the rules for intuitionist negation are *in harmony*. The grounds for and consequences of asserting $\neg A$ balance each other:

$$\neg I\colon \dfrac{\begin{array}{c}\overline{A}^{\,i}\\ \Pi\\ \bot\end{array}}{\neg A}\,i \qquad \neg E\colon \dfrac{\neg A \quad A}{\bot} \qquad \bot E\colon \dfrac{\bot}{B}$$

By applying $\neg E$, we only get out of an assertion of $\neg A$ what is required for an application of $\neg$I: a deduction of $\bot$ from A. Everything follows from $\bot$, so it has no grounds for its assertion: $\bot E$ is harmonious with the lack of an introduction rule for $\bot$.

The rules for classical negation are not in harmony, as we need to add, e.g., one of the following:

$$\dfrac{\begin{array}{c}\overline{\neg A}^{\,i}\\ \Pi\\ \bot\end{array}}{A}\,i \qquad \dfrac{\neg\neg A}{A} \qquad \dfrac{\begin{array}{cc}\overline{A}^{\,i} & \overline{\neg A}^{\,i}\\ \Pi & \Xi\\ C & C\end{array}}{C}\,i \qquad \dfrac{}{A \vee \neg A}$$

This creates a misbalance between the consequences of asserting $\neg\neg A$ and the grounds for asserting it: we get more out of $\neg\neg A$ than we put in.[1]

There are axioms and rules involving neither $\bot$ nor $\neg$ that also have the effect of resulting in classical logic when added to intuitionist logic, such as Peirce's Law

[1] Each of these four classical negation rules has its champion. Gentzen (1934): 190 opts for the fourth, later Gentzen (1936): 515 opts for the second, Prawitz (1965): 20 chooses the first, Tennant (1978) the third. Kürbis (2015) discusses intuitionist and classical negation from the perspective of a theory of meaning in Dummett's sense.

$\vdash ((A \supset B) \supset A) \supset A$, $\vdash A \vee (A \supset B)$, or $\vdash (A \supset B) \vee (B \supset C)$.[2] Putting these axioms into rule form eliminates the appeal to disjunction:

$$\dfrac{\dfrac{}{A \supset B}\,{}^{i}}{\dfrac{A}{A}\,{}^{i}\ \Pi} \qquad \dfrac{\dfrac{}{A}\,{}^{i} \quad \dfrac{}{A \supset B}\,{}^{i}}{\dfrac{C \qquad C}{C}\,{}^{i}\ \Pi\ \Sigma} \qquad \dfrac{\dfrac{}{A \supset B}\,{}^{i} \quad \dfrac{}{B \supset C}\,{}^{i}}{\dfrac{D \qquad D}{D}\,{}^{i}\ \Pi\ \Sigma}$$

If we define $\neg A$ as $A \supset \bot$, then the misbalance in the classical rules is one between the grounds and the consequences of assertions of formulas of the form $A \supset B$: the classical rules allow us to get more out of some assertions of the form $A \supset B$ than we put into them, given the introduction rule for $\supset$. The problematic classical rules for negation and implication have in common that they introduce additional options for the *discharge of hypotheses*.

The canonical ground for the assertion of $A \supset B$ is that under assumption A, I can derive B. A derivation of B from A allows discharge of A and derivation of $A \supset B$. In their antecedents, conditionals contain information about the discharge of assumptions. Peirce's Rule can be understood in the following way: if A can be derived under the assumption that A can be discharged, that is if A is the premises of some deduction, where any B will do as the conclusion, then infer A and discharge the assumption that it can be discharged. More pithily: If A can be derived under the assumption that it can be discharged, then A is true regardless.[3] The other two rules allow for analogous interpretations: if C can be derived from A and the assumption that A can be discharged, then C is true regardless; if D can be derived on the assumption that B can be concluded and that B can be discharged, then D is true regardless.

We could say that the difference between classical and intuitionist logic is located in the notion of discharge of hypotheses. This raises a question: harmony is a relation between the grounds and consequences of formulas. Grounds and consequences are complementary notions, related by the notion of harmony. Harmony of rules for a connective $*$ is a relation between $*I$ and $*E$. What the characteristically classical rules add to the harmonious intuitionist ones are further options for the *discharge of hypotheses* with $*$ as main operator. It is this wider notion of discharge that is captured by the classical conditional and principles such as Peirce's Rule. The intuitionist logician recognises only two ways of manipulating formulas with a main operator $*$ in deductions for which harmony is salient. There are, however, at least three ways: introduction, elimination and discharge of formulas with $*$ as main operator. This observation opens up a path that allows the classicist to resist the charge that classical principles governing $\supset$ such as Peirce's rule upset the harmony that holds between its introduction and elimination rules: to demand an extension of the notion of harmony such that it relates not only the introduction and elimination rules for a connective $*$, but also rules allowing the

[2] Another option is $\vdash (A \supset (B \vee C)) \supset ((A \supset B) \vee C)$.

[3] This reading of Peirce's Rule was suggested to me by Wilfried Meyer-Viol in conversation.

discharge of formulas with $*$ as main operator with a suitable complementary way of manipulating formulas in deductions. Can we extend the notion of harmony in such a way that it lets us specify something harmonious to rules such as Peirce's Law? The question is whether there is such a notion: what might it be that stands discharge of formulas as introduction stands to elimination? This requires a fourth way of manipulation formulas in deductions. One that immediately comes to mind is the making of assumptions. I leave it as an open question for further research whether the notion of harmony can be fruitfully extended in the way suggested here.

2 Classical Bilateralism

In *bilateral logic* we find a wholesale revision of what it is that is assumed and manipulated by rules of inference in deductions: rules apply to speech acts – assertions and denials – rather than propositions. In a bilateralist system of natural deduction, motivated by Price (1983) (see also Price (1990) and Price (2016)) and formalised by Rumfitt (2000), the meanings of the logical constants are specified in terms of two primitive speech acts, assertion and denial. Now the situation appears to be reversed: the rules for classical negation are in harmony, and the misbalance occurs between the grounds for and consequences of denying negated formulas in intuitionist bilateral logic.

Rules of bilateral logics apply to *signed formulas*: asserted formulas are signed by $+$, denied ones by $-$. Lower case Greek letters range over signed formulas. α^* designates the *conjugate* of α, the result of 'reversing' its sign from $+$ to $-$, and from $-$ to $+$. Rumfitt's system contains rules that specify primitively, for each connective, the grounds for asserting/denying any complex formulas and the consequences of asserting/denying them. Call the following list of rules $\mathfrak{AD}$:

$$+\&I: \quad \frac{+A \quad +B}{+A\&B} \qquad +\&E: \quad \frac{+A\&B}{+A} \quad \frac{+A\&B}{+B}$$

$$-\&I: \quad \frac{-A}{-A\&B} \quad \frac{-B}{-A\&B} \qquad -\&E: \quad \frac{\begin{array}{ccc} & \overline{-A}^{\,i} & \overline{-B}^{\,i} \\ & \Pi & \Sigma \\ -A\&B & \phi & \phi \end{array}}{\phi}\, i$$

$$+\vee I: \quad \frac{+A}{+A\vee B} \quad \frac{+B}{+A\vee B} \qquad +\vee E: \quad \frac{\begin{array}{ccc} & \overline{+A}^{\,i} & \overline{+B}^{\,i} \\ & \Pi & \Sigma \\ +A\vee B & \phi & \phi \end{array}}{\phi}\, i$$

$-\vee I$: $\dfrac{-A \quad -B}{-A \vee B}$ $\qquad$ $-\vee E$: $\dfrac{-A \vee B}{-A} \quad \dfrac{-A \vee B}{-B}$

$+\supset I$: $\dfrac{\begin{array}{c}\overline{+A}^{\,i} \\ \Pi \\ +B\end{array}}{+A \supset B}\,^{i}$ $\qquad$ $+\supset E$: $\dfrac{+A \supset B \quad +A}{+B}$

$-\supset I$: $\dfrac{+A \quad -B}{-A \supset B}$ $\qquad$ $-\supset E$: $\dfrac{-A \supset B}{+A} \quad \dfrac{-A \supset B}{-B}$

$+\neg I$: $\dfrac{-A}{+\neg A}$ $\qquad$ $+\neg E$: $\dfrac{+\neg A}{-A}$

$-\neg I$: $\dfrac{+A}{-\neg A}$ $\qquad$ $-\neg E$: $\dfrac{-\neg A}{+A}$

The four rules for negation evidently exhibit some kind of harmony. An intuitionist must reject $-\neg E$. This creates a misbalance between the grounds for denying $\neg A$ as specified by $-\neg I$ and the consequences of denying it: we get less out of denying $\neg A$ than we put in.

Gibbard (2002) points out that $\mathfrak{AD}$ is constructive logic with strong negation: double negation elimination and DeMorgan's Laws hold, but the laws of excluded middle and non-contradiction do not. Rumfitt (2002) responds that $\mathfrak{AD}$ is intended to be supplemented by a form of *reductio* he names after Timothy Smiley:

Smiley: If $\Gamma, \alpha \vdash \beta$ and $\Gamma, \alpha \vdash \beta^*$, then $\Gamma \vdash \alpha^*$

Adding Smiley to $\mathfrak{AD}$ gives a system of classical bilateral logic which I call $\mathfrak{B}$.

Notice that, in line with observations of the previous section, what needs to be added to $\mathfrak{AD}$ in a formalisation of classical bilateral logic is once more a rule that allows further options for the discharge of hypotheses. Classical logic requires a stronger notion of discharge than $\mathfrak{AD}$, constructive logic with strong negation, allows. So once more, we can locate the difference between the two systems in a difference of the rules for the discharge of hypotheses.

Alternatively to adding Smiley, Rumfitt can appeal to a notion of incompatibility between the speech acts assertion and denial, registered by $\perp$, and add the following two rules to $\mathfrak{AD}$:

Non-Contradiction: From α, α^*, infer $\perp$

Reductio: If $\Gamma, \alpha \vdash \perp$, then $\Gamma \vdash \alpha^*$

Adding these two rules to $\mathfrak{AD}$ gives a system essentially equivalent to $\mathfrak{B}$.

3 Intuitionist Bilateralism

Rumfitt claims that bilateralism is superior to unilateralism, the usual approach to natural deduction, as it allows us to justify classical and rule out intuitionist logic. In bilateral logic, classical negation is governed by harmonious rules, while intuitionist negation is not governed by harmonious rules. Closer inspection of the resources for formulating rules of inference provided by the bilateral framework shows, however, that the claim does not stand.

It is undeniable that dropping $-\neg E$ or weakening it somehow while keeping $-\neg I$ creates a misbalance. However, if we weaken *both* rules we can formulate intuitionistically acceptable, harmonious rules for the denials of negated formulas. To formalise $\mathfrak{Int}$-$\mathfrak{B}$, an intuitionist system of bilateral logic, we adopt the assertive rules of $\mathfrak{LD}$, change the rules for &, $\supset$ and $\neg$, restrict Smiley and add a version of *ex contradictione quodlibet*:

Reductio$_{Int}$: If $\Gamma, +A \vdash \beta$ and $\Gamma, +A \vdash \beta^*$, then $\Gamma \vdash -A$

ex contradictione quodlibet: $\alpha, \alpha^* \vdash \beta$
(more economically, $+A, -A \vdash +B$ suffices)

$$-\&I_{Int}: \quad \dfrac{\begin{array}{c}\overline{+A}^{\,i}\\ \Pi\\ -B\end{array}}{-A\&B}^{\,i} \qquad -\&E_{Int}: \quad \dfrac{-A\&B}{-B} \quad \dfrac{+A}{}$$

$$-\supset I_{Int}: \quad \dfrac{-B \quad \begin{array}{c}\overline{-A}^{\,i}\\ \Pi\\ \alpha\end{array} \quad \begin{array}{c}\overline{-A}^{\,i}\\ \Xi\\ \alpha^*\end{array}}{-A \supset B}^{\,i} \qquad -\supset E_{Int}: \quad \dfrac{-A \supset B \quad -A}{\beta} \quad \dfrac{-A \supset B}{-B}$$

$$-\neg I_{Int}: \quad \dfrac{\begin{array}{cc}\overline{-A}^{\,i} & \overline{-A}^{\,i}\\ \Pi & \Xi\\ \alpha & \alpha^*\end{array}}{-\neg A}^{\,i} \qquad -\neg E_{Int}: \quad \dfrac{-\neg A \quad -A}{\beta}$$

α and β can be restricted to atomic signed formulas in any rule. The rules exhibit harmony in Dummett's and Prawitz' sense: grounds and consequences of rejected formulas balance each other, and we can prove a normalisation theorem. They are also harmonious in Tennant's sense, approved by Rumfitt, where 'an introduction rule I is in *harmony* with an elimination rule E when (a) E's major premiss expresses the weakest proposition that can be eliminated when using E, with I taken as given, and (b) I's conclusion expresses the strongest proposition that can be introduced using I, with E taken as given.' (Rumfitt (2000): 790) Hence the claim that the negation rules of a bilateral intuitionist logic cannot be harmonious is incorrect.

$\mathfrak{Int}$-$\mathfrak{B}$ shares all the meaning-theoretically relevant properties of $\mathfrak{B}$ and meets all formal requirements Rumfitt imposes on satisfactory systems of bilateral logic. (See Kürbis (2016) for details.) Thus, just as $\mathfrak{B}$ according to Rumfitt, $\mathfrak{Int}$-$\mathfrak{B}$ specifies the senses of the connectives bilaterally. This time, however, that sense is intuitionist. There is nothing specifically classicist about bilateralism.

Reductio may display a symmetry that $\textit{Reductio}_{Int}$ + *ex contradictione quodlibet* does not display. This is no objection. According to Rumfitt, *Reductio* is a structural rule and not subject to considerations of harmony. It holds by stipulation: 'as a matter of simple definition, then, quite independently of the soundness of double negation elimination, the double conjugate α^{**} is strictly identical with α itself.' (Rumfitt (2000): 804) The intuitionist can adopt an analogous attitude.

Bilateralism fails where unilateralism succeeds. On Dummett's and Prawitz' unilateralist account, classical logic is anomalous, but intuitionist logic is not, and so classical logic is ruled out by proof-theoretic considerations. On the bilateral account, intuitionist logic is not anomalous, hence not ruled out by proof-theoretic considerations. As a consequences, the methodological complications introduced by bilateralism cannot be justified by claiming that their introduction allows us to meet a well-known Dummettian challenge.

Adding the other half of Smiley to $\mathfrak{Int}$-$\mathfrak{B}$ gives a system equivalent to $\mathfrak{B}$. Thus it looks as if on the bilateral account, whether a logic is classical or intuitionist depends on which version *reductio* is adopted. Looking back to the discussion of the different roles of discharge of hypotheses in classical and intuitionist logic, Smiley allows additional cases of discharge of assumptions of the form $-A$ that $\textit{Reductio}_{Int}$ does not allow. That lack of options for the discharge of denied formulas may give the impression of some kind of misbalance. Once more, however, harmony as it stands has nothing to say about what it is that might balance discharge of assumptions, and so an independent argument would be needed to establish that something is amiss about $\textit{Reductio}_{Int}$.

An extended notion of harmony that also applies to the discharge of hypotheses might get the classical bilateralist on the way to addressing the issue of how to exclude $\mathfrak{Int}$-$\mathfrak{B}$. But we don't know until it's on the table.

Even if we had an argument for excluding $\textit{Reductio}_{Int}$, whether it is based on an extended account of harmony or not, this would not yet show that there is something wrong with constructive logic with strong negation, if we do not add Smiley to $\mathfrak{AD}$, or an intuitionist version thereof, should anyone want it, if we drop $\textit{Reductio}_{Int}$ and *ex contradictione quodlibet* from $\mathfrak{Int}$-$\mathfrak{B}$. There is a more general question whether there are principled reasons for deciding between these options from the bilateralist perspective. So far, no one has given any.

4 A Problem about the Status of Hypotheses in Bilateral Logic

The formal framework of bilateral logic has no advantage over the ordinary approach to proof-theory when it comes to the question whether we should adopt

classical or intuitionist logic. In this section I will argue that it may in fact have disadvantages.

In bilateral logic, the premises, discharged assumptions and conclusions of rules of inference are supposed to be asserted or denied formulas. Many, and Rumfitt amongst them, accept the view that speech acts cannot be embedded in other speech acts. Thus, the formulas in Rumfitt's system cannot be understood as being prefixed by 'It is assertible that' and 'It is deniable that', as these are sentential operators that can be embedded.

Assertion and denial are *activities*. Ordinary proof-theory is normally understood not to be about activities but about *propositions*. Ordinary proof-theory is concerned with such activities only in a derivative sense. If I have asserted that A, and there is a deduction of B from A, then I can assert B, should assert B, or, failing that, retract my assertion of A or of some other proposition I asserted and that the deduction of B depends on. But deductions can equally be carried out independently of any assertions, even if, when all assumptions are discharged, we reach a propositions we should accept and assert as true.

How are we to understand the $+$'s and $-$'s of Rumfitt's logic? They cannot mean that some assertions or denials have actually been made. This is irrelevant for logic. Maybe no one ever asserted that he is being deceived by a most powerful and evil demon, but nevertheless we may assume that proposition and see what consequences it has. The making of assumptions is essential to logic. What is it to make an assumption in Rumfitt's system? Rumfitt often paraphrases $+A$ as 'It is correctly assertible that' and $-A$ as 'It is correctly deniable that A'. Although Rumfitt accepts 'that whenever it is correct to assert a sentence, that sentence is true; and that whenever it is correct to deny a sentence, that sentence is false' (Rumfitt (2002): 314), he does not accept the converses. 'To say that it is (objectively) correct to assert (or to deny) a sentence A is to say that knowledge is (tenselessly) available which, were a speaker to apprehend it, would warrant him in asserting (or in denying) A' (Rumfitt (2002): 313) Thus to assume that it is correctly assertible that A is a stronger assumption than the assumption that A is true, and equally to assume that it is correctly deniable that A is a stronger assumption than the assumption that A is false. To assume that A is correctly assertible is to assume that something about our epistemic state in addition to the mere truth of A, and to assume that A is correctly deniable is to assume something about our epistemic state in addition to the mere falsehood of A. The problem now is that in Rumfitt's bilateral system, all formulas are prefaced with $+$ or $-$. Thus it would appear that all assumptions in the system correspond to the stronger assumptions about our epistemic status, and nothing corresponds to the weaker assumptions of the mere truth or falsity of a formula. But it is those latter assumptions that logic is concerned with.

Weiss (2017) argues that Rumfitt's system does not allow him to draw a distinction between the truth and the assertibility of a sentence. That distinction, however, turns out to be crucial not only to Rumfitt's classicist allegiance, but to

the entire bilateralist approach, on the basis of which the classicist allegiance was supposed to be justified. A core tenet of Rumfitt's approach is that denial and assertion conditions are independent of each other in the sense that they cannot be derived from each other: in a bilateralist theory of meaning that the denial and assertion conditions of a sentence must be stipulated independently. Weiss argues that as a consequence of there being no viable distinction between truth and assertibility in the system, 'denial conditions follow from failure of assertion conditions, or, more strictly, from assertion that assertion conditions fail'. Weiss appeals to very plausible principles governing the operators 'It is assertible that' and 'It is deniable that':

(A.1) From $+ A$ infer $+$ (It is assertible that A)
(A.2) From $+$ (It is assertible that A) infer $+ A$

(D.1) From $- A$ infer $+$ (It is deniable that A)
(D.2) From $+$ (It is deniable that A) infer $- A$

Weiss argues as follows. Suppose $+$ (It is not assertible that A). If $- \neg A$, then, by $-\neg E$, $+ A$, so by (A.1), $+$ (It is assertible that A). Hence $+ \neg A$ by Smiley, and so $- A$, by $+\neg E$. Hence from (A.1), which encapsulates the lack of a distinction between truth and assertibility in Rumfitt's account, it follows that $+$ (It is not assertible that A) entails $- A$.

Rumfitt should accept (A.2), as he accepts that if a sentence is assertible, then it is true, and (A.2) is the only possibility to express this in his logic. Rumfitt should also accept (D.2), as he accepts that if a sentence is deniable, then it is false, so its negation is true, and (D.2) is the only possibility to express this in his logic. Rumfitt wants to reject (A.1), as he accepts that there may be sentences that are true but not assertible, and reject (D.1), as he accepts that there are sentences that are false but not deniable. But, as Weiss points out, he can hardly do either of them, as his logic does not allow him to reason from sentences, but only from sentences that are asserted or denied.

It is possible to back up Weiss's account by the following observation. Presumably it is inconsistent to assert A and deny that it is assertible that A, and it is inconsistent to deny A and deny that it is deniable that A. We should expect to have:

$$\frac{+ A \qquad - (\text{It is assertible that } A)}{\bot} \qquad \frac{- A \qquad - (\text{It is deniable that } A)}{\bot}$$

(A.1) and (D.1) now follow by *Reductio*. Thus we can prove Weiss's principles on the basis of what may be considered even simpler ones.

To draw the discussion to a close, there is something even worse for bilateralism than what has already been said. There lurks a danger for the coherence of Rumfitt's entire framework. It is essential to rules such as $+\supset I$ and *reductio* that

their application allows the discharge of assumptions. We may wonder what it could mean to discharge a speech act? Is it like making an assertion that is no longer needed, like having informed someone yesterday that it looks like it is raining and it is best to take an umbrella, or an assertion that is retracted, like having informed someone yesterday when it looked like it was going to rain that it will be best to take an umbrella today, but now the sun is shining? If an assertion has been made, it is not possible to make it "unhappen". Assertions are activities that have effects external to any reasoning we might do on the basis of them that cannot be made undone. Making an assumption and discharging it in no way commits a reasoner to the proposition nor any consequences that the making of the assumption might have, apart from what follows by an application of a rule that discharges it. That's the point of making an assumption. But it is worse than that. What does it mean to assume $+A$ and $-A$? It is plausible that making an assumption is a particular speech act, as argued by Dummett (1981): 309ff. $+A$ and $-A$ are supposed to represent speech acts. Rumfitt accepts that speech acts cannot be embedded in other speech acts. But then it is meaningless to assume $+A$ or $-A$.

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Ecumenism : a new perspective on the relation between logics

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Ecletism

Ecletism is not a position available to an intuitionist mathematician/logician "of faith". The classical mathematician/logician may even consider the intuitionist position quite interesting, since constructive proofs, although usually longer, are more informative than indirect classical proofs, since they have an algorithmic nature and satisfy interesting informative properties such as the disjunction property and the property of the existential quantifier.

Ecletism

To the intuitionist mathematician/logician, however, there seems to be no alternative but to revise and revoke the universal validity of certain classical principles of reasoning; for the intuitionist, mathematics must be constructed exclusively on constructively valid forms of argument. From the point of view of the classical mathematician, the intuitionist position, if taken seriously, means a mutilation of the mathematical corpus; for the intuitionist it is simply the only correct way of doing mathematics. (we cannot lose what we do not have!).

Revisionism in Logic

A possible way of disqualifying the conflict between classical and intuitionistic logic is to seek to extend to the domains of logic and mathematics a *nihilistic* view usually associated to "value nihilism". Just as in the case of basic moral values and principles, we could not argue for or against basic logical rules and principles.

Revisionism in Logic

Once an option is made for basic rules and principles, we can rightly discuss what follows and what belongs to the system determined by our basic choices, but the basis itself cannot be justified or criticized. An immediate consequence of this nihilistic position is that in accepting it, we must give up the possibility of rationally discussing the logical choices we make.

Revisionism in Logic

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Revisionism in Logic

Another standard form of disqualifying the conflict between the classical logic and the intuitionistic standpoint is based on the somewhat reasonable idea that the *litigants* are talking about distinct things (or speaking different things), and that if they are talking about different things, there is not "the same thing" - a rule or a principle - on which they diverge and dispute.

According to this position, it is as if the participants of the conflict spoke different languages and did not realize it.

A problem for revisionism in Logic

An easy argument (Quine, 1970):

1. If the deviant/revisionist logician does not accept the general validity of a classical principle of reasoning, then he gives new meanings to the concepts used in the formulation of the principle.
2. If the deviant logician gives new meanings to the concepts used in the formulation of the principle, then the deviant logician and the classical logician are not talking about the same thing (principle).
3. If they are are talking about different things, they cannot disagree!!!
4. The deviant logician does not accept the general validity of the principle.

Thus, they do not disagree!!!!

It is for no other reason that the participants (of the conflict) seek a common ground, a common minimal language, where discussion and conflict may occur.

1. A common ground
2. The finite
3. The decidable
4. Translations
5. Fragments

The Ecumenical approach

The ecumenical view

Prawitz 2015

Dowek 2015

Krauss 1992

An alternative is to use the idea of Hilbert and Poincaré that axioms and deduction rules define the meaning of the symbols of the language and it is then possible to explain that some judge the proposition $(P \vee \neg P)$ true and others do not because they do not assign the same meaning to the symbols $\vee$, $\neg$, etc.
(Dowek [2015])

The need to distinguish several meanings of a common word is usual in mathematics. For instance the proposition "there exists a number x such that $2x = 1$" is true or false depending on whether the word "number" means "natural number" or "real number". Even for logical connectives, the word "or' has to be disambiguated into inclusive and exclusive. (Dowek [2015])

Taking this idea seriously, we should not say that the proposition $(P \vee \neg P)$ has a classical proof but no constructive proof, but we should say that the proposition $(P \vee_c \neg_c P)$ has a proof and the proposition $(P \vee \neg P)$ does not, that is we should introduce two symbols for each connective and quantifier, for instance a symbol $\vee$ for the constructive disjunction and a symbol $\vee_c$ for the classical one, instead of introducing two judgments: "has a classical proof" and "has a constructive proof' (Dowek [2015])

Dag Prawitz seems to agree with Quine when he says:

"When the classical and intuitionistic codifications attach different meanings to a constant, we need to use different symbols, and I shall use a subscript c for the classical meaning and i for the intuitionistic. The classical and intuitionistic constants can then have *a peaceful coexistence* in a language that contains both."
(Prawitz [2015])

What's Prawitz' main idea?

The same meaning explanation for classical logic and intuitionistic logic.

But this does not seem possible!

Gentzen's introduction rule for disjunction (and for implication and the existential quantifier) is too strong! It cannot give the meaning of classical disjunction.

Solution: different introduction rules for classical disjunction

Interesting: two disjunctions, but the same idea of meaning explanation.

The Ecumenical logic
(Propositional Part)

Ec

The language of Ec is defined as follow:
Alphabet

1. Propositional letters: p, q, r, ...
2. logical constants: $\bot$, $\wedge$, $\neg$, $\vee_i$, $\vee_c$, $\rightarrow_i$ and $\rightarrow_c$.
3. Auxiliary signs: (,) .

The grammar of Ec is the usual.

The Natural Deduction Ecumenical system

NEc

The Natural Deduction system NEc defined by Prawitz has the following rules of inference:

1. The rules for $\wedge$, $\neg$ and for the intuitionistic operators are the usual Gentzen-Prawitz introduction and elimination for these operators.
2. The intuitionistic absurd rule:

$$\frac{\bot}{A}$$

3. The rules for classical disjunction and classical implication are defined as follows:

$$\frac{\begin{array}{c}[A] \qquad [\neg B] \\ \Pi_1 \\ \bot\end{array}}{A \to_c B} \to_c\text{-Int}$$

$$\frac{A \to_c B \qquad A \qquad \neg B}{\bot} \to_c\text{-Elim}$$

$$\dfrac{\begin{array}{c}[\neg A] \\ \Pi_1 \\ \bot\end{array}}{A \vee_c B}\ \vee_c\text{-Int} \qquad \dfrac{A \vee_c B \qquad \neg A \qquad \neg B}{\bot}\ \vee_c\text{-Elim}$$

A brief remark on classical implication and negation

(1) Classical implication: Contrary to what we could expect from any reasonable concept of conditional judgements (hypothetical judgement), the operator $\rightarrow_c$ does not satisfy *modus ponens*. This is due to the fact that the introduction rule for $\rightarrow_c$ is weaker than the introduction for $\rightarrow_i$, since the classical logician is allowed to assert $(A \rightarrow_c B)$ in cases where the intuitionistic logician is not. It is interesting to observe that the general validity of *modus ponens* for $\rightarrow_c$ would not depend solely on the meaning of $\rightarrow_c$, but would also depend on a concept of negation that is not determined by the introduction rule for negation. The classical implication $\rightarrow_c$ clearly satisfies a weak form of *modus ponens*: {A, (A $\rightarrow_c$ B)} $\vdash \neg\neg B$.
(2) Negation: It would be natural to ask why the system has just one negation, given that it has two negations and the $\bot$. The point is that $(A \rightarrow_i \bot)$ is equivalent to $(A \rightarrow_c \bot)$, both classically and intuitionistically. In a more general sense, one could also say that there is just one way to assert a negated proposition: assume the proposition and show that this assumption leads to a contradiction.

Some interesting theorems

1. $\vdash_{NEc} (A \to_i B) \Rightarrow_i (A \to_c B)$
2. $\vdash_{NEc} (A \wedge B) \Leftrightarrow_i \neg(\neg A \vee_c \neg B)$
3. $\vdash_{NEc} (A \wedge B) \Leftrightarrow_i \neg(A \to_c \neg B)$
4. $\vdash_{NEc} \neg(\neg A \wedge \neg B) \Leftrightarrow_i (A \vee_c B)$
5. $\vdash_{NEc} \neg(A \wedge \neg B) \Leftrightarrow_i (A \to_c B)$

Definition

A formula B is called *classical* if and only if its main operator is classical (we sometimes indicate that B is classical with the notation B^c)

Some more interesting theorems

1. $\vdash_{NEc} (A \to_c B^c) \to_i (A \to_i B^c)$
2. $\{A, (A \to_c B^c)\} \vdash_{NEc} B^c\}$

Interesting remark: The system NEc does not satisfy the deduction theorem!

Lot's of things to be done!

1. Proof theory
2. Truth-theoretical semantics.

Proof Theory

Reductions

The reductions for the intuitionistic operators are the usual Prawitz' reductions.
The reductions for the classical operators are defined below:

$$\dfrac{\dfrac{\dfrac{[A] \quad [\neg B]}{\Pi_1}}{\dfrac{\bot}{A \to_c B}} \quad \Pi_2 \quad \Pi_3}{\dfrac{A \quad \neg B}{\bot}}$$

Reduces to:

$$\dfrac{\Pi_2 \quad \Pi_3}{\dfrac{[A] \quad [\neg B]}{\dfrac{\Pi_1}{\bot}}}$$

$$\dfrac{\dfrac{\dfrac{[\neg A] \quad [\neg B]}{\Pi_1}}{\dfrac{\bot}{A \vee_c B}} \quad \Pi_2 \quad \Pi_3}{\dfrac{\neg A \quad \neg B}{\bot}}$$

Reduces to:

$$\dfrac{\Pi_2 \quad \Pi_3}{\dfrac{[\neg A] \quad [\neg B]}{\dfrac{\Pi_1}{\bot}}}$$

Problem for normalization: inductive measures!

Solution: new measures of complexity!

In the case of the new reductions we immediately see that through the elimination of a maximum formula, new maximum formulas of the same degree may be produced, and because of this the usual normalization strategy does not work anymore. An easy way to solve this difficult is through the modification of the usual definition of the degree of a formula as the number of occurrences logical operators in the formula.

It is clear that in the case of classical disjunction and classical implication there are some *hidden negations*, and that any definition of the complexity of a formula must take this point in consideration. The new measure of complexity of a formula A will be called the ecumenical degree of A, ed(A), and is defined as follows:

E-degree

1. $\mathrm{ed}(\phi) = 0$
2. $\mathrm{ed}(\neg A) = \mathrm{ed}(A) + 1$
3. $\mathrm{ed}(A \Box B) = \mathrm{ed}(A) + \mathrm{ed}(B) + 1$, if $\Box$ is $\wedge$ or an intuitionistic operator.
4. $\mathrm{ed}(A \vee_c B) = \mathrm{ed}(A) + \mathrm{ed}(B) + 4$
5. $\mathrm{ed}(A \to_c B) = \mathrm{ed}(A) + \mathrm{ed}(B) + 3$

The Normalization Theorem

The main definitions are standard.

Definition

A formula A in a derivation Π is a maximum formula if and only if:

1. A is the conclusion of an application of an α-introduction rule and at the same time the major premisse of an α-elimination rule in Π.
   or
2. A is the conclusion of an application of the $\bot$-rule and at the same time the major premisse of an elimination rule in Π.
   or
3. A is the conclusion of an application of the $\vee_i$-elimination rule and at the same time the major premisse of an elimination rule in Π.

Definition

The ecumenical degree of a derivation Π, $ed[\Pi]$ is defined as the max$\{$ ed[A] s. t. A is a maximum formula in Π $\}$.

Definition

A derivation Π is called *critical* iff

1. Π ends with an elimination rule α;
2. The major premiss of α is a maximum formula;
3. For every proper sub-derivation Π' of Π, $ed[\Pi'] \leq ed[\Pi]$.

Lemma

Let $\Pi 1$ / A and A/$\Pi 2$ be two derivations in NEc such that d($\Pi 1$) = n1 and d($\Pi 2$) = n2. Then, d($\Pi 1$/[A]/$\Pi 2$) = max(d[A], n1, n2).

Lemma

Let Π be a critical derivation of $\Gamma \vdash_{NEc} A$. Then, Π reduces to a derivation Π' of $\Delta \subseteq \Gamma \vdash_{NEc} A$ such that d(Π') < $d(\Pi)$.

Lemma

Let Π be a derivation of $\Gamma \vdash_{NEc} A$. Then, Π reduces to a derivation Π' of $\Delta \subseteq \Gamma \vdash_{NEc} A$ such that d(Π') < $d(\Pi)$.

Proof.

Directly from the previous lemma using induction on the length of Π. □

Theorem

Let Π be a derivation of $\Gamma \vdash_{NEc} A$. Then, Π reduces to a normal derivation Π' of $\Delta \subseteq \Gamma \vdash_{NEc} A$.

The sequent calculus LEc

The sequent calculus LEc for Prawitz' ecumenical logic is defined in the standard way. Sequents are expressions of the form $\Gamma \Rightarrow \Delta$ where Γ and Δ are multiset of formulas.

$\Gamma 1$: Innitial sequents are expressions of the form $A \Rightarrow A$, where A is an atomic formula.

$\Gamma 2$: The structural rules, the rules for $\wedge$, $\neg$, and the rules for the intuitionistic operators are the structural rules and the rules for these operators in LJ.

$\Gamma 3$: The rules for $\rightarrow_c$ and $\vee_c$ are as follows:

$$\frac{\Gamma,A,\neg B \Rightarrow}{\Gamma \Rightarrow A \rightarrow_c B} : \Rightarrow \rightarrow_c \qquad \frac{\Gamma \Rightarrow A \;\; \Delta \Rightarrow \neg B}{\Gamma,\Delta,A \rightarrow_c B \Rightarrow} : \rightarrow_c \Rightarrow$$

$$\frac{\Gamma,\neg A,\neg B \Rightarrow}{\Gamma \Rightarrow A \vee_c B} : \Rightarrow \vee_c \qquad \frac{\Gamma \Rightarrow \neg A \;\; \Delta \Rightarrow \neg B}{\Gamma,\Delta,A \vee_c B \Rightarrow} : \vee_c \Rightarrow$$

Cut-elimination for LEc

The proof of the cut-elimination theorem for LEc follows the standard Gentzen strategy:

G1 We replace the cut-rule by the mix-rule (as in the case of Gentzen's proof)

G2 We prove the following main lemma:
Let Π be a derivation of $\Gamma \Rightarrow \Delta$ in LEc, such that Π ends with an application of the mix-rule and that this is the only application of the mix-rule in Π. Then, Π can be transformed into a mix-free derivation Π' of $\Gamma \Rightarrow \Delta$.

Proof.
The proof of this main lemma is carried out by induction on the pair $< n, m >$, where n is the degree of the mix-formula and n is its Gentzen-rank. □

G3 We then prove the main result:

Cut-elimination for LEc

Theorem
The system LEc satisfies cut-elimination.

Proof.
The result now follows directly by induction on the number of cuts occurring in a derivation and the lemma above. □

Kripke semantics for Ec

Definition

An ecumenical intuitionistic Kripke model is a pair $\langle \mathcal{F}, e \rangle$ where $\mathcal{F} = \langle W, \leq, \rangle$ is an intuitionistic Kripke frame and e is a intuitionistic valuation, i.e., e is a mapping associating with each propositional variable p and element of $U_{\leq}$, where $U_{\leq}$ is the set of all uppersets of $\langle W, \leq \rangle$, i.e. all sets $U \subseteq W$ such that if $\omega \in U$ and $\omega \leq \nu$ then also $\nu \in U_{\leq}$.

Pereira and Rodriguez

Let $\mathcal{M} = \langle \mathcal{F}, e \rangle$ be an ecumenical Kripke model and $\omega \in W$. By induction in the construction of a formula φ we define the relation $(\mathcal{M}, \omega) \models \varphi$ as follows:

$(\mathcal{M}, \omega) \models p$ iff $\omega \in e(p)$;
$(\mathcal{M}, \nu) \not\models \bot$
$(\mathcal{M}, \omega) \models \neg\psi$ iff for all $\nu \in W$ such that $\omega \leq \nu : (\mathcal{M}, \nu) \not\models \psi$
$(\mathcal{M}, \omega) \models \varphi \wedge \psi$ iff $(\mathcal{M}, \omega) \models \varphi$ and $(\mathcal{M}, \omega) \models \psi$;
$(\mathcal{M}, \omega) \models \varphi \vee_i \psi$ iff $(\mathcal{M}, \omega) \models \varphi$ or $(\mathcal{M}, \omega) \models \psi$;
$(\mathcal{M}, \omega) \models \varphi \vee_c \psi$ iff $(\mathcal{M}, \omega) \models \neg(\neg\varphi \wedge \neg\psi)$;
$(\mathcal{M}, \omega) \models \varphi \rightarrow \psi$ iff $\forall\, \nu \in W$ s. t. $\omega \leq \nu : (\mathcal{M}, \nu) \models \varphi \Rightarrow (\mathcal{M}, \nu) \models \psi$
$(\mathcal{M}, \omega) \models \varphi \rightarrow_c \psi$ iff $(\mathcal{M}, \omega) \models \neg(\varphi \wedge \neg\psi)$;

Soundness

Theorem (Soundness)

$\Gamma \vdash_{NEc} A \Longrightarrow \Gamma \models_{Ec} A$

Proof.

The proof of soundness is carried out, as usual, by induction on the length of a derivation Π of $\Gamma \vdash_{NEc} A$. We shall consider just one of the possible cases, the other cases being treated in a similar way. □

Soundness

1. The last rule applied in Π is $\vee_c - int$. The derivation Π is:

$$\dfrac{\begin{array}{cc}[\neg A] & [\neg B] \\ & \Gamma \\ & \bot\end{array}}{A \vee_c B}$$

Soundness

By inductive hypothesis, we know that $\{\Gamma, \neg A, \neg B\} \models_E \bot$, i.e., that there is no world ω such that $(\mathcal{M}, \omega) \models_E \Gamma \cup \{\neg A\} \cup \{\neg B\}$. Assume now that $\Gamma \not\models_E (A \vee_c B)$, which is equivalent to say, by definition, that $\Gamma \not\models_E \neg(\neg A \wedge \neg B)$. According to the definition of semantical consequence, $\Gamma \not\models_E \neg(\neg A \wedge \neg B)$ if and only if there exists a world ν such that $(\mathcal{M}, \nu) \models_E \Gamma$ and $(\mathcal{M}, \nu) \models_E (\neg A \wedge \neg B)$. But, this leads to a contradiction, because $(\mathcal{M}, \nu) \models_E \Gamma \cup \{\neg A\} \cup \{\neg B\}$.

Completeness

In order to prove the completeness theorem for the calculus NEc, we shall define a translation Pr from the language of Ec into the language of intuitionistic propositional logic.

1. $Pr[\varphi] = \varphi$, if φ is a propositional variable.
2. $Pr[\bot] = \bot$
3. $Pr[\neg A] = \neg Pr[A]$
4. $Pr[(A \wedge B)] = (Pr[A] \wedge Pr[B])$.
5. $Pr[(A \vee_i B)] = (Pr[A] \vee_i Pr[B])$.
6. $Pr[(A \to_i B)] = (Pr[A] \to_i Pr[B])$.
7. $Pr[(A \vee_c B)] = \neg(\neg Pr[A] \wedge \neg Pr[B])$.
8. $Pr[(A \to_c B)] = \neg(Pr[A] \wedge \neg Pr[B])$.

Completeness

Lemma

$\Gamma \vdash_{NE_{ic}} A$ if and only $\Pr[\Gamma] \vdash_{I_p} Pr[A]$, where $\vdash_{I_p}$ is the usual derivability relation in the propositional fragment of Prawitz' natural deduction intuitionistic system I.

Proof.

From left to right: By induction over the length of the derivation Π of $\Gamma \vdash_{NE_{ic}} A$. We shall only consider the case where the last rule applied in Π is $\to_c$-introduction, the other cases being treated in a similar way. Let Π be the following derivation in NEc:

$$\dfrac{\begin{array}{ccc}[A] & [\neg B] & \Gamma \\ & \Pi & \\ & \bot & \end{array}}{A \to_c B}$$

□

Completeness

We can construct the following derivation:

$$\dfrac{\dfrac{[(Pr[A] \wedge \neg Pr[B])]}{Pr[A]} \qquad \dfrac{\dfrac{[(Pr[A] \wedge \neg Pr[B])]}{\neg Pr[B]}}{Pr[\neg B]} \qquad Pr[\Gamma]}{\dfrac{\dfrac{Pr[\Pi] \quad \bot}{\neg(Pr[A] \wedge \neg Pr[B])}}{A \rightarrow_c B}}$$

Completeness

In other to prove the other direction, we shall prove an auxiliary lemma:

Lemma

$\vdash_{NEc} A \leftrightarrow_i Pr[A])$

Proof.

By induction on the ecumenical degree of A. □

Completeness

Lemma

$\Gamma \vDash_{Ec} A$ *if and only* $\Pr[\Gamma] \vDash_{I_p} Pr[A]$, *where* $\vDash_{I_p}$ *is the usual consequence relation in the propositional intuitionistic logic.*

Proof.

Directly from the previous lemma and soundness. □

Completeness

Theorem

Let $\Gamma \vDash_{Ec} A$. *Then* $\Gamma \vdash_{NEc} A$

Proof.

Assume that $\Gamma \vDash_{Ec} A$. By the previous lemma, $\Pr[\Gamma] \vDash_{I_p} Pr[A]$. By the completeness of the system I_p, we have $\Pr[\Gamma] \vdash_{I_p} Pr[A]$. The result now follows directly from the auxiliary lemma. □

Related work

The idea of using different signs for the different meanings attached to intuitionistic and classical operators is not new, it was used by P. Krauss in 1992. The same idea was used again in 2015 by Gilles Dowek. Both Krauss and Dowek have classical versions for $\wedge$ and $\forall$. It is interesting to observe that [1] $\wedge_c$ does not satisfy (in general) projections and is not idempotent and that [2] $\forall_c$ does not (in general) satisfy universal instantiation. The main motivation of both Krauss and Dowek was to explore the possibility of *hybrid* readings of axioms of mathematical theories. The example discussed by Krauss is the axiom of choice and the example discussed by Dowek is also taken from set theory. The whole point is, in Dowek's own words, to consider that "which mathematical results have a classical formulation that can be proved from the axioms of constructive set theory or constructive type theory and which require a classical formulation of these axioms and a classical notion of entailment remais to be investigated".

Future work

1. We have just indicated the way to obtain a normalization for NEc. Clearly there are lots of things to be done with respect to the *proof theory* of NEc. We know that we do not have as a corollary of normalization the sub-formula principle in its usual form. But can we have a weak sub-formula principle based on the intended meaning of the classical operators? Can we have confluence? Strong Normalization?
2. It would be interesting to explore the intended meaning of the classical operators in order to obtain a Curry-Howard type of result.
3. As we mentioned above, an interesting application of ecumenical systems is related to the analysis of mathematical results that depend on ecumenical readings of axioms (see Krauss and Dowek). It would certainly be interesting to pursue the investigation of other axiomatic theories. .
4. We are also planning to define a sequent calculus and a tableaux system for the Ecumenical modal logic S4.

Further Reading I


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Normality beyond logic

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Beyond Logic

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Cerisy

- How should one characterize reductions and their associated normal forms once one steps outside the realm of "standard" systems of natural deduction ?
- On which criteria it is possible to rely in order to provide such a characterization ?
- Present framework: the proof-theoretic analysis of paradoxes.

Outline

1. Proof theory and paradoxes
2. Paradoxes in normal form
3. Conclusion

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1. Proof theory and paradoxes
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Redundancies in Natural Deduction

- Natural deduction has the following introduction and elimination rules for implication

$$\frac{\begin{array}{c}[A]^n\\ B\end{array}}{A \to B} \to I, n \qquad \frac{A \to B \quad A}{B} \to E$$

- Negation is defined as implication of absurdity

$$\neg A =_{def} A \to \bot$$

- The application in a derivation of an introduction rule followed immediately by an application of the corresponding elimination rule constitutes a redundancy. Redundancies can be eliminated by a $\to$-reduction.

$$\frac{\dfrac{\begin{array}{c}[A]^n\\ \mathcal{D}\\ B\end{array}}{A \to B} \to I, n \quad \begin{array}{c}\mathcal{D}'\\ A\end{array}}{B} \to E \qquad \rightsquigarrow_{\to} \qquad \begin{array}{c}\mathcal{D}'\\ [A]\\ \mathcal{D}\\ B\end{array}$$

Prawitz on paradoxes (I)

- Prawitz [1965] considered a system for naive set theory by extending minimal logic with an introduction and elimination rule for formulas of the form $t \in \{x : A\}$ for set-theoretical comprehension

$$\frac{A[t/x]}{t \in \{x : A\}} \in I \qquad \frac{t \in \{x : A\}}{A[t/x]} \in E$$

- An application of $\in I$ immediately followed by $\in E$ constitutes a redundancy which can be eliminated by a $\in$-reduction

$$\frac{\dfrac{\begin{array}{c}\mathcal{D}\\ A[t/x]\end{array}}{t \in \{x : A\}}}{A[t/x]} \qquad \rightsquigarrow_{\in} \qquad \begin{array}{c}\mathcal{D}\\ A[t/x]\end{array}$$

- Take λ to be $r \in r$, where r is the Russell term $\{x : x \notin x\}$.
  An application of $\in E$ allows one to pass from $\neg\lambda$ to λ; an application of $\in I$ allows one to pass from λ to $\neg\lambda$.

Prawitz on paradoxes (II)

- Russell's paradox has the form of the following derivation of absurdity

$$\dfrac{\dfrac{\dfrac{[\lambda]^n}{\neg\lambda} \in E \qquad [\lambda]^n}{\dfrac{\bot}{\neg\lambda} \to I, n} \to E \qquad \dfrac{\dfrac{\dfrac{[\lambda]^m}{\neg\lambda} \in E \qquad [\lambda]^m}{\dfrac{\bot}{\neg\lambda} \to I, m} \to E}{\dfrac{\neg\lambda}{\lambda} \in I}}{\lambda} \to E}{\bot}$$

- By applying an implication reduction $\rightsquigarrow_{\to}$, one obtains the following

$$\dfrac{\dfrac{\dfrac{\dfrac{[\lambda]^n}{\neg\lambda} \in E \qquad [\lambda]^n}{\dfrac{\bot}{\neg\lambda} \to I, n} \to E}{\dfrac{\neg\lambda}{\lambda} \in I}}{\dfrac{\lambda}{\neg\lambda} \in E} \qquad \dfrac{\dfrac{\dfrac{[\lambda]^m}{\neg\lambda} \in E \qquad [\lambda]^m}{\dfrac{\bot}{\neg\lambda} \to I, m} \to E}{\dfrac{\neg\lambda}{\lambda} \in I}}{\lambda} \to E}{\bot}$$

- By applying an $\in$ reduction $\rightsquigarrow_{\in}$, one obtains the first derivation.
  All possible reduction sequences loop.

Tennant on paradoxes

- Tennant [1982] considered a wide range of examples and showed that all prominent mathematical and logical paradoxes follow this pattern. The steps playing the role of $\in I$ and $\in E$ are called *id est* inferences, as they result from extra-logical principles.
- He conjectures that the reduction loops are the distinguishing feature of these paradoxes and proposes the test of non-terminating reduction sequences as the criterion for paradoxicality.

> *[...] enumerate proofs of absurdity; start normalizing those that are not in normal form; and check to see whether the reduction sequences ever enter loops, or manifest any other conclusive evidence that they will not terminate. As soon as a reduction sequence does enter a loop, or manifest such evidence, one can check off the proof concerned as a 'paradoxical' proof.*

- Tennant [1995] broadens the test to non-terminating reduction sequences, which covers paradoxes such as Yablo's.

Paradoxes *vs* inconsistencies

- Tennant [1982] distinguishes between the derivation of $\bot$ generated with the rules of λ, and other derivations of $\bot$, such as

$$\dfrac{\dfrac{A \wedge \neg A}{A} \quad \dfrac{A \wedge \neg A}{\neg A}}{\bot}$$

  which is already normal.

- Whereas a derivation of $\bot$ with a looping reduction sequence shows that the sentences involved in its *id est* inferences are paradoxical, a normalizable open derivation of $\bot$ shows the inconsistency of its assumptions.

  1. Non-normalizing derivations of $\bot$ (if no undischarged assumptions, then a *pure* paradox);
  2. Normal derivations of $\bot$ (the conclusion depends always on at least one open assumption).

Challenges to Tennant's view

1. Looping is not a *necessary* condition: von Plato [2000] / Rogerson [2007] (a normal paradox in classical logic).
2. Looping is not a *sufficient* condition: Schroeder-Heister/Tranchini [2016] (on Ekman's paradox). One must require:
   - that reduction does not trivializes identity of proof;
   - *id est* inferences provide an *isomorphism* between λ and $\neg\lambda$.

But...

1. Rogerson's counterexample only works in the $\vee, \exists$-free fragment of (classical) natural deduction (SH/Tranchini [2016]);
2. The isomorphism criterion presupposes that any paradox has the form of a Russell-like antinomy.

An amended criterion

- Following Tennant [2016], Russell's paradox... is not a paradox!

  > *[...] rules need to be parallelized, in order to enable the construction of a normal disproof for Russell's 'Paradox', thereby depriving it of the status of a genuine paradox.*

- On the other hand

  > *[...] the Liar Paradox remains genuinely paradoxical according to our current modification of my earlier proof-theoretic test - unlike Russell's Paradox.*

- The upshot: a neat distinction between set-theoretic and semantical paradoxes.

Outline

1 Proof theory and paradoxes

2 Paradoxes in normal form

3 Conclusion

Kreisel/Takeuti's trick

- Kreisel-Takeuti [1974]: a simple technique to swallow cuts in sequents of the form $\Gamma, \forall X(X \to X) \vdash \Delta$:

$$\dfrac{\Gamma, \forall X(X \to X) \vdash A, \Delta \qquad \Gamma', A \vdash \Delta'}{\Gamma, \Gamma', \forall X(X \to X) \vdash \Delta, \Delta'}\ cut$$

reduces to

$$\dfrac{\dfrac{\dfrac{\Gamma, \forall X(X \to X) \vdash A, \Delta \qquad \Gamma', A \vdash \Delta'}{\Gamma, \Gamma', \forall X(X \to X), A \to A \vdash \Delta, \Delta'}\ L\to}{\Gamma, \Gamma', \forall X(X \to X), \forall X(X \to X) \vdash \Delta, \Delta'}\ \forall L}{\Gamma, \Gamma', \forall X(X \to X) \vdash \Delta, \Delta'}\ C$$

In natural deduction

- In a ND setting, KT's trick allows to normalize derivations with *ad hoc* open assumptions:

$$\dfrac{\dfrac{\dfrac{[A]^n}{\vdots}}{\dfrac{B}{A \to B}\to I, n} \qquad \dfrac{\vdots}{A}}{B}\to E$$

$$\vdots$$

is transformed into

$$\dfrac{\dfrac{[A \to A]^m \qquad \dfrac{\vdots}{A}}{A}\to E}{\vdots}$$

$$B$$

$$\vdots$$

Back to paradoxes (I)

- Tennant's account retains two essential features of a paradoxical derivation. A paradox is a closed derivation $\mathcal{D}$ of the absurdity ($\bot$), such that:

  (i) $\mathcal{D}$ employs *id est* rules;

  (ii) $\mathcal{D}$ has no normal form.

- Condition (ii) is a straightforward consequence of the fact that Tennant characterizes a paradox as a derivation of absurdity whose reduction sequence does not terminate.

- Thus, following Tennant, the lack of a normal form is a necessary condition for paradoxicality.

Back to paradoxes (II)

- Because of (ii), a paradox contains at least one redundancy.

- Since all derivations in standard natural deduction without *id est* inferences have a normal form, the reduction of this redundancy must produce a derivation containing a redundancy which introduces and immediately eliminates a formula λ whose behavior is determined by the *id est* inferences in (i).

- Under Tennant's hypotheses, a paradoxical derivation $\mathcal{D}$, possibly after some standard reductions, can be depicted as follows:

$$\dfrac{\dfrac{\vdots}{\lambda}\,\lambda I}{\ddots\ \vdots\ \iddots}\,\lambda E$$
$$\bot$$

Paradoxes in normal form (I)

- Idea: by exploiting KT technique, a *normal* paradoxical derivation can be constructed.
- The critical redundancy formed by the *id est* rules can be blocked by introducing a new assumption of the form $\lambda \to \lambda$, a trivially valid formula. The new configuration is the following:

$$\dfrac{\dfrac{[\lambda \to \lambda]^n \qquad \dfrac{\vdots}{\lambda}\,\lambda I}{\lambda}\,\to E}{\dfrac{\lambda}{\vdots}\,\lambda E}$$
$$\dfrac{\bot}{(\lambda \to \lambda) \to \bot}\,\to I, n$$

- By discharging such an assumption at the end of the derivation via a $\to$-introduction rule, one obtains a closed derivation in normal form of a formula which is false in every interpretation (i.e. a contradiction).

Paradoxes in normal form (II)

- This technique can be applied uniformly to all the paradoxes analyzed by Tennant, both for derivations with standard and general (or parallelized) rules.
- In particular, it can be applied uniformly both to semantical and mathematical paradoxes, blurring the distinction between these two groups.
- This shows that there is a problem with Tennant's condition (*ii*): the lack of normal form cannot be considered as a necessary condition for characterizing a paradox.

An objection (I)

- The argument just presented does not go through when a more general notion of redundancy is considered.
- Tennant (1995) proposes adding, to the reductions eliminating redundancies

> *[...] other abbreviatory procedures σ, which have the general form of 'shrinking', to a single occurrence of A, any logically circular segments of branches (within the proof) of the form shown below to the left:*

$$\begin{array}{c} A \\ \hline B_1 \\ \vdots \\ \dfrac{B_n}{A} \end{array} \qquad \Rightarrow_\sigma \qquad A$$

> *One thereby identifies the top occurrence of A with the bottom occurrence of A, and gets rid of the intervening occurrences of $B_1, ..., B_n$, that form the filling of this unwanted sandwich.*

An objection (II)

- Under this stronger notion of reduction, the new derivation schema is not in normal form.
- Tennant's reduction can be applied to the subderivation of hypothesis and conclusion λ, reproducing the original derivation schema and causing the paradoxical derivation to enter a loop.
- Hence, Tennant's proof theoretic test would be satisfied also in this case.

A possible reply

- Tennant's reduction raise a question about
  - what constitutes a normal form beyond the standard logical systems;
  - what are the criteria for extending the reduction procedures.
- The objection just presented overlooks an important property of proofs.
- If reduction procedures are taken to eliminate inessential or redundant parts in the proof, it is natural to require that the *identity of proof* relation must be invariant with respect to them.
  - If two proofs can be reduced to the same proof by eliminating their redundancies, then those two proofs should be regarded as identical.
- It is a remarkable and well-known fact that the identity of proof relation arising from standard reductions in intuitionistic logic is not trivial, that is, it does not identify all proofs.

Trivialization of identity of proofs

- The identity of proof arising from Tennant's general reduction (also in the intuitionistic case) is trivial: Tennant's general reduction forces the identification of all derivations.

$$\dfrac{\dfrac{\vdots g}{B \to A} \qquad \dfrac{\vdots f \qquad \vdots p}{\dfrac{A \to B \qquad A}{B}}}{A} \quad \rightsquigarrow \quad \begin{array}{c} \vdots p \\ A \end{array}$$

- I.e. $g(f(p)) = p$, for all f and p.

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$$\dfrac{\dfrac{[A]^n}{A \to A}\, n \qquad \dfrac{\dfrac{\vdots u_1}{A} \qquad \dfrac{\vdots u_2}{A}}{A \to A} \qquad A}{A} \quad \rightsquigarrow \quad \begin{array}{c}\vdots u_1 \\ A\end{array} \;\nearrow \;\searrow\; \begin{array}{c}\vdots u_2 \\ A\end{array}$$

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- I.e. $g(f(p)) = p$, for all f and p.
- Thus, the portions of proofs which are eliminated by this procedure are not inessential nor redundant.
- Accepting the general reduction would preclude the very possibility of a proof theoretic analysis of paradoxes.

Normalization

- Add to usual natural deduction a new formula λ and
  - either assume that *every* derivation has an open assumption $[\neg\lambda \to \neg\lambda]^n$;
  - or take $\neg\lambda \to \neg\lambda$ as an axiom
- Then, given usual rules for λ:

$$\frac{\neg\lambda}{\lambda}\,\lambda I \qquad\qquad \frac{\lambda}{\neg\lambda}\,\lambda E$$

we can define an *ad hoc* reduction designed to block the detours on the formula $\neg\lambda$:

$$\dfrac{\dfrac{\dfrac{\vdots}{\neg\lambda}\,\lambda I}{\lambda}\,\lambda E}{\neg\lambda}\ \vdots \qquad \rightsquigarrow \qquad \dfrac{[\neg\lambda \to \neg\lambda] \qquad \dfrac{\vdots}{\neg\lambda}}{\neg\lambda}\to E \ \vdots$$

Normalizing Russell's paradox

$$\dfrac{\dfrac{\dfrac{[\lambda]^n}{\neg\lambda}\,\lambda E \qquad [\lambda]^n}{\bot}\to E}{\dfrac{\bot}{\neg\lambda}\to I, n} \qquad \dfrac{\dfrac{\dfrac{[\lambda]^m}{\neg\lambda}\,\lambda E \qquad [\lambda]^m}{\bot}\to E}{\dfrac{\neg\lambda}{\lambda}\,\lambda I}\to I, m$$

$$\bot \qquad \to E$$

Normalizing Russell's paradox

$$\dfrac{\dfrac{\dfrac{[\lambda]^n}{\neg\lambda}\,\lambda E \qquad [\lambda]^n}{\bot}\to E}{\dfrac{\bot}{\neg\lambda}\to I, n} \qquad \dfrac{\dfrac{\dfrac{[\lambda]^m}{\neg\lambda}\,\lambda E \qquad [\lambda]^m}{\bot}\to E}{\dfrac{\bot}{\neg\lambda}\to I, m}$$

$$\dfrac{\dfrac{\neg\lambda}{\lambda}\,\lambda I}{\bot}\to E$$

$$\dfrac{\bot}{(\neg\lambda\to\neg\lambda)\to\bot}\to I, p$$

Normalizing Russell's paradox

$$\dfrac{\dfrac{\dfrac{[\lambda]^n}{\neg\lambda}\,\lambda E \qquad [\lambda]^n}{\bot}\to E}{\dfrac{\bot}{\neg\lambda}\to I, n} \qquad \dfrac{\dfrac{\dfrac{[\lambda]^m}{\neg\lambda}\,\lambda E \qquad [\lambda]^m}{\bot}\to E}{\dfrac{\bot}{\neg\lambda}\to I, m}$$

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Normalizing Russell's paradox

$$\dfrac{\dfrac{\dfrac{\dfrac{[\lambda]^n}{\neg\lambda}\,\lambda E \qquad [\lambda]^n}{\bot}\to E}{\neg\lambda}\to I,n \qquad \dfrac{\dfrac{\dfrac{[\lambda]^m}{\neg\lambda}\,\lambda E \qquad [\lambda]^m}{\bot}\to E}{\neg\lambda}\to I,m}{\dfrac{[\neg\lambda\to\neg\lambda]^p \qquad \lambda}{\bot}\to E}\,\lambda I \qquad \to E$$

Normalizing Russell's paradox

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{[\lambda]^n}{\neg\lambda}\,\lambda E \qquad [\lambda]^n}{\bot}\to E}{\neg\lambda}\to I,n \qquad \dfrac{\dfrac{\dfrac{[\lambda]^m}{\neg\lambda}\,\lambda E \qquad [\lambda]^m}{\bot}\to E}{\neg\lambda}\to I,m}{\dfrac{[\neg\lambda\to\neg\lambda]^p \qquad \lambda}{\bot}\to E}\,\lambda I \qquad \to E}{(\neg\lambda\to\neg\lambda)\to\bot}\to I,p$$

An analysis of the Liar with GER

- Given I/GE rules for the truth predicate T:

$$\frac{\phi}{T\phi}\,T_I \qquad\qquad \frac{T\phi \qquad \begin{array}{c}[\phi]^n\\ \vdots\\ \theta\end{array}}{\theta}\,T_E, n$$

- and their associated reduction procedure

$$\dfrac{\begin{array}{c}\vdots\\ \phi\end{array} \qquad \begin{array}{c}[\phi]^n\\ \vdots\\ \theta\end{array}}{\theta} \quad\rightsquigarrow\quad \begin{array}{c}\vdots\\ \phi\\ \vdots\\ \theta\end{array}$$

An analysis of the Liar with GER

- The new I/GE *id est* rules for λ become

$$\frac{\begin{array}{c}[T\lambda]^n\\ \vdots\\ \bot\end{array}}{\lambda}\,\lambda_I, n \qquad\qquad \frac{\lambda \qquad \begin{array}{c}[\neg T\lambda]^n\\ \vdots\\ \theta\end{array}}{\theta}\,\lambda_E, n$$

- we can define an *ad hoc* reduction designed to block the detours on the formula $\neg T\lambda$:

$$\dfrac{\begin{array}{c}[T\lambda]^n\\ \vdots\\ \bot\end{array}}{\lambda} \qquad \begin{array}{c}[\neg T\lambda]^n\\ \vdots\\ \theta\end{array} \quad\rightsquigarrow_{KT}\quad \dfrac{[\neg T\lambda \rightarrow \neg T\lambda] \qquad \dfrac{\begin{array}{c}[T\lambda]^n\\ \vdots\\ \bot\end{array}}{\neg T\lambda}\,n \qquad \begin{array}{c}[\neg T\lambda]^m\\ \vdots\\ \theta\end{array}}{\theta}\,{\rightarrow}_{E(gen)}, m$$

Normalizing the Liar

$$\dfrac{\dfrac{[T\lambda]^o \qquad \dfrac{\dfrac{[\lambda]^n \qquad \dfrac{[\neg T\lambda]^m \qquad [T\lambda]^o}{\bot}\to_E}{\bot}\lambda_E, m}{\bot}T_E, n}{\dfrac{\bot}{\lambda}\lambda_I, o} \qquad \dfrac{[\neg T\lambda]^p \qquad \dfrac{\lambda}{T\lambda}T_I}{\bot}\to_E}{\bot}\lambda_E, p$$

Normalizing the Liar

Π $\nwarrow$

$$\dfrac{\dfrac{[T\lambda]^o \qquad \dfrac{\dfrac{[\lambda]^n \qquad \dfrac{[\neg T\lambda]^m \qquad [T\lambda]^o}{\bot}\to_E}{\bot}\lambda_E, m}{\bot}T_E, n}{\dfrac{\bot}{\lambda}\lambda_I, o} \qquad \dfrac{[\neg T\lambda]^p \qquad \dfrac{\lambda}{T\lambda}T_I}{\bot}\to_E}{\bot}\lambda_E, p$$

Normalizing the Liar

$$\dfrac{[\neg T\lambda \to \neg T\lambda] \qquad \dfrac{\dfrac{\dfrac{[\neg T\lambda]^m \qquad [T\lambda]^o}{\bot}\to_E}{\lambda_E, m}\qquad [\lambda]^n}{\bot}}{\dfrac{\bot}{\neg T\lambda}\to_I, o}\; T_E, n \qquad \dfrac{[\neg T\lambda]^p \qquad \dfrac{\Pi}{T\lambda}\, T_I}{\bot}\to_E}{\bot}\to_{E(gen)}, p$$

Evaluating the amended criterion

- The schema for the generalized KT reduction is

$$\begin{array}{c}\vdots\\ \dfrac{\neg\lambda}{\lambda}\lambda I\\ \dfrac{\lambda}{\neg\lambda}\lambda E\\ \vdots\\ \bot\end{array} \quad \rightsquigarrow \quad \dfrac{[\lambda\to\lambda] \qquad \begin{array}{c}[\lambda]^n\\ \vdots\\ \lambda\end{array} \qquad \begin{array}{c}[\lambda]^n\\ \vdots\\ \bot\end{array}}{\bot}\to E(gen), n$$

- It is not possible to apply the generalized reduction $\rightsquigarrow_\sigma$ to this derivation schema: the two derivations of λ are unrelated.
- Hence
  - Tennant's amended test fails;
  - This version of the paradox is immune to the objection on generalized reduction;
  - The distinction between semantic and mathematical paradoxes is blurred.

Properties of the KT reduction

The system $\mathcal{NJ} + \{A \to A | A \textit{ formula}\}$ is strongly normalizing in linear time with the following reduction:

$$\dfrac{\dfrac{\begin{array}{c}[A]^n\\ \vdots\\ B\end{array}}{A \to B} \to I, n \qquad \begin{array}{c}\vdots\\ A\end{array}}{B} \to E \quad \rightsquigarrow \quad \dfrac{\dfrac{\begin{array}{c}\vdots\\ A \to A\end{array} \qquad \begin{array}{c}\vdots\\ A\end{array}}{A} \to E}{\begin{array}{c}\vdots\\ B\end{array}}$$

- From the Curry-Howard point of view, this corresponds to adding new *constants* $k_A : A \to A$ with the reduction rule

$$(\lambda x^A.t^B)u^A \rightsquigarrow t^B[(k_A)u^A/x]$$

- This modification is *ad hoc* but is innocuous in the following sense:
  - from a logical point of view, we just add some valid formulas as axioms, hence consistency is preserved;
  - from a computational point of view, we add a reduction which does not violate normalization nor computational consistency:identity of proofs is not trivialized.

32 / 36

Outline

1 Proof theory and paradoxes

2 Paradoxes in normal form

3 Conclusion

33 / 36

Taking stock

- We have shown that Tennant's account faces a dilemma:
  1. either one denies that the closed normal derivation obtained by the construction just presented is a genuine paradox,
  2. or one must acknowledge that there is a problem with Tennant's characterization of paradoxes.
- (1) seems difficult to maintain: the conclusion $(\lambda \to \lambda) \to \bot$ of the new derivation schema is intuitionistically (and classically) equivalent to the absurdity.
  We conclude that the lack of normal form cannot be considered as a necessary condition for characterizing a paradox.
- An objection to the fact that the new derivation schema can be considered in normal form led us to consider a general problem about the criteria for defining normal forms and reductions beyond standard logical systems.
- We investigated the KT reduction both for standard and GE rules.
  We showed that such a reduction avoids loops but still does not trivialize the identity of proof relation.

Solving the paradox? (I)

- One can wonder whether these normal paradoxes can be ruled out by considerations in terms of *proof theoretic validity* (Prawitz [1971])
- One must consider
  - either validity for open derivations: in this case one enters into a loop in the justification procedure, as the reduction always produces *new* open assumptions to be closed...hence one cannot prove its validity.
  - or validity for derivations containing axioms: one looses introduction property but the paradox becomes valid!
- Hence it seems that considerations in terms of validity do not help solving this issue.

Solving the paradox? (II)

- Restrict identity axioms to atomic formulas (PSH).
  Issue of "naturality": $A \to A$ where A has n atoms, a trivial proof would take (almost) n steps.
- Following the criterion put forward in Schroeder-Heister/Tranchini [2016], our "paradoxes in normal form" are not genuine paradoxes (the isomorphism condition is not satisfied).
  - If looping is all that matters in order to characterize a paradox, a pathological example can be produced. Therefore some other property (e.g. isomorphism) must be invoked in order to reject our example and characterize genuine paradoxes.
- The isomorphism condition presupposes that the *id-est* inferences have a very specific form: the (unique) premiss of the introduction rule must be the conclusion of the elimination rule.
- These normal paradoxes seem a "stable anomaly" in the proof-theoretic analysis of paradoxes.